

## NOTES ON NON-COMMUTATIVE IWASAWA THEORY\*

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**Abstract.** We discuss two topics in non-commutative Iwasawa theory. One is on the ranks of the dual of the Selmer groups over Iwasawa algebras. Another is a new proof for a result of Ochi-Venjakob.

**Key words.** Selmer groups, elliptic curves, Iwasawa algebras.

**AMS subject classifications.** Primary 11R23; Secondary 11G05, 16E65

**1. Introduction.** Let  $E$  be an elliptic curve defined over a number field  $k$  of finite degree and  $p$  an odd prime number. Let  $k_\infty/k$  be a Galois extension and denote the Galois group  $\text{Gal}(k_\infty/k)$  by  $G$ . We assume that  $k_\infty/k$  is unramified outside a finite set of primes of  $k$  and  $G$  is a compact  $p$ -adic Lie group. We are interested in the case when  $G$  is non-commutative. We investigate the Selmer group of  $E$  over  $k_\infty$ ,

$$\text{Sel}(E/k_\infty) := \ker \left( H^1(k_\infty, E[p^\infty]) \rightarrow \prod_w H^1(k_{\infty,w}, E)[p^\infty] \right)$$

and its Pontrjagin dual

$$\text{Sel}(E/k_\infty)^\vee := \text{Hom}_{\mathbb{Z}_p}(\text{Sel}(E/k_\infty), \mathbb{Q}_p/\mathbb{Z}_p).$$

We can endow this group with a natural left action of the Iwasawa algebra

$$\Lambda(G) = \varprojlim_U \mathbb{Z}_p[G/U]$$

of  $G$ . Here,  $U$  runs over the set of normal open subgroups of  $G$ . It is known that  $\text{Sel}(E/k_\infty)^\vee$  is finitely generated over  $\Lambda(G)$ .

In this paper, we first give a result on the  $\Lambda(G)$ -rank of  $\text{Sel}(E/k_\infty)^\vee$  in the case when  $G$  is uniformly powerful and soluble (Theorem 2.3 in §2). Then, in §3, we will give a new (and simple) proof for a result of Ochi-Venjakob (cf. [OV1]) on the non-existence of non-trivial pseudo-null submodule of  $\text{Sel}(E/k_\infty)^\vee$ .

**2.  $\Lambda(G)$ -ranks of Selmer groups.** Let  $E/k$ ,  $p$ ,  $k_\infty$  and  $G$  be as in §1. In this section, we assume always that  $G$  is a pro- $p$  group without  $p$ -torsion elements. This assumption assures that  $\Lambda(G)$  is a Noetherian ring which has no non-zero zero-divisor, and hence that  $\Lambda(G)$  has a skew field of fraction  $Q(G)$ . For a finitely generated left  $\Lambda(G)$ -module  $M$ , we define its  $\Lambda(G)$ -rank by

$$\text{rank}_{\Lambda(G)} M = \dim_{Q(G)} Q(G) \otimes_{\Lambda(G)} M.$$

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We denote by  $S_p^{\text{ss}}$  the set of primes of  $k$  above  $p$  where  $E$  has potentially supersingular reduction and put

$$s(E/k) := \sum_{v \in S_p^{\text{ss}}} [k_v : \mathbb{Q}_p].$$

Let  $k_{\text{cyc}}$  be the cyclotomic  $\mathbb{Z}_p$ -extension of  $k$ . With these notations, we have the following conjecture.

