

EMBEDDING PROPERTY OF J -HOLOMORPHIC CURVES IN CALABI-YAU MANIFOLDS FOR GENERIC J^*

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Abstract. In this paper, we prove that for a generic choice of tame (or compatible) almost complex structures J on a symplectic manifold (M^{2n}, ω) with $n \geq 3$ and with its first Chern class $c_1(M, \omega) = 0$, all somewhere injective J -holomorphic maps from any closed smooth Riemann surface into M are *embedded*. We derive this result as a consequence of the general optimal 1-jet evaluation transversality result of J -holomorphic maps in general symplectic manifolds that we also prove in this paper.

Key words. 1-jet evaluation transversality, somewhere injective, embedded J -holomorphic curves, Calabi-Yau manifolds.

AMS subject classifications. 53D35, 14H10

1. Introduction. Let (M, ω) be a symplectic manifold of real dimension $2n$. We denote by J an almost complex structure tame to ω and by \mathcal{J}_ω the set of tame almost complex structures. It is a classical fact [G], [M] that for a generic choice of J , any somewhere injective J -holomorphic curve is a *smooth* point in the moduli space of J -holomorphic curves : A J -holomorphic curve $u : \Sigma \rightarrow M$ is called somewhere injective if there is a point $z \in \Sigma$ such that

$$du(z) \neq 0 \quad \text{and} \quad u^{-1}(u(z)) = \{z\}.$$

This fact has been a fundamental point in the definition of Gromov-Witten invariants and the counting problem of J -holomorphic curves. Recent development in the Gromov-Witten theory unravels necessity of finer structure theorem on the image of J -holomorphic curves. In particular a conjectural mathematical definition of Gopakuma-Vafa invariant of BPS-count is closely related to the number of *embedded* J -holomorphic curves in Calabi-Yau three-folds for a generic choice of J [K], [P].

Now description of the main results of this paper is in order.

Let Σ be a connected closed smooth surface of genus g . We denote by j a complex structure on Σ and denote by $\mathcal{M}_g = \mathcal{M}(\Sigma)$ the moduli space of complex structures on Σ . We call a pair $((\Sigma, j), u)$ a J -holomorphic map if u is (j, J) -holomorphic, i.e., if it satisfies

$$J \circ du = du \circ j.$$

We say that $((\Sigma, j), u)$ is Fredholm regular if the linearization of the map

$$\bar{\partial}_J : (j, u) \mapsto \frac{du + J \circ du \circ j}{2}$$

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is surjective at $((\Sigma, j), u)$. We have the index formula given by

$$\text{Index } D_{(j,u)}\bar{\partial}_J = \begin{cases} 2(c_1(M, \omega)(\beta) + (n - 3)(1 - g)) & \text{for } g \geq 2 \\ 2(c_1(M, \omega)(\beta) + 1) & \text{for } g = 1 \\ 2(c_1(M, \omega)(\beta) + n) & \text{for } g = 0 \end{cases} \quad (1.1)$$

for the maps u with $[u] = \beta \in H_2(\mathbb{Z})$, and hence the virtual dimension of the associated moduli space $\mathcal{M}_g(M, J; \beta)$ is given by

$$2(c_1(M, \omega)(\beta) + (n - 3)(1 - g)) \quad (1.2)$$

in all cases.

In this paper, we prove the following theorem.

THEOREM 1.1. *Assume $n \geq 3$. Let (Σ, j) be a closed smooth Riemann surface of any genus g . Then there exists a subset $\mathcal{J}_\omega^{emb} \subset \mathcal{J}_\omega$ of second category such that for $J \in \mathcal{J}_\omega^{emb}$, for any complex structure j on Σ , all somewhere injective (j, J) -holomorphic maps $u : \Sigma \rightarrow M$ are Fredholm regular and embedded whenever $c_1(M, \omega)([u]) \leq 0$.*

We would like to point out that for $n > 3$ this theorem has any content only for the case $g = 0, 1$ for otherwise the dimension formula (1.2) shows that higher genus somewhere injective J -holomorphic curves cannot exist for a generic J when $n > 3$.

