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$\mathbb{Q}$-UNIVERSAL DESINGULARIZATION*

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To Professor Heisuke Hironaka with warmest wishes on his eightieth birthday

Abstract. We prove that the algorithm for desingularization of algebraic varieties in characteristic zero of the first two authors is functorial with respect to regular morphisms. For this purpose, we show that, in characteristic zero, a regular morphism with connected affine source can be factored into a smooth morphism, a ground-field extension and a generic-fibre embedding. Every variety of characteristic zero admits a regular morphism to a $\mathbb{Q}$-variety. The desingularization algorithm is therefore $\mathbb{Q}$-*universal* or *absolute* in the sense that it is induced from its restriction to varieties over $\mathbb{Q}$. As a consequence, for example, the algorithm extends functorially to localizations and Henselizations of varieties.

Key words. Resolution of singularities, functorial, canonical, marked ideal.

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1. Introduction. Our main result is the following.

THEOREM 1.1. *Every algebraic variety in characteristic zero admits (strong) resolution of singularities that is functorial with respect to regular morphisms.*

More precisely, we show that the desingularization algorithm of [BM2, BM4] is functorial with respect to regular morphisms. (See Theorem and Addendum 6.1 below for a precise statement of Theorem 1.1. "Strong" means in particular that the desingularization is by blowings-up along smooth subvarieties.) The assertion of Theorem 1.1 is called $\mathbb{Q}$-*universal* resolution of singularities by Hironaka [Hi] because any algebraic variety X in characteristic zero admits a regular morphism to a variety Y defined over the rational numbers $\mathbb{Q}$ (see Theorem 3.1 below), so that resolution of singularities of X is induced by that of Y. In [Hi], Hironaka writes that $\mathbb{Q}$-universal desingularization will be proved in a subsequent paper, but a proof has not appeared before as far as we know (see Remark 5.5).

An *(algebraic) variety* means a scheme X which admits a morphism of finite type $X \to \operatorname{Spec} \underline{k}$, where $\underline{k}$ is a field. (It will be convenient to extend this definition to schemes that are finite disjoint unions of such; see Remarks 4.7(2).) If a morphism $X \to \operatorname{Spec} \underline{k}$ is fixed, we will say that X is a variety with *ground field* $\underline{k}$, or a $\underline{k}$*-variety*.

A morphism of schemes $f : X \to Y$ is *regular* if f is flat and all fibres of f are *geometrically regular*; equivalently, if f is flat and, for every morphism $T \to Y$ of finite type, all fibres of $X \times_Y T \to T$ are regular [Ma, §§28, 33]. If f is of finite type, then f is regular if and only if it is smooth [Ha, Thm. 10.2]. Thus regularity is a generalization of smoothness to morphisms that are not necessarily of finite type.

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Theorem 1.2. *A regular morphism $f : X \to Y$, where X is a connected affine variety and Y is a variety over a field $\underline{k}$ of characteristic zero, can be factored as*

$$X \cong Z_\eta \times_{\operatorname{Spec} \underline{m}} \operatorname{Spec} \underline{l} \xrightarrow{f_{\underline{l}}} Z_\eta \xrightarrow{f_{\underline{m}}} Z \xrightarrow{f_{\underline{k}}} Y, \tag{1.1}$$

where $f_{\underline{k}}$ is a smooth morphism of $\underline{k}$-varieties, $f_{\underline{l}}$ is a ground-field extension and $f_{\underline{m}} : Z_\eta \to Z$ is a generic-fibre embedding.

A *generic-fibre embedding* $f_{\underline{m}} : Z_\eta \to Z$ means there is a dominant $\underline{k}$-morphism $Z \to T$ to an integral $\underline{k}$-variety T, and $f_{\underline{m}} : Z_\eta \to Z$ is the canonical morphism from the *generic fibre* $Z_\eta = Z \times_T \eta$ (where $\eta = \operatorname{Spec} \underline{m}$ is the generic point of T. See Sections 2 and 4.)

For example, $\operatorname{Spec} \mathbb{Q}(x)[y] \to \operatorname{Spec} \mathbb{Q}[x, y]$ is a generic-fibre embedding; it is a regular morphism that is not a composite of smooth morphisms and ground-field extensions.

Functoriality with respect to smooth morphisms and ground field extensions in the strong desingularization algorithm for varieties [BM2] is proved in [BM4]. ([W] and [K] provide versions of weak desingularization of varieties that are also functorial with respect to smooth morphisms and ground field extensions.) In Section 6, we deduce Theorem 1.1 from the previous results, using Theorem 1.2 and functoriality with respect to generic-fibre embeddings (see §4.3 and Proposition 6.3).

Note, however, that all previous results on functoriality seem to make a tacit assumption that the smooth morphisms have constant relative dimension. We impose no such restriction in Theorem 1.1, so we also have to show that the desingularization algorithms of [BM2, BM4] are functorial with respect to arbitrary smooth morphisms. We are grateful to Ofer Gabber for raising this issue; see §6.3.

Functorial desingularization involves important local–global issues. For example, even if a variety has several connected components (so that resolutions of singularities of different components are independent), functoriality depends on the order in which the components are blown up. Such issues intervene throughout the article (see §4.3 and Remarks 5.1, 6.2).

The algorithm for strong resolution of singularities of [BM2, BM4] is based on a desingularization algorithm for a *marked ideal* (as presented in [BM4]). The proofs of the theorems involve a notion of *equivalence* of marked ideals. (The meanings of these notions are recalled in §5 below; for details we refer to [BM4].)

Theorem 1.3. *The algorithm for resolution of singularities of marked ideals in characteristic zero (of [BM4]) is functorial with respect to equivalence classes (of marked ideals of a given dimension; see §5.1) and with respect to regular morphisms.*

In Section 5, we obtain Theorem 1.3 from previous functoriality results (with respect to smooth morphisms and ground field extensions) again using Theorem 1.2, §4.3 and functoriality with respect to generic-fibre embeddings (Propostion 5.3). If $Z_\eta \to Z$ is a generic-fibre embedding in characteristic zero, where Z is smooth, then equivalent marked ideals on Z pull back to equivalent marked ideals on Z_η (see Lemma 5.2).

An algorithm for principalization of an ideal that is functorial with respect to regular morphisms also follows from Theorem 1.3.

For Proposition 5.3, we follow the proof in [BM4] step-by-step. The only point that is not immediate involves passage from a marked ideal to a (local) *coefficient ideal* (Step I in [BM4]). Suppose that $\psi : Z_\eta \to Z$ is a generic-fibre embedding as above, where Z_η, Z are smooth. Let $\underline{\mathcal{I}}_\eta$ denote the pullback $\psi^*(\underline{\mathcal{I}})$ to Z_η of a marked ideal $\underline{\mathcal{I}}$ on Z. It follows from Lemma 5.1 that the coefficient ideal of $\underline{\mathcal{I}}$ pulls back to a marked ideal which is equivalent to $\underline{\mathcal{I}}_\eta$ (see Lemma 5.6).

REMARK 1.4. To explain the significance of the latter, let us recall that the local coefficient ideal for $\underline{\mathcal{I}}$ is defined using ideals of derivatives of $\underline{\mathcal{I}}$. If $\mathcal{I} \subset \mathcal{O}_X$ is a coherent ideal on a regular variety X, then the *derivative ideal* $\mathcal{D}(\mathcal{I})$ is generated by all first derivatives of local sections of $\mathcal{I}$. If X is an $\underline{m}$-variety, this means that $\mathcal{D}(\mathcal{I})$ is the image of the natural morphism $\mathrm{Der}_X \times \mathcal{I} \to \mathcal{O}_X$, where Der_X denotes the sheaf of $\underline{m}$-derivations $\mathrm{Der}_{\underline{m}}(\mathcal{O}_X, \mathcal{O}_X)$; i.e., $\underline{m}$-linear homomorphisms $\mathcal{O}_X \to \mathcal{O}_X$ that satisfy Leibniz's rule (hence vanish on $\underline{m}$).

A coefficient ideal of a marked ideal on Z involves $\underline{k}$-derivations, while on Z_η a coefficient ideal involves $\underline{m}$-derivations. Derivative ideals defined using $\underline{k}$-derivations (or $\mathbb{Q}$-derivations) may be much larger than those defined over $\underline{m}$ because they involve derivatives along "constants" (elements of $\underline{m}$ that are transcendental over $\underline{k}$ or $\mathbb{Q}$). $\mathbb{Q}$-*universal resolution of singularities* means that the derivative ideals defined using $\mathbb{Q}$-derivations nevertheless do not result in smaller centres of blowing up, so we can run the desingularization algorithm for a variety defined over a field or characteristic zero, in general, using derivatives defined over $\mathbb{Q}$.

In Section 7, we present an alternative (though less explicit) approach to universal desingularization algorithms based on approximation methods of [EGA IV, §8] that we use in our proof of the factorization theorem 1.2. We can start with any desingularization algorithm for varieties over $\mathbb{Q}$ that is functorial with respect to smooth morphisms, and extend it to a class of schemes over $\mathbb{Q}$ that includes all varieties of characteristic zero as well as their localizations and Henselizations along closed subvarieties. The resulting desingularization algorithm is again functorial with respect to regular morphisms.

2. The generic fibre. Let $\pi : Z \to T$ denote a dominant morphism of $\underline{k}$-varieties, where $\underline{k}$ is a field and T is integral. Let η denote the generic point of T; i.e., $\eta = \operatorname{Spec} \underline{m}$, where $\underline{m}$ is the function field $K(T)$ of T. There is a fibred-product diagram

$$\begin{array}{ccc} Z \times_T \eta & \xrightarrow{\psi} & Z \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \eta & \longrightarrow & T \end{array}$$

in which all morphisms are dominant. The *generic fibre* Z_η of π denotes the $\underline{m}$-variety $Z \times_T \eta$.

Suppose that Z and T are affine. Then $\underline{m}$ is the field of fractions $\underline{k}(T)$ of the coordinate ring $\underline{k}[T]$; by definition, $\underline{k}(T)$ is the localization $\underline{k}[T]_S$, where $S = S(T)$ is the multiplicative subset $\underline{k}[T]\backslash\{0\}$ of $\underline{k}[T]$. The morphism π induces an injection $\underline{k}[T] \hookrightarrow \underline{k}[Z]$, so that the coordinate ring of Z_η,

$$\underline{m}[Z_\eta] = \underline{k}(T) \otimes_{\underline{k}[T]} \underline{k}[Z] \cong \underline{k}[Z]_S.$$

REMARK 2.1. We recall that if A is a ring and S is a multiplicative subset of A, then an ideal of A_S is prime if and only if it is of the form $\mathfrak{p} \cdot A_S$, where $\mathfrak{p}$ is a prime ideal of A disjoint from S.

If $a \in Z$, we write $\mathfrak{m}_{Z,a}$ for the maximal ideal of $\mathcal{O}_{Z,a}$ and $\kappa(a)$ for the residue field $\mathcal{O}_{Z,a}/\mathfrak{m}_{Z,a}$. Let b be a point of Z_η and let $a = \psi(b) \in Z$. (a is not necessarily closed, even if b is closed.) Let $\psi_b^* : \mathcal{O}_{Z,a} \to \mathcal{O}_{Z_\eta,b}$ denote the homomorphism of local rings induced by ψ.