CONJECTURE 2.1. *If  $k_\infty$  contains  $k_{\text{cyc}}$ , then*

$$\text{rank}_{\Lambda(G)} \text{Sel}(E/k_\infty)^\vee = s(E/k).$$

Recall the following fact:

PROPOSITION 2.2. (cf. [OV2]) *Assume  $E$  has good reduction at all primes above  $p$ , and  $k_\infty$  contains  $k_{\text{cyc}}$ . Then  $\text{rank}_{\Lambda(G)} \text{Sel}(E/k_\infty)^\vee \geq s(E/k)$ .*

Although this is well known, let us review an outline of the proof briefly. Let  $S$  be a finite set of  $k$  which contains all infinite primes, all primes dividing  $p$ , all primes which are ramified in  $k_\infty/k$  and the primes where  $E/k$  has bad reduction. Denote by  $k_S$  the maximal extension of  $k$  unramified outside  $S$ . Note that  $k_\infty \subset k_S$ . For a prime  $v$  of  $k$ , let

$$J_v(E/k_\infty) := \varinjlim_F \bigoplus_{u|v} H^1(F_u, E(\overline{k_v}))[p^\infty].$$

Here,  $F$  runs over all finite subextensions in  $k_\infty/k$ . Then we have an exact sequence

$$(2.1) \quad 0 \rightarrow \text{Sel}(E/k_\infty) \rightarrow H^1(k_S/k_\infty, E[p^\infty]) \xrightarrow{\varphi} \bigoplus_{v \in S} J_v(E/k_\infty).$$

Proposition 2.2 follows from the following two facts:

$$(2.2) \quad \text{rank}_{\Lambda(G)} H^1(k_S/k_\infty, E[p^\infty])^\vee - \text{rank}_{\Lambda(G)} H^2(k_S/k_\infty, E[p^\infty])^\vee = [k : \mathbb{Q}]$$

and

$$(2.3) \quad \text{rank}_{\Lambda(G)} J_v(E/k_\infty)^\vee = \begin{cases} [k_v : \mathbb{Q}_p] & \text{if } v|p \text{ and } v \notin S_p^{\text{ss}}, \\ 0 & \text{otherwise.} \end{cases}$$

Here,  $*^\vee$  denotes the Pontrjagin dual. See [HV] Proposition 7.4 for a proof of (2.2). For (2.3), we first see that

$$J_v(E/k_\infty)^\vee \cong \Lambda(G) \hat{\otimes}_{\Lambda(G_v)} (H^1(k_{\infty,w}, E(\overline{k_v}))[p^\infty])^\vee$$

and hence  $\text{rank}_{\Lambda(G)} J_v(E/k_\infty)^\vee = \text{rank}_{\Lambda(G_v)} (H^1(k_{\infty,w}, E(\overline{k_v}))[p^\infty])^\vee$ . Here,  $w$  is a prime above  $v$  and  $G_v = \text{Gal}(k_{\infty,w}/k_v)$ . Note that  $\dim G_v \geq 1$  for all  $v$  and that if  $v|p$  then  $v$  is deeply ramified in  $k_\infty/k$ , since  $k_\infty \supset k_{\text{cyc}}$ . For  $v \nmid p$ , we have  $\text{rank}_{\Lambda(G_v)} (H^1(k_{\infty,w}, E(\overline{k_v}))[p^\infty])^\vee = 0$  (cf. [OV1] Theorem 4.1). For  $v|p$ ,

$$H^1(k_{\infty,w}, E(\overline{k_v}))[p^\infty] \cong H^1(k_{\infty,w}, \tilde{E}_v[p^\infty])$$

([CG] Proposition 4.3, 4.8). Here,  $\tilde{E}_v$  denotes the reduction of  $E$  modulo  $v$ . From this, we have  $H^1(k_{\infty, w}, E(\overline{k_v}))[\mathfrak{p}^\infty] = 0$  for  $v \in S_p^{\text{ss}}$ . For  $v \notin S_p^{\text{ss}}$ , we have

$$\sum_{i=0}^2 (-1)^i \text{rank}_{\Lambda(G_v)} H^i(k_{\infty, w}, \tilde{E}_v[\mathfrak{p}^\infty])^\vee = -[k_v : \mathbb{Q}_p]$$

by the same method as the proof for [HV] Proposition 7.4. Since  $\dim G_v \geq 1$ , we have  $\text{rank}_{\Lambda(G_v)} H^i(k_{\infty, w}, \tilde{E}_v[\mathfrak{p}^\infty])^\vee = 0$  for  $i = 0$  and  $2$  and hence we have (2.3).