In fact, we prove the following general theorem which immediately gives rise to Theorem 1.1. We prove this theorem by establishing a transversality result for the 1-jet evaluation map (see Proposition 2.3) and then by a dimension counting argument.

THEOREM 1.2. *Let (M^{2n}, ω) be any symplectic manifold and $\beta \in H_2(M, \mathbb{Z})$. There exists a subset $\mathcal{J}_\omega^{imm} \subset \mathcal{J}_\omega$ of second category such that for $J \in \mathcal{J}_\omega^{imm}$, all somewhere injective (j, J) -holomorphic maps $u : (\Sigma, j) \rightarrow (M, J)$ in class β are immersed for any $j \in \mathcal{M}_g$, provided*

$$c_1(\beta) + (3 - n)(g - 1) < n - 1. \quad (1.3)$$

And there exists another subset $\mathcal{J}_\omega^{emb} \subset \mathcal{J}_\omega^{imm} \subset \mathcal{J}_\omega$ of second category such that for $J \in \mathcal{J}_\omega^{emb}$ all somewhere injective curves (j, J) -holomorphic curves are embedded, provided

$$c_1(\beta) + (3 - n)(g - 1) < n - 2. \quad (1.4)$$

An immediate corollary of Theorem 1.1 is the following specialization to the symplectic Calabi-Yau manifolds with $n \geq 3$.

COROLLARY 1.3. *Let (M, ω) be symplectic Calabi-Yau, i.e., (M, ω) symplectic and $c_1(M, \omega) = 0$. Assume $n \geq 3$. Let (Σ, j) be a closed smooth Riemann surface of any genus g . Then there exists a subset $\mathcal{J}_\omega^{emb} \subset \mathcal{J}_\omega$ of second category such that for $J \in \mathcal{J}_\omega^{emb}$, for any complex structure j on Σ , all somewhere injective J -holomorphic map $u : (\Sigma, j) \rightarrow (M, J)$ are Fredholm regular and embedded.*

An immediate consequence of this corollary is the following classification result of stable maps in symplectic Calabi-Yau threefolds.

THEOREM 1.4. *Suppose $c_1(M, \omega) = 0$ and $n = 3$. Then there exists a subset $\mathcal{J}_\omega^{nodal} \subset \mathcal{J}_\omega^{emb} \subset \mathcal{J}_\omega$ of second category such that for $J \in \mathcal{J}_\omega^{nodal}$ any stable J -holomorphic map in Calabi-Yau threefolds is one of the following three types :*

1. *it is either smooth and embedded or*
2. *it has smooth domain and factors through the composition*

$$u = u' \circ \phi : \Sigma \rightarrow C \hookrightarrow M$$

for some embedding $u' : C \rightarrow M$ and a ramified covering $\phi : \Sigma \rightarrow C$ or

3. *it is a stable map of the type such that all of its irreducible components have the same locus of images as C , an embedded J -holomorphic curve in M , and are ramified over the domain of C , except those of constant components.*

The latter two cases can occur only when $\beta = [u]$ is of the form $\beta = d\gamma$ for some positive integer d and homology class $\gamma \in H_2(M, \mathbb{Z})$.

A brief outline of this paper is in order.

In section 2, we establish the generic immersion property of J -holomorphic curves (Theorem 2.6) and prove the first half of Theorem 1.2. This section is the most novel and essential part of this paper which involves the 1-jet evaluation of the map u . Consideration of the 1-jet evaluation map in turn forces us to work with the Fredholm setting of $W^{k,p}$ for $k \geq 3$ so that the 1-jet evaluation map becomes differentiable with respect to the variation of evaluation points, which involves taking two derivatives of the map. With this choice of Sobolev spaces however, the actual Fredholm analysis involving the one-jet evaluation map is rather delicate partly because one has to overcome the fact that the evaluation of an L^p -map is not defined pointwise. One novelty of our proof is a judicious usage of the structure theorem of distributions with point supports. See the proof of Lemma 2.5.