If T is affine as above, then $\mathcal{O}_{Z_\eta,b}$ can be identified with the localization $(\mathcal{O}_{Z,a})_S$, where $S = S(T)$; thus $\mathfrak{m}_{Z,a} \cap S = \emptyset$, by Remark 2.1, so that $\mathcal{O}_{Z_\eta,b} \cong (\mathcal{O}_{Z,a})_S = \mathcal{O}_{Z,a}$. Therefore, (1) $\kappa(a) = \kappa(b)$; (2) $\mathcal{O}_{Z_\eta,b}$ is regular if and only if $\mathcal{O}_{Z,a}$ is regular; (3) if $f \in \mathcal{O}_{Z,a}$, then the order $\operatorname{ord} f = \operatorname{ord} \psi_b^*(f)$.

Suppose that W is an integral subvariety of Z. It follows from Remark 2.1 that $\pi|_W : W \to T$ is dominant if and only if there is a point $b \in Z_\eta$ such that W is the closure $\overline{a}$ of $a = \psi(b)$.

Suppose that b is a closed point of Z_η. Then $\kappa(b)/\underline{m}$ is a finite field extension, by the Nullstellensatz [E, Th. 4.19]. Let $a = \psi(b)$ and let $W = \overline{a}$. Then $\kappa(b) = \kappa(a) = K(W)$, so that $K(W)/K(T)$ is a finite extension, and $\pi|_W$ is a generically finite morphism.

In general, if $\pi|_W$ is not dominant, then $\psi^{-1}(W) = \emptyset$, and if $\pi|_W$ is dominant, then $\psi^{-1}(W) = W_\eta$. In the latter case, if $\underline{k}$ is perfect, then W_η is smooth if and only if there is an open subset of T over which W is smooth (and the restriction of π is a smooth morphism).

3. Embedding as the generic fibre of a $\mathbb{Q}$-variety. Every variety X in characteristic zero can be obtained by a base change from a variety which admits a generic fibre embedding $Z_\eta \to Z$ into a variety Z over $\mathbb{Q}$. This is a well-known result which is a special case of Theorem 1.2. We outline a proof for completeness and also to illustrate a technique developed in great generality by Grothendieck [EGA IV, Th. 8.8.2, Prop. 8.13.1] that we will use to prove Theorem 1.2.

THEOREM 3.1. *Let X denote a variety over a field $\underline{k}$ of characteristic zero. Then there exists a $\mathbb{Q}$-variety Z and a dominant morphism $\pi : Z \to T$, where T is an integral $\mathbb{Q}$-variety, such that X is obtained from the generic fibre Z_η of π by base extension; i.e., $X = Z_\eta \times_{\operatorname{Spec} \underline{m}} \operatorname{Spec} \underline{k}$, where $\underline{m} = K(T)$. If X is smooth, then we can take Z smooth.*

Proof. Our $\underline{k}$-variety X can be constructed by glueing together finitely many affine varieties X_i along open subsets. (See [Ha, Ch. II, Exercise 2.12].) Let $\underline{m}$ denote the subfield of $\underline{k}$ obtained by extending $\mathbb{Q}$ by the coefficients of the polynomials comprising (finite) generating sets for the ideals I_i, where $X_i = \operatorname{Spec} \underline{k}[y]/I_i$, for all i, together with the coefficients of the polynomials needed to present the glueing data. In other words: Let $\{c_j\} \subset \underline{k}$ denote the (finite) set of all coefficients above, and consider the ring homomorphism $\gamma : \mathbb{Q}[x] \to \underline{k}$ given by $\gamma(x_j) = c_j$, where $\mathbb{Q}[x]$ denotes the ring of polynomials over $\mathbb{Q}$ in indeterminates x_j. The $\ker \gamma$ is a prime ideal $\mathfrak{p}$, and $\underline{m}$ denotes the field of fractions of $\mathbb{Q}[x]/\mathfrak{p}$.

The field $\underline{m}$ is an extension of $\mathbb{Q}$ of finite type. Our variety X can be considered also as a variety $Z_{\underline{m}}$ defined over $\underline{m}$. (As a $\underline{k}$-variety, X is obtained from $Z_{\underline{m}}$ by base extension $\operatorname{Spec} \underline{k} \to \operatorname{Spec} \underline{m}$.)

Let $T := \operatorname{Spec} \mathbb{Q}[x]/\mathfrak{p}$. For each affine chart $X_i = \operatorname{Spec} \underline{k}[y]/I_i$ above, let $J_i \subset \mathbb{Q}[x, y]$ denote the ideal with generators obtained from those above by replacing each coefficient c_j by x_j. Then there is a $\mathbb{Q}$-variety Z constructed by glueing together the affine varieties $Z_i = \operatorname{Spec} \mathbb{Q}[x, y]/(J_i, \mathfrak{p})$ (where $(J_i, \mathfrak{p})$ denotes the ideal generated by J_i and $\mathfrak{p}$) using glueing morphisms obtained in the same way from those for $X = \cup X_i$.

Clearly, there is a dominant morphism $\pi : Z \to T$, and $Z_{\underline{m}}$ can be identified with the generic fibre Z_η of π (where $\eta = \operatorname{Spec} \underline{m}$ is the generic point of T.)

If X is smooth, then $Z_{\underline{m}}$ is smooth. The variety Z is *a priori* singular, but we can restrict to an open subset of T over which it is smooth (as in §2). □

4. Factorization of a regular morphism.

4.1. Ground fields. A variety X may admit many different structures of a $\underline{k}$-variety (i.e., many different morphisms of finite type $f : X \to \operatorname{Spec} \underline{k}$, even for a fixed field $\underline{k}$). This is usually the case when X is not reduced. (A simpler possibility is that $\underline{k}$ is finite over a subfield isomorphic to $\underline{k}$ itself). Nevertheless, a connected reduced variety possesses a unique maximal ground field, so there is a natural choice of ground field.

LEMMA 4.1. *Let X be a connected reduced variety. Then the ring $\mathcal{O}_X(X)$ contains a maximal subfield $\underline{k}$ containing any other subfield. In particular, any morphism $X \to \operatorname{Spec} \underline{l}$, where $\underline{l}$ is a field, factors through the morphism $X \to \operatorname{Spec} \underline{k}$ corresponding to the embedding $\underline{k} \hookrightarrow \mathcal{O}_X(X)$.*

Proof. Let $\mathbb{F}$ be the prime subfield contained in $\mathcal{O} := \mathcal{O}_X(X)$ and let $\underline{k}$ be the set of elements $f \in \mathcal{O}$ such that $\mathcal{O}$ contains the subfield $\mathbb{F}(f)$. It suffices to prove that $\underline{k}$ is a subfield of $\mathcal{O}$, since it is then clear that $\underline{k}$ is as required. Fix a structure $X \to \operatorname{Spec} \underline{l}$ of an $\underline{l}$-variety on X. Then any element $f \in \mathcal{O}$ induces a morphism $F : X \to \mathbb{A}^1_{\underline{l}} = \operatorname{Spec} \underline{l}[T]$ whose image Z is constructible, by Chevalley's theorem [Ha, Ex. 3.19]. Since Z is connected, either it is a point or it omits at most finitely many points. In the latter case, $f \notin \underline{k}$ because $\mathbb{F}[T]$ has infinitely many primes. On the other hand, in the first case, the map F is constant on X and equal to an element of the algebraic closure of $\underline{l}$, hence f annihilates an irreducible polynomial over $\underline{l}$ and so $\underline{l}[f]$ is a subfield of $\mathcal{O}$. This proves that F is constant if and only if $f \in \underline{k}$. It follows that $\underline{k}$ is a ring, and we have also seen that $\underline{l}[f]$ is a field for any element $f \in \underline{k}$. Thus $\underline{k}$ is a field as required. (We have actually proved that it coincides with the integral closure of $\underline{l}$ in $\mathcal{O}$.) □

The lemma following shows that to a general connected affine variety X, we can assign a uniquely defined maximal ground field $\underline{k}$ just by taking the maximal ground field of its reduction. However, in sharp contrast to the reduced case above, the structure morphism $X \to \operatorname{Spec} \underline{k}$ is absolutely not canonical. In particular, non-isomorphic $\underline{k}$-varieties can be isomorphic as abstract schemes.

LEMMA 4.2. *Let X be an affine variety with reduction X_0. Then any morphism $f_0 : X_0 \to \operatorname{Spec} \underline{k}$, where $\underline{k}$ is a field, extends to a morphism $f : X \to \operatorname{Spec} \underline{k}$ in the sense that f_0 is the composition of f with the reduction morphism $X_0 \to X$. Moreover, suppose that $\underline{l}$ is a subfield of $\underline{k}$ such that $\underline{k}/\underline{l}$ is separable, and fix an extension $f' : X \to \operatorname{Spec} \underline{l}$ of the morphism $f'_0 : X_0 \to \operatorname{Spec} \underline{k} \to \operatorname{Spec} \underline{l}$. Then we can choose f compatible with f'.*

Proof. Let $\mathbb{F}$ be the prime subfield of $\underline{k}$. Then $\underline{k}/\mathbb{F}$ is separable and X admits a unique morphism to $\operatorname{Spec} \mathbb{F}$. Let $\underline{l}$ be a subfield of $\underline{k}$ such that $\underline{k}/\underline{l}$ is separable (for example, $\underline{l} = \mathbb{F}$ to get the first assertion of the lemma). Since $\underline{k}/\underline{l}$ is separable, the morphism $\operatorname{Spec} \underline{k} \to \operatorname{Spec} \underline{l}$ is formally smooth, by [Ma, Prop. 28.I]. In particular, the $\underline{l}$-morphism $X_0 \to \operatorname{Spec} \underline{k}$ extends to an $\underline{l}$-morphism $X \to \operatorname{Spec} \underline{k}$. □

Combining Lemmas 4.1 and 4.2, we obtain the following corollary.

Corollary 4.3. *Let X be a connected affine variety. Then any morphism $X \to \operatorname{Spec} \underline{l}$, where $\underline{l}$ is a perfect field, factors through a morphism $X \to \operatorname{Spec} \underline{k}$ of finite type, where $\underline{k}$ is a field.*

The following example shows why we have to assume that X is connected in Corollary 4.3.

Example 4.4. Let $\underline{k}$ be a field which admits an endomorphism $\varphi : \underline{k} \to \underline{k}$ with $[\underline{k} : \varphi(\underline{k})] = \infty$; for example, $\underline{k} = \mathbb{C}$. Set $S := \operatorname{Spec} \underline{k}$ and let $X = S_1 \coprod S_2$ be the disjoint union of two copies of S. Then the morphism $X \to S$ which restricts to the identity on S_1 and to $\operatorname{Spec} \varphi$ on S_2 cannot be factored as in Corollary 4.3.