Now we state our result. We need the following three assumptions:

- (A1)  $k_\infty$  contains  $k_{\text{cyc}}$ .
- (A2)  $G$  is uniformly powerful (see [DdMS] for the definition).
- (A3)  $G$  is soluble.

Note  $G$  is pro- $p$  with no torsion elements by the assumption (A2). Put  $\Gamma := \text{Gal}(k_{\text{cyc}}/k)$  and denote by  $\Lambda(\Gamma)$  the Iwasawa algebra of  $\Gamma$ .

**THEOREM 2.3.** *Let  $E/k$  be an elliptic curve which has good reduction at all primes above  $p$ . Assume that  $k_\infty/k$  and  $G$  satisfy the above assumptions (A1), (A2) and (A3). Then, we have  $\text{rank}_{\Lambda(G)} \text{Sel}(E/k_\infty)^\vee = s(E/k)$  if  $\text{rank}_{\Lambda(\Gamma)} \text{Sel}(E/k_{\text{cyc}})^\vee = s(E/k)$ .*

**REMARK 2.4.** *In the case when  $\text{Gal}(k_\infty/k_{\text{cyc}}) \cong \mathbb{Z}_p$ , Theorem 2.3 is proven in [HV]. The condition  $\text{rank}_{\Lambda(\Gamma)} \text{Sel}(E/k_{\text{cyc}})^\vee = s(E/k)$  is known to be true if  $E$  is defined over  $\mathbb{Q}$  and  $k/\mathbb{Q}$  is an abelian extension by Kato, Rubin and Rohrlich.*

Let us give a proof of the Theorem. By Proposition 2.2, it is enough to show the other inequality. Put  $H := \text{Gal}(k_\infty/k_{\text{cyc}})$ . Then  $G$  satisfies the following condition:

- (A4)  $G$  contains a closed normal subgroup  $H$  satisfying  $\Gamma = G/H \cong \mathbb{Z}_p$ .

It is shown by a standard argument in Iwasawa theory combined with [Hr] Lemma 2.5.1 that the kernel and cokernel of the natural restriction map

$$\text{Sel}(E/k_{\text{cyc}}) \rightarrow \text{Sel}(E/k_\infty)^H$$

are cofinitely generated  $\mathbb{Z}_p$ -modules (cf. [HV] Theorem 3.1). This implies that  $\text{rank}_{\Lambda(\Gamma)} (\text{Sel}(E/k_\infty)^\vee)_H = \text{rank}_{\Lambda(\Gamma)} \text{Sel}(E/k_{\text{cyc}})^\vee$ . Here,  $M_H$  denotes the  $H$ -coinvariant of  $M$  for a  $\Lambda(H)$ -module  $M$ . Thus, it is enough to show the following Lemma:

**LEMMA 2.5.** *Let  $G$  be a group satisfying (A2), (A3) and (A4). For a finitely generated  $\Lambda(G)$ -module  $M$ , we have  $\text{rank}_{\Lambda(G)} M \leq \text{rank}_{\Lambda(\Gamma)} M_H$ .*

The proof of this lemma heavily depends on the results in [BH]. First we show:

**LEMMA 2.6** (Balister-Howson [BH]). *Assume the same assumptions on  $G$  as Lemma 2.5. If  $M_H$  is  $\Lambda(\Gamma)$ -torsion then  $M$  is  $\Lambda(G)$ -torsion.*

*Proof.* This fact is not explicitly stated but almost the whole of the proof can be found in [BH]. We review the proof briefly. We prove the assertion by the induction on the dimension of  $G$ . There is nothing to prove when  $\dim G = 1$  since  $H = \{1\}$  and  $G = \Gamma$ . Let  $\dim G > 1$  and suppose the lemma holds for any  $G'$  satisfying  $\dim G' < \dim G$  and the assumptions (A2), (A3) and (A4). We claim that there exists a closed normal subgroup  $N$  of  $G$  satisfying

- (i)  $N \cong \mathbb{Z}_p^r$  for some  $r > 0$ ,
- (ii)  $G/N$  is uniformly powerful and soluble where  $\dim G/N < \dim G$ , and
- (iii)  $N \subset H$ .