The scheme of our proof is motivated by a similar theorem of the authors in [OZ] which sketched the proof of immersion property of nodal Floer trajectory curves. The latter is in turn partly motivated by Hutchings and Taubes' proof of Theorem 4.1 [HT] which concerns the immersion property of the case of 4 ($n = 2$) dimension in a different context : More specifically see the proof of Lemma 4.2 [HT]. There is also a Corollary 3.17 in [Wen] which also concerns immersion property of somewhere injective J -holomorphic curves for the moduli spaces of dimension 0 and 1. In their proofs, both papers utilize the fact that they concern a low dimensional moduli space of J -holomorphic curves. One comparison between Theorem 1.2 and Corollary 3.17 [Wen] (with the case $\partial\Sigma = \emptyset$) is that the condition in our theorem is optimal and corresponds to

$$\text{Index } D_{(j,u)} \bar{\partial}_J < 2(n - 1)$$

(for $n \geq 2$) while Wendl's would correspond to

$$\text{Index } D_{(j,u)} \bar{\partial}_J < 2.$$

It is, however, conceivable that their proofs, with some modifications, could be generalized to higher dimensional moduli spaces, which we did not check.

Aside from establishing the immersion property, our *natural* Fredholm framework for the proof of 1-jet evaluation transversality used in section 2 and 3 has its own merit and suits well for the generalization to the study of higher jet evaluation transversality. We hope to come back to the study of this higher order transversality elsewhere.

In section 3, we establish generic one-one property and prove the second half of Theorem 1.2. Theorem 1.1 stated above then follows by a dimension counting argument.

In section 4, we prove some key lemma, a type of removable singularity theorem for which we utilize a structure theorem of distributions with point supports (see [GS] for example).

In section 5, we discuss some implication of our results to the Gromov-Witten theory of Calabi-Yau threefolds and derive Theorem 1.4.

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2. The 1-jet evaluation transversality. In this subsection, we will give the proof of immersion property. This is the most novel and essential part of the present work. Except the proof of this immersion property, the arguments used in other parts are all standard and well-known in the study of pseudo-holomorphic curves.

2.1. Fredholm setting. We first provide some informal discussion to motivate the necessary Fredholm set-up for the study of immersion property. We will provide the precise analytical framework in the end of this discussion.

We consider a triple $(J, (j, u), z)$ of compatible J and $u : (\Sigma, j) \rightarrow (M, J)$ a (j, J) -holomorphic map and $z \in \Sigma$. Define a map Υ by

$$\Upsilon(J, (j, u), z) = (\bar{\partial}(J, (j, u)); \partial(J, (j, u))(z)) \tag{2.1}$$

where we denote

$$\begin{aligned} \bar{\partial}(J, (j, u)) &:= \bar{\partial}_{(j, J)}(u) = (du)_{(j, J)}^{(0,1)} = \frac{du + Jduj}{2} \\ \partial(J, (j, u)) &:= \partial_{(j, J)}(u) = (du)_{(j, J)}^{(1,0)} = \frac{du - Jduj}{2}. \end{aligned}$$

We now identify the domain and the target of the map Υ . For any given (j, J) , consider the bundles over $\Sigma \times M$

$$\begin{aligned} H_{(j, J)}^{(0,1)}(\Sigma \times M) &:= \bigcup_{(z, x)} Hom''_{(j_z, J_x)}(T_z \Sigma, T_x M) \\ H_{(j, J)}^{(1,0)}(\Sigma \times M) &:= \bigcup_{(z, x)} Hom'_{(j_z, J_x)}(T_z \Sigma, T_x M), \end{aligned}$$

where the above unions are taken for all (z, x) of $(\Sigma \times M)$. Over any (z, x) , the fibers are the (j, J) -anti-linear and (j, J) -linear parts of $Hom(T_z \Sigma, T_x M)$, denoted by $Hom''(T_z \Sigma, T_x M)$ and $Hom'(T_z \Sigma, T_x M)$ respectively. We denote

$$\Lambda_{(j_z, J_x)}^{(1,0)}(T_x M) = Hom'_{(j_z, J_x)}(T_z \Sigma, T_x M)$$

as usual.

We now introduce the necessary framework for the Fredholm theory needed to prove the main theorem. Let $\beta \in H_2(M, \mathbb{Z})$ be given and consider the off-shell function space

$$\mathcal{F}(\Sigma, M; \beta) = \{((\Sigma, j), u) \mid j \in \mathcal{M}(\Sigma), u : \Sigma \rightarrow M, [u] = \beta\}$$

hosting the operator $\bar{\partial}_J : (j, u) \mapsto \bar{\partial}_{(j, J)}(u)$.