4.2. Regular morphisms. The assertion of our factorization theorem 1.2 is included in the following result.

Theorem 4.5. *Let $\underline{k}$ denote a perfect field, and let $f : X \to Y$ denote a morphism, where X is a connected affine variety and Y is a $\underline{k}$-variety.*

1. *Then f can be factored as*

$$X \cong Z_\eta \times_{\operatorname{Spec} \underline{m}} \operatorname{Spec} \underline{l} \xrightarrow{f_{\underline{l}}} Z_\eta \xrightarrow{f_{\underline{m}}} Z \xrightarrow{f_{\underline{k}}} Y, \tag{4.1}$$

   where $f_{\underline{k}}$ is a morphism of $\underline{k}$-varieties, $f_{\underline{l}}$ is a ground-field extension and $f_{\underline{m}}$ is a generic-fibre embedding.
2. *Assume that* $\operatorname{char} \underline{k} = 0$*. Then f is regular if and only if the morphism $f_{\underline{k}}$ in (4.1) is smooth on a neighbourhood U of Z_η. (So, if f is regular, then we get (4.1) with f_k smooth by restricting to U.)*

Proof. By Corollary 4.3 the morphism $X \xrightarrow{f} Y \to \operatorname{Spec} \underline{k}$ extends to a morphism $X \to \operatorname{Spec} \underline{l}$ of finite type. We construct Z_η by approximating X with a variety defined over a finitely generated $\underline{k}$-field $\underline{m}$. The $\underline{k}$-scheme $\operatorname{Spec} \underline{l}$ is the projective limit of the $\underline{k}$-schemes $\operatorname{Spec} \underline{m}_i$ where $\underline{m}_i$ runs over all subfields of $\underline{l}$ that contain $\underline{k}$ and are finitely generated over $\underline{k}$. By [EGA IV, Thm. 8.8.2(ii)], there exist $\underline{m} = \underline{m}_i$ and an $\underline{m}$-variety Z_η which induces X in the sense that $X \xrightarrow{\cong} Z_\eta \otimes_{\underline{m}} \underline{l}$. (The latter is an abbreviation for $Z_\eta \times_{\operatorname{Spec} \underline{m}} \operatorname{Spec} \underline{l}$.) Moreover, X is the projective limit of the $\underline{k}$-schemes $X \otimes_{\underline{m}} \underline{m}_i$, for the $\underline{m}_i$ which contain $\underline{m}$. By [EGA IV, Prop. 8.13.1], after replacing $\underline{m}$ with a larger m_i if necessary, there is a $\underline{k}$-morphism $g : Z_\eta \to Y$ which induces the natural $\underline{k}$-morphism $X \to Y$ in the sense that the latter factors through g. In particular, we obtain a factorization $X \xrightarrow{\cong} Z_\eta \otimes_{\underline{m}} \underline{l} \to Z_\eta \to Y$.

Now we construct Z by approximating Z_η with a $\underline{k}$-variety. Take an integral $\underline{k}$-variety M_0 with field of fractions $\underline{m}$. Then $\operatorname{Spec} \underline{m}$ is the projective limit of all open subvarieties $M_i \hookrightarrow M_0$. By [EGA IV, Thm. 8.8.2(ii)], there exist i and a morphism $Z_i \to M_i$ of finite type such that $Z_\eta = Z_i \times_{M_i} \operatorname{Spec} \underline{m}$; then Z_η is the projective limit of the schemes $Z_j = Z_i \times_{M_i} M_j$ for $M_j \hookrightarrow M_i$. Obviously, each morphism $Z_\eta \to Z_i$ is a generic-fibre embedding. By [EGA IV, Prop. 8.13.1], the $\underline{k}$-morphism $Z_\eta \to Y$ is induced by a morphism $Z \to Y$ for an appropriate choice of $Z = Z_j$, i.e. $Z_\eta \to Y$ factors through a morphism of $\underline{k}$-varieties $Z \to Y$. This proves (1).

Now we prove (2). Note that $f_{\underline{l}}$ is the base change obtained from $h : \operatorname{Spec} \underline{l} \to \operatorname{Spec} \underline{m}$; hence $f_{\underline{l}}$ is faithfully flat and $f_{\underline{l}}$ is regular if and only if h is regular. The latter condition is automatic in characteristic zero. Note also that $f_{\underline{m}}$ is regular because it is a *pro-open immersion* in the sense of [T1, §2.1] (i.e. $f_{\underline{m}}$ is a projective limit of open immersions; in particular, it is injective and $\mathcal{O}_Z|_{Z_\eta} = \mathcal{O}_{Z_\eta}$) , and $f_{\underline{k}}$ is of finite type, hence it is regular if and only if it is smooth. Since $f = f_{\underline{k}} \circ f_{\underline{m}} \circ f_{\underline{l}}$ and regularity is preserved by composition [Ma, Lemma 33.B], we see that f is regular provided that $f_{\underline{k}}$ is smooth, and clearly then provided that $f_{\underline{k}}$ is smooth on a neighborhood of $Z_\eta \hookrightarrow Z$. Conversely, suppose that f is regular. By [Ma, Lemma 33.B], since $f_{\underline{l}}$ is faithfully flat, the morphism $Z_\eta \to Y$ is regular. Let $T \subset Z$ be the non-smooth locus of $f_{\underline{k}}$; then a point $z \in Z$ lies in T if and only if the morphism $f_z : \operatorname{Spec} \mathcal{O}_{Z,z} \to \operatorname{Spec} \mathcal{O}_{Y, f_{\underline{k}}(z)}$ is not regular. Consider $z \in Z_\eta$. Then the local ring of z in Z_η is the same as its local ring in Z, because $\mathcal{O}_{Z_\eta} = \mathcal{O}_Z|_{Z_\eta}$. Therefore, f_z is regular because $Z_\eta \to Y$ is regular. So $z \notin T$; thus Z_η is disjoint from the closed set T and $f_{\underline{k}}$ is smooth on $U := Z \setminus T$, as required. □

4.3. Application to functorial resolution of singularities.

COROLLARY 4.6. *A desingularization algorithm for algebraic varieties in characteristic zero is functorial with respect to regular morphisms if and only if it is functorial with respect to smooth morphisms, ground-field extensions and generic-fibre embeddings.*

REMARKS 4.7. (1) A *functorial* desingularization algorithm associates to a variety X a sequence of blowings-up $\mathcal{F}(X)$ with the property that, if $f : X \to Y$ is an allowed morphism (e.g., a regular morphism in Corollary 4.6), then the desingularization sequence $\mathcal{F}(Y)$ pulls back to $\mathcal{F}(X)$, after perhaps deleting isomorphisms in the pulled-back sequence *when f is not surjective.* (For example, if f is an open immersion, then the centre of a given blowing-up in the resolution sequence for Y may have no points over X. See also [T2, §§2.3.3–2.3.6].)

(2) Suppose that Y is a $\underline{k}$-variety, where $\operatorname{char} \underline{k} = 0$, and that $f : X \to Y$ is a regular morphism of varieties. Consider a finite covering $\{X_i\}$ of X by connected open affine subvarieties. For each i, let $\gamma_i : X_i \hookrightarrow X$ denote the inclusion and set $f_i := f \circ \gamma_i : X_i \to Y$. For each i, there is a morphism of finite type $X_i \to \operatorname{Spec} \underline{l}_i$ (by Corollary 4.3) and f_i factors according to Theorem 1.2; let us denote the factorization as

$$X_i \cong (Z_i)_{\eta_i} \times_{\operatorname{Spec} \underline{m}_i} \operatorname{Spec} \underline{l}_i \xrightarrow{(f_i)_{\underline{l}_i}} (Z_i)_{\eta_i} \xrightarrow{(f_i)_{\underline{m}_i}} Z_i \xrightarrow{(f_i)_{\underline{k}}} Y.$$

There are induced morphisms

$$\coprod_i X_i \to \coprod_i (Z_i)_{\eta_i} \to \coprod_i Z_i \to Y, \tag{4.2}$$

where $\coprod$ denotes disjoint union. The proof below involves functoriality with respect to these morphisms. But $\coprod_i (Z_i)_{\eta_i}$ is not necessarily a variety since it does not necessarily admit a morphism of finite type to Spec of a fixed field. It is therefore convenient to extend our desingularization theorems to schemes that are finite disjoint unions of varieties. (We will extend the use of *variety* to include such schemes.) The desingularization theorems of [BM4] extend trivially to this larger category.

Proof of Corollary 4.6. Assume we have a desingularization algorithm that is functorial with respect to smooth morphisms, ground-field extensions and generic-fibre embeddings. Let Y be a $\underline{k}$-variety, where $\operatorname{char} \underline{k} = 0$, and let $f : X \to Y$ denote a regular morphism of varieties. We have to show that the desingularization sequence for Y pulls back to that for X (modulo trivial blowings-up in the pull-back sequence; cf. Remarks 4.7(1)). We use the notation of Remarks 4.7(2). The morphism

$\gamma : \coprod X_i \to X$ induced by the inclusions $\gamma_i : X_i \hookrightarrow X$ is étale and surjective. By functoriality with respect to smooth (and hence, in particular, étale) morphisms, it is therefore enough to show that the desingularization algorithm commutes with pullback by the composite of the three morphisms in (5.2). This is true by the assumption. □

REMARK 4.8. Another approach to functoriality (and, in particular, to the assertion of Corollary 4.6) of origin in [BM2] involves proving a stronger desingularization theorem where the centres of blowings-up of a variety X are given by the maximum loci of an upper-semicontinuous *desingularization invariant* (see [BM4, §7]). Functoriality of the algorithm with respect to smooth morphisms, ground-field extensions and generic-fibre embeddings then implies functoriality with respect to regular morphisms, directly by Theorem 1.2.

Both Corollary 4.6 and Remark 4.8 have analogues for desingularization of marked ideals that can be obtained in the same way.

5. Functoriality of desingularization of a marked ideal with respect to generic-fibre embeddings. In this section we prove that the desingularization algorithm for marked ideals [BM4, §5] is functorial with respect to generic-fibre embeddings (Proposition 5.3 below). Theorem 1.3 then follows from Corollary 4.6 together with functoriality with respect to ground-field extensions and with respect to smooth morphisms ([BM4, §7]; see Remark 5.1 below).

Let $\pi : Z \to T$ denote a dominant morphism of $\underline{k}$-varieties, where $\underline{k}$ is a field of characteristic zero. Let $\psi : Z_\eta \to Z$ denote the embedding of the generic fibre of π, as in Section 2. (Z_η is an $\underline{m}$-variety, where $\underline{m} = K(T)$.) If $\mathcal{I} \subset \mathcal{O}_Z$ is an ideal (i.e., a coherent sheaf of ideals), let $\mathcal{I}_\eta \subset \mathcal{O}_{Z_\eta}$ denote the inverse image (pullback) $\psi^*(\mathcal{I})$. Then $\mathcal{I}_\eta$ is a coherent sheaf of ideals on Z_η.

Every subvariety of Z_η is of the form C_η, where C is a subvariety of Z such that $\pi|_C : C \to T$ is dominant. Moreover C_η is smooth if and only if there is an open subset of T over which C (and also $\pi|_C$) is smooth (Section 2). Let $\mathrm{Bl}_C Z \to Z$ denote the blowing-up with centre a subvariety C of Z. By functoriality of blowing-up with respect to flat base extension,

$$(\mathrm{Bl}_C Z)_\eta = \mathrm{Bl}_{C_\eta} Z_\eta \tag{5.1}$$

(where, if $\pi|_C$ is not dominant, then we understand $C_\eta = \emptyset$ and $\mathrm{Bl}_{C_\eta} Z_\eta = Z_\eta$).

