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ON INSTANTONS ON NEARLY KÄHLER 6-MANIFOLDS*

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Abstract. We study ω-instantons on nearly Kähler 6-manifolds. These are defined as connections A whose curvatures F satisfy $*F = -\omega \wedge F$. First, we show these connections enjoy nice properties: they are Yang-Mills and variational. Second, we discuss their relation with instantons over the G_2 cones. Third, we derive a Weitzenböck formula for the infinitesimal deformation and derive some rigidity results. Fourth, we construct some $SO(4)$-invariant examples over open sets of S^6.

Key words. Nearly Kähler, Yang-Mills connections, Instantons.

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Introduction. The notion of anti-self-dual instantons plays an important role in Donaldson's theory of 4-manifolds ([7]). This concept has been generalized to higher dimensions (e.g., [8] and [11]). To motivate the generalization, we first recall the 4-dimensional theory.

Suppose M is an oriented 4-dimensional Riemannian 4-manifold. It is well known that the space of 2-forms splits into self-dual and anti-self-dual parts, corresponding respectively to ± 1-eigenspaces of Hodge $*$ operator. A connection A on a certain principal bundle over M is said to be an *anti-self-dual* instanton if its curvature F, when viewed as a vector-bundle valued two-form, satisfies $*F = -F$. Of course, this definition does not generalize directly to higher dimensions. If, moreover, M is almost Hermitian, i.e., endowed with an almost complex structure compatible with the Riemannian structure, we can formulate the notion in another way. This is based on the observation that anti-self-dual 2-forms are exactly ω-trace free $(1,1)$-forms. Thus, in the almost Hermitian case, we can equally define anti-self-dual instantons to be those connections A satisfying

$$F^{2,0} = \mathrm{tr}_\omega F = 0. \tag{1}$$

The latter description obviously allows generalizations to higher dimensional almost Hermitian manifolds. We will also call connections satisfying (1) *pseudo-Hermitian-Yang-Mills* by slight abuse of terminology (compare [3], for example).

When the dimension is 6, we can formulate (1) in yet another way. Notice that the operator $*(\omega \wedge \cdot)$ maps the space of two forms into itself. It can also be shown that the space of ω-trace free $(1,1)$-forms is exactly the -1 eigenspace of $*(\omega \wedge \cdot)$. Thus, we can rewrite the equation (1) as

$$\omega \wedge F = - * F. \tag{2}$$

For this reason, we also call pseudo-Hermitian-Yang-Mills connections ω-anti-self-dual instantons.

Now, (2) makes sense in even more general contexts. Suppose that M is endowed with an $n-4$ form Ω. Then the operator $*(\Omega \wedge \cdot)$ maps 2-forms into 2-forms. We

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can define Ω-*anti-self-dual instantons* to be those connections A whose curvatures F satisfy

$$\Omega \wedge F = - * F. \tag{3}$$

This definition behaves the best when M has a special structure such as $SU(3)$, G_2 or $Spin(7)$. In this situation, Ω is naturally defined, i.e., Ω is the Kähler form for an $SU(3)$-structure, the defining 3-form for a G_2-structure, or the defining 4-form for a $Spin(7)$-structure.

However, even when Ω is parallel, (3) is in general overdetermined. It is natural to ask when (3) has solutions, even locally, and how general they are. In dimension 6, R. Bryant showed in [3] that there is a large class of almost Hermitian structures, called *quasi-integrable*, for which the differential system for pseudo-Hermitian-Yang-Mills $SU(n)$-connections is involutive. Thus the theory behaves well in quasi-integrable case. It is interesting to ask under what conditions other instanton differential systems will be involutive.

In this paper, we are mainly interested in ω-anti-self-dual instantons on a nearly Kähler 6-manifold and Ω-anti-self-dual instantons on its G_2-cone. We first show that ω-anti-self-dual instantons are automatically Yang-Mills, i.e., are critical points of the Yang-Mills functional. We prove the involutivity of the ω-anti-self-dual instanton system. We construct a Chern-Simons type functional on nearly Kähler 6-manifold. This is an $\mathbf{R}$-valued functional, rather than $\mathbf{R}/\mathbf{Z}$-valued as in 3-manifold case. We show that its critical connections are exactly the ω-anti-self-dual instantons. We compute its gradient flow and discuss its relation with Ω-instantons on the G_2-cone. Second, we derive a Weitzenböck formula for an elliptic operator on nearly Kähler manifolds and apply it to study deformations of ω-anti-self-dual instantons. Finally, we construct a class of instantons on S^6 and $\mathbf{R}^7$ that display interesting singularities.

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1. Some linear algebra in 6 and 7 dimensions. In this section, we clarify notational convention of inner product spaces in 6 and 7 dimensions with emphasis on representation theory of $SU(3)$ and G_2. The interplay between Hodge star operations will be important in later discussions.

Suppose V is an n-dimensional oriented inner product space and let $\{e_i\}_{i=1}^n$ be a oriented orthonormal basis. The inner product on V induces an inner product $\langle , \rangle$ on its dual V^* with the dual basis denoted by $\{dx_i\}$. By taking the convention that $\{dx_{i_1} \wedge \cdots \wedge dx_{i_k}\}$ be orthonormal, we make $\Lambda^* V^*$ an inner product space. We define *Hodge star* $*$ on $\Lambda^* V^*$ by the following rule. Let $\phi \in \Lambda^* V^*$ and its Hodge star $*\phi$ is determined by

$$*\phi \wedge \psi = \langle \phi, \psi \rangle \mathrm{vol}_V, \tag{4}$$

for any $\psi \in \Lambda^* V^*$ where $\mathrm{vol}_V = dx_1 \wedge \cdots \wedge dx_n$ is the volume form on V.

REMARK 1.1. *Through the inner product, we identify vectors and 1-forms. We will not distinguish between them. Thus for example, an linear operator defined on vectors may be thought of as an operator on 1-forms. No confusion should be caused.*

1.1. Dimension 6. In dimension 6, we suppose further that V is endowed with a complex structure and a complex volume form Ψ. The complex structure coupled with the inner product determines a symplectic form ω on V. We normalize these quantities so that $\frac{1}{6}\omega^3 = \frac{i}{8}\Psi\wedge\overline{\Psi} = \mathrm{vol}_V$. It is now natural to complexify V^* and its various exterior powers. Denote $V^*_{\mathbf{C}}$ the space of complex linear forms on V. Then $V^*\otimes\mathbf{C} = V^*_{\mathbf{C}}\oplus\overline{V^*_{\mathbf{C}}}$. We extend the inner product and Hodge star operation complex linearly to $V\otimes\mathbf{C}$.

We pick an orthonormal basis $\{dx_i, dy_i\}_{i=1}^3$ for V^* such that $dz_i = dx_i+\sqrt{-1}dy_i$ is complex linear and that

$$\omega = \frac{\sqrt{-1}}{2}(dz_1\wedge\overline{dz_1}+dz_2\wedge\overline{dz_2}+dz_3\wedge\overline{dz_3}),\quad \Psi = dz_1\wedge dz_2\wedge dz_3.$$

1.1.1. $SU(3)$-representations. The subgroup of $SO(6)$ preserving both ω and ψ is the special unitary group $SU(3)$. Under the action of $SU(3)$, $\Lambda^*V^*\otimes\mathbf{C}$ may be decomposed into irreducible pieces

$$V^*\otimes\mathbf{C} = V^*_{\mathbf{C}}\oplus\overline{V^*_{\mathbf{C}}}$$

$$\Lambda^2V^*\otimes\mathbf{C} = \wedge^2V^*_{\mathbf{C}}\oplus\wedge^2\overline{V^*_{\mathbf{C}}}\oplus\mathbf{C}\cdot\omega\oplus V^{(1,1)}$$

$$\Lambda^3V^*\otimes\mathbf{C} = \mathbf{C}\cdot\Psi\oplus\mathbf{C}\cdot\overline{\Psi}\oplus V^{(2,0)}\oplus V^{(0,2)}\oplus V^*_{\mathbf{C}}\wedge\omega\oplus\overline{V^*_{\mathbf{C}}}\wedge\omega$$

$$\Lambda^4V^*\otimes\mathbf{C} = \overline{V^*_{\mathbf{C}}}\wedge\Psi\oplus V^*_{\mathbf{C}}\wedge\overline{\Psi}\oplus\mathbf{C}\omega^2\oplus V^{(1,1)}_{\mathbf{C}}\wedge\omega$$

$$\Lambda^5V\otimes\mathbf{C} = V^*_{\mathbf{C}}\wedge\omega^2\oplus\overline{V^*_{\mathbf{C}}}\wedge\omega^2,$$

where $V^{(1,1)}$ denotes the representation of the highest weight $(1,1)$, which consists of $(1,1)$-forms whose inner product with ω is zero, $V^{(0,2)}\simeq\mathrm{sym}^2V^*_{\mathbf{C}}$ is the representation of the highest weight $(0,2)$ and $V^{(0,2)}\simeq\overline{V^{(2,0)}}$. The decomposition of 2-forms and 4-forms will be the most important for us. Note that the wedge product with ω gives an isomorphism between the irreducible pieces in Λ^2 and Λ^4 as outlined above. Another isomorphism is given by Hodge star. These two isomorphisms will be fundamental in the definition of anti-self-dual instantons later, so we examine their relation carefully below.

1.1.2. Hodge star. It is easy to compute that

$$*(dz_1\wedge dz_2) = \frac{\sqrt{-1}}{2}dz_1\wedge dz_2\wedge dz_3\wedge\overline{dz_3}$$

$$*(dz_2\wedge dz_3) = \frac{\sqrt{-1}}{2}dz_1\wedge dz_2\wedge dz_3\wedge\overline{dz_1}$$

and

$$*(dz_3\wedge dz_1) = \frac{\sqrt{-1}}{2}dz_1\wedge dz_2\wedge dz_3\wedge\overline{dz_2}.$$

Also

$$\omega \wedge dz_1 \wedge dz_2 = \frac{i}{2} dz_1 \wedge dz_2 \wedge dz_3 \wedge d\overline{z_3}$$

and similarly for $dz_2 \wedge dz_3$, $dz_3 \wedge dz_1$. Thus we have

$$*\alpha = \omega \wedge \alpha \tag{5}$$

for any $\alpha \in \wedge^2 V^*_{\mathbf{C}} \oplus \wedge^2 \overline{V^*_{\mathbf{C}}}$.

Moreover,

$$*\omega = \frac{1}{2}\omega^2. \tag{6}$$

On the other hand,

$$*(dz_1 \wedge d\overline{z_2}) = -\frac{\sqrt{-1}}{2} dz_1 \wedge d\overline{z_2} \wedge dz_3 \wedge d\overline{z_3} = -\omega \wedge (dz_1 \wedge d\overline{z_2}).$$

More generally, we have

$$*\alpha = -\omega \wedge \alpha \tag{7}$$

for any $\alpha \in V^{(1,1)}$.

To conclude, the irreducible (real) $SU(3)$-modules in $\Lambda^2 V^*$ are indexed by the eigenvalues of the operator $*(\omega \wedge)$ (note $*^2 = 1$ on 2-forms).

The other chain of isomorphic $SU(3)$-representations consists of $V^*_{\mathbf{C}}$, $\wedge^2 \overline{V^*_{\mathbf{C}}}$ and various Hodge star images. Again, there are many isomorphisms among these spaces given by compositions of Hodge star, wedge product with the Ψ and with ω. We exploit some of them.

First, we compute that

$$*(dz_3) = \frac{\sqrt{-1}}{4} dz_1 \wedge dz_2 \wedge dz_3 \wedge d\overline{z_1} \wedge d\overline{z_2},$$

and thus,

$$\frac{\sqrt{-1}}{4} * (dz_1 \wedge dz_2 \wedge dz_3 \wedge d\overline{z_1} \wedge d\overline{z_2}) = -dz_3.$$

On the other hand

$$\mathrm{Im}\Psi \wedge d\overline{z_1} \wedge d\overline{z_2} = \frac{\sqrt{-1}}{2} (dz_1 \wedge dz_2 \wedge dz_3 \wedge d\overline{z_1} \wedge d\overline{z_2}).$$

Thus we have

$$\mathrm{Im}\Psi \wedge *(\mathrm{Im}\Psi \wedge d\overline{z_1} \wedge d\overline{z_2}) = -\sqrt{-1} d\overline{z_1} \wedge d\overline{z_2} \wedge d\overline{z_3} \wedge dz_3.$$

It is easy to see

$$\omega \wedge d\overline{z_1} \wedge d\overline{z_2} = -\frac{\sqrt{-1}}{2} \overline{dz_1 \wedge dz_2 \wedge dz_3} \wedge dz_3$$

and thus

$$\mathrm{Im}\Psi \wedge *(\mathrm{Im}\Psi \wedge d\overline{z_1} \wedge d\overline{z_2}) = 2\omega \wedge d\overline{z_1} \wedge d\overline{z_2}.$$

Because $\wedge^2 \overline{V^*_{\mathbf{C}}}$ is an irreducible $SU(3)$-representation, it must hold that

$$\mathrm{Im}\Psi \wedge *(\mathrm{Im}\Psi \wedge \alpha) = 2\omega \wedge \alpha. \tag{8}$$

1.1.3. Some linear operators. We use the $SU(3)$-representation theory to describe several useful linear operators. Some of them are standard, but we hope to fix notation.