For each given $(J, (j, u), z)$, we associate a $2n$ -dimensional vector space

$$H_{(J,(j,u),z)}^{(1,0)} := \Lambda_{(j,J)}^{(1,0)}(u^*TM)|_z = \Lambda_{(jz,Ju(z))}^{(1,0)}(T_{u(z)}M)$$

and define the vector bundle of rank $2n$

$$H^{(1,0)} = \bigcup_{(J,(j,u),z)} \Lambda_{(j,J)}^{(1,0)}(u^*TM)|_z$$

over the space $\mathcal{F}_1(\Sigma, M; \beta)$ defined by

$$\mathcal{F}_1(\Sigma, M; \beta) = \{((\Sigma, j), u, z) \mid ((\Sigma, j), u) \in \mathcal{F}(\Sigma, M; \beta), z \in \Sigma\}.$$

We denote the corresponding moduli space of marked J -holomorphic curves $((\Sigma, j), u, z)$ by $\mathcal{M}_1(\Sigma, M; \beta)$.

REMARK 2.1. In this paper, the domain complex structure j does not play much role in our study. Especially it does not play any role throughout our calculations except that it appears as a parameter.

We introduce the standard bundle

$$\mathcal{H}'' = \bigcup_{(J,(j,u))} \mathcal{H}''_{((j,u),J)}, \quad \mathcal{H}''_{(J,(j,u))} = \Omega_{(j,J)}^{(0,1)}(u^*TM).$$

Then we have the map

$$\Upsilon : \mathcal{J}_\omega \times \mathcal{F}_1(\Sigma, M; \beta) \rightarrow \mathcal{H}'' \times H^{(1,0)}; (J, (j, u), z) \mapsto (\bar{\partial}_{(j,J)}u, (\partial_{(j,J)}u)(z))$$

where $\mathcal{H}'' \times H^{(1,0)}$ is the fiber product of the two bundles

$$\pi_1 : \mathcal{H}'' \rightarrow \mathcal{J}_\omega \times \mathcal{F}(\Sigma, M; \beta)$$

and

$$\pi_2 : H^{(1,0)} \rightarrow \mathcal{J}_\omega \times \mathcal{F}_1(\Sigma, M; \beta) \rightarrow \mathcal{J}_\omega \times \mathcal{F}(\Sigma, M; \beta).$$

More explicitly we can express the fiber product as

$$\mathcal{H}'' \times H^{(1,0)} := \left\{ (\eta, \zeta_0; J, (j, u), z) \mid \eta \in \mathcal{H}''_{(J,(j,u))}, \zeta_0 \in H_{(J,(j,u),z)}^{(1,0)} \right\}$$

We regard this fiber product as a vector bundle over $\mathcal{J}_\omega \times \mathcal{F}_1(\Sigma, M; \beta)$,

$$(\eta, \zeta_0; J, (j, u), z) \mapsto (J, (j, u), z)$$

whose fiber at $(J, (j, u), z)$ is given by

$$\mathcal{H}''_{(J,(j,u))} \times H_{(J,(j,u),z)}^{(1,0)}.$$

Then the above map Υ will become a smooth *section* of this vector bundle.

The union of standard moduli spaces $\mathcal{M}_1(M, J; \beta)$ over $J \in \mathcal{J}_\omega$ is nothing but

$$\Upsilon^{-1}(o_{\mathcal{H}''} \times H^{(1,0)}) / \text{Aut}(\Sigma) \tag{2.2}$$

where $o_{\mathcal{H}''}$ is the zero section of the bundle \mathcal{H}'' defined above, and $\text{Aut}(\Sigma)$ acts on $((\Sigma, j), u)$ by conformal equivalence for any j . We also denote

$$\begin{aligned} \widetilde{\mathcal{M}}_1(M; \beta) &= \Upsilon^{-1}(o_{\mathcal{H}''} \times H^{(1,0)}) \\ \widetilde{\mathcal{M}}_1(M, J; \beta) &= \widetilde{\mathcal{M}}_1(M; \beta) \cap \pi_2^{-1}(J). \end{aligned}$$

The following characterization of the critical point is obvious to see, which however is a key ingredient for the Fredholm framework used in our proof of immersion property.