5.1. Marked ideals. A *marked ideal* $\underline{\mathcal{I}}$ is a quintuple $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$, where: $Z \supset N$ are smooth varieties, $E = \sum_{i=1}^{s} H_i$ is a simple normal crossings divisor on Z which is tranverse to N and *ordered* (the H_i are smooth hypersurfaces in Z, not necessarily irreducible, with ordered index set as indicated), $\mathcal{I} \subset \mathcal{O}_N$ is an ideal, and $d \in \mathbb{N}$. The *cosupport* of $\underline{\mathcal{I}}$,

$$\operatorname{cosupp} \underline{\mathcal{I}} := \{x \in N : \operatorname{ord}_x \mathcal{I} \geq d\}.$$

We say that $\underline{\mathcal{I}}$ is of *maximal order* if $d = \max\{\operatorname{ord}_x \mathcal{I} : x \in \operatorname{cosupp} \underline{\mathcal{I}}\}$. The *dimension* $\dim \underline{\mathcal{I}}$ denotes $\dim N$.

A blowing-up $\sigma : Z' = \mathrm{Bl}_C Z \to Z$ (with smooth centre C) is *admissible* for $\underline{\mathcal{I}}$ if $C \subset \operatorname{cosupp} \underline{\mathcal{I}}$, and C, E have only normal crossings. The *(controlled) transform* of $\underline{\mathcal{I}}$ by an admissible blowing-up $\sigma : Z' \to Z$ is the marked ideal $\underline{\mathcal{I}}' = (Z', N', E', \mathcal{I}', d' = d)$, where N' is the strict transform of N by σ, $E' = \sum_{i=1}^{s+1} H_i'$, where H_i' denotes the strict transform of H_i, for each $i = 1, \ldots, s$, and $H_{s+1}' := \sigma^{-1}(C)$ (the exceptional divisor of σ, introduced as the last member of E'), and $\mathcal{I}' := \mathcal{I}_{\sigma^{-1}(C)}^{-d} \cdot \sigma^*(\mathcal{I})$ (where $\mathcal{I}_{\sigma^{-1}(C)} \subset \mathcal{O}_{N'}$ denotes the ideal of $\sigma^{-1}(C)$). In this definition, note that $\sigma^*(\mathcal{I})$ is divisible by $\mathcal{I}_{\sigma^{-1}(C)}^{d}$ and E' is a normal crossings divisor transverse to N', because σ is admissible.

We define a *resolution of singularities* of a marked ideal $\underline{\mathcal{I}}$ as a finite sequence of admissible blowings-up after which $\operatorname{cosupp} \underline{\mathcal{I}} = \emptyset$.

Let $\underline{\mathcal{I}}$ be a marked ideal as above, and let $\varphi : Y \to Z$ denote a regular morphism. We define the *inverse image* (or *pullback*) $\varphi^*(\underline{\mathcal{I}})$ as the marked ideal $(Y, \varphi^{-1}(N), \varphi^{-1}(E), \varphi^*(\mathcal{I}), d)$ (where $\varphi^{-1}(E)$ inherits the ordering of E). If $\psi : Z_\eta \to Z$ is a generic-fibre embedding as above, then $\varphi^*(\underline{\mathcal{I}}) = \underline{\mathcal{I}}_\eta$, where the latter denotes the marked ideal $(Z_\eta, N_\eta, E_\eta, \mathcal{I}_\eta, d)$. ($N_\eta$ is empty (so $\operatorname{cosupp} \underline{\mathcal{I}}_\eta = \emptyset$) unless $\pi|_N$ is dominant, and E_η is empty if no component of E dominates T.)

Remark 5.1. There are two proofs of functoriality of the desingularization algorithm for a marked ideal with respect to étale or smooth morphisms in [BM4, §7]. The first proof comes from Kollar [K, Prop. 3.37]. For this argument, we assume that our marked ideals are equidimensional and that the smooth or regular morphisms considered have constant relative dimension; it seems inconvenient to carry out the proof without these assumptions. The proof is by induction on dimension. It uses functoriality in the inductive step in a way that necessitates working with marked ideals $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$ where Z may have several components (cf. Remarks 4.7(2)). Although the blowings-up of different components are independent, a functorial algorithm depends on which we take first, second, etc. (In dimension 1, the algorithm dictates blowing up the points of maximum order of $\mathcal{I}$ at each step.)

The second proof of functoriality with respect to smooth morphisms involves proving the stronger desingularization theorem [BM4, Thm. 7.1], where the centres of blowing up are given by the maximum loci of an upper-semicontinuous *desingularization invariant* (cf. Remark 4.8). The invariant is defined by induction on $\dim \underline{\mathcal{I}}$. This functorial desingularization algorithm does not require an equidimensionality assumption on the marked ideals, and applies to smooth or regular morphisms that are not necessarily of constant relative dimension.

5.2. Test transformations and equivalence. Let $\underline{\mathcal{I}}$ denote a marked ideal as above. Sequences of test transformations are introduced to test for invariance of local numerical characters of $\underline{\mathcal{I}}$ (see [BM4, §6]). *Test transformations* are transformations of a marked ideal by morphisms of three possible kinds: admissible blowings-up, projections from products with an affine line, and exceptional blowings-up:

Product with a line. Let $Z' := Z \times \mathbb{A}^1$, and let $\pi : Z' \to Z$ denote the projection. We define the *transform* $\underline{\mathcal{I}}'$ of $\underline{\mathcal{I}}$ by π as the marked ideal $\underline{\mathcal{I}}' = (Z', N', E', \mathcal{I}', d' = d)$, where $N' := \pi^{-1}(N)$, $\mathcal{I}' := \pi^*(\mathcal{I})$, but $E' = \sum_{i=1}^{s+1} H_i'$, where $H_i' := \pi^{-1}(H_i)$, for each $i = 1, \dots, s$, and H_{s+1}' denotes the *horizontal divisor* $D := Z \times \{0\}$ (included as the last member of E').

Exceptional blowing-up. A blowing-up $\sigma : Z' \to Z$ is called an *exceptional blowing-up* for $\underline{\mathcal{I}}$ if its centre C is an intersection $H_i \cap H_j$ of distinct hypersurfaces $H_i, H_j \in E$. We define the *transform* $\underline{\mathcal{I}}' = (Z', N', E', \mathcal{I}', d')$ of $\underline{\mathcal{I}}$ by σ as the marked ideal in the same way as for an admissible blowing-up. (In the case of an exceptional blowing-up, $N' = \sigma^{-1}(N)$ and $\mathcal{I}' = \sigma^*(\mathcal{I})$.)

A *test sequence* for $\underline{\mathcal{I}}_0 = \underline{\mathcal{I}}$ means a sequence of morphisms

$$Z = Z_0 \xleftarrow{\sigma_1} Z_1 \longleftarrow \cdots \xleftarrow{\sigma_t} Z_t \,,$$

where each successive σ_{j+1} is either an admissible blowing-up, the projection from a product with a line, or an exceptional blowing-up.

We say that two marked ideals $\underline{\mathcal{I}}$ and $\underline{\mathcal{I}}_1$ (with the same ambient variety Z and the same normal crossings divisor E) are *equivalent* if they have the same test sequences (i.e., every test sequence for one is a test sequence for the other).

For example, coefficient ideals defined on different local maximal contact hypersurfaces for a marked ideal $\underline{\mathcal{I}}$ are equivalent because they are each equivalent to $\underline{\mathcal{I}}$ (see §5.4 below). This is the reason why it is very convenient, for the purpose of induction on dimension, to formulate a stronger statement of desingularization of marked ideals that includes functoriality with respect to equivalence classes (Theorem 1.3; see also [BM4, §7]).

LEMMA 5.2. *Let $Z_\eta \to Z$ denote a generic-fibre embedding as above. Suppose that $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$ and $\underline{\mathcal{J}} = (Z, P, E, \mathcal{J}, e)$ are marked ideals on Z. If $\underline{\mathcal{I}}$ is equivalent to $\underline{\mathcal{J}}$, then $\underline{\mathcal{I}}_\eta$ is equivalent to $\underline{\mathcal{J}}_\eta$.*

Proof. This follows directly from the definitions, together with the fact that any test sequence for $\underline{\mathcal{I}}_\eta$ lifts to a test sequence for $\underline{\mathcal{I}}$ over some neighbourhood of Z_η in Z (cf. (5.1)) and the fact that if $b \in N_\eta$ and $a = \psi(b)$, then $\mathrm{ord}_a \mathcal{I} = \mathrm{ord}_b \mathcal{I}_\eta$ (see Section 2). □

5.3. Functoriality with respect to generic-fibre embeddings.

PROPOSITION 5.3. *Let $\psi : Z_\eta \to Z$ denote a generic-fibre embedding and let $\underline{\mathcal{I}}$ denote a marked ideal on Z, as above. Then the sequence of blowings-up involved in the desingularization algorithm [BM4, §5] for $\underline{\mathcal{I}}_\eta$ is the pullback of the desingularization sequence for $\underline{\mathcal{I}}$.*

REMARK 5.4. In the blowing-up sequence for $\underline{\mathcal{I}}$, any centre of blowing up that does not dominate T pulls back to an empty centre, so that the corresponding blowing-up over Z_η is the identity morphism.

Proof of Proposition 5.3. The proof consists simply of following that in [BM4, §5] step-by-step. The proof is by induction on $\dim \underline{\mathcal{I}}$. We will not go through the entire process. The proof (or the algorithm) as presented in [BM4, §5] has two main steps, each of which involves an important construction: in Step I, passage from a marked ideal $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$ of maximal order to a *coefficient ideal* $\underline{\mathcal{C}}(\underline{\mathcal{I}})$ on an open subset U of Z, to decrease the dimension by 1 (for induction), and in Step II, passage from a general marked ideal $\underline{\mathcal{I}}$ to a *companion ideal* $\underline{\mathcal{G}}(\underline{\mathcal{I}})$ of maximal order, to reduce to Step I.

The companion ideal $\underline{\mathcal{G}}(\underline{\mathcal{I}})$ is constructed from a factorization of $\mathcal{I}$ into a *monomial ideal* (generated by a product of ideals of components of the exceptional divisor) and a *nonmonomial* or *residual ideal* (divisible by no such component). See [BM4, §5, Step II.B] for a precise definition.

It is easy to see from the definition that $\underline{\mathcal{G}}$ commutes with $\underline{\mathcal{I}} \mapsto \underline{\mathcal{I}}_\eta$; we leave the details to the reader. For $\underline{\mathcal{C}}$, commutativity with respect to $\underline{\mathcal{I}} \mapsto \underline{\mathcal{I}}_\eta$ is true only on the level of equivalence classes. This is proved in the following subsection. Our proposition follows from these two results. □

REMARK 5.5. Step II in [BM4, §5] involves proving that the equivalence class of $\underline{\mathcal{G}}(\underline{\mathcal{I}})$ depends only on the equivalence class (and dimension) of $\underline{\mathcal{I}}$. This result is proved using the fact that two local numerical characters of a marked ideal, $\mathrm{ord}_a \mathcal{I}/d$ and $\mathrm{ord}_{H,a} \mathcal{I}/d$, $H \in E$ (where $\mathrm{ord}_{H,a}$ denotes the order along H) are invariants of the equivalence class. (See [BM4, Cor. 5.3, Thms. 6.1, 6.2].) In [Hi], Hironaka proposes to prove Theorem 1.3 above using a weaker notion of equivalence where test sequences involve only admissible blowings-up and product with an affine line. Although $\mathrm{ord}_a\mathcal{I}/d$ is an invariant of the weaker equivalence class, $\mathrm{ord}_{H,a}\mathcal{I}/d$ is not [BM3, Ex. 5.14].