First, we describe $\lrcorner$. For any 1-form $v \in V^*$, $v\lrcorner : \Lambda^k V^* \to \Lambda^{k-1} V^*$ is defined as

$$v\lrcorner(\alpha_1 \wedge \cdots \alpha_k) = \sum_i (-1)^{i-1} \langle v, \alpha_i \rangle \alpha_1 \wedge \cdots \hat{\alpha_i} \wedge \cdots \wedge \alpha_k$$

where $\langle\rangle$ is the inner product. Note that this is adjoint to the wedge product in the sense that

$$\langle v\lrcorner\alpha, \beta \rangle = \langle \alpha, v \wedge \beta \rangle.$$

We extend $\lrcorner$ complex linearly to $V^* \otimes \mathbf{C}$ and $\Lambda^*_{\mathbf{C}}$. Something must be cautioned. For instance

$$dz_1 \lrcorner dz_1 = 0.$$

Next, we use $\lrcorner$ to identify $\Lambda^* V^*$ inside $\mathfrak{so}(V^*)$ by

$$\beta : \alpha \mapsto \alpha\lrcorner\beta.$$

The inverse map is given by for any $A \in \mathfrak{so}(V^*)$

$$A :\mapsto \frac{1}{2} \sum \omega_i \wedge A(\omega_i)$$

where ω_i is an orthonormal basis.

Now, if a linear map commutes with the complex structure on V^*, i.e., maps $V^*_{\mathbf{C}}$ to itself, then it is easy to see that viewed as a 2-form, A lies in the space $\Lambda^{1,1}$. In fact, the corresponding 2-form is given by

$$\frac{1}{2} d\overline{z_i} \wedge A(dz_i).$$

Since Ψ is $SU(3)$-invariant, any linear combination r of maps $v \mapsto v\lrcorner\mathrm{Re}(\Psi)$ and $v \mapsto v\lrcorner\mathrm{Im}\Psi$ gives an $SU(3)$-equivariant map $V^* \to \wedge^2 V^*$. The image $r(v)$ may be viewed as a map $V^* \to V^*$. Skewsymmetrizing $r(v)$ gives a map $\wedge^2 V^* \to \wedge^2 V^*$

$$r(v) : \alpha \wedge \beta \mapsto r(v)(\alpha) \wedge \beta + \alpha \wedge r(v)(\beta).$$

We still denote the map by $r(v)$. From $SU(3)$-equivariance of r, we see that

$$r(g(v))(\alpha) = g(r(v)(g^{-1}\alpha)), \tag{9}$$

for any $g \in SU(3)$, $v \in V$ and $\alpha \in \wedge^2 V^*$. We define

$$\begin{aligned} \pi_{(2,0)}(\beta) \quad = \quad & \tfrac{1}{4}\langle \beta, \overline{dz_1 \wedge dz_2} \rangle dz_1 \wedge dz_2 + \tfrac{1}{4}\langle \beta, dz_1 \wedge dz_2 \rangle \overline{dz_1 \wedge dz_2} \\ & + \tfrac{1}{4}\langle \beta, \overline{dz_1 \wedge dz_3} \rangle dz_1 \wedge dz_3 + \tfrac{1}{4}\langle \beta, dz_1 \wedge dz_3 \rangle \overline{dz_1 \wedge dz_3} \\ & + \tfrac{1}{4}\langle \beta, \overline{dz_2 \wedge dz_3} \rangle dz_2 \wedge dz_3 + \tfrac{1}{4}\langle \beta, dz_2 \wedge dz_3 \rangle \overline{dz_2 \wedge dz_3}, \end{aligned} \tag{10}$$

dually

$$-\sqrt{-1}\pi_{(0,2)}(\beta) \quad = \quad \tfrac{1}{4}\langle\beta, \overline{dz_1\wedge dz_2}\rangle dz_1\wedge dz_2 - \tfrac{1}{4}\langle\beta, dz_1\wedge dz_2\rangle\overline{dz_1\wedge dz_2}$$
$$+\tfrac{1}{4}\langle\beta, \overline{dz_1\wedge dz_3}\rangle dz_1\wedge dz_3 - \tfrac{1}{4}\langle\beta, dz_1\wedge dz_3\rangle\overline{dz_1\wedge dz_3} \tag{11}$$
$$+\tfrac{1}{4}\langle\beta, \overline{dz_2\wedge dz_3}\rangle dz_2\wedge dz_3 - \tfrac{1}{4}\langle\beta, dz_2\wedge dz_3\rangle\overline{dz_2\wedge dz_3}.$$

and

$$\pi_\omega(\beta) = \frac{1}{3}\langle\beta,\omega\rangle\omega \tag{12}$$

where the bracket is the complex extension of the inner product. Note that both $\pi_{(2,0)}$ and $\pi_{(0,2)}$ are real operators. Also define the projection onto ω-trace free 2-forms

$$\pi_0^{1,1} = I - \pi_{(2,0)} - \pi_\omega. \tag{13}$$

Note that $\pi_{(2,0)}$ are identity on forms of type $(2,0)$ and type $(0,2)$. While $\pi_{(0,2)}$ is multiplication by $\sqrt{-1}$ on $(2,0)$ forms and $-\sqrt{-1}$ on $(0,2)$ forms. Both of them are clearly $SU(3)$ equivariant. In fact, if we think of the diagonal elements in $\Lambda^2V^*\oplus\Lambda^2\overline{V^*}$ as a real representation of $SU(3)$, the space of $SU(3)$ equivariant homomorphisms is real 2-dimensional, spanned by $\pi_{(2,0)}$ and $\pi_{(0,2)}$. They satisfy the relation

$$\pi_{(2,0)}^2 = \pi_{(2,0)}, \quad \pi_{(0,2)}^2 = -\pi_{(2,0)}. \tag{14}$$

In particular, $\pi_{(2,0)}$ is a projection but $\pi_{(0,2)}$ is not.

Denote

$$P = \lambda\pi_{(2,0)} + \mu\pi_\omega \tag{15}$$

where λ and μ are real constants. Clearly, P is a real operator and commutes with the action of $SU(3)$. Moreover, $P^2 = \lambda^2\pi_{(2,0)} + \mu^2\pi_\omega$.

Let $\{v_i\}_{i=1}^6$ be a orthonormal basis of V and ω_i dual basis. We define a map by

$$B(\alpha) = \sum_i \omega_i \lrcorner [r(v_i), P^2](\alpha) \tag{16}$$

where $\alpha \in \wedge^2V^*$. Note that the definition of B does not depend on the choice of the orthonormal basis. We have the following result concerning the B.

Proposition 1.2. *The operator B factors through a (possibly complex) linear combination of $\pi_{(2,0)}$ and $\pi_{(0,2)}$.*

Proof. For any $\alpha \in \wedge^2V^*$ and $g \in SU(3)$, we have

$$\begin{aligned} B(g\alpha) &= \sum_i \omega_i \lrcorner [r(v_i), P^2](g\alpha) \\ &= \sum_i \omega_i \lrcorner g([r(g^{-1}(v_i)), P^2](\alpha)) \\ &= g(\sum_i g^{-1}(\omega_i)\lrcorner([r(g^{-1}(v_i)), P^2](\alpha)) \\ &= g(B(\alpha)), \end{aligned}$$

where the second equality is due to (9) as well as the commutativity of P and $SU(3)$, the third is because $g(v \lrcorner \alpha) = g(v) \lrcorner g(\alpha)$ and the last is because of the independence of orthonormal coframes in the definition of B. So B gives a $SU(3)$-equivariant map from $\wedge^2 V^* \to V^*$. Since as a $SU(3)$-space, $\wedge^2 V^*$ contains only a copy of the irreducible representation isomorphic to V^*, namely, $(\wedge^2 V^*_{\mathbf{C}} \oplus \overline{\wedge^2 V^*_{\mathbf{C}}})_{\mathbf{R}}$, we know from Schur's Lemma, B must factor through a linear combination of $\pi_{(2,0)}$ and $\pi_{(0,2)}$. ☐

Next, for each i, j, we consider the operator on V^*,

$$L(\omega_i, \omega_j)(\alpha) = \omega_i \wedge (\omega_j \lrcorner \alpha) + \omega_i \lrcorner P^2(\omega_j \wedge \alpha). \tag{17}$$

We have the following result concerning L.

PROPOSITION 1.3. *Let $\lambda = \sqrt{2}$ and $\mu = \sqrt{3}$ and thus*

$$P = \sqrt{2}\pi_{(2,0)} + \sqrt{3}\pi_\omega. \tag{18}$$

Then the operator L satisfies the Clifford relations, i.e.,

$$L(\omega_i, \omega_j) + L(\omega_j, \omega_i) = 2\delta_{ij}.$$

Moreover, we define an operator $M : \wedge^2 \mathbf{R}^6 \to \mathrm{End}(\mathbf{R}^6)$ by linearly extending $L(\omega_i, \omega_j)$ for $i \neq j$. Then M is an $SU(3)$ equivariant map from $\Lambda^2 V^*$ to $V \otimes V^*$. In fact, we have

$$M(\beta)(v) = v \lrcorner (-2\pi_0^{1,1}\beta + \pi_\omega \beta).$$

Proof. Since L is real, it suffices to prove the proposition for $(1,0)$ forms. Without loss of generality, we check for $L(dx_1, dx_1)$, $L(dx_1, dy_1)$ and $L(dx_1, dx_2)$. Let $\alpha_i = dx_i + \sqrt{-1}dy_i$. These form a basis for $V^*_{\mathbf{C}}$. Then

$$\begin{aligned}
L(dx_1, dx_1)(\alpha_1) &= dx_1 \wedge (dx_1 \lrcorner \alpha_1) + dx_1 \lrcorner P^2(dx_1 \wedge \alpha_1) \\
&= dx_1 + dx_1 \lrcorner (\lambda^2 \pi_{(2,0)} + \mu^2 \pi_\omega)(\frac{1}{2}\overline{dz_1} \wedge dz_1) \\
&= dx_1 + dx_1 \lrcorner \frac{\sqrt{-1}}{3}\mu^2 \omega \\
&= dx_1 + \sqrt{-1}dy_1
\end{aligned}$$

because $\mu^2 = 3$.

Using $\lambda^2 = 2$, one can similarly compute, for $i = 2, 3$

$$L(dx_1, dx_1)(\alpha_i) = \alpha_i.$$

This proves the first equality.

Now consider $L(dx_1, dy_1)$. We compute

$$\begin{aligned}
L(dx_1, dy_1)(\alpha_1) &= \sqrt{-1}dx_1 + dy_1 \lrcorner (\lambda^2 \pi_{(2,0)} + \mu^2 \pi_\omega)(\frac{-1}{2\sqrt{-1}}\overline{dz_1} \wedge dz_1) \\
&= \sqrt{-1}dx_1 + dx_1 \lrcorner \frac{\mu^2}{3}\sqrt{-1}\omega \\
&= \sqrt{-1}dx_1 - \frac{1}{3}\mu^2 dx_1 \lrcorner \omega \\
&= \sqrt{-1}dx_1 - dy_1 \\
&= \sqrt{-1}(dz_1)
\end{aligned}$$

and

$$\begin{aligned}
L(dy_1, dx_1)(\alpha_1) &= dy_1 + dy_1 \lrcorner (\lambda^2 \pi_{(2,0)} + \mu^2 \pi_\omega)(\frac{1}{2} d\overline{z_1} \wedge dz_1) \\
&= dy_1 + dy_1 \lrcorner \mu^2 \pi_\omega (\frac{1}{2} d\overline{z_1} \wedge dz_1) \\
&= dy_1 + dy_1 \lrcorner \mu^2 \frac{\sqrt{-1}}{3} \omega \\
&= dy_1 - \sqrt{-1} dx_1 \\
&= -\sqrt{-1}(dx_1 + \sqrt{-1} dy_1).
\end{aligned}$$

Thus

$$L(dx_1, dy_1)(\alpha_1) + L(dy_1, dx_1)(\alpha_1) = 0.$$

Also,

$$\begin{aligned}
L(dx_1, dy_1)(\alpha_2) &= 0 + dx_1 \lrcorner (\lambda^2 \pi_{(2,0)} + \mu^2 \pi_\omega)(\frac{1}{2\sqrt{-1}}(dz_1 \wedge dz_2 - d\overline{z_1} \wedge dz_2)) \\
&= -\sqrt{-1} dx_1 \lrcorner (dz_1 \wedge dz_2) \\
&= -\sqrt{-1} dz_2
\end{aligned}$$

and

$$\begin{aligned}
L(dy_1, dx_1)(\alpha_2) &= dy_1 \lrcorner (\lambda^2 \pi_{2,0})(\frac{1}{2} dz_1 \wedge dz_2) \\
&= \sqrt{-1} dz_2.
\end{aligned}$$

Thus

$$L(dx_1, dy_1)(\alpha_2) + L(dy_1, dx_1)(\alpha_2) = 0.$$

Similarly

$$L(dx_1, dy_1)(\alpha_3) + L(dy_1, dx_1)(\alpha_3) = 0.$$

Next we consider $L(dx_1, dx_2)$.