LEMMA 2.2. *For any $((j, u), z) \in \widetilde{\mathcal{M}}_1(M, J; \beta)$, since $\bar{\partial}_{J,j}u = 0$, we have*

$$du(z) = 0 \quad \text{if and only if} \quad \partial_{(j,J)}u(z) = 0. \tag{2.3}$$

Some remarks concerning the necessary Banach manifold set-up of the map Υ are now in order :

1. To make evaluating ∂u at a point $z \in \Sigma$ make sense, we need to take at least $W^{2,p}$ -completion with $p > 2$ of $\mathcal{F}(\Sigma, M; \beta)$ so that $\bar{\partial}_{(j,J)}u$ lies in $W^{1,p}$ which is then continuous. We actually need to take $W^{k,p}$ -completion of $\mathcal{F}(\Sigma, M; \beta)$ with $k \geq 3$ so that the section Υ , especially the evaluation map, is differentiable (see (2.6)). We denote the corresponding completion of $\mathcal{F}(\Sigma, M; \beta)$ by

$$\mathcal{F}^{k,p} = \mathcal{F}^{k,p}(\Sigma, M; \beta).$$

2. We provide the \mathcal{H}'' the topology of a $W^{k-1,p}$ Banach bundle, with each fiber of class $W^{k-1,p}$. The choice of k will also depend on the index of the linearization of $D\bar{\partial}_{(j,J)}$ on $\mathcal{F}(\Sigma, M; \beta)$ and should be chosen sufficiently large so that one can apply Sard-Smale theorem [Sm].
3. We also need to provide some Banach manifold structure on \mathcal{J}_ω . We can borrow Floer's scheme [F] for this whose details we refer readers thereto. Also see Remark 3.2.7 [MS].

We will assume these settings during the proof of Proposition 2.3 without explicit mentioning unless it is absolutely necessary. At fixed $(J, (u, j), z_0)$ where we do linearization of Υ , we will write

$$\begin{aligned} \Omega_{k,p}^0(u^*TM) &:= W^{k,p}(u^*TM) = T_u\mathcal{F}^{k,p}(\Sigma, M; \beta) \\ \Omega_{k-1,p}^{(0,1)}(u^*TM) &:= W^{k-1,p}\left(\Lambda_{(j,J)}^{(0,1)}(u^*TM)\right) \end{aligned}$$

for the simplicity of notations. Let $o_{H^{(1,0)}}$ be the zero section of $H^{(1,0)}$.

2.2. Proof of generic immersion property. We now prove the following proposition by linearizing the section Υ .

PROPOSITION 2.3. *The section Υ is transverse to the zero section*

$$o_{\mathcal{H}'' \times H^{(1,0)}} = o_{\mathcal{H}''} \times o_{H^{(1,0)}} \subset \mathcal{H}'' \times H^{(1,0)}. \tag{2.4}$$

In particular the set

$$\Upsilon^{-1}(o_{\mathcal{H}''} \times o_{H^{(1,0)}})$$

is a submanifold of $\widetilde{\mathcal{M}}_1(M; \beta)$ of codimension $2n$.

Proof. Recall that the subset

$$o\mathcal{H}'' \times o_{H^{(1,0)}} \subset o\mathcal{H}'' \times H^{(1,0)} \tag{2.5}$$

is a submanifold of codimension $2n$.