5.4. Coefficient ideals. Let $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$ denote a marked ideal as above. Recall from Section 1 that the derivative ideal $\mathcal{D}(\mathcal{I})$ is the image of the natural morphism $\mathrm{Der}_N \times \mathcal{I} \to \mathcal{O}_N$. Let $\mathcal{D}_E(\mathcal{I}) \subset \mathcal{O}_N$ denote the ideal generated by all local sections of $\mathcal{I}$ and all derivations that preserve the ideal $\mathcal{I}_E$ of E. Higher-derivative ideals are defined inductively by

$$\mathcal{D}_E^{j+1}(\mathcal{I}) := \mathcal{D}_E(\mathcal{D}_E^j(\mathcal{I})), \quad j = 1, \dots .$$

We define marked ideals

$$\underline{\mathcal{D}}_E^j(\underline{\mathcal{I}}) := (M, N, E, \mathcal{D}_E^j(\mathcal{I}), d-j), \quad j = 1, \dots, d-1,$$

and a weighted sum of marked ideals

$$\underline{\mathcal{C}}_E^k(\underline{\mathcal{I}}) := \sum_{j=0}^{k} \underline{\mathcal{D}}_E^j(\underline{\mathcal{I}}), \quad k \le d-1$$

(see [BM4, §3.3]); $\underline{\mathcal{C}}_E^k(\underline{\mathcal{I}})$ is a marked ideal $\left(M, N, E, \mathcal{C}_E^k(\underline{\mathcal{I}}), d_{\underline{\mathcal{C}}_E^k(\underline{\mathcal{I}})}\right)$. The marked ideals $\underline{\mathcal{I}}$ and $\underline{\mathcal{C}}_E^k(\underline{\mathcal{I}})$, $k \le d-1$, are equivalent [BM4, Cor. 3.1].

Suppose that $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$ is of maximal order. Then every point of $\mathrm{cosupp}\,\underline{\mathcal{I}}$ has an open neighbourhood $U \subset Z$ in which $\underline{\mathcal{I}}$ has a *maximal contact hypersurface* $P \subset N|_U$ [BM4, §4]. The corresponding *coefficient ideal* is defined as

$$\underline{\mathcal{C}}_{E,P}(\underline{\mathcal{I}}) := \left(U, P, E, \mathcal{C}_E^{d-1}(\underline{\mathcal{I}})|_P, d_{\underline{\mathcal{C}}_E^{d-1}(\underline{\mathcal{I}})}\right).$$

It follows that $\underline{\mathcal{C}}_{E,P}(\underline{\mathcal{I}})$ is equivalent to $\underline{\mathcal{I}}|_U$ [BM4, Cor. 4.1].

LEMMA 5.6. *Suppose that $\psi : Z_\eta \to Z$ is a generic fibre embedding as above. Then $\underline{\mathcal{C}}_{E,P}(\underline{\mathcal{I}})_\eta = \psi^*\underline{\mathcal{C}}_{E,P}(\underline{\mathcal{I}})$ is equivalent to $\underline{\mathcal{C}}_{E_\eta,P_\eta}(\underline{\mathcal{I}}_\eta)$.*

Proof. This is immediate from the preceding equivalence and Lemma 5.2. □

REMARK 5.7. Lemma 5.6 can be understood also in another way. The marked ideal $\underline{\mathcal{C}}_{E,P}(\underline{\mathcal{I}})$ is equivalent to a smaller coefficient ideal $\underline{\mathcal{C}}_{z,P}(\underline{\mathcal{I}})$ defined using only derivatives in the normal direction to the hypersurface $P \subset N$ (i.e., derivatives with respect to a local generator z of the ideal of P in $\mathcal{O}_N$) [BM4, Ex. 4.4(1)]. For this variant of the coefficient ideal, $\psi^*\underline{\mathcal{C}}_{z,P}(\underline{\mathcal{I}}) = \underline{\mathcal{C}}_{z|_{N_\eta},P_\eta}(\underline{\mathcal{I}}_\eta)$. (Derivatives along elements of $\underline{m}$ that are transcendental over $\underline{k}$ are not explicitly involved; cf. Remark 1.4.) Lemma 5.6 follows also from the latter, [BM4, Ex. 4.4(1)], and Lemma 5.2.

6. Functoriality of desingularization of a variety. We begin with a precise statement of Theorem 1.1.

THEOREM 6.1. *Given a variety X over a field $\underline{k}$ of characteristic zero, there is finite sequence of blowings-up $\sigma_{j+1} : X_{j+1} \to X_j$ with smooth centres,*

$$X = X_0 \xleftarrow{\sigma_1} X_1 \longleftarrow \cdots \xleftarrow{\sigma_t} X_t\,, \tag{6.1}$$

such that:

1. *X_t is smooth and the exceptional divisor in X_t has only normal crossings.*
2. *All centres of blowing up are disjoint from the preimages of $X \setminus \operatorname{Sing} X$.*
3. *The resolution morphism $\sigma_X : X_t \to X$ given by the composite of the σ_j (or the entire sequence of blowings-up (6.1)) is associated to X in a way that is functorial with respect to regular morphisms. (See Remarks 4.7(1).)*

This theorem can be proved with the following stronger version of the condition (2): For each j, let $C_j \subset X_j$ denote the centre of the blowing-up $\sigma_{j+1} : X_{j+1} \to X_j$. Then either $C_j \subset \operatorname{Sing} X_j$ or X_j is smooth and C_j lies in the support of the exceptional divisor of $\sigma_1 \circ \cdots \circ \sigma_j$. In fact, we prove Theorem 6.1 together with the following addendum (where (3), (4) should again be understood modulo trivial blowings-up as in Remarks 4.7(1)).

Theorem 6.1 Addendum. *Given any embedding (i.e., closed immersion) $X|_U \hookrightarrow Z$, where U is an open subset of X and Z is smooth, there is a sequence of blowings-up $\tau_{j+1} : Z_{j+1} \to Z_j$,*

$$Z = Z_0 \xleftarrow{\tau_1} Z_1 \longleftarrow \cdots \xleftarrow{\tau_t} Z_t\,, \tag{6.2}$$

which satisfies the following conditions. Set $Y_0 := X|_U$. For each j, let C_j denote the centre of τ_{j+1}, let E_{j+1} denote the exceptional divisor of $\tau_1 \circ \cdots \circ \tau_{j+1}$, and define Y_{j+1} inductively as the strict transform of Y_j by τ_{j+1}. Then:

1. *Each C_j is smooth and has only normal crossings with respect to E_j.*
2. *For each j, either $C_j \subset \operatorname{Sing} Y_j$ or Y_j is smooth and $C_j \subset Y_j \cap \operatorname{supp} E_j$.*
3. *Each $X_j|_U = Y_j$ and, over U, the resolution sequence (6.1) is given by the restriction of (6.2) to the Y_j.*
4. *The sequence of blowings-up (6.2) is associated to $X|_U \hookrightarrow Z$ in a way that is functorial with respect to regular morphisms (of the ambient variety Z).*

A weaker version of Theorem 6.1 (and the Addendum) can be obtained directly from Theorem 1.3 applied to the marked ideal $\underline{\mathcal{I}} = (Z, Z, \emptyset, \mathcal{I}_X, 1)$, where $X \hookrightarrow Z$ is a (local) embedding of X in a smooth $\underline{k}$-variety Z and $\mathcal{I}_X \subset \mathcal{O}_Z$ is the ideal of X (see [BM4, §1.1]. In this version, the resolution sequence (6.1) is given by restricting the blowing-up sequence provided by Theorem 1.3 to the successive strict transforms X_j of X — the intersections of the centres C_j with the X_j will not necessarily be smooth, nor will condition (2) of the Addendum necessarily hold.

Theorem 6.1 (including the Addendum) as stated except for weaker functoriality conditions — with respect to smooth morphisms and base-field extensions — is proved in [BM2, BM4] (under the tacit assumption that the smooth morphisms have constant relative dimension. We show how to remove this restriction in §6.3 below.) The proof involves the *Hilbert-Samuel function* $H_{X,a} \in \mathbb{N}^{\mathbb{N}}$, $a \in X$, and desingularization of an associated marked ideal that we call a *presentation* of the Hilbert-Samuel function. We will use commutativity of a presentation with respect to generic-fibre embeddings (Proposition 6.3 below) together with Theorem 1.2 and Remark 4.8 to deduce Theorem and Addendum 6.1 in full (see §6.4).

6.1. The Hilbert-Samuel function. If a is a closed point, then the *Hilbert-Samuel function* $H_{X,a}$ is defined as

$$H_{X,a}(k) := \operatorname{length} \frac{\mathcal{O}_{X,a}}{\mathfrak{m}_{X,a}^{k+1}}\,, \quad k \in \mathbb{N}\,.$$

Thus $H_{X,a} \in \mathbb{N}^{\mathbb{N}}$. We can extend the definition to arbitrary points of X so that $a \mapsto H_{X,a}$ will be upper-semicontinuous, where the order on $\mathbb{N}^{\mathbb{N}}$ is the usual order on functions (i.e., $H_1 \leq H_2$ means that $H_1(k) \leq H_2(k)$, for all $k \in \mathbb{N}$):

In general, define $\Lambda : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by

$$\Lambda(F)(k) = \sum_{j=0}^{k} F(j)\,, \qquad k \in \mathbb{N}\,,$$

where $F \in \mathbb{N}^{\mathbb{N}}$. Define $\Lambda^j : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, $j \geq 1$, inductively by $\Lambda^j(F) = \Lambda(\Lambda^{j-1}(F))$. Suppose that R is a Noetherian local ring with maximal ideal $\mathfrak{m}$. Define $H^{(0)}(R) \in \mathbb{N}^{\mathbb{N}}$ by

$$H^{(0)}(R)(k) := \text{length}\, \frac{R}{\mathfrak{m}^{k+1}}\,, \quad k \in \mathbb{N}\,.$$

For each $j \in \mathbb{N}$, let $H^{(j)}(R)$ denote $\Lambda^j(H^{(0)}(R))$ (cf. [Be, 0(1.3)]). If $a \in X$, then we define $H^{(j)}_{X,a} := H^{(j)}(\mathcal{O}_{X,a})$, for all $j \in \mathbb{N}$, and we let $H_{X,a}$ denote $H^{(l)}_{X,a}$, where l denotes the transcendence degree of the residue field $\kappa(a)$ over $\underline{k}$ (i.e., the dimension of the closure of a).

The Hilbert-Samuel function $H_{X,a}$ determines the *minimal embedding dimension* $e_{X,a}$ of X at a (in a smooth affine $\underline{k}$-variety): $e_{X,a} = H_{X,a}(1) - 1$.

The Hilbert-Samuel function $H_{X,\cdot} : X \to \mathbb{N}^{\mathbb{N}}$ has the following basic properties, established by Bennett [Be] (see [BM1, §5], [BM2, §§7, 9] for simple proofs): (1) $H_{X,\cdot}$ distinguishes smooth and singular points. (2) $H_{X,\cdot}$ is (Zariski) upper-semicontinuous. (3) $H_{X,\cdot}$ is *infinitesimally upper-semicontinuous* (i.e., $H_{X,\cdot}$ cannot increase after blowing-up with centre on which it is constant). (4) Any decreasing sequence in the value set of the Hilbert-Samuel function stabilizes.