$$\begin{aligned}
L(dx_1, dx_2)(dz_1) &= 0 + dx_1 \lrcorner (\lambda^2 \pi_{(2,0)} + \mu^2 \pi_\omega)(\frac{1}{2} dz_2 \wedge dz_1 + \frac{1}{2} d\overline{z_2} \wedge dz_1) \\
&= dx_1 \lrcorner \lambda^2 \pi_{(2,0)} (\frac{1}{2} dz_2 \wedge dz_1) \\
&= dx_1 \lrcorner (dz_2 \wedge dz_1) \\
&= -dz_2
\end{aligned}$$

and

$$\begin{aligned}
L(dx_2, dx_1)(dz_1) &= dx_2 + dx_2 \lrcorner (\lambda^2 \pi_{(2,0)} + \mu^2 \pi_\omega)(\frac{1}{2} d\overline{z_1} \wedge dz_1) \\
&= dx_2 + \mu^2 \pi_\omega (\frac{1}{2} d\overline{z_1} \wedge dz_1) \\
&= dx_2 + \sqrt{-1} dx_2 \lrcorner \omega \\
&= dx_2 + \sqrt{-1} dy_2 = dz_2
\end{aligned}$$

Thus

$$L(dx_1, dx_2)(dz_1) + L(dx_2, dx_1)(dz_1) = 0.$$

Similarly

$$L(dx_1, dx_2)(dz_2) + L(dx_2, dx_1)(dz_2) = 0.$$

Moreover,

$$\begin{aligned} L(dx_1, dx_2)(dz_3) &= dx_1 \wedge (dx_2 \lrcorner dz_3) + dx_1 \lrcorner P^2(dx_2 \wedge dz_3) \\ &= 0 + dx_1 \lrcorner (\lambda^2 \pi_{(2,0)} + \mu^2 \pi_\omega)(\frac{1}{2} dz_2 \wedge dz_3 + \frac{1}{2} d\overline{z_2} \wedge dz_3) \\ &= dx_1 \lrcorner \lambda^2 \pi_{(2,0)} \frac{1}{2} dz_2 \wedge dz_3 \\ &= 0. \end{aligned}$$

and

$$\begin{aligned} L(dx_2, dx_1)(dz_3) &= dx_2 \wedge (dx_1 \lrcorner dz_3) + dx_2 \lrcorner P^2(dx_1 \lrcorner dz_3) \\ &= 0 + dx_2 \lrcorner \lambda^2 \frac{1}{2} dz_1 \wedge dz_3 \\ &= 0. \end{aligned}$$

Thus

$$L(dx_1, dx_2)(dz_3) + L(dx_2, dx_1)(dz_3) = 0.$$

So far we have proved that

$$L(dx_1, dx_1) = 1$$

and

$$L(dx_1, dy_1) + L(dy_1, dx_1) = L(dx_1, dx_2) + L(dx_2, dx_1) = 0.$$

By symmetry and the linearity of L we see that

$$L(dx_i, dx_i) = L(dy_i, dy_i) = 1,$$

$$L(dx_i, dx_j) + L(dx_j, dx_i) = 0, i \neq j$$

and

$$L(dx_i, dy_j) + L(dy_j, dx_i) = 0.$$

For instance, in order to show

$$L(dx_1, dy_2) + L(dy_2, dx_1) = 0$$

we replace dx_2 by dy_2 and dy_2 by $-dx_2$. Then it follows from the calculation on $L(dx_1, dx_2)$.

Now for arbitray orthonormal basis ω_i, the Clifford relations follow from the fact that the orthogonal transformations act transitively on coframes. If they hold for a particular coframe, they hold for all.

Suppose $g \in SU(3)$. We have for any 1-form α,

$$\begin{aligned} L(\omega_i, \omega_j)(g\alpha) &= \omega_i \wedge \omega_j \lrcorner g(\alpha) + \omega_i \lrcorner P^2(\omega_j \wedge g(\alpha)) \\ &= g[(g^{-1}\omega_i) \wedge (g^{-1}\omega_j) \lrcorner \alpha + (g^{-1}\omega_i) \lrcorner P^2(g^{-1}(\omega_j) \wedge \alpha)] \\ &= gL(g^{-1}\omega_i, g^{-1}\omega_j)(\alpha). \end{aligned}$$

Now by the definition of M we have for any $\alpha = a_i\omega_i$ and $\beta = b_j\omega_j$,

$$M(\alpha, \beta) = \frac{1}{2}(a_ib_j - a_jb_i)L(\omega_i, \omega_j).$$

Thus,

$$\begin{aligned} M(g\alpha, g\beta)(v) &= \tfrac{1}{2}(a_ib_j - a_jb_i)L(g\omega_i, g\omega_j)(v) \\ &= \tfrac{1}{2}(a_ib_j - a_jb_i)gL(\omega_i, \omega_j)g^{-1}(v) \\ &= gM(\alpha, \beta)g^{-1}(v) \end{aligned}$$

i.e., M is $SU(3)$ equivariant.

Note from the above computations, $M(\beta)$ maps $(1,0)$ forms to $(1,0)$ forms for any two-form β. Moreover, since P^2 is self-adjoint, $M(\beta)$ also preserves the inner product. Thus $M(\beta)$, when identified as a two-form, takes value in $\Lambda^{1,1}$. Combined with the $SU(3)$-equivariance, M gives a $SU(3)$-equivariant map from Λ^2V^* to $\Lambda^{1,1}$. Since both of the two irreducible components $\Lambda_0^{1,1}$ and $\mathbf{R}\omega$ are real, $Hom_{SU(3)}(\Lambda^2, \Lambda^{1,1})$ is real 2-dimensional. In other words, there exist two constants a, b so that

$$M(\beta) = a\pi_0^{1,1}(\beta) + b\pi_\omega(\beta).$$

It is a matter of computing examples to determine the constants.

If we take $\beta = dx_1 \wedge dy_1$, then by the convention described above,

$$\begin{aligned} M(\beta) &= \frac{1}{2}(d\overline{z_1} \wedge M(\beta)(dz_1) + d\overline{z_2} \wedge M(\beta)(dz_2) + d\overline{z_3} \wedge M(\beta)(dz_3)) \\ &= \frac{\sqrt{-1}}{2}(d\overline{z_1} \wedge dz_1 - d\overline{z_2} \wedge dz_2 - d\overline{z_3} \wedge dz_3) \\ &= -(dx_1 \wedge dy_1 - dx_2 \wedge dy_2 - dx_3 \wedge dy_3). \end{aligned}$$

On the other hand

$$\pi_\omega(dx_1 \wedge dy_1) = \frac{1}{3}\omega.$$

and

$$\pi_0^{1,1}(dx_1 \wedge dy_1) = \frac{1}{3}(2dx_1 \wedge dy_1 - dx_2 \wedge dy_2 - dx_3 \wedge dy_3).$$

Consequently $a = -2,\ b = 1$. Thus M is of the desired form. $\square$

1.2. Dimension 7. Now we assume that $\dim V = 7$ and we pick an oriented orthonormal basis for V^* denoted by $\{dx_1, dy_1, dx_2, dy_2, dx_3, dy_3, du\}$. For later use, let

$$dz_i = dx_i + \sqrt{-1}dy_i$$

and define ω and Ψ as in §2.1. We introduce a special three form

$$\begin{aligned} \Omega &= du \wedge \omega + Im\Psi \\ &= du \wedge (dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3) \\ &\quad + dx_1 \wedge dx_2 \wedge dy_3 - dy_1 \wedge dy_2 \wedge dy_3 \\ &\quad + dy_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge dy_2 \wedge dx_3. \end{aligned} \tag{19}$$

Due to [4], it is now well-known that the exceptional Lie group G_2 may be defined as the stabilizers of Ω. For this reason, we call Ω the *fundamental 3-form.* We embed $\mathbf{R}^6$ considered in the last section into V to be the hyperplane $du = 0$. We also let $SU(3)$ act on V by identity on the line $dx_i = dy_i = 0$ and the standard action on $du = 0$. Clearly, $SU(3)$ preserves Ω, so it embeds into G_2 as a Lie subgroup.

1.2.1. G_2-representations. A good resource on this part is [5]. We recall some basic facts. The standard V^* is irreducible with the highest weight $(1, 0)$. The most important part for us is $\Lambda^2 V^*$. It decomposes as the sum of two irreducible pieces $V^{(1,0)} \oplus V^{(1,1)}$ where $V^{(a,b)}$ is the irreducible representation of G_2 with the highest weight (a, b). The subspace $V^{(1,0)}$ is 7-dimensional, consisting of 2-forms $v \lrcorner \Omega$ for any $v \in V^*$. The other one $V^{(1,1)}$ is isomorphic to the Lie algebra $\mathfrak{g}_2$.

The space $\Lambda^5 V^*$ is isomorphic to Λ^2 as G_2-modules either by wedge product with Ω or by the Hodge star operation. Again the interplay between these two isomorphisms will be important in defining anti-self-dual instantons in dimesion 7.

1.2.2. Hodge star. We only consider the Hodge star on Λ^2. Now we may compute that

$$*\alpha = \frac{1}{2}\Omega \wedge \alpha \tag{20}$$

for all $\alpha \in V^{(1,0)} \subset \Lambda^2$ and that

$$*\alpha = -\Omega \wedge \alpha \tag{21}$$

for $\alpha \in \mathfrak{g}_2$. These may be checked for special forms (e. g., $\alpha = du \lrcorner \Omega \in V^{(1,0)}$ and $\alpha = dz_1 \wedge d\overline{z_2} \in \mathfrak{su}(3) \subset \mathfrak{g}_2$). Then, since these spaces are irreducible and both $*$ and $\Omega\wedge$ commutes with G_2 action, we know these relations must be true for the whole spaces. Thus, these irreducible subspaces are indexed by the eigenvalues of $*(\Omega\wedge)$.

2. Anti-self-dual instantons on nearly Kähler 6-manifolds and G_2-cones. Let G be a compact Lie group. Suppose X^n is a smooth manifold endowed with an (n-4)-form Υ (for our purposes, $X = M$ is nearly Kähler and Υ is the $(1, 1)$-form ω, or $X = N$ has G_2 holonomy and Υ is the fundamental 3-form Ω). Suppose also Υ is a (n-4)-form on M and $\mathbf{P}$ is a principal G-bundle over X. A connection A on $\mathbf{P}$ is called Υ-*instanton* if its curvature F_A satisfies

$$\Upsilon \wedge F_A = - *_X F_A. \tag{22}$$

Remark 2.1. *When G is a unitary group, our definition is different from the one used in [11] (see Remark 1 in its §1.2, however). When G is a special unitary group, these two definitions coincide. This is the group we will use mostly.*

Note that if Υ is closed, an Υ-instanton A is Yang-Mills, i.e., it satisfies the Euler-Lagrange equation of the Yang-Mills functional, since

$$d_A *_X F = -d\Upsilon \wedge F \pm \Upsilon \wedge d_A F = 0$$

because of the Bianchi identity. Thus, an Ω-instanton on a manifold with holonomy in G_2 is Yang-Mills since Ω is closed. Remarkably, as we will show later, when $X = M$ is nearly Kähler, although ω is not closed, an ω-anti-self-dual instanton is still Yang-Mills.

2.1. Nearly Kähler 6-manifolds. In this subsection, we collect basic facts about nearly Kähler 6-manifolds. The concept was first introduced and studied by A. Gray in [9]. Later on, N. Hitchin [10] found that it is a critical point of a diffeomorphism invariant functional and thus put it in a more natural context.

An $SU(3)$ structure on a 6-manifold M is a reduction of the total coframe bundle to an $SU(3)$ subbundle. It may be specified by a real two-form ω of type $(1,1)$ and a $(3,0)$-form Ψ normalized so that $\frac{1}{6}\omega^3 = \frac{i}{8}\Psi \wedge \overline{\Psi}$. A nearly Kähler structure is an $SU(3)$-structure for which

$$d\omega = 3c\mathrm{Im}\Psi, \quad d\Psi = 2c\omega^2. \tag{23}$$

for some real constant c.

When $c = 0$, the underlying almost complex structure is integrable. In fact, M is Calabi-Yau. When $c \neq 0$, by scaling the metric, we can always assume $c = 1$. In this situation, M is usually called *strictly nearly Kähler*. In this chapter, we assume from now on that $c = 1$ and we speak of this as *nearly Kähler* without danger of confusion.