Here, an important point is that  $N$  can satisfy (iii). This is the only fact which is not explicitly written in [BH]. If this claim holds,  $G/N$  is uniformly powerful and soluble with a subgroup  $H/N$  satisfying  $(G/N)/(H/N) \cong \Gamma$ . This means that  $G/N$  satisfies  $\dim G/N < \dim G$  and the assumptions (A2), (A3) and (A4). Thus we have  $M_N$  is  $\Lambda(G/N)$ -torsion if  $(M_N)_{H/N} = M_H$  is  $\Lambda(\Gamma)$ -torsion. By tracing the proof of the last Theorem in [BH] almost words by words, we can prove the following fact: If  $M_N$  is  $\Lambda(G/N)$ -torsion then  $M$  is  $\Lambda(G)$ -torsion. This proves the Lemma.

We show the claim mentioned above. If  $G$  is abelian, we may take  $N = H$ . Thus, we assume  $G$  is not abelian. Set  $D^{(0)}(G) := G$  and  $D^{(n+1)}(G) := \overline{[D^{(n)}(G), D^{(n)}(G)]}$ . Then  $D^{(m+1)}(G) = 0$  but  $D^{(m)}(G) \neq 0$  for some  $m \geq 1$  since  $G$  is soluble and non-abelian. Let

$$N := \{g \in G \mid g^{p^k} \in D^{(m)}(G) \text{ for some } k\}.$$

Then the proof of (3) of the first Proposition in §4 of [BH] shows  $N$  satisfies (i) and (ii). (iii) is shown as follows: Note that  $D^{(m)}(G) \subset D^{(1)}(G) \subset H$ . Take an element  $g$  in  $N$ . Then  $g^{p^k}$  is contained in  $D^{(m)}(G)$ , hence in  $H$ . This means that the image of  $g$  in  $G/H$  is  $p$ -torsion. But  $G/H$  is  $p$ -torsionfree, the image of  $g$  in  $G/H$  should be zero, i.e.  $g \in H$ . Hence  $N \subset H$ .  $\square$

We return to the proof of Lemma 2.5. Assume  $r = \text{rank}_{\Lambda(G)} M > \text{rank}_{\Lambda(\Gamma)} M_H = s$ . Take  $s$  elements  $x_1, x_2, \dots, x_s$  in  $M_H$  which generates  $Q(\Gamma) \otimes_{\Lambda(\Gamma)} M_H$ . Take their lifts  $y_1, y_2, \dots, y_s$  in  $M$ . Then we have the exact sequence

$$\Lambda(G)^{\oplus s} \rightarrow M \rightarrow C \rightarrow 0$$

by sending  $e_i$  to  $y_i$  where  $\{e_1, e_2, \dots, e_s\}$  is the canonical basis of  $\Lambda(G)^{\oplus s}$ . Then we can see that  $\text{rank}_{\Lambda(G)} C \geq r - s > 0$  but that  $\text{rank}_{\Lambda(\Gamma)} C_H = 0$ . This contradicts to Lemma 2.6 and proves Lemma 2.5.

*REMARK 2.7. We stress that Lemma 2.6 (hence Lemma 2.5) does not hold in general if  $G$  is not soluble. See the arguments in [BH].*

**3. Non-existence of pseudo-null submodules.** Let  $E/k$ ,  $p$ ,  $k_\infty$  and  $G$  be again as in §1. In this section, we need not to assume that  $G = \text{Gal}(k_\infty/k)$  is pro- $p$ . We assume only that  $G$  has no  $p$ -torsion elements. This assures that  $\Lambda(G)$  is a left and right Noetherian Auslander regular ring with the global dimension  $\dim G + 1$  (cf. [V] Theorem 3.26). For a left (or right)  $\Lambda(G)$ -module  $M$  and an integer  $i \geq 0$ , we put  $E^i(M) := \text{Ext}_{\Lambda(G)}^i(M, \Lambda(G))$ . If  $M$  is a left (resp. right)  $\Lambda(G)$ -module, then  $E^i(M)$  has a natural structure of a right (resp. left)  $\Lambda(G)$ -module.