6.2. Presentation of the Hilbert-Samuel function. The Hilbert-Samuel function $H_{X,a}$ is a local invariant that plays the same role with respect to strict transform of a variety X as the order plays with respect to (weak) transform of a (marked) ideal. More precisely, for all $a \in X$, there is an embedding $X|_U \hookrightarrow Z$, where U is a neighbourhood of a and Z is smooth, and a marked ideal $\underline{\mathcal{I}} = (Z, N, \emptyset, \mathcal{I}, d)$ which has the same *test sequences* as $\underline{X} := (Z, \emptyset, X|_U, H)$, where $H = H_{X,a}$. (We define a *test sequence* for $\underline{X} = (M, E, X, H)$ by analogy with that for a marked ideal (§5.2), but where a blowing-up $\sigma : Z' \to Z$ with smooth centre C is *admissible* if $C \subset \operatorname{supp} \underline{X} := \{x \in X : H_{X,x} \geq H\}$ and C, E have only normal crossings, and where X transforms by *strict transform.*) We call $\underline{\mathcal{I}}$ a *presentation* of $H_{X,\cdot}$ at a.

A construction of a presentation is given in [BM2, Ch. III] and some familiarity with the latter will be needed to understand the results of this section in detail. The essential point needed is that $\mathcal{I}$ is generated by suitable powers of a special system of generators of $\underline{\mathcal{I}}_X \subset \mathcal{O}_Z$ at a which is determined by the *vertices* of the *diagram of initial exponents* $\mathfrak{N}(\mathcal{I}_{X,a})$ with respect to a local coordinate system for Z at a (see §6.5 below).

We can choose a presentation $\underline{\mathcal{I}} = (Z, N, \emptyset, \mathcal{I}, d)$ of $H_{X,\cdot}$ at a so that Z is a smooth minimal embedding variety for X at a. Given $\underline{\mathcal{I}}$, there is an étale morphism $\varphi : Z' \to Z$ onto a neighbourhood of a such that $\varphi^*(\underline{\mathcal{I}})$ is equivalent to a marked ideal $\underline{\mathcal{J}} = (Z', N', \emptyset, \mathcal{J}, e)$ of maximal order and codimension zero (i.e., $N' = Z'$)

6.3. Functoriality with respect to smooth morphisms. If a is a maximum point of $H_{X,\cdot}$ and $\underline{\mathcal{I}}$ is a presentation of $H_{X,\cdot}$ at a, then the corresponding maximal value of $H_{X,\cdot}$ decreases after desingularization of $\underline{\mathcal{I}}$. This is the main point of a presentation of the Hilbert-Samuel function, needed to prove the strong desingularization theorem for a variety using functorial desingularization of a marked ideal together with the basic properties of the Hilbert-Samuel function in §6.1 above.

In [BM4], versions of Theorem 6.1, where the functoriality assertion is with respect to smooth morphisms and base-field extensions, are proved using a presentation of the Hilbert-Samuel function, again following either of the two schemes recalled in Remark 5.1 above. (See Theorems 1.1, 1.3 and Sections 1.3, 7 of [BM4].)

However, there are equidimensionality issues for smooth morphisms that are not treated in previous works, even using a desingularization invariant. (These issues were raised in a letter from Ofer Gabber to the third author.) We can deal with them only using the second of the two methods in Remark 5.1, which again involves proving a stronger desingularization theorem where the centres of blowing up are given by the maximum loci of a desingularization invariant inv_X defined inductively over a sequence of admissible blowings-up. Since the marked ideal $\underline{\mathcal{J}}$ above is of maximal order, the desingularization invariant $\mathrm{inv}_{\underline{\mathcal{J}}}$ for $\underline{\mathcal{J}}$ is a finite sequence whose first term is 1 throughout the cosupports of the successive transforms of $\underline{\mathcal{J}}$ (see [BM4, §7.2]); inv_X is defined at the corresponding points of X and its successive strict transforms by replacing this first term by $H_{X,a}$. For details we refer to [BM2, BM3, BM4]. (It is important to begin with a presentation $\underline{\mathcal{J}}$ as above so that the desingularization invariant will be independent of the choice of a local embedding variety for X.)

As shown in [BM4], blowing up with centre = maximum locus of inv_X gives an algorithm for resolution of singularities of arbitrary X, functorial with respect to smooth morphisms of constant relative dimension.

In order to prove functoriality with respect to arbitrary smooth morphisms, we first note that (in the notation of §6.1), $\Lambda^k(H_{X,a}) = H_{X\times\mathbb{A}^k,(a,0)}$. (See also §6.5 below.) Moreover a presentation $\underline{\mathcal{I}} = (Z, N, \emptyset, \mathcal{I}, d)$ of $H_{X,\cdot}$ at a induces a presentation of $H_{X\times\mathbb{A}^k,\cdot}$ at $(a,0)$ by pull-back by the projection $Z\times\mathbb{A}^k \to Z$.

Suppose that X is locally equidimensional. Define a modified invariant inv_X^* by replacing the first term $H_{X,x}$ of $\mathrm{inv}_X(x)$ at each point x by $\Lambda^{d-q}(H_{X,x})$, where $d = \dim X$ and $q = q(x)$ denotes the dimension of (the irreducible components of) X at x. Then blowing up with centre = maximum locus of inv_X^* gives an algorithm for resolution of singularities of locally equidimensional varieties X, *functorial with respect to arbitrary smooth morphisms.*

We can use the preceding idea together with a suggestion of Gabber (in the letter cited above) to define a modified invariant $\mathrm{inv}_X^\#$ such that blowing up with centre = maximum locus of $\mathrm{inv}_X^\#$ gives an algorithm for *resolution of singularities of arbitrary varieties X, functorial with respect to arbitrary smooth morphisms*: Let $q(x)$ denote the smallest dimension of the irreducible components of X at x. Define $\mathrm{inv}_X^\#$ by replacing the first term $H_{X,x}$ of $\mathrm{inv}_X(x)$ at each point x by $\Lambda^{d-q(x)}(H_{X,x})$, where $d = \dim X$. It is easy to see that a marked ideal is a presentation of the Hilbert-Samuel function at x if and only if it is a presentation of $\Lambda^{d-q(\cdot)}(H_{X,\cdot})$. The assertion follows.

Remark 6.2. The fact that the invariants $H_{X,\cdot}$ and $\Lambda^{d-q(\cdot)}(H_{X,\cdot})$ share a common presentation at every point means, in particular, that every component of a constant locus of one of these invariants is also a component of a constant locus of the other. It follows that the maximum loci of the two invariants inv_X and $\mathrm{inv}_X^\#$ are each unions of closed components of constant loci of the invariant inv_X, but not necessarily of the same closed components in each case — i.e., the order in which we blow up these components may not be the same. The invariant $\Lambda^{d-q(\cdot)}(H_{X,\cdot}))$ is contrived to force us to blow up components in an order that gives functoriality with respect to arbitrary smooth morphisms.

6.4. Functoriality with respect to generic-fibre embeddings. In order to deduce Theorem 6.1 (and its Addendum) with the full version of functoriality, i.e., with respect to regular morphisms in general, using Theorem 1.2 and Remark 4.8, it is now again enough to prove functoriality with respect to generic-fibre embeddings. For the latter, because of Lemma 5.2 and Proposition 5.3, it is enough to prove Proposition 6.3 following.

Let $\psi : X_\eta \to X$ denote a generic-fibre embedding, corresponding to a dominant morphism of $\underline{k}$-varieties $\pi : X \to T$, where T is integral. Let b denote a closed point of X_η and let $a = \psi(b) \in X$. Then there is a neighbourhood U of a in X so that $X|_U$ embeds in a smooth $\underline{k}$-variety Z such that π extends to a (dominant) morphism $Z \to T$. We can choose U and Z such that Z is a minimal embedding variety for X at a.

For simplicity of notation, we will write simply X instead of $X|_U$, and X_η for the generic fibre of the latter. Then $X_\eta = X \times_Z Z_\eta$ and there is a commutative diagram

$$\begin{array}{ccc} X_\eta & \xrightarrow{\psi} & X \\ \downarrow & & \downarrow \\ Z_\eta & \longrightarrow & Z \end{array}$$

where the horizontal arrows are the generic-fibre embeddings and the right (respectively, left) vertical arrow is a morphism of $\underline{k}$-varieties (respectively, $\underline{m}$-varieties, where $\underline{m} = K(T)$).

PROPOSITION 6.3. *With the notation preceding, there is a neighbourhood V of a in Z and a presentation $\underline{\mathcal{I}} = (Z|_V, N, \emptyset, \mathcal{I}, d)$ of $H_{X,\cdot}$ at a such that $\underline{\mathcal{I}}_\eta$ is a presentation of $H_{X_\eta,\cdot}$ at b.*

More precisely, we claim that, for suitable local coordinates for Z, the presentation constructed in [BM2, Ch. III] has the required properties. Since we do not want to repeat the construction, we will give only the new ingredients needed by a reader who is familiar with the latter to verify our claim in a straightforward way.

6.5. The diagram of initial exponents. The construction of a presentation in [BM2, Ch. III] depends on the way that the Hilbert-Samuel function can be computed using the diagram of initial exponents of the ideal $\mathcal{I}_X \subset \mathcal{O}_Z$ of X with respect to local coordinates for Z at a point of X.

Consider a ring of formal power series $R = K[\![X]\!] = K[\![X_1, \ldots, X_n]\!]$ over a field K. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, put $|\alpha| = \alpha_1 + \ldots + \alpha_n$. We totally order $\mathbb{N}^n$ by using the lexicographic ordering of $(n+1)$-tuples $(|\alpha|, \alpha_1, \ldots, \alpha_n)$. Consider $F = \sum_{\alpha \in \mathbb{N}^n} F_\alpha X^\alpha \in K[\![X]\!]$, where $X^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. Let $\operatorname{supp} F := \{\alpha : F_\alpha \neq 0\}$. The *initial exponent* $\exp F$ means the smallest element of $\operatorname{supp} F$. ($\exp F := \infty$ if $F = 0$.)

Let I be an ideal in R. The *diagram of initial exponents* $\mathfrak{N}(I) \subset \mathbb{N}^n$ is defined as

$$\mathfrak{N}(I) := \{\exp F : F \in I \setminus \{0\}\}.$$

Clearly, $\mathfrak{N}(I) + \mathbb{N}^n = \mathfrak{N}(I)$. It follows that there is a smallest finite subset $\mathfrak{V}$ of $\mathfrak{N}(I)$ (the *vertices* of $\mathfrak{N}(I)$) such that $\mathfrak{N}(I) = \mathfrak{V} + \mathbb{N}^n$. ($\mathfrak{V} = \{\alpha \in \mathfrak{N}(I) : (\mathfrak{N}(I) \setminus \{\alpha\}) + \mathbb{N}^n \neq \mathfrak{N}(I)\}$.)

Given $\mathfrak{N} \subset \mathbb{N}^n$ such that $\mathfrak{N} + \mathbb{N}^n = \mathfrak{N}$, let $H_{\mathfrak{N}} \in \mathbb{N}^{\mathbb{N}}$ denote the function

$$H_{\mathfrak{N}}(k) = \#\{\alpha \in \mathbb{N}^n \setminus \mathfrak{N} : |\alpha| \le k\}, \quad k \in \mathbb{N}$$

(where $\#S$ denotes the number of elements in a finite set S). Then $H^{(0)}(R/I) = H_{\mathfrak{N}(I)}$ (see §6.1). It is easy to see that, if $\mathfrak{N}$ is a product $\mathfrak{N} = \mathbb{N}^p \times \mathfrak{N}^*$, then $H_{\mathfrak{N}} = \Lambda^p(H_{\mathfrak{N}^*})$.