2.1.1. Structure equations. Let $\alpha_i, i = 1, \cdots, 3$ be a local special unitary coframe, i.e., α_i is complex linear and

$$\omega = \frac{\sqrt{-1}}{2}(\alpha_1 \wedge \overline{\alpha_1} + \alpha_2 \wedge \overline{\alpha_2} + \alpha_3 \wedge \overline{\alpha_3}), \quad \Psi = \alpha_1 \wedge \alpha_2 \wedge \alpha_3.$$

There exists a unique $\mathfrak{su}(3)$-valued 1-form $(\kappa_{i\bar{j}})$ so that

$$d\alpha_i = -\kappa_{i\bar{j}} \wedge \alpha_j + \epsilon_{ijk}\overline{\alpha_j \wedge \alpha_k} \tag{24}$$

where summation is understood when repeated barred and unbarred indices appear. Differentiate this and we get the curvature of κ:

$$d\kappa_{i\bar{j}} + \kappa_{i\bar{k}} \wedge \kappa_{k\bar{j}} = \frac{1}{4}(3\alpha_i \wedge \overline{\alpha_j} - \delta_{i\bar{j}}\alpha_l \wedge \overline{\alpha_l}) + K_{i\bar{j}p\bar{q}}\alpha_q \wedge \overline{\alpha_p}, \tag{25}$$

where $K_{i\bar{j}p\bar{q}} = K_{p\bar{j}i\bar{q}} = K_{i\bar{q}p\bar{j}} = \overline{K_{j\bar{i}q\bar{p}}}$ and $K_{i\bar{i}p\bar{q}} = 0$.

It follows from the structure equations that κ is a pseudo-Hermitian-Yang-Mills connection on the complex tangent bundle of M.

Compact nearly Kähler examples include the standard S^6, the flag manifold $SU(3)/T^2$, $S^3 \times S^3$, and $\mathbf{CP}^3$ (with an unusual almost complex structure). All these examples are homogeneous. On the other hand, it remains open to find non-homogeneous compact examples.

EXAMPLE 2.2 (G_2-invariant S^6). *The standard G_2 invariant almost complex structure on S^6 is perhaps the best known non-integrable almost complex structure. As a subgroup of $SO(7)$, G_2 acts transitively on S^6 and the stabilizer of any point is isomorphic to $SU(3) \subset G_2$. Thus, G_2 preserves an $SU(3)$-structure on S^6. This $SU(3)$ structure is in fact nearly Kähler. Using Maurer-Cartan forms on G_2, we write the nearly Kähler structure equations*

$$d\alpha_i = -\kappa_{i\bar{j}} \wedge \alpha_j + \epsilon_{ijk}\overline{\alpha_j \wedge \alpha_k} \tag{26}$$

$$d\kappa_{i\bar{j}} + \kappa_{i\bar{k}} \wedge \kappa_{k\bar{j}} = \frac{1}{4}(3\alpha_i \wedge \overline{\alpha_j} - \delta_{i\bar{j}}\alpha_l \wedge \overline{\alpha_l}). \tag{27}$$

2.2. G_2-cones over a nearly Kähler 6-manifold. A G_2-structure on a 7-manifold N^7 is a reduction of the total coframe bundle to a G_2-subbundle. Suppose N^7 has such a G_2 structure. Then, on N, there exists a fundamental 3-form Ω characterized by the property that at each point x, there exists a linear isomorphism $u : T_xN \to \mathbf{R}^7$ so that $\Omega_x = u^*(\Omega_0)$. Conversely, given such a fundamental 3-form Ω on N, the set of such linear isomorphisms forms a G_2 subbundle of the total coframe bundle and thus defines a G_2-structure on N.

Associated with any G_2-structure, N has a metric g. The Levi-Civita connection of g has its holonomy group contained in G_2 if and only if $d\Omega = d(*\Omega) = 0$. R. Bryant constructed the first metric with holonomy G_2 [4]. It was the cone metric over $\mathbf{R}_+ \times SU(3)/T^2$. It is now well-known that if M^5 is nearly Kähler with the metric g_M, the cone metric on $N = \mathbf{R}_+ \times M^6$ defined as

$$g_N = dt^2 + t^2 g_M$$

has holonomy in G_2. The fundamental 3-form is

$$\Omega = t^2 dt \wedge \omega + t^3 \mathrm{Im}\Psi.$$

Such conical G_2-singularities were used by string physicists recently to construct string models with chiral matter fields (see [2], [1]). For us, the case $M = S^6$ is especially important. Then the cone has a removable singularity and in fact $N = \mathbf{R}^7$. When studying anti-self-dual instantons on manifolds with G_2 holonomy, $\mathbf{R}^7$ plays the natural role of an infinitesimal model.

2.2.1. Hodge star on 2-forms. Suppose $\omega_i (i = 1, \cdots, 6)$ is an oriented local orthonormal coframe for M. Then, $dt, t\omega_i (i = 1, \cdots, 6)$ form an oriented local orthonormal coframe for the cone N. Denote $*_M$ and $*_N$ Hodge star operations on M and N respectively. It is easy to show that

$$t^2 dt \wedge *_M(\omega_i \wedge \omega_j) = *_N(\omega_i \wedge \omega_j)$$

and

$$*_N(dt \wedge \omega_i) = t^4 *_M (\omega_i).$$

Consequently, if a 2-form α on N satisfies $\frac{\partial}{\partial t} \lrcorner \alpha = 0$, its Hodge star may be computed by

$$*_N\alpha = t^2 dt \wedge *_M(\alpha), \tag{28}$$

and if $\alpha = dt \wedge \beta$ with $\frac{\partial}{\partial t} \lrcorner \beta = 0$,

$$*_N(dt \wedge \beta) = t^4 *_M (\beta), \tag{29}$$

where we extend $*_M$ linearly across functions on N. The formula (28), (29) will be important below.

2.3. ω **anti-self-dual instantons .** If the underlying manifold is almost Hermitian with the Kähler form ω, we may decompose the curvature as

$$F = F^{2,0} + \overline{F^{2,0}} + (F^\circ)^{1,1} + H\omega,$$

where $F^{2,0}$ is of type $(2,0)$ and $(F^\circ)^{1,1}$ is of type $(1,1)$ but with zero ω-trace. Now the ω-anti-self-dual instanton condition (22) is equivalent to

$$F^{2,0} = H = 0,$$

or, in terms of the operator defined in Proposition 1.3,

$$P(F) = 0.$$

REMARK 2.3. *In the case G is a special unitary group, the above argument implies that an ω-instanton is the same as a pseudo-Hermitian-Yang-Mills connection on the canonically associated complex vector bundle.*

If, moreover, we are working on a nearly Kähler manifold, this condition may be simplified.

LEMMA 2.4. *Suppose A is a connection on nearly Kähler M^6 and F is its curvature. The following are equivalent:*

a. $F \wedge \mathrm{Im}\Psi = 0$.
b. $F \wedge \Psi = 0$.
c. $F \wedge \mathrm{Re}\Psi = 0$.
d. A is an ω-anti-self-dual instanton.

Consequently, if F is of type $(1,1)$, A is an ω-anti-self-dual instanton.

Proof.

1. a$\Longrightarrow$b. We write $F = F^{2,0} + \overline{F^{2,0}} + (F^\circ)^{1,1} + H\omega$. Then $F \wedge \mathrm{Im}\Psi = 0$ gives

$$F \wedge (\Psi - \overline{\Psi}) = 0,$$

i.e.,

$$F^{2,0} \wedge \overline{\Psi} = 0.$$

It follows then that

$$F \wedge \overline{\Psi} = 0$$

and hence $F \wedge \Psi = 0$.

2. b$\Longrightarrow$c is obvious.

3. c$\Longrightarrow$d. As mentioned before, A is ω-anti-self-dual if and only if $F^{2,0} = 0$ and $H = 0$ in the above decomposition. Now

$$F \wedge \mathrm{Re}\Psi = \frac{1}{2} F \wedge (\Psi + \overline{\Psi}) = \frac{1}{2} F^{2,0} \wedge \overline{\Psi} + \overline{F^{2,0}} \wedge \Psi.$$

Thus c gives $F^{2,0} = 0$. Differentiating c gives

$$\begin{aligned} 0 &= d_A(F \wedge \mathrm{Re}\Psi) \\ &= d_A F \wedge \mathrm{Re}\Psi + F \wedge d\mathrm{Re}\Psi \\ &= 0 + 2F \wedge \omega^2 \\ &= 2H\omega^3 \end{aligned}$$

where the last equality uses (23) and the Bianchi identity. Hence

$$H = 0.$$

4. d$\Longrightarrow$a is obvious.

∎

This lemma says that we could have defined an ω-anti-self-dual as $F^{2,0} = 0$. This reduces the indeterminacy and will be useful later when we construct concrete examples.

Remark 2.5. *The same result holds for a more general class of almost complex manifolds, called strictly quasi-integrable in [3]. We leave it for the reader to carry out the details. In fact, this has already been observed in [3] for unitary instantons.*

2.3.1. Generality. We now address the problem of the involutivity problem of the instanton equations. First, the instanton equations may be rephrased as

$$F \wedge \Psi = 0 \tag{30}$$

and

$$F \wedge \omega^2 = 0. \tag{31}$$

Since the problem is local, we assume that the bundle is trivial, and the connection is simply a $\mathfrak{g}$-valued 1-form A. The differential system we need to analyze is

$$\mathbf{I} = \langle F \wedge \Psi, F \wedge \omega^2 \rangle,$$

defined on $M \times \mathfrak{g}$ where $F = dA + \frac{1}{2}[A, A]$. We have

Lemma 2.6. *The system* $\mathbf{I}$ *is involutive with Cartan characters*

$$(s_0, s_1, s_2, s_3, s_4, s_5, s_6) = (0, 0, 0, 0, 2d, 3d, d).$$

where $d = \dim G$*.*

Proof. Note that

$$d(F \wedge \Psi) = [A, F] \wedge \Psi + F \wedge d\Psi \equiv 0, \quad \mathrm{mod}\ \mathbf{I}.$$

because of Bianchi identity and the nearly Kähler condition $d\Psi = 2\omega^2$. It is now routine to check the system is involutive with displayed characters. ∎

Some remarks are in order.

REMARK 2.7. *More generally, Lemma 2.6 holds for similarly defined ω-instantons on a quasi-integrable $U(3)$-structure (see [3] for the definition). We leave the details for the interested readers. When $G = U(r)$ is a unitary group, it is treated in [3] .*

REMARK 2.8. *For nearly Kähler M, we could have used the differential system $\langle F \wedge \mathrm{Im}\Psi \rangle$ by Lemma 2.4. The reader can check that this is involutive with the last Cartan character also equal to d. The advantage of the original system is that it applies to more general almost complex manifolds.*

REMARK 2.9. *The last character is d, due to the fact that gauge transformations depend on d functions of 6 variables and that instanton equations are gauge-invariant. We leave for the interested reader to impose a symmetry breaking condition.*

2.3.2. Instantons are Yang-Mills. Now we compute

$$d_A *_M F = -d\omega \wedge F - \omega \wedge d_A F = -3\mathrm{Im}\Psi \wedge F = 0$$

because F is of type $(1,1)$.

PROPOSITION 2.10. *An ω-instanton on a nearly Kähler 6-manifold is Yang-Mills.*

A consequence is some removable singularity results for instantons on nearly Kähler 6-manifolds.

COROLLARY 2.11. *Suppose that all representations of $\pi_1(M) \to G$ are trivial and that E is a trivial smooth bundle over M. Assume that A is a ω-instanton on E with a closed singular set S whose $n-4$ Hausdorff measure is locally finite. Then there exists $\epsilon = \epsilon(G, M)$ such that if*

$$\| F_A \|_\infty \leq \epsilon,$$

then the singularity of A is removable.

COROLLARY 2.12. *Suppose that all representations of $\pi_1(M) \to G$ are trivial and that E is a trivial smooth G bundle over M. Assume that A is a ω-instanton on E whose singular set is a closed smooth submanifold of codimension at least 4. Then there exists $\epsilon = \epsilon(G, M)$ such that if*

$$\| F_A \|_{L^{\frac{6}{2}}(M)} \leq \epsilon,$$

then the singularity of A is removable.

Both are proved by employing the results in [12].

2.3.3. Instantons as critical points of a Chern-Simons functional. Consider the functional

$$CS(A) = \int_M \mathrm{tr}(F_A^2) \wedge \omega. \tag{32}$$

On a Kähler manifold, since ω is closed, CS is a topological constant. However, on a nearly Kähler manifold, this gives more interesting information.

It is easy to compute that the first variation of CS is

$$\delta CS = 2\int_M \mathrm{tr}(F_A \wedge d_A \delta A) \wedge \omega.$$

Integration by parts gives

$$\delta CS = 2\int_M \mathrm{tr}[d_A(F_A \wedge \omega) \wedge \delta A].$$

Thus the Euler-Lagrange equation for CS is

$$d_A(F_A \wedge \omega) = 0.$$

Using Bianchi Identity, we see this is equivalent to

$$F \wedge \mathrm{Im}\Psi = 0. \tag{33}$$

It follows from Lemma 2.4 that

Proposition 2.13. *An ω-anti-self-dual instanton is equivalent to a critical connection of the Chern-Simons functional CS.*

This makes it possible to use variational methods to study ω-anti-self-dual instantons on nearly Kähler 6-manifolds.

It also follows that the gradient flow of CS takes the form

$$\frac{d}{dt}A = *_M(F \wedge \mathrm{Im}\Psi). \tag{34}$$

Remark 2.14. *To illustrate, we assume that the principal bundle under consideration is topologically trivial. Using $d\omega = 3\mathrm{Im}\Psi$ and transgression formula, it can be shown that up to a constant*

$$CS(A) = \int_M \mathrm{tr}(F \wedge A - \frac{1}{3}A \wedge A \wedge A) \wedge \mathrm{Im}\Psi. \tag{35}$$

Here we regard G as a matrix Lie group. This formulation is more similar to the Chern-Simons functional on 3-manifolds.