**DEFINITION 3.1.** *A left  $\Lambda(G)$ -module  $M$  is pseudo-null if  $E^0(M) = E^1(M) = 0$ .*

Note that for general rings, we use a different definition from this (cf. [CSS]), but the above definition is equivalent to that if the ring is Auslander regular (cf. [CSS] Lemma 2.4, [V] Proposition 3.5). The following properties are known:

- (1) Any  $\Lambda(G)$ -module  $M$  has a unique maximal pseudo-null submodule  $M'$  (cf. [V]). Any pseudo-null submodule  $M''$  of  $M$  is contained in  $M'$ .

(2) Any submodules and quotient modules of a pseudo-null module are pseudo-null.

(3) For an exact sequence

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0,$$

$M_2$  is pseudo-null if so are  $M_1$  and  $M_3$ .

The condition  $E^0(M) = \text{Hom}_{\Lambda(G)}(M, \Lambda(G)) = 0$  is equivalent to the condition that  $M$  is  $\Lambda(G)$ -torsion, i.e., every element  $m$  in  $M$  is killed by some element in  $\Lambda(G)$  which is not a zero-divisor. For such  $\Lambda(G)$ -torsionness, it is well known that the similar properties as (1), (2) and (3) above hold. (Replace “pseudo-null” with “ $\Lambda(G)$ -torsion”.)

For a prime  $v$  of  $k$ , denote by  $\tilde{E}_v$  the modulo  $v$  reduction of  $E$ . By fixing a prime  $w$  of  $k_\infty$  over  $v$ , we put  $G_v = \text{Gal}(k_{\infty,w}/k_v) \subset G$  and let  $\kappa_{\infty,w}$  be the residue field of  $k_{\infty,w}$ . Now we state a theorem of Ochi and Venjakob. We assume  $p$  is an odd prime. We assume the following five assumptions (i) to (v):

(i)  $\dim(G_v) \geq 2$  for any bad prime  $v$  of  $E$ ,

(ii) all primes above  $p$  are deeply ramified in  $k_\infty/k$ ,

(iii)  $\dim(G_v) > 2$  or  $\dim(G_v) = 2$  and  $\#\tilde{E}_v(\kappa_{\infty,w})[p^\infty] < \infty$  for any prime  $v|p$  of  $k$  which has good ordinary reduction,

(iv) the Weak Leopoldt’s conjecture  $H^2(k_S/k_\infty, E[p^\infty]) = 0$  holds,

(v) the map  $\varphi$  in (2.1) is surjective.

**THEOREM 3.2** (cf. [OV1] Theorem 5.5, [HV] Theorem 2.6, [O-y]). *Assume  $E$  has good reduction at all primes of  $k$  above  $p$ . If we assume the assumptions (i)–(v) above, then  $\text{Sel}(E/k_\infty)^\vee$  has no non-trivial pseudo-null  $\Lambda(G)$ -submodule.*

The purpose of this section is to give a different proof of this Theorem which is much simpler from the original paper. Let  $S$  be the set of primes of  $k$  which exactly contains all the infinite primes, the primes above  $p$ , the primes which is ramified in  $k_\infty/k$  and the primes where  $E/k$  has bad reduction. Taking the Pontrjagin dual of the sequence (2.1), we have an exact sequence

$$0 \rightarrow \bigoplus_{v \in S} J_v(E/k_\infty)^\vee \rightarrow H^1(k_S/k_\infty, E[p^\infty])^\vee \rightarrow \text{Sel}(E/k_\infty)^\vee \rightarrow 0$$

because of the condition (v). The following is obtained by Ochi-Venjakob and used also in the original proof.