Suppose that $(x_1, \dots, x_n)$ is a coordinate system (system of parameters) on an open subset U of Z. If c is a closed point in U, then there is a unique isomorphism $\widehat{\mathcal{O}}_{Z,c} \xrightarrow{\cong} \kappa(c)[\![X_1, \dots, X_n]\!]$ such that each $x_i \mapsto x_i(c) + X_i$, where $x_i(c)$ denotes the image of x_i in the residue field $\kappa(c) = \mathcal{O}_{Z,c}/\mathfrak{m}_{Z,c}$. If $\mathcal{I}_X \subset \mathcal{O}_Z$ denotes the ideal of X, then the *diagram of initial exponents* $\mathfrak{N}(\mathcal{I}_{X,c})$ of $\mathcal{I}_{X,c}$ *with respect to the given coordinate system* denotes $\mathfrak{N}(I)$, where $I \subset \kappa(c)[\![X]\!]$ is the ideal induced by $\mathcal{I}_{X,c}$.

We totally order $\{\mathfrak{N} \subset \mathbb{N}^n : \mathfrak{N} + \mathbb{N}^n = \mathfrak{N}\}$ by giving each $\mathfrak{N}$ the lexicographic order of the sequence of its vertices (in increasing order). Each point of X admits a coordinate neighbourhood in Z in which the associated diagram $\mathfrak{N}(\mathcal{I}_{X,c})$ can be extended to arbitrary points so that $c \mapsto \mathfrak{N}(\mathcal{I}_{X,c})$ is upper-semicontinuous.

6.6. Proof of Proposition 6.3. By restricting Z and T to suitable affine open neighbourhoods of the point a and its image in T, we can assume (by the Jacobian criterion for smoothness) that:

1. T is a subvariety $V(P)$ of $\mathbb{A}^{p+q}_{\underline{k}}$ determined by an ideal $(P) \subset \underline{k}[y,z] = \underline{k}[y_1, \dots, y_p, z_1, \dots, z_q]$ generated by polynomials $P_1(y,z), \dots, P_q(y,z)$, where the determinant J_P of the Jacobian matrix $\partial P/\partial z = (\partial P_i/\partial z_j)$ is nonvanishing on T.
2. $Z = V(P,G) \subset \mathbb{A}^{n+m}_{\underline{k}}$, where $n \ge p$, $m \ge q$, (P,G) is an ideal in $\underline{k}[x,w]$,
$$\begin{aligned} x &= (u,v) = (u_1, \dots, u_p, v_1, \dots, v_{n-p}), \\ w &= (s,t) = (s_1, \dots, s_q, t_1, \dots, t_{m-q}), \end{aligned}$$
(P,G) is generated by $P_1(u,s), \dots, P_q(u,s)$ (from (1)) together with polynomials $G_1(x,w), \dots, G_{m-q}(x,w)$, and the determinant $J_{(P,G)}$ of $\partial(P,G)/\partial(s,t)$ is nonvanishing on Z.
3. The morphism $Z \to T$ is induced by the inclusion $\underline{k}[y,z] \hookrightarrow \underline{k}[x,w]$ given by $u = y$, $s = z$.

It follows that

(4) $Z_\eta = V(G_\eta) \subset \mathbb{A}^{(n-p)+(m-q)}_{\underline{m}}$, where $(G_\eta) \subset \underline{m}[v,t] = K(T)[v,t]$ is the ideal generated by the polynomials $G_{j,\eta}(v,t)$ which are induced by the $G_j(u,v,s,t)$, $j = 1, \dots, m-q$.

Since $J_{(P,G)} = J_P \cdot J_G$, where $J_G = \det(\partial G/\partial t)$, we see that $\det(\partial G_\eta/\partial t)$ is nonvanishing on Z_η.

Therefore, $y = (y_1, \dots, y_p)$, $x = (x_1, \dots, x_n) = (u_1, \dots, u_p, v_1, \dots, v_{n-p})$ and $v = (v_1, \dots, v_{n-p})$ (respectively) induce local coordinates (regular parameters) on T, Z and Z_η (respectively).

Let W denote the closure of $a = \psi(b)$ in X. Then there is an open subset V of W on which the projection to T is étale (see §2), so that $y = (y_1, \dots, y_p)$ is a system of coordinates on V. Given a closed point c of V, let $X_{\pi(c)}$ denote the fibre $X \times_T \pi(c)$ over $\pi(c)$ and let $\mathcal{I}_{X_{\pi(c)}}$ denote the ideal of $X_{\pi(c)} \subset Z_{\pi(c)}$. Let $\mathfrak{N}(\mathcal{I}_{X,c})$ and $\mathfrak{N}(\mathcal{I}_{X_{\pi(c)},c})$ denote the diagrams of initial exponents with respect to the coordinates x and v (respectively) for Z and $Z_{\pi(c)}$ (respectively). By semicontinuity of the diagram of initial exponents, we can assume that $\mathfrak{N}(\mathcal{I}_{X,c})$ and $\mathfrak{N}(\mathcal{I}_{X_{\pi(c)},c})$ are constants, say

$\mathfrak{N} \subset \mathbb{N}^n$ and $\mathfrak{N}^* \subset \mathbb{N}^{n-p}$ (respectively), on the closed points c of V. It follows in a simple way that $\mathfrak{N} = \mathbb{N}^p \times \mathfrak{N}^*$, and $\mathfrak{N}(\mathcal{I}_{X_\eta,b}) = \mathfrak{N}^*$, where $\mathfrak{N}(\mathcal{I}_{X_\eta,b})$ is the diagram with respect to the coordinates $v = (v_1, \dots, v_{n-p})$ for N_η. (Compare with [BM3, Proof of Th. 6.18].) In particular, $H_{X,a} = H^{(p)}_{X_\eta,b}$. ($p$ is the transcendence degree of $\underline{m}$ over $\underline{k}$.)

A presentation $\underline{\mathcal{I}}$ of the Hilbert-Samuel function $H_{X,\cdot}$ at a with respect to the coordinates $x = (u, v)$, as constructed in [BM2, Ch. III], is characterized by certain formal properties [BM2, (7.2)] related to the vertices of $\mathfrak{N}(\mathcal{I}_{X,c})$ above. Because of the product structure $\mathfrak{N} = \mathbb{N}^p \times \mathfrak{N}^*$ of this diagram, it is easy to verify that, if these properties are satisfied at every closed point of an open subset of W, then they are satisfied by the induced marked ideal $\underline{\mathcal{I}}_\eta$ at b. The details are left to the reader. ∎

7. Absolute desingularization. In this section, we apply the same approximation methods of [EGA IV, §8] that we used in the proof of Theorem 4.5 to show that any desingularization algorithm for $\mathbb{Q}$-varieties that is functorial with respect to smooth morphisms extends uniquely to a desingularization algorithm for a class $\mathfrak{C}$ of schemes over $\mathbb{Q}$ which includes all varieties of characteristic zero as well as their localizations and Henselizations along closed subvarieties. Moreover, the algorithm for $\mathfrak{C}$ will be functorial with respect to all regular morphisms between schemes in $\mathfrak{C}$.

We refer to [EGA IV, §§18.6, 18.8] for definitions of *Henselization* and *strict Henselization.*

Remarks 7.1. (1) One of our main motivations here is to extend the desingularization algorithm for a variety X to its Henselization X^h_Z along a closed subvariety Z. Since $X^h_Z \to X$ is a regular morphism, we could just pull back the desingularization sequence from X, but it would not be clear that the induced desingularization sequence depends only on the scheme X^h_Z. The problem is that, while the ground field morphism $X \to \mathrm{Spec}\,(\underline{k})$ is more or less unique (by §4.1), the morphism $X^h_Z \to \mathrm{Spec}\,(k)$ admits many deformations in general.

For example, even in the case $X = \mathbb{A}^1_{\mathbb{C}}$, if x is the origin, then the homomorphism $\phi : \mathcal{O}^h_{X,x} \to \kappa(x) \xrightarrow{\cong} \mathbb{C}$ from the Henselization $\mathcal{O}^h_{X,x}$ of $\mathcal{O}_{X,x}$ admits many different sections $s : \mathbb{C} \hookrightarrow \mathcal{O}^h_{X,x}$ besides the structure homomorphism. Indeed, choose any transcendence basis $T = \{T_i\}_{i\in I}$ of $\mathbb{C}$ over $\mathbb{Q}$ and take $s(T_i)$ to be any element of $\phi^{-1}(T_i)$. Then the embedding $s : k[T] \hookrightarrow \mathcal{O}_{X,x}$ extends to $s : k(T) \hookrightarrow \mathcal{O}_{X,x}$ because $\mathcal{O}_{X,x}$ is local, and the induced embedding $s : k(T) \hookrightarrow \mathcal{O}^h_{X,x}$ extends further to $s : \mathbb{C} = \overline{k(T)} \to \mathcal{O}^h_{X,x}$ because $\mathcal{O}^h_{X,x}$ is Henselian.

Therefore, given a desingularization algorithm for $\underline{k}$-varieties, the blowing-up sequence for X^h_Z obtained by pulling back that of X might depend on the morphism $X^h_Z \to \mathrm{Spec}\,(\underline{k})$. We overcome this obstacle by descending to $\mathbb{Q}$ — we show that an absolute desingularization algorithm for varieties defined over $\mathbb{Q}$ induces a desingularization algorithm for Henselian varieties (and certain other schemes) that depends only on the schemes.

(2) Our Henselian result will be used in [T2] to construct a canonical desingularization of rig-regular formal varieties in characteristic zero (independent of algebraization). The class includes, for example, the formal completion of a variety along its singular locus.

It seems to be an interesting open question whether the algorithm of [BM2] extends to functorial desingularization of formal varieties in general. It is true that, if X and Y are varieties (over perhaps different ground fields) and $\widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathcal{O}}_{Y,y}$, for some $x \in X$, $y \in Y$, then the desingularizations of X, Y induce the same sequences of formal blowings-up of $\widehat{X}_x := \operatorname{Spec} \widehat{\mathcal{O}}_{X,x}$ and $\widehat{Y}_y$. We can show this using a formal presentation of the Hilbert-Samuel function as given in [BM2, §§7, 9] together with the marked ideal techniques of [BM4] and commutativity of blowing up and formal completion [T1, Lemma 2.1.8].

Definition 7.2. Consider a filtered projective family $\{X_i\}_{i\in I}$ of $\mathbb{Q}$-varieties with smooth affine transition morphisms $f_{ji} : X_j \to X_i$. The projective limit $X = \operatorname{proj\,lim}_{i\in I} X_i$ exists in the category of schemes, by [EGA IV, Prop. 8.2.3]. Assume that X is Noetherian. Since the morphisms $X_j \to X_i$, $j \geq i$, are regular, each projection $f_i : X \to X_i$ is regular. (See, for example, [S, Lemma 1.4]. The same argument shows, moreover, that X is Noetherian provided that $\dim X_i$ is bounded.) Let $\mathfrak{C}_{\text{loc}}$ denote the family of all such schemes X, and let $\mathfrak{C}$ we denote the class of schemes each obtained by gluing together finitely many elements of $\mathfrak{C}_{\text{loc}}$.