Next we compute the second variation Q of CS. Suppose that $A(s,t)$ (for small s,t) are a two parameter family of connections such that $A = A(0,0)$ is an instanton. Let $a = \frac{\partial A}{\partial s}|_{s=0,t=0}$, $\quad b = \frac{\partial A}{\partial t}|_{s=0,t=0}$.Then by definition

$$Q(a,b) = \frac{\partial^2}{\partial s \partial t}|_{s=0,t=0} CS(A)$$

We have essentially computed that

$$\frac{\partial}{\partial t}CS(A) = -6\int_M \mathrm{tr}(F_A \wedge \mathrm{Im}\Psi \wedge \frac{\partial}{\partial t}A(s,t)).$$

Thus the second derivative is (remember $F_A \wedge \mathrm{Im}\Psi = 0$)

$$Q(a,b) = -6\int_M \mathrm{tr}(d_A(\frac{\partial A}{\partial s}|_{s=0,t=0}) \wedge \mathrm{Im}\Psi \wedge \frac{\partial A}{\partial t}|_{s=0,t=0}) \tag{36}$$

$$= -6\int_M \mathrm{tr}(d_A a \wedge \mathrm{Im}\Psi \wedge b) \tag{37}$$

Clearly, this is a symmetric bilinear form.

The null space of Q consists of a so that

$$d_A a \wedge \mathrm{Im}\Psi = 0.$$

This implies $d_A a$ is of type $(1,1)$ and hence

$$d_A a \wedge \Psi = 0.$$

Differentiating once and using Bianchi Identity gives one more equation

$$d_A a \wedge \omega^2 = 0.$$

This is exactly the infinitesimal deformation of the instanton equation.

2.4. Ω-anti-self-dual instantons on the G_2-cone. We investigate the relation between ω-anti-self-dual instantons on M and Ω-instantons on N.

First, note that any principal G-bundle over N is isomorphic to a bundle $P \times \mathbf{R}^+ \to M \times \mathbf{R}^+$ for a G-bundle P over M. Thus, without loss of generality, we assume that the G-bundle we are working on is a pull-back from M and we use the same letter P to denote these two bundles.

Suppose A is an Ω-instanton. A priori, A involves a dt-term $a \cdot dt$. However, we may perform a gauge transformation $A \mapsto g^{-1}Ag + g^{-1}dg$ to eliminate the dt-term. It is easy to see that we can simply take g as a solution to the differential equation

$$g^{-1}agdt + g^{-1}dg = 0.$$

Thus, we assume that A has no dt-term. We regard A as a family of connections on P parametrized by t and denote $\dot{A} = \frac{d}{dt}A$. Now the curvature may be computed

$$F^N = dA + \frac{1}{2}[A,A] = dt \wedge \dot{A} + F^M,$$

where $F^M = d_M A + \frac{1}{2}[A,A]$. The Ω-instanton condition with the formulae (28) and (29) gives

$$t *_M \alpha = -\mathrm{Im}\Psi \wedge F^M$$

and

$$\omega \wedge F^M - t\mathrm{Im}\Psi \wedge \alpha = - *_M F^M.$$

We denote the $(1,1)$-part (with coefficients depending on t) of F^M by F_0^M and $F_1^M = F^M - F_0^M$. By type decomposition in the above two equations we have

$$t *_M \dot{A} = -\mathrm{Im}\Psi \wedge F_1^M,$$

$$\omega \wedge F_0^M = - *_M F_0^M, \tag{38}$$

and

$$\omega \wedge F_1^M - t\mathrm{Im}\Psi \wedge \dot{A} = - *_M F_1^M.$$

By taking Hodge star of both sides, we see that the first equation is equivalent to

$$t\dot{A} = *(\mathrm{Im}\Psi \wedge F_1^M). \tag{39}$$

Combining (8) and (5) we see that the last equation is implied by (39).

The equation (38) looks very much like the ω-anti-self-dual instanton equation on M. The only problem is that F_0^M is not necessarily the curvature of a well-defined connection.

The equation (39) is exactly the gradient flow of the Chern-Simons functional CS. It would be interesting to analyze this equation coupled with (38). The first natural question is whether we could evolve through (39) in the class of ω-anti-self-dual instantons on M to get a Ω-anti-self-dual instanton on N. Unfortunately, this is impossible. An ω-instanton has its curvature of type $(1,1)$. If $A(t)$ stays ω-anti-self-dual for all t, the evolution equation (39) will imply that $\frac{d}{dt}A = 0$, i.e., A is constant in t. On the other hand, if A is constant in t and ω-anti-self-dual, it is Ω-anti-self-dual when pulled back to the cone N. These give a class of special solutions.

Lemma 2.15. *Suppose A is an ω-anti-self-dual connection on the nearly Kähler 6-manifold M and extend it to the G_2-cone N by constant in t. Then A is a Ω-anti-self-dual connection on N.*

Remark 2.16. *When $M = S^6$, in order that the principal bundle extend through the origin in $\mathbf{R}^7$, P has to be trivial over M. Even when this is true, the extended Ω-anti-self-dual connection on $\mathbf{R}^7\backslash\{0\}$ described in the above Lemma does not necessarily extend through origin. It is interesting to ask under what condition this singularity is removable after a gauge transformation.*

3. A Weitzenböck formula. In this section, we derive a Weitzenböck formula for nearly Kähler 6-manifolds and describe its application to the deformation of ω-anti-self-dual instantons.

3.1. The general formula. Let E be a vector bundle over M. Suppose E is equipped with a metric and a metric-compatible connection A. Suppose also that A is an ω-instanton. Consider the following complex

$$0 \to \Gamma(E) \xrightarrow{d_A} \Gamma(E \otimes T^*M) \xrightarrow{Pd_A} \Gamma(E \otimes (\Lambda^{(2,0)}T^*M)_{\mathbf{R}} \oplus \mathbf{R}\omega),$$

where the operator d_A is induced from d and the connection A and P is defined Proposition 1.3 in §1.1.3, the projection onto the orthogonal complement of ω-trace free $(1,1)$-forms. This complex is elliptic at the middle term. It could be extended to an elliptic complex, but we will not need the full sequence.

The 0th cohomology group consists of parallel sections of E. We are mainly interested in the 1st cohomology group. A well-known result in Hodge Theory states that this group can be represented by harmonic sections, i.e., the kernel of the elliptic operator

$$\Delta_A = (d_A^* \oplus Pd_A)^*(d_A^* \oplus Pd_A) = d_A d_A^* + d_A^* P^2 d_A.$$

As usual, we will compare Δ_A with a certain rough Laplacian of a connection. Note that, on $E \otimes T^*M$, there are several connections, e.g., A, coupled with the $\mathfrak{su}(3)$-connection on T^*M, denoted by $\hat{D}$ as well as A with the Levi-Civita connection, denoted by D. After many trials, we choose D. However, $\hat{D}$ will be useful.

Suppose $x \in M$ is a fixed point. Let $\{e_i\}_{i=1}^6$ be a local orthonormal frames centered at x whose covariant derivatives with respect to the Levi-Civita connection vanish at x. Let $\{\omega_i\}$ be the coframe. The Hodge Laplacian may be computed

$$\begin{aligned}
\Delta_A &= d_A d_A^* + d_A^* P^2 d_A \\
&= (\sum_{i=1}^6 \omega_i \wedge D_{e_i}) \circ (-\sum_{j=1}^6 \omega_j \lrcorner D_{e_j}) - (\sum_{i=1}^6 \omega_i \lrcorner D_{e_i}) \circ P^2 \circ (\sum_{j=1}^6 \omega_j \wedge D_{e_j}) \\
&= (\sum_{i=1}^6 \omega_i \wedge D_{e_i}) \circ (-\sum_{j=1}^6 \omega_j \lrcorner D_{e_j}) - (\sum_{i=1}^6 \omega_i \lrcorner \circ P^2 \circ D_{e_i}) \circ (\sum_{j=1}^6 \omega_j \wedge D_{e_j}) \\
&\quad -(\sum_{i=1}^6 \omega_i \lrcorner \circ [D_{e_i}, P^2]) \circ (\sum_{j=1}^6 \omega_j \wedge D_{e_j}) \\
&= -\sum_{i,j=1}^6 (\omega_i \wedge \circ \omega_j \lrcorner + \omega_i \lrcorner \circ P^2 \circ \omega_j \wedge) D_{e_i} D_{e_j} - (\sum_{i=1}^6 \omega_i \lrcorner \circ [D_{e_i}, P^2]) \circ d_A \\
&= -\sum_{i=1}^6 (\omega_i \wedge \circ \omega_i \lrcorner + \omega_i \lrcorner \circ P^2 \circ \omega_i \wedge) D_{e_i} D_{e_i} \\
&\quad -\sum_{i \neq j} (\omega_i \wedge \circ \omega_j \lrcorner + \omega_i \lrcorner \circ P^2 \circ \omega_j \wedge) D_{e_i} D_{e_j} - (\sum_{i=1}^6 \omega_i \lrcorner \circ [D_{e_i}, P^2]) \circ d_A \\
&= -\sum_{i=1}^6 D_{e_i} D_{e_i} - (\sum_{i=1}^6 \omega_i \lrcorner \circ [D_{e_i}, P^2]) \circ d_A \\
&\quad -\sum_{i<j} M(\omega_i \wedge \omega_j) \circ (D_{e_i} D_{e_j} - D_{e_j} D_{e_i})
\end{aligned}$$

where the operator M is defined in §1.1.3.

Recall $De_i|_x = 0$. Thus at x,

$$-\sum_{i=1}^6 D_{e_i} D_{e_i} = D^* D$$

the rough Laplacian. For the same reason, $D_{e_i} D_{e_j} - D_{e_j} D_{e_i}$ is the curvature on $T^*M \otimes E$. The curvature has two parts $R \otimes Id_E + Id_{T^*M} \otimes F^E$ where R is the Riemannian curvature of M. We write $R = \frac{1}{4} R_{klij} \omega_l \wedge \omega_k \otimes \omega_i \wedge \omega_j$. Given a 1-form α and two vectors X and Y

$$D_X D_Y \alpha - D_Y D_X \alpha - D_{[X,Y]} \alpha = \frac{1}{4} R_{klij} \omega_i \wedge \omega_j (X, Y) \alpha \lrcorner (\omega_l \wedge \omega_k).$$

Now consider the term in the formula $\sum_{i=1}^6 \omega_i \lrcorner \circ [D_{e_i}, P^2]$. Note that the $\mathfrak{su}(3)$-connection $\hat{D}$ commutes with P^2, i.e., $[\hat{D}_{e_i}, P^2] = 0$ for any e_i. Moreover, the difference $r(e_i) = D_{e_i} - \hat{D}_{e_i}$ is exactly the $\mathfrak{su}(3)$-torsion up to a constant. Here, the nearly Kähler structure plays the central role. By definition, this torsion r is covariantly constant with respect to the $\mathfrak{su}(3)$-connection. Hence, r satisfies (9) and the operator

$$\sum_{i=1}^6 \omega_i \lrcorner \circ [D_{e_i}, P^2] = \sum_{i=1}^6 \omega_i \lrcorner \circ [r(e_i), P^2] = B$$

factors through a linear combination of $\pi_{(2,0)}$ and $\pi_{(0,2)}$ according to (1.2).

In summary, we have the following Weitzenböck formula.

$$\begin{aligned}\Delta_A = D_A^* D_A & \\ -B \circ d_A & \\ -\frac{1}{2}\sum_{i,j} M(\omega_i \wedge \omega_j) \circ R_{ij} \otimes Id_E - \frac{1}{2}\sum_{ij} M(\omega_i \wedge \omega_j) \otimes F_{ij}.\end{aligned} \tag{40}$$

We have written the formula in three separate lines to indicate explicitly the leading part, the first-order part, and the curvature part, respectively. A routine consequence of this formula is the following:

Lemma 3.1. *Suppose M is a compact nearly Kähler 6-manifold. Suppose the curvature term in (40) is non-negative as an operator on $T^*M \otimes E$. Then the first cohomology group is of dimension at most $6 \cdot \operatorname{rank} E$. If, moreover, the curvature is positive somewhere, the first cohomology group vanishes.*

Proof. The key observation is that for any harmonic section s of the elliptic sequence representing an element in the first cohomology group, we have

$$Bd_A(s) = 0.$$

The rest of the proof parallels the argument in usual Bochner Technique. ∎

Remark 3.2. *It is not difficult to work out the explicit formula for the curvature term in (40) (the last line) using $SU(3)$-representation theory. We will discuss this for $M = S^6$ and leave the general case as an exercise for the interested reader.*

3.2. Deformation of ω-anti-self-dual instantons. Suppose $\mathbf{P}$ is a principal G-bundle with G a compact Lie group. As said before, a connection A on $\mathbf{P}$ is an ω-anti-self-dual instanton if and only if its curvature F satisfies

$$P(F) = 0. \tag{41}$$

Unless G is Abelian, this equation is nonlinear in A. Moreover, it is invariant under the action of the gauge transformations of $\mathbf{P}$.