**THEOREM 3.3** (Ochi-Venjakob). (1) ([OV2] Theorem 4.6). *Under the assumption (iv),  $H^1(k_S/k_\infty, E[p^\infty])^\vee$  has no non-trivial pseudo-null submodule.*

(2) ([OV2] Lemma 5.4, [HV] Proposition 2.3). *Under the assumptions (i), (ii) and*

(iii),  $\bigoplus_{v \in S} J_v(E/k_\infty)^\vee$  *is a reflexive module.*

Here, a  $\Lambda(G)$ -module  $M$  is said to be reflexive if the natural map  $M \rightarrow E^0 E^0(M)$  is an isomorphism. Note that a reflexive module has no  $\Lambda(G)$ -torsion submodule since  $E^0(N)$  has no  $\Lambda(G)$ -torsion for any module  $N$ . The following is also by Ochi-Venjakob.

**PROPOSITION 3.4** (cf. [OV1] Lemma 3.1, Proposition 3.3). *For a finitely generated left  $\Lambda(G)$ -module  $M$ ,  $W = E^0 E^0(M)$  is a reflexive module. The kernel of the natural map  $M \rightarrow W$  is the maximal  $\Lambda(G)$ -torsion submodule of  $M$  and the cokernel is pseudo-null.*

So the proof of the theorem is done if we show the following Proposition, which is a new part of the proof:

**PROPOSITION 3.5.** *Let  $0 \rightarrow U \rightarrow V \rightarrow M \rightarrow 0$  be an exact sequence of  $\Lambda(G)$ -modules. Assume that  $U$  is reflexive and  $V$  is a module which has no non-trivial pseudo-null submodule. Then,  $M$  has no nontrivial pseudo-null  $\Lambda(G)$ -submodule.*

*Proof.* Take any pseudo-null submodule  $N$  of  $M$ . Let  $V'$  be the inverse image of  $N$  in  $V$ . Then

$$0 \rightarrow U \rightarrow V' \rightarrow N \rightarrow 0$$

is exact. Since  $U$  is reflexive, it has no  $\Lambda(G)$ -torsion submodule (see Proposition 3.4). This implies that the maximal  $\Lambda(G)$ -torsion submodule of  $V'$  must be pseudo-null because  $N$  is pseudo-null. But since  $V$  has no pseudo-null submodules, it should be 0. Proposition 3.4 tells us that there exist a reflexive module  $W$  and an injection  $V' \rightarrow W$  whose cokernel is pseudo-null. Therefore the the cokernel  $N'$  of the map  $U \rightarrow W$  which is obtained by the composition is again pseudo-null. If we show that the map  $U \rightarrow W$  is an isomorphism, we see that  $N$  is forced to be 0, which proves the proposition. Now we consider the sequence

$$0 \rightarrow U \rightarrow W \rightarrow N' \rightarrow 0.$$

We have the long exact sequence

$$0 \rightarrow E^0(N') \rightarrow E^0(W) \rightarrow E^0(U) \rightarrow E^1(N')$$

Since  $N'$  is pseudo-null, we have that  $E^0(N') = E^1(N') = 0$  and  $E^0(W) \rightarrow E^0(U)$  is an isomorphism. Hence  $E^0 E^0(U) \rightarrow E^0 E^0(W)$  is also an isomorphism. But since both  $U$  and  $V$  are reflexive, this map is nothing other than the original map  $U \rightarrow W$ . Therefore the map is an isomorphism, which is what we want.  $\square$

This proof simplifies the latter half of the proof of Theorem 5.2 in [OV1] (after Lemma 5.6 of that paper). The theorem has been proved by showing that  $E^i E^i(\text{Sel}(E/k_\infty)^\vee) = 0$  for all  $i \geq 2$ , which is an equivalent condition for the non-existence of pseudo-null submodules in the all previously known proofs. We modify the proof of the Lemma in p. 123 of [Gr] and adapt it to the new definition of pseudo-null modules (see also [O-t] Lemma 8.7).

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