So it is easy to check the statement on the codimension once we prove Υ is transverse to the submanifold $o\mathcal{H}'' \times o_{H^{(1,0)}} \subset \mathcal{H}'' \times H^{(1,0)}$. Let $(J, (j, u), z) \in \Upsilon^{-1}(o\mathcal{H}'' \times o_{H^{(1,0)}})$. The linearization of Υ at $(J, (j, u), z)$

$$D_{(J, (j, u), z)} \Upsilon : T_J \mathcal{J}_\omega \times T_{((j, u), z)} \mathcal{F}_1^{k,p}(\Sigma, M; \beta) \rightarrow \mathcal{H}''_{(J, (j, u))} \times H^{(1,0)}_{(J, (j, u), z)}$$

is given by the formula

$$(B, (b, \xi), v) \mapsto \left(D_{J, (j, u)} \bar{\partial}(B, (b, \xi)), D_{J, (j, u)} \partial(B, (b, \xi))(z) + \nabla_v(\partial_{(j, J)} u)(z) \right) \tag{2.6}$$

for $B \in T_J \mathcal{J}_\omega, b \in T_j \mathcal{M}(\Sigma), v \in T_z \Sigma$ and $\xi \in \Omega_{k,p}^0(u^*TM)$. Recall that u is in $W^{k,p}$ with $k \geq 3$ (in fact, u is smooth by elliptic regularity since $\bar{\partial}_{(j, J)} u = 0$) so $D_{J, (j, u)} \partial(B, (b, \xi))$ and $\nabla_v(\partial_{(j, J)} u)$ are in $W^{k-2,p}$ where $k-2 \geq 1$. Therefore their evaluations at z are well-defined.

We need to prove that at each $(J, (j, u), z_0) \in \Upsilon^{-1}(o\mathcal{H}'' \times o_{H^{(1,0)}})$, the system of equations

$$D_{J, (j, u)} \bar{\partial}(B, (b, \xi)) = \gamma \tag{2.7}$$

$$D_{J, (j, u)} \partial(B, (b, \xi))(z_0) + \nabla_v(\partial_{(j, J)} u) = \zeta_0 \tag{2.8}$$

has a solution $(B, (b, \xi), v)$ for each given data

$$\gamma \in \Omega_{k-1,p}^{(0,1)}(u^*TM), \quad \zeta_0 \in H^{(1,0)}_{(J, (j, u), z_0)}.$$

It will be enough to consider the triple with $b = 0$ and $v = 0$ which we will assume from now on.

In general, a well-known computation shows

$$D_{J, (j, u)} \partial(B, (0, \xi)) = (\nabla_{du} \xi)_{(j, J)}^{(1,0)} + T_{(j, J)}^{(1,0)}(du, \xi) + \frac{1}{2} B \circ du \circ j \tag{2.9}$$

with respect to a J -complex connection, i.e., ∇ with $\nabla J = 0$ and its torsion tensor T . Here we denote

$$T_{(j, J)}^{(1,0)}(du, \xi) = \frac{1}{2} (T(du, \xi) + JT(du \circ j, \xi)).$$

However if $u \in \Upsilon^{-1}(o\mathcal{H}'' \times o_{H^{(1,0)}})$, we have $du(z_0) = 0$ and hence

$$T_{(j, J)}^{(1,0)}(du(z_0), \xi(z_0)) = 0 = \frac{1}{2} B(u(z_0)) \circ du(z_0) \circ j_{z_0}$$

for any ξ . If we just want to solve (2.8) at z_0 , then (2.8) is reduced to

$$(\nabla_{du} \xi)_{(j, J)}^{(1,0)}(z_0) = \zeta_0. \tag{2.10}$$

Now we study solvability of (2.7)-(2.8) by applying the Fredholm alternative. For this purpose, we make the following crucial remark

REMARK 2.4. We emphasize that for the map

$$(z_0, v) \mapsto D_{J,(j,u)}\partial(B, (b, \xi))(z_0) + \nabla_v(\partial_{(j,J)}u)$$

to be defined as a continuous map to $H_{(J,(j,u),z_0)}^{(1,0)} = \Lambda^{(1,0)}(T_{u(z_0)}M)$, the map u must be at least $W^{2+\epsilon,p}$ for $\epsilon > 0$: On $W^{2,p}$, the map $D_{J,(j,u)}\partial(B, (b, \xi))$ will be only in L^p for which the evaluation at a point is not defined in general, let alone being continuous. However, the evaluation map

$$z_0 \mapsto D_{J,(j,u)}\partial(B, (0, \xi))(z_0) \quad (2.11)$$

is well-defined and continuous on $W^{2,p}$ as shown by the explicit formula (2.9), which involves only *one* derivative of the section ξ . This reduction from $W^{k,p}$ to $W^{2,p}$ of the regularity requirement in the study of the map (2.11), which can be achieved after restricting to $b = 0$, $v = 0$, will play a crucial role in our proof. See the proof of Lemma 2.5.