The linearization of (41) at an ω-anti-self-dual instanton A is given by

$$Pd_A\alpha = 0$$

for $\alpha \in T^*M \otimes \mathbf{P} \times_G \mathfrak{g}$. Of course, one would like to divide by the infinitesimal gauge transformation since (41) is gauge-invariant. These infinitesimal gauge transformations are given by the image of $d_A : \mathbf{P} \times_G \mathfrak{g} \to T^*M \otimes \mathbf{P} \times_G \mathfrak{g}$. Thus, in fact, the essential infinitesimal deformations of the ω-anti-self-dual instanton A correspond to the elements of the first cohomology group of the following sequence

$$0 \to \Gamma(E) \xrightarrow{d_A} \Gamma(E \otimes T^*M) \xrightarrow{Pd_A} \Gamma(E \otimes (\Lambda^{(2,0)}T^*M)_{\mathbf{R}} \oplus \mathbf{R}\omega) \to 0,$$

where $E = \mathbf{P} \times_G \mathfrak{g}$.

It follows that, all discussion in the previous section applies to instanton deformations. We will illustrate this by analyzing S^6.

3.2.1. Applications to S^6. For S^6, the Riemannian curvature simplifies greatly

$$R = \frac{1}{2}\omega_j \wedge \omega_i \otimes \omega_i \wedge \omega_j.$$

We have identities

$$\begin{aligned} &\frac{1}{4}\sum M(\omega_i \wedge \omega_j)(\omega_1 \lrcorner (\omega_j \wedge \omega_i)) \\ =& \frac{1}{4}\sum M(\omega_i \wedge \omega_j)(\delta_{1j}\omega_i - \delta_{1i}\omega_j) \\ =& \frac{1}{4}(\sum_{i=1}^{6} M(\omega_i \wedge \omega_1)(\omega_i) - \sum_{j=1}^{6} M(\omega_1 \wedge \omega_j)(\omega_j)) \\ =& -\frac{5}{2}\omega_1. \end{aligned}$$

By symmetry, it holds that

$$\frac{1}{4}\sum M(\omega_i \wedge \omega_j)(\alpha \lrcorner (\omega_j \wedge \omega_i)) = -\frac{5}{2}\alpha$$

for any one-form α. Thus the first curvature term in the Weitzenbock formula

$$-\frac{1}{2}\sum_{i,j} M(\omega_i \wedge \omega_j) \circ R_{ij} \otimes Id_E = \frac{5}{2}.$$

An easy consequence is

Theorem 3.3. *A flat ω-instanton on S^6 is rigid.*

As another application, we consider the $\mathfrak{su}(3)$-connection on the standard structure bundle $G_2 \to S^6$. We need to describe the $\mathfrak{su}(3)$ connection a bit. Recall the connection 1-form $\kappa_{i\bar{j}}$ in (26). Through this, the connection on $(1,0)$-forms is

$$D_X \alpha_i = -\alpha_j \kappa_{i\bar{j}}(X)$$

for any vector field X. Correspondingly the curvature is give by

$$D_X D_Y - D_Y D_X - D_{[X,Y]}(\alpha_i) = -\alpha_j (d\kappa_{i\bar{j}} + \kappa_{i\bar{k}} \wedge \kappa_{k\bar{j}})(X, Y).$$

Thus the action of F on $(1,0)$ forms is given by

$$\alpha_i \mapsto -\alpha_j \otimes (d\kappa_{i\bar{j}} + \kappa_{i\bar{k}} \wedge \kappa_{k\bar{j}}).$$

More generally,

$$F : \alpha \mapsto \alpha \lrcorner - \frac{1}{2}(\overline{\alpha_i} \wedge \alpha_j) \otimes (d\kappa_{i\bar{j}} + \kappa_{i\bar{k}} \wedge \kappa_{k\bar{j}}).$$

Denote $\Omega_{i\bar{j}} = d\kappa_{i\bar{j}} + \kappa_{i\bar{k}} \wedge \kappa_{k\bar{j}} = \frac{1}{4}(3\alpha_i \wedge \overline{\alpha_j} - \delta_{i\bar{j}}\alpha_l \wedge \overline{\alpha_l})$. For each k, l, F_{kl} in the curvature term is

$$F_{kl} : \alpha \mapsto \alpha \lrcorner - \frac{1}{2}(\overline{\alpha_i} \wedge \alpha_j)\Omega_{i\bar{j}}(e_k, e_l).$$

Then the second curvature term in this context when $E = T^{1,0}$ is

$$\begin{aligned}
&-\frac{1}{2}M(\omega_k \wedge \omega_l)(v) \otimes F_{kl}(\alpha) \\
&= -\frac{1}{2}M(\omega_k \wedge \omega_l)(v) \otimes \alpha \lrcorner (-\frac{1}{2})(\overline{\alpha_i} \wedge \alpha_j)\Omega_{i\bar{j}}(e_k, e_l) \\
&= \frac{1}{4}M(\Omega_{i\bar{j}})(v) \otimes \alpha \lrcorner (\overline{\alpha_i} \wedge \alpha_j) \\
&= -\frac{1}{2}v \lrcorner \Omega_{i\bar{j}} \otimes \alpha \lrcorner (\overline{\alpha_i} \wedge \alpha_j)
\end{aligned}$$

because $\Omega_{i\bar{j}}$ is in $\Lambda_0^{1,1}$. This is not exactly what we want when we study the deformation of $\mathfrak{su}(3)$ connection on G_2. However, all we need is to replace α by a section of $ad_{G_2} \simeq \Lambda_0^{1,1}$ where the identification has been defined before. The action of $\overline{\alpha_i} \wedge \alpha_j$ will be the Lie bracket whose meaning should be clear via the aforementioned identification.

Denote

$$B(s,t) = \langle -\frac{1}{2}M(\omega_i, \omega_j) \otimes F_{ij}(s), t\rangle$$

where s, t are sections of $T^* \otimes ad_{G_2}$. Note that if $s = \phi \otimes X$ and $t = \psi \otimes Y$ we have

$$\begin{aligned}
B(s,t) &= -\frac{1}{2}\langle M(\omega_i \wedge \omega_j)(\phi) \otimes [F_{ij}, X], \psi \otimes Y\rangle \\
&= -\frac{1}{2}\langle M(\omega_i \wedge \omega_j)(\phi), \psi\rangle\langle [F_{ij}, X], Y\rangle \\
&= -\frac{1}{2}\langle M(\omega_i \wedge \omega_j), \phi \wedge \psi\rangle\langle F_{ij}, [X, Y]\rangle \\
&= \langle -\frac{1}{2}M(\omega_i \wedge \omega_j) \otimes F_{ij}, [s,t]\rangle.
\end{aligned}$$

where we view $M(\omega_i \wedge \omega_j)$ as a 2-form.

Then since F is $SU(3)$-invariant, B is a $SU(3)$-invariant symmetric bilinear form on $\mathbf{R}^6 \otimes \mathfrak{su}(3)$. We study the space of $SU(3)$ invariant symmetric bilinear forms on $\mathbf{R}^6 \otimes \mathfrak{u}(3)$. One candidate is obvious, the $SU(3)$ invariant inner product, denoted by B_0. For others we apply representation theory of $SU(3)$. Then complexified representation $(\mathbf{R}^6 \otimes \mathfrak{su}(3)) \otimes \mathbf{C} \cong (\mathbf{C}^3 \oplus \overline{\mathbf{C}^3}) \otimes_{\mathbf{C}} sl_3(\mathbf{C})$ decomposes as

$$(V^{(1,0)} \oplus \overline{V^{(1,0)}}) \oplus V^{(2,0)} \oplus \overline{V^{(2,0)}} \oplus V^{(2,1)} \oplus \overline{V^{(2,1)}},$$

where $V^{(a,b)}$ denotes the irreducible complex representation of $SU(3)$ with the highest weight (a,b). The representation $V^{(a,b)}$ is real (i.e., $V^{(a,b)} \cong \overline{V^{(a,b)}}$) if and only if $a = b$. Thus the original $\mathbf{R}^6 \otimes \mathfrak{u}(3)$ decomposes as

$$(V^{(1,0)})_{\mathbf{R}} \oplus (V^{(2,0)})_{\mathbf{R}} \oplus (V^{(2,1)})_{\mathbf{R}},$$

where $V_{\mathbf{R}}$ means the real representation by forgetting the complex structure of V. The irreducible pieces are $6, 12, 30$ dimensional respectively. One of them is known as $V^{(1,0)} = \mathbf{C}^3$. Coupled with the standard inner product, every $SU(3)$-invariant bilinear form on $\mathbf{R}^6 \otimes \mathfrak{u}(3)$ will give rise to a $SU(3)$-invariant endomorphism. The space of such endomorphisms is 6 dimensional, 2 for each irreducible component. Of the two independent bilinear forms on every irreducible component, one can be taken as the $SU(3)$-invariant inner product and the other is symplectic. Thus the space of $SU(3)$-invariant symmetric bilinear forms is 3-dimensional, represented by linear combinations of inner products of various components.

We will construct a basis for this 3-dimensional space. We already have one– the inner product of the whole space B_0. To construct two more, we need more information about the irreducible components.

First consider the map

$$T_1 : \mathbf{R}^6 \otimes \mathfrak{su}(3) \to \mathbf{R}^6$$

defined by $v \otimes \alpha \mapsto v \lrcorner \alpha$. This map is clearly $SU(3)$-equivariant so $B_1(u,v) = \langle T_1 u, T_1 v\rangle$ is clearly $SU(3)$-invariant. Moreover, since $\mathbf{R}^6 \otimes \mathfrak{su}(3)$ contains only one copy of $\mathbf{R}^6$ and T_1 is nonzero, by Schur's Lemma, T_1 and thus B_1 is zero on $(V^{(2,0)})_{\mathbf{R}} \oplus (V^{(2,1)})_{\mathbf{R}}$.

For later estimate, we need the right inverse of T_1. Define the operatore

$$S_1 : \mathbf{R}^6 \to \mathbf{R}^6 \otimes \mathfrak{su}(3)$$

by

$$v \to \frac{3}{16}\sum_i \alpha_i \otimes \pi_0^{1,1}(\overline{\alpha_i} \wedge v) + \frac{3}{16}\sum_i \overline{\alpha_i} \otimes \pi_0^{1,1}(\alpha_i \wedge v).$$

It is clearly $SU(3)$ equivariant. Since $\mathbf{R}^6$ is irreducible, S_1 maps onto the irreducible components $V_{\mathbf{R}}^{(1,0)} \in \mathbf{R}^6 \otimes \mathfrak{su}(3)$.

Then the composition $T_1 \circ S_1$ must be a linear combination of Id and J(the almost complex structure). However, it may be computed that

$$\begin{aligned} S_1(\alpha_1) &= \frac{3}{16}(\alpha_1 \otimes \pi_0^{1,1}(\overline{\alpha_1} \wedge \alpha_1) + \alpha_2 \otimes \pi_0^{1,1}(\overline{\alpha_2} \wedge \alpha_1) + \alpha_3 \otimes \pi_0^{1,1}(\overline{\alpha_3} \wedge \alpha_1) \\ &= \frac{3}{16}(\alpha_1 \otimes \frac{1}{3}(2\overline{\alpha_1} \wedge \alpha_1 + \alpha_2 \wedge \overline{\alpha_2} + \alpha_3 \wedge \overline{\alpha_3}) + \alpha_2 \otimes \overline{\alpha_2} \wedge \alpha_1 + \alpha_3 \otimes \overline{\alpha_3} \wedge \alpha_1) \end{aligned}$$

Thus $T_1 S_1(\alpha_1) = \alpha_1$ and hence $T_1 S_1 = Id$.

Meanwhile, it is easy to compute that

$$B_0(S_1(\alpha), S_1(\overline{\alpha})) = \frac{3}{4} B_1(S_1(\alpha), S_1(\overline{\alpha})). \tag{42}$$

Second consider the map

$$T_2 : \mathbf{R}^6 \otimes \mathfrak{su}(3) \to \wedge^3 \mathbf{R}^6 \to (\mathbf{R}^6 \wedge \omega)^{\perp}.$$

defined by $v \otimes \alpha \mapsto v \wedge \alpha$ followed by the projection onto the orghogonal complement of $\mathbf{R}^6 \wedge \omega$. Define $B_2(u,v) = \langle T_2 u, T_2 v\rangle$. Then T_2 is $SU(3)$ equivariant and B_2 is $SU(3)$ invariant. The image of T_2 lies in the space of type $(2,1)+(1,2)$ forms.

We also need the partial inverse of T_2. Define

$$S_2 : \psi \mapsto \frac{1}{4}(\alpha_i \otimes \pi_0^{1,1}(\overline{\alpha_i} \lrcorner \psi) + \overline{\alpha_i} \otimes \pi_0^{1,1}(\alpha_i \lrcorner \psi)).$$

It is clearly $SU(3)$ equivariant. The image under S_2 of $(2,1)+(1,2)$ forms orthogonal to $\mathbf{R}^6 \wedge \omega$ is $V^{(2,0)}$. It is easy to compute

$$S_2(\alpha_1 \wedge \alpha_2 \wedge \overline{\alpha_3}) = \frac{1}{4}(2\alpha_1 \otimes \alpha_2 \wedge \overline{\alpha_3} - 2\alpha_2 \otimes \alpha_1 \wedge \overline{\alpha_3}).$$

Consequently, $T_2S_2(\alpha_1 \wedge \alpha_2 \wedge \overline{\alpha_3}) = \alpha_1 \wedge \alpha_2 \wedge \overline{\alpha_3}$. Thus $T_2S_2 = 1$.