Remarks 7.3. (1) A simple argument in the proof of Theorem 7.5 below shows that we could consider only families of affine varieties X_i in Definition 7.2 — we would get a smaller category $\mathfrak{C}_{\text{loc}}$, while the category $\mathfrak{C}$ would not change.

(2) Any noetherian (or even quasi-compact quasi-separated) scheme in characteristic zero is a projective limit of $\mathbb{Q}$-varieties, by a noetherian approximation theorem of Thomason [Th, C.9], but the transition morphisms are not smooth (or even flat) in general.

For example (in positive characteristic), if K is a perfect field of positive transcendence degree over $\mathbb{F}_p$, then $\operatorname{Spec}(K)$ is not a projective limit of a filtered family of $\mathbb{F}_p$-varieties with smooth transition morphisms. This follows from the fact that K is not separable over any finitely generated subfield of positive absolute transcendence degree.

This example implies that, although any regular morphism $X \to Y$ is a projective limit of smooth morphisms $X_i \to Y$, by Popescu's desingularization theorem (see [P] or [S]), the transition morphisms $X_j \to X_i$ and the projections $X \to X_i$ cannot be made regular in general.

Theorem 7.4. *Any desingularization algorithm for $\mathbb{Q}$-varieties that is functorial with respect to smooth morphisms extends uniquely to a desingularization algorithm on $\mathfrak{C}$ that is functorial with respect to all regular morphisms. Moreover, if the original algorithm satisfies the stronger conditions of Theorem 6.1, then the extended algorithm satisfies the same conditions.*

Proof. Fix a desingularization algorithm $\mathcal{F}$ for $\mathbb{Q}$-varieties. First we extend $\mathcal{F}$ to $\mathfrak{C}_{\text{loc}}$. Let X be an element of $\mathfrak{C}_{\text{loc}}$ and let $X = \operatorname{proj\,lim}_{i\in I} X_i$ denote a representation of X as a projective limit of $\mathbb{Q}$-varieties with smooth affine transition morphisms. Then the desingularizations $\mathcal{F}(X_i)$ are compatible, so that each of them induces the same desingularization sequence for X, which we denote $\mathcal{F}(X)$. Moreover, if the $\mathcal{F}(X_i)$ satisfy the conditions of Theorem 6.1, then $\mathcal{F}(X)$ also satisfies them.

We have to prove that $\mathcal{F}(X)$ is independent of the choice of the projective limit representation and that this extension of $\mathcal{F}$ to $\mathfrak{C}_{\text{loc}}$ is compatible with all regular morphisms. For both tasks, it is enough to prove that, given another family $\{Y_j\}_{j\in J}$ of $\mathbb{Q}$-varieties with smooth affine transition morphisms and limit Y, and given a regular morphism $h : Y \to X$, there exist $i \in I$, $j \in J$ and a regular morphism

$h_{ji} : Y_j \to X_i$ compatible with h in the sense that the following diagram commutes.

$$\begin{array}{ccc} Y & \longrightarrow & Y_j \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle h_{ji}} \\ X & \longrightarrow & X_i \end{array}$$

Indeed, if we have morphisms such that the diagram commutes, then, on the one hand, $\mathcal{F}(Y)$ is induced by $\mathcal{F}(Y_j)$ and hence by $\mathcal{F}(X_i)$ (since $\mathcal{F}$ is compatible with the smooth morphisms h_{ji}), and, on the other hand, $\mathcal{F}(X)$ is induced by $\mathcal{F}(X_i)$. Therefore, $\mathcal{F}(Y)$ is induced by $\mathcal{F}(X)$, thus proving compatibility with regular morphisms. The fact that $\mathcal{F}(X)$ is well defined is then obtained by applying the preceding argument to an isomorphism $X \xrightarrow{\cong} X$ and two representations of X as a projective limit.

To find a regular h_{ji} as above: Fix i. By [EGA IV, Cor. 8.13.2], the morphism $Y \to X_i$ factors through Y_j, for some j. So we get a morphism h_{ji} and it remains to show only that it can be chosen regular. We claim that the image of Y in Y_j lies in the maximal open subscheme U of Y_j such that $h_{ji}|_U$ is smooth. Indeed, let $y \in Y$ and let $y_j \in Y_j$, $x_i \in X_i$ denote its images. Then the local homomorphisms $\psi : \mathcal{O}_{Y_j,y_j} \to \mathcal{O}_{Y,y}$ and $\mathcal{O}_{X_i,x_i} \to \mathcal{O}_{Y,y}$ are regular (since the morphisms $Y \to Y_j$ and $Y \to X \to X_i$ are regular); hence the homomorphism $\mathcal{O}_{X_i,x_i} \to \mathcal{O}_{Y_j,y_j}$ is regular, by [Ma, Lemma 33.B] (where all we need to know about ψ is that it is faithfully flat). Thus $y_j \in U$ and therefore Y lies in U. Applying [EGA IV, Cor. 8.13.2] again, we see that the morphism $Y \to U$ factors through Y_k for some $k \geq j$. Then the morphism $h_{ki} : Y_k \to U \to X_i$ is smooth since $Y_k \to Y_j$ and $U \to X_i$ are smooth.

Finally, we can extend $\mathcal{F}$ from $\mathfrak{C}_{\mathrm{loc}}$ to $\mathfrak{C}$ using the gluing argument of [K, Prop. 3.37]: Given X in $\mathfrak{C}$, take a covering of X by open subschemes $X_i \in \mathfrak{C}_{\mathrm{loc}}$, $i = 1, \dots k$. Then the disjoint unions $\coprod_i X_i$ and $\coprod_{i \leq j} X_i \cap X_j$ belong to $\mathfrak{C}_{\mathrm{loc}}$, and we have a commutative diagram

$$\begin{array}{ccc} \coprod_{i\leq j} X_i \cap X_j & \longrightarrow & \coprod_i X_i \\ \big\downarrow & & \big\downarrow \\ \coprod_i X_i & \longrightarrow & X \end{array}$$

where the left and top arrows are induced by $X_i \cap X_j \hookrightarrow X_i$ and $X_i \cap X_j \hookrightarrow X_j$, respectively. These arrows are étale, hence smooth, so that $\mathcal{F}$ on $\mathfrak{C}_{\mathrm{loc}}$ is compatible with them. It follows that the blowing-up sequences $\mathcal{F}(X_i)$ glue together to give a desingularization $\mathcal{F}(X)$. Clearly, $\mathcal{F}$ is compatible with regular morphisms among members of $\mathfrak{C}$. □

THEOREM 7.5. *(1) The class $\mathfrak{C}_{\mathrm{loc}}$ contains all separated varieties of characteristic zero, as well as their localizations, Henselizations and strict Henselizations along closed subvarieties. As a result, the class $\mathfrak{C}$ contains analogous classes of schemes (that are not necessarily separated).*

(2) If $\{X_i\}_{i\in I}$ is a filtered projective family of separated schemes in $\mathfrak{C}_{\mathrm{loc}}$ with regular affine transition morphisms then $X = \operatorname{proj\,lim}_{i\in I} X_i$ belongs to $\mathfrak{C}_{\mathrm{loc}}$.

Proof. We start with (2). For each X_i fix a representation $X_i \xrightarrow{\cong} \operatorname{proj\,lim}_{j\in J_i} X_{ij}$ with smooth affine transition morphisms between $\mathbb{Q}$-varieties X_{ij}. Using [Th, C.7] we can assume that all X_{ij} are separated. We recall that the *schematic image* X'_{ij} of X_i in X_{ij} means the smallest closed subscheme of X_{ij} through which the morphism

$X_i \to X_{ij}$ factors. The morphism $X_i \to X'_{ij}$ is regular and each transition morphism $X_{ij} \to X_{ik}$ restricts to a regular morphism $X'_{ij} \to X'_{ik}$; hence, by replacing X_{ij} with X'_{ij}, for all j, we can assume that the projections $X_i \to X_{ij}$ are schematically dominant.

We will now add certain transition morphisms $X_{ij} \to X_{kl}$ with $i \geq k$, which are regular, affine, and moreover make the entire family $\{X_{ij}\}_{i\in I, j\in J_i}$ into a filtered family with projective limit X. This will prove (2). For each $i' \geq i$ and $j \in J_i$ let $f_{i'ij} : X_{i'} \to X_{ij}$ denote the morphism obtained by composing $X_{i'} \to X_i$ and $X_i \to X_{ij}$. Note that if $f_{i'ij}$ factors through a morphism $f_{i'j'ij} : X_{i'j'} \to X_{ij}$, then $f_{i'j'ij}$ is unique because $X_{i'} \to X_{i'j'}$ is schematically dominant and X_{ij} is separated. If such $f_{i'j'ij}$ exists and is affine and regular, then we declare that $(i'j') \geq (ij)$. Affineness and regularity are preserved by composition, so this defines an order on the set $J := \coprod_{i\in I} J_i$. Moreover, this makes J into a filtered ordered set because the argument from the proof of Theorem 7.4 shows that for each $i' \geq i$ and $j \in J_i$ the morphism $f_{i'j'ij}$ exists and is regular and affine for $j' \geq j'_0(i, i', j)$. Since $X \xrightarrow{\cong} \operatorname{proj\,lim}_{i\in I} \operatorname{proj\,lim}_{j\in J_i} X_{ij} \xrightarrow{\cong} \operatorname{proj\,lim}_{(ij)\in J} X_{ij}$, the family $\{X_{ij}\}_{(ij)\in J}$ is as required, and X is in $\mathfrak{C}_{\mathrm{loc}}$.

The assertion (1) follows from (2): Indeed, suppose that Y is a variety over a field $\underline{l}$ that is finitely generated over $\mathbb{Q}$. Then Y is a pro-open subscheme of a $\mathbb{Q}$-variety; hence Y is the projective limit of all its open neighborhoods and the transition morphisms are open immersions. So Y is in $\mathfrak{C}_{\mathrm{loc}}$. An arbitrary variety X is of the form $Y \otimes_{\underline{l}} \underline{k} := Y \times_{\operatorname{Spec} \underline{l}} \operatorname{Spec} \underline{k}$ with Y and $\underline{l}$ as above (see Theorem 3.1), so X is the projective limit of varieties $X_i = Y \otimes_{\underline{l}} \underline{k}_i$ where $\underline{k}_i$ is a finitely generated $\underline{l}$-subfield of $\underline{k}$. The transition morphisms are regular by the characteristic zero assumption, and if X is separated then each X_i is a separated $\underline{k}_i$-variety. Then all $X_i \in \mathfrak{C}_{\mathrm{loc}}$ and hence $X \in \mathfrak{C}_{\mathrm{loc}}$, by (2). Finally, the strict Henselization (respectively, Henselization, or localization) of X along a closed subvariety Z is a projective limit of a family of X-étale schemes X_j. Since the X_j are separated $\underline{k}$-varieties, we get (1) using (2) again. □

REMARKS 7.6. (1) It is interesting to ask whether the category $\mathfrak{C}$ can be naturally extended further. (See, for example, Remark 7.1(2).)

(2) In principle, if X admits a regular morphism f to a variety Y, we could induce a desingularization of X from a desingularization of Y (even though X might not be quasi-excellent!). We do not know if this desingularizaton would be independent of f; even a tool as strong as Popescu's theorem would seem to be of no help here. (See also remark 7.3(2).)

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