Utilizing this remark, we will first show that the image of the map (2.6) restricted to the elements of the form $(B, (0, \xi), 0)$ is onto as a map

$$T_J\mathcal{J}_\omega \times \Omega_{2,p}^0(u^*TM) \rightarrow \Omega_{1,p}^{(0,1)}(u^*TM) \times H_{(J,(j,u),z_0)}^{(1,0)}$$

where (u, j, z_0, J) lies in $o_{\mathcal{H}'} \times o_{H^{(1,0)}}$. In the end of the proof, we will establish solvability of (2.7)-(2.8) on $W^{k,p}$ for $\gamma \in W^{k-1,p}$ by applying an elliptic regularity result of the map (2.7).

We regard

$$\Omega_{1,p}^{(0,1)}(u^*TM) \times H_{(J,(j,u),z_0)}^{(1,0)} := \mathcal{B}$$

as a Banach space with the norm

$$\|\cdot\|_{1,p} + |\cdot|$$

where $|\cdot|$ any norm induced by an inner product on

$$H_{(J,(j,u),z_0)}^{(1,0)} = \Lambda_{(j,J)}^{(1,0)}(u^*TM)_{z_0} \cong \mathbb{C}^n.$$

For the clarification of notations, we denote the natural pairing

$$\Omega_{1,p}^{(0,1)}(u^*TM) \times \left(\Omega_{1,p}^{(0,1)}(u^*TM)\right)^* \rightarrow \mathbb{R}$$

by $\langle \cdot, \cdot \rangle$ and the inner product on $H_{(J,(j,u),z_0)}^{(1,0)}$ by $(\cdot, \cdot)_{z_0}$.

We will first prove that the image is dense in \mathcal{B} .

Let $(\eta, \alpha_{z_0}) \in \left(\Omega_{1,p}^{(0,1)}(u^*TM)\right)^* \times H_{(J,(j,u),z_0)}^{(1,0)}$ such that

$$\langle D_{J,(j,u)}\bar{\partial}_{(j,J)}(B, (0, \xi)), \eta \rangle + (D_{J,(j,u)}\partial_{(j,J)}(B, (0, \xi))(z_0), \alpha_{z_0})_{z_0} = 0 \quad (2.12)$$

for all $\xi \in \Omega_{2,p}^0(u^*TM)$ and B . Without loss of any generality, we may assume that ξ is smooth since $C^\infty(u^*TM) \hookrightarrow \Omega_{2,p}^0(u^*TM)$ is dense. Under this assumption, we would like to show that $\eta = 0 = \alpha_{z_0}$.

Now we simplify the expression of $D_{J,(j,u)}\partial_{(j,J)}(B, (0, \xi))(z_0)$ in complex coordinates z at z_0 . Let $x_0 = u(z_0)$, and identify a neighborhood of z_0 with an open subset of \mathbb{C} and a neighborhood of x_0 with an open set in $T_{x_0}M$. We now introduce the linear operator q_{J,x_0} defined by

$$q_{J,x_0}(x) = (J_{x_0} + J(x))^{-1}(J_{x_0} - J(x))$$

for x such that $d(x, x_0) < \delta$ for $\delta > 0$ depending only on (M, ω, J) but independent of x_0 . q_{J,x_0} satisfies $q_{J,x_0}(x_0) = 0$. (See [Si].) Then if we identify $(T_{x_0}M, J_{x_0}) \cong \mathbb{C}^n$, we can write the operator

$$\begin{aligned} (\nabla_{du}\xi)_{(j,J)}^{(1,0)} &= \partial\xi - q_{J,x_0}(u)\bar{\partial}\xi + C \cdot \xi \\ &= \partial\xi - A \cdot \bar{\partial}\xi + C \cdot \xi, \end{aligned}$$

where in a neighborhood of z_0 , $\partial, \bar{\partial}$ are the standard Cauchy-Riemann operators on \mathbb{C}^n and A, C are smooth pointwise (matrix) multiplication operators with