It is also easy to verify that

$$B_0(S_2(\psi), S_2(\overline{\psi})) = \frac{1}{2}B_2(S_2(\psi), S_2(\overline{\psi})). \tag{43}$$

On the other hand, it may be computed that

$$T_2S_1 = 0, \quad T_1S_2 = 0. \tag{44}$$

The 3 symmetric bilinear forms B_0, B_1, B_2 are clearly linearly independent. Thus there exist constants λ_i such that $B = \lambda_0B_0+\lambda_1B_1+\lambda_2B_2$. We will compute examples to determine these constants.

Set

$$u_1 = \alpha_1 \otimes \sqrt{-1}(2\alpha_1 \wedge \overline{\alpha_1} - \alpha_2 \wedge \overline{\alpha_2} - \alpha_3 \wedge \overline{\alpha_3}),$$

$$u_2 = \alpha_1 \otimes (\alpha_2 \wedge \overline{\alpha_3} + \overline{\alpha_2} \wedge \alpha_3).$$

and

$$u_3 = \alpha_1 \otimes \alpha_1 \wedge \overline{\alpha_2}.$$

It is easy to see that $[u_1, \overline{u_1}] = [u_2, \overline{u_2}] = 0$. Thus

$$0 = B(u_1, \overline{u_1}) = \lambda_0B_0(u_1, \overline{u_1}) + \lambda_1B_1(u_1, \overline{u_1}) = (\lambda_0 + \frac{4}{3}\lambda_1)B_0(u_1, \overline{u_1}),$$

$$0 = B(u_2, \overline{u_2}) = \lambda_0B_0(u_2, \overline{u_2}) + \lambda_2B_2(u_2, \overline{u_2}) = (\lambda_0 + 2\lambda_2)B_0(u_2, \overline{u_2}),$$

and

$$B(u_3, \overline{u_3}) = \lambda_0B_0(u_3, \overline{u_3}).$$

Hence

$$\lambda_1 = -\frac{3}{4}\lambda_0, \ \ \lambda_2 = -\frac{1}{2}\lambda_0$$

and

$$\lambda_0 = \frac{B(u_3, \overline{u_3})}{B_0(u_3, \overline{u_3})}.$$

The curvature $F = -\frac{1}{8}(3\alpha_i \wedge \overline{\alpha_j} - \delta_{ij}\alpha_l \wedge \overline{\alpha_l}) \otimes_{\mathbf{C}} \overline{\alpha_i} \wedge \alpha_j$. Thus

$$\begin{aligned} B(u_3, \overline{u_3}) &= \langle F, [u_3, \overline{u_3}]\rangle \\ &= \langle F, \alpha_1 \wedge \overline{\alpha_1} \otimes (-2\overline{\alpha_1} \wedge \alpha_1 + 2\overline{\alpha_2} \wedge \alpha_2)\rangle \\ &= 12. \end{aligned}$$

Thus $\lambda_0 = \frac{3}{2}$. Consequently,

$$B = \frac{3}{2}(B_0 - \frac{3}{4}B_1 - \frac{1}{2}B_2).$$

Lemma 3.4. *There holds*

$$B \geq 0.$$

Proof. Let $\psi \in \mathbf{R}^6 \otimes \mathfrak{su}(3)$ be real. Write $\psi = S_1 T_1(\psi) + S_2 T_2(\psi) + \hat{\psi}$. Note that $\hat{\psi} \in \ker T_1 \cap \ker T_2$. Thus, in fact $\psi \in V_{\mathbf{R}}^{(2,1)}$. These three different components are thus pairwise perpendicular, since they lie in different irreducible pieces. It follows that

$$\frac{2}{3} B(\psi, \psi) = B_0(\hat{\psi}, \hat{\psi}).$$

□

The contribution from the second curvature term is nonnegative. All together the curvature part is strictly positive.

To summarize, we have the following result.

Theorem 3.5. *The $\mathfrak{su}(3)$-connection on $G_2 \to S^6$ in (26) is a rigid $SU(3)$ instanton.*

4. $SO(4)$-invariant examples. We construct cohomogeneity one $SU(2)(S^3)$ anti-self-dual instantons (equivalent to pseudo-Hermitian-Yang-Mills here) on S^6. The idea is to impose symmetries to reduce the instanton equations to ODEs. We regard $SU(2) = S^3$ as the set of unit quaternions whose Lie algebra is the tangent space at 1 consisting of imaginary quaternions for which we use I, J, K to denote the standard basis for imaginary quaternions. A remark on the notation is necessary. Throughout this section, we use $\sqrt{-1}$ to represent complex numbers to avoid confusion with quaternions. It should be cautioned that when complex numbers are regarded as coefficients in the complexified Lie algebra, they commute with I, J, K rather than following the usual rule of multiplication with quaternions. Hopefully, this will be clear from context.

4.1. A dense open subset U of S^6. More precisely, $U = S^6 \setminus (S^2 \cup S^3)$ is parametrized by $S^2 \times S^3 \times (0, \frac{\pi}{2})$ as

$$(x, y, t) \mapsto v = (x \cos t, y \sin t))$$

where we think of $x \in S^3 \subset \mathbf{R}^4$ as a unit 4-vector and $y \in S^2 \subset \mathbf{R}^3$ as a unit 3-vector. Actually, if we extend the map to the closed interval $[0, \frac{\pi}{2}]$, we cover the whole S^6. Reverse the picture and we get a map $t : S^6 \to [0, \frac{\pi}{2}]$ which is roughly the distance function from the totally geodesic pseudo-holomorphic $S^2 = \{t = 0\}$. A generic level set is a scaled $S^2 \times S^3$ and $\{t = \frac{\pi}{2}\}$ is a totally geodesic, special Lagrangian S^3.

For later use,

$$S^3 \times S^2 = S^3 \times S^3 / S^1$$

as a homogeneous space via $(p, q) \sim (pz, qz)$ for $(p, q) \in S^3 \times S^3$ and $z \in S^1$. Composing this quotient with the map $(x, y, t) \mapsto v$, we have a map $S^3 \times S^3 \times (0, \frac{\pi}{2}) \to U \subset S^6$ by

$$(p, q) \mapsto (pI\overline{p} \cos t, qp^{-1} \sin t).$$

Denote by $\omega = \omega_1 I + \omega_2 J + \omega_3 K$ and $\psi = \psi_1 I + \psi_2 J + \psi_3 K$ the left-invariant Maurer-Cartan forms on the two copies of S^3, respectively. Then, $dt, \omega_2, \omega_3, \psi_2, \psi_3$ and $\tau = \omega_1 - \psi_1$ form a basis of semibasic 1-forms for the projection $S^3 \times S^3 \times (0, \frac{\pi}{2}) \to U$. We use this to describe the nearly Kähler structures on U induced from S^6.

Recall that the G_2-invariant almost complex structure J on $T_v S^6$ is given by the left Cayley multiplication by v when we we regard both v and tangent vectors as Cayley numbers in $\mathbf{R}^8 = \mathbb{O}$. In other words,

$$J : dv \mapsto v \cdot dv. \tag{45}$$

The standard metric and J determines the Kähler 2-form $\omega = \langle J dv, dv \rangle$.

Using (45) and Cayley-Dickson rule of Cayley multiplication, we can establish the following

$$\begin{aligned}
J(dt) &= \sin t\tau, \\
J(2\cos t\omega_3) &= (2\cos^2 t - \sin^2 t)\omega_2 + \sin^2 t\psi_2, \\
J(-2\cos t\omega_2) &= (2\cos^2 t - \sin^2 t)\omega_3 + \sin^2 t\psi_3.
\end{aligned}$$

The Kähler form ω is determined by

$$\begin{aligned}
-\omega = \langle v, Jv \rangle = {} & 2\sin t\psi_1 \wedge dt - 2\sin t\omega_1 \wedge dt \\
& + 2\cos t(9\cos^2 t - 5)\omega_3 \wedge \omega_2 + 6\sin^2 t\cos t\omega_3 \wedge \psi_2 \\
& - 6\cos t\sin^2 t\omega_2 \wedge \psi_3 + 2\sin^2 t\cos t\psi_2 \wedge \psi_3.
\end{aligned}$$

4.2. Bundle constructions and $SO(4)$-invariant connections.

4.2.1. S^3-bundles. We now describe the principal S^3-bundles on which to construct instantons. First, note that $S^3 \times S^3 \times (0, \frac{\pi}{2}) \to S^3 \times S^2 \times (0, \frac{\pi}{2})$ in §4.1 is a principal S^1-bundle. The principal S^3-bundles are obtained by extending the structure group through the group homomorphisms

$$z \mapsto z^l$$

for $z \in S^1$. More explicitly, denote

$$B_l = S^3 \times S^3 \times S^3 \times (0, \frac{\pi}{2}) / \sim$$

where

$$(p, q, r, t) \sim (pz, qz, rz^{-l}, t)$$

for any $(p, q, r, t) \in S^3 \times S^3 \times S^3 \times (0, \frac{\pi}{2})$, $t \in (0, \frac{\pi}{2})$ and $z \in S^1$. The structure group S^3 acts on B_l by

$$[p, q, r, t] \mapsto [p, q, rg, t]$$

for any $g \in S^3$. Clearly this is well-defined. Then the projection

$$[p, q, r, t] \mapsto (pIp^{-1}\cos t, qp^{-1}\sin t)$$

makes B_l a principal S^3-bundle over U.

Remark 4.1 (on the symmetry of B_l). *Note that if we let* $[g_1, g_2] \in SO(4) = S^3 \times S^3/\mathbb{Z}_2$ *act on* B_l *by*

$$[p, q, r, t] \to [g_1 p, g_2 q, r, t]$$

and on S^6 *by*

$$(x, y) \mapsto (pap^{-1}, qbp^{-1}),$$

this action commutes with the bundle projection. In other words, the principal bundle B_l *over* U *has an* $SO(4)$*-symmetry. It is well-known that the action on* S^6 *is induced from the embedding of* $SO(4)$ *into* G_2 *and has cohomogeneity 1. We will construct* $SO(4)$*-invariant instantons, i.e., instantons of cohomogeneity one.*

Remark 4.2 (on the topology of B_l). *A priori,* B_l *is only defined on* U*. However, note that* B_l *is actually the pullback of a* S^3*-bundle from* S^2 *obtained by extending the structure group of a Hopf circle bundle. Since* $\pi_1(S^3)$ *is trivial, every* S^3*-bundle over* S^2 *must be trivial. As a consequence,* B_l *is also trivial. In other words, it is possible to make gauge transformations so that* $B_l \sim U \times S^3$*. Thus this bundle has natural extension to the whole* S^6*, and, for later use, to the whole* $\mathbf{R}^7$*. The former description has the advantage that it makes the* $SO(4)$*-symmetry clear.*

Remark 4.3 (on the numbers l). *A priori, this construction only makes sense for integer* l*. However, we will see that it is more interesting if we think of* l *as real valued.*

We will carry out computations on $S^3 \times S^3 \times S^3 \times (0, \frac{\pi}{2})$. We will continue with the notation in §4.1 on the left-invariant forms on the first two copies of S^3. However, for the last S^3, we need use the *right*- invariant Maurer-Cartan form $drr^{-1} = \beta = \beta_1 I + \beta_2 J + \beta_3 K$. The left invariant Maurer-Cartan form is $r^{-1}dr = r^{-1}\beta r$. Of course, the following Maurer-Cartan equations hold

$$d\omega = -\omega \wedge \omega,$$

$$d\psi = -\psi \wedge \psi,$$

and

$$d\beta = \beta \wedge \beta.$$

More explicitly

$$d\omega_1 = -2\omega_2 \wedge \omega_3, \quad d\omega_2 = -2\omega_3 \wedge \omega_1, \quad d\omega_3 = -2\omega_1 \wedge \omega_2,$$

similarly for ψ_i and

$$d\beta_1 = 2\beta_2 \wedge \beta_3, \quad d\beta_2 = 2\beta_3 \wedge \beta_1, \quad d\beta_3 = 2\beta_1 \wedge \beta_2.$$

The space of semibasic 1-forms for the projection $S^3 \times S^3 \times S^3 \times (0, \frac{\pi}{2})$ is spanned by $dt, \omega_2, \omega_3, \psi_2, \psi_3, \beta_2, \beta_3, \omega_1 - \psi_1$ and $l\psi_1 + \beta_1$.

4.2.2. Invariant connections. Now suppose A is an $SO(4)$-invariant connection on B_l. We pull back A to $S^3 \times S^3 \times S^3 \times (0, \frac{\pi}{2})$ and denote it by the same letter. Then, since A is semibasic with respect to the projection $S^3 \times S^3 \times S^3 \times (0, \frac{\pi}{2}) \to B_l$, we can write

$$A = A_0\tau + A_1(l\psi_1 + \beta_1) + A_2\omega_2 + A_3\omega_3 + B_2\psi_2 + B_3\psi_3 + C_2\beta_2 + C_3\beta_3 + B_0 dt$$

with A_i, B_i, C_i valued in $Lie(S^3)$. Since A is $SO(4)$-invariant and the 1-forms listed are also $SO(4)$-invariant, the coefficients do not depend on (p, q), i.e., they are functions only in t and r. Moreover, A has to satisfy the following properties:

1. A must be right S^3-equivariant where we let S^3 act on $S^3 \times S^3 \times S^3 \times (0, \frac{\pi}{2})$ and B_l by right multiplication on the last S^3 factor.
2. A restricts to the last S^3 factor to be the Maurer-Cartan left invariant form $r^{-1}\beta r$.
3. The differential dA must be semibasic.

We investigate the consequences of these conditions.

1. Since all the forms listed in A are S^3 right-invariant, this condition is equivalent to

$$A_i(t, r) = r^{-1}A(t, 1)r, B_i(t, r) = r^{-1}B_i(t, 1)r, C_i = r^{-1}C_i(t, 1)r.$$

   To save notation, we will, from now on, write

$$A = r^{-1}(A_0\tau + A_1(l\psi_1 + \beta_1) + A_2\omega_2 + A_3\omega_3 + B_2\psi_2 + B_3\psi_3 + C_2\beta_2 + C_3\beta_3 + B_0 dt)r$$

   where A_i, B_i, C_i are functions of t.
2. This condition says that

$$A_1 = I, \quad C_2 = J, \quad C_3 = K.$$

   Thus we may further reduce A to

$$A = r^{-1}(A_0\tau + Il\psi_1 + A_2\omega_2 + A_3\omega_3 + B_2\psi_2 + B_3\psi_3 + B_0 dt)r + r^{-1}\beta r$$

3. It can be computed from Maurer-Cartan equations that

$$\begin{aligned} rdAr^{-1} \equiv\ & -l[B_0, I]\psi_1 \wedge dt - l[A_0, I]\psi_1 \wedge \tau \\ & -(l[A_2, I] + 2A_3)\psi_1 \wedge \omega_2 - (l[A_3, I] + 2A_2)\psi \wedge \omega_3 \\ & -(l[B_2, I] + 2B_3)\psi_1 \wedge \psi_2 - (l[B_3, I] + 2B_2)\psi_1 \wedge \psi_3 \end{aligned}$$

   mod semibasic 2-forms. Thus this condition is equivalent to the following algebraic equations

$$\begin{aligned} & l[A_0, I] = 0, && l[B_0, I] = 0, \\ & l[B_2, I] + 2B_3 = 0, && l[B_3, I] + 2B_2 = 0, \\ & l[A_2, I] + 2A_3 = 0, && l[A_3, I] + 2A_2 = 0. \end{aligned} \tag{46}$$

Hence, we solve the algebraic equations (46). We divide the solutions into several cases according to different values of l.

1. Case $l = 0$. We have $B_2 = B_3 = A_2 = A_3 = 0$ but (46) puts no restrictions on A_0 and B_0. Therefore A is reduced to

$$A = r^{-1}(A_0\tau + B_0 dt)r + r^{-1}\beta r.$$

2. Case $l = 1$. We have

$$A_0 = a_0 I, \quad B_0 = b_0 I,$$

$$A_2 = u_1 J + u_2 K, \quad A_3 = -u_2 J + u_1 K,$$

$$B_2 = v_1 J + v_2 K, \quad B_3 = -v_2 J + v_1 K,$$

for a_0, b_0, u_i, v_i functions of t.

3. Case $l = -1$. We have

$$A_0 = a_0 I, \quad B_0 = b_0 I,$$

$$A_2 = u_1 J + u_2 K, \quad A_3 = u_2 J - u_1 K,$$

$$B_2 = v_1 J + v_2 K, \quad B_3 = v_2 J - v_1 K,$$

for a_0, b_0, u_i, v_i functions of t.

4. Case $l \neq 0, \pm 1$. We have

$$A_0 = a_0 I, \quad B_0 = b_0 I,$$

$$A_2 = A_3 = B_2 = B_3 = 0.$$

4.3. $SO(4)$**-invariant instantons.** Now we take instanton conditions into consideration. As mentioned before, A is an ω-anti-self-dual instanton if and only if its curvature F satisfies

$$F^{2,0} = \mathrm{tr}_\omega F = 0.$$

It is easily seen that, restricted to U, this equivalent to

$$F \wedge \sigma_0 \wedge \sigma_1 \wedge \sigma_2 = 0, \tag{47}$$

and

$$F \wedge \omega^2 = 0. \tag{48}$$

According to Lemma 2.4, (48) is implied by (47), so we only care about (47). This simplifies the problem greatly. We consider four different cases according to the four different values of l in the last section.

4.3.1. $l = 0$**.** It may be computed that

$$F \;=\; r^{-1}\{(\dot{A}_0 + [B_0, A_0])dt \wedge \tau - 2A_0(\omega_2 \wedge \omega_3 - \psi_2 \wedge \psi_3)\}r,$$

where $\dot{A}_0 = \frac{d}{dt}A_0$. The equation (47) gives

$$32\sqrt{2}A\cos^2 t \sin t\omega_3 \wedge \omega_2 \wedge \psi_2 \wedge \psi_3 \wedge (dt - \sqrt{-1}\sin t\tau) = 0.$$

The only solution is $A_0 = 0$, which is the trivial connection

$$A = r^{-1}dr.$$

This is a case of little interest.

4.3.2. $l \neq 0, \pm 1$**.** It may be computed that

$$F = \{\dot{a_0} dt \wedge \tau - 2a_0(\omega_2 \wedge \omega_3 - \psi_2 \wedge \psi_3) - 2l\psi_2 \wedge \psi_3\} r^{-1} I r.$$

The equation (47) gives

$$8a_0 \cos^2 t + l - 9l \cos^2 t = 0.$$

It is solved by

$$a_0 = \frac{l}{8} \frac{9\cos^2 t - 1}{\cos^2 t}.$$

For safety, one can check that, in fact, a_0 also satisfies the equation (48) which, in this case, is

$$-4a_0 + 5l + 8\cos^2 t a_0 - 9\cos^2 t l + 2\sin t \cos t \dot{a_0} = 0.$$

We arrive at the corresponding instanton, pulled back to $S^3 \times S^3 \times S^3 \times (0, \frac{\pi}{2})$,

$$A = r^{-1} l I r \left(\frac{1}{8} \frac{9\cos^2 t - 1}{\cos^2 t} \tau + \psi_1 + b dt \right) + r^{-1} dr. \tag{49}$$

THEOREM 4.4. *(49) defines for each $l \in \mathbf{Z}$ a singular Hermitian-Yang-Mill connection on S^6.*

REMARK 4.5 (on singularity). *The coordinate system is not extendable through the submanifolds $S^2 = \{t = 0\}$ and $S^3 = \{t = \frac{\pi}{2}\}$. However, the connection A has different behavior when t approaches 0 and $\frac{\pi}{2}$. When $t \to \frac{\pi}{2}$, the curvature F blows up. However, for $t = 0$, the connection is bounded. It might be possible to remove the singularity by (2.11), we can extend the connection to the locus $t = 0$. In other words, this might be a singularity due to unwise choice of coordinates, rather than a singularity of the instanton A itself.*

REMARK 4.6 (on reducibility). *A cautious reader may have noticed that, A has its holonomy in S^1, so it is reducible. If we restrict the connection to the generic level sets of t, we obtain the standard Hopf connection up to a constant.*

REMARK 4.7 (on b). *Note that b is not essential. We could have applied a gauge transformation in the t direction to A at the beginning to remove the dt component. The same remark applies to the next subsection.*

4.3.3. $l = \pm 1$**.** We only deal with the case $l = 1$. The other case is similar.

According to Case 2 in §5.2.2, the curvature is computed to be

$$\begin{aligned}
rFr^{-1} = {} & \dot{a_0} I dt \wedge \tau - 2I\psi_2 \wedge \psi_3 \\
& + (\dot{u_1} J + \dot{u_2} K) dt \wedge \omega_2 + (-\dot{u_2} J + \dot{u_1} K) dt \wedge \omega_3 \\
& + (\dot{v_1} J + \dot{v_2} K) dt \wedge \psi_2 + (-\dot{v_2} J + \dot{v_1} K) dt \wedge \psi_3 \\
& - 2a_0 I(\omega_2 \wedge \omega_3 - \psi_2 \wedge \psi_3) \\
& - 2(u_1 J + u_2 K)\omega_3 \wedge \tau - 2(-u_2 J + u_1 K)\tau \wedge \omega_2 \\
& + 2a_0(u_1 K - u_2 J)\tau \wedge \omega_2 + 2a_0(-u_2 K - u_1 J)\tau \wedge \omega_3 \\
& + 2a_0(v_1 K - v_2 J)\tau \wedge \psi_2 + 2a_0(-v_2 K - v_1 J)\tau \wedge \psi_3 \\
& + 2(u_1^2 + u_2^2) I\omega_2 \wedge \omega_3 + 2(v_1^2 + v_2^2) I\psi_2 \wedge \psi_3 \\
& + 2(u_1 v_2 - u_2 v_1) I(\omega_2 \wedge \psi_2 + \omega_3 \wedge \psi_3) \\
& + 2(u_1 v_1 + u_2 v_2) I(\omega_2 \wedge \psi_3 - \omega_3 \wedge \psi_2)
\end{aligned}$$

where, again, $\dot{}$ means $\frac{d}{dt}$.

A tedious computation shows that the equation (47) amounts to the following

$$\sin t(1-3\cos^2 t)\dot{v_1} - \sin^3 t\dot{u_1} + 4a\cos t v_1 = 0,$$

$$\sin t(1-3\cos^2 t)\dot{v_2} - \sin^3 t\dot{u_2} + 4a\cos t v_2 = 0,$$

$$\sin t\cos t\dot{v_1} + u_1(1-a)\sin^2 t + a(3\cos^2 t - 1)v_1 = 0,$$

$$\sin t\cos t\dot{v_2} + u_2(1-a)\sin^2 t + a(3\cos^2 t - 1)v_2 = 0,$$

$$u_1 v_2 = u_2 v_1,$$

and

$$\begin{aligned} -9\cos^2 t + 1 + 8a\cos^2 t - \sin^2 t(u_1^2 + u_2^2) &+ \\ (9\cos^2 t - 1)(v_1^2 + v_2^2) &+ \\ (6\cos^2 t - 2)(u_1 v_1 + u_2 v_2) &= 0. \end{aligned}$$

We may assume that

$$u_2 = \lambda u_1, \quad v_2 = \lambda v_1$$

with λ necessarily constant. It can be shown that by a substitution like $(u_1, v_1) \mapsto \sqrt{1+\lambda^2}(u_1, v_1)$, we may simply assume that $v_2 = u_2 = 0$.

The system reduces to

$$\sin t(1-3\cos^2 t)\dot{v_1} - \sin^3 t\dot{u_1} + 4a\cos t v_1 = 0,$$

$$\sin t\cos t\dot{v_1} + u_1(1-a)\sin^2 t + a(3\cos^2 t - 1)v_1 = 0,$$

$$-9\cos^2 t + 1 + 8a\cos^2 t - \sin^2 t u_1^2 + (9\cos^2 t - 1)v_1^2 + (6\cos^2 t - 2)u_1 v_1 = 0$$

which is now determined and thus solvable.

It is easy to see that any solution must be of the form

$$u_1 = U(\sin t), v_1 = V(\sin t), a = W(\sin t),$$

where the functions $U(x), V(x)$ and $W(x)$ defined on $[0,1]$ satisfy

$$x(-2+3x^2)\frac{d}{dx}V - x^3\frac{d}{dx}U + 4WV = 0$$

$$x(1-x^2)\frac{d}{dx}V + x^2 U(1-W) + (2-3x^2)WV = 0$$

$$-8 + 9x^2 + 8(1-x^2)W - x^2U^2 + (8-9x^2)V^2 + (4-6x^2)UV = 0.$$

We rewrite the ODEs as

$$x(1-x^2)\frac{d}{dx}V = -x^2U(1-W) - (2-3x^2)WV \tag{50}$$

$$x^3(1-x^2)\frac{d}{dx}U = x^2(2-3x^2)U(1-W) + (8-16x^2+9x^4) \tag{51}$$

$$9x^2 + 8(1-x^2)W - x^2U^2 + (8-9x^2)V^2 + (4-6x^2)UV = 8. \tag{52}$$

It is clear that the system (50), (51), (52) has many solutions which have possible singularities along $x = 0$ and $x = 1$.

THEOREM 4.8. *Each solution of the ode system (50), (51), (52) and a real number λ determine a unique Hermitian-Yang-Mills connection on the trivial $SU(2)$ bundle over S^6, with possible singularities along submanifolds S^2 and S^3.*

REMARK 4.9. *It is interesting to ask whether we could apply Corollary (2.11) or (2.12) to remove the possible singularities along S^2. This should be doable by analyzing the singular behavior of the above ODE system along $x = 0$.*

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