$$A(z_0) = C(z_0) = 0. \tag{2.13}$$

Therefore we have

$$D_{J,(j,u)}\partial_{(j,J)}(B, (0, \xi))(z_0) = (\nabla_{du}\xi)_{(j,J)}^{(1,0)} = (\partial\xi - A \cdot \bar{\partial}\xi + C \cdot \xi)(z_0) \tag{2.14}$$

at the given point z_0 for any given ξ_0 . Since we just need ξ to satisfy (2.14) at z_0 , by the condition of A and C at z_0 , we have shown

$$D_{J,(j,u)}\partial_{(j,J)}(B, (0, \xi))(z_0) = \partial\xi(z_0). \tag{2.15}$$

By the above discussion on $D_{J,(j,u)}\bar{\partial}_{(j,J)}(B, (0, \xi))$ and $D_{J,(j,u)}\partial_{(j,J)}(B, (0, \xi))(z_0)$, (2.12) is equivalent to

$$\langle D_u\bar{\partial}_{(j,J)}\xi + \frac{1}{2}B \circ du \circ j, \eta \rangle + \langle \partial\xi, \delta_{z_0}\alpha_{z_0} \rangle = 0 \tag{2.16}$$

for all B and ξ of C^∞ where δ_{z_0} is the Dirac-delta function.

Taking $B = 0$ in (2.16), we obtain

$$\langle D_u\bar{\partial}_{(j,J)}\xi, \eta \rangle + \langle \partial\xi, \delta_{z_0}\alpha_{z_0} \rangle = 0 \quad \text{for all } \xi \text{ of } C^\infty. \tag{2.17}$$

Therefore by definition of the distribution derivatives, η satisfies

$$(D_u\bar{\partial}_{(j,J)})^\dagger\eta - \bar{\partial}(\delta_{z_0}\alpha_{z_0}) = 0$$

as a distribution, i.e.,

$$(D_u\bar{\partial}_{(j,J)})^\dagger\eta = \bar{\partial}(\delta_{z_0}\alpha_{z_0})$$

where $(D_u\bar{\partial}_{(j,J)})^\dagger$ is the formal adjoint of $D_u\bar{\partial}_{(j,J)}$ whose symbol is the same as $D_u\partial_{(j,J)}$ and so is an elliptic first order differential operator. We also recall that $\partial^\dagger = -\bar{\partial}$. Since $\text{supp } \bar{\partial}(\delta_{z_0})\alpha_{z_0} \subset \{z_0\}$, we have $(D_u\bar{\partial}_{(j,J)})^\dagger\eta = 0$ on $\Sigma \setminus \{z_0\}$ as a distribution. Then by the elliptic regularity (see Theorem 13.4.1 [Ho], for example), η must be smooth on $\Sigma \setminus \{z_0\}$.

On the other hand, by setting $\xi = 0$ in (2.16), we get

$$\left\langle \frac{1}{2}B \circ du \circ j, \eta \right\rangle = 0 \tag{2.18}$$

for all $B \in T_J\mathcal{J}_\omega$. From this identity, standard argument from [F], [M] shows that $\eta = 0$ in a small neighborhood of any somewhere injective point in $\Sigma \setminus \{z_0\}$. Such a somewhere injective point exists by the hypothesis of u being somewhere injective and the fact that the set of somewhere injective points is open and dense in the domain under the hypothesis (see [M]). Then by the unique continuation theorem, we conclude that $\eta = 0$ on $\Sigma \setminus \{z_0\}$ and so the support of η as a distribution on Σ is contained at the one-point subset $\{z_0\}$ of Σ .

We will postpone the proof of the following lemma till section 3, in which reduction of the regularity requirement for u mentioned in Remark 2.4 will play an essential role.

LEMMA 2.5. *η is a distributional solution of $(D_u\bar{\partial}_{(j,J)})^\dagger\eta = 0$ on Σ and so continuous. In particular, we have $\eta = 0$ in $\left(\Omega_{(1,p)}^{(0,1)}(u^*TM)\right)^*$.*

Once we know $\eta = 0$, the equation (2.12) is reduced to

$$(D_{J,(j,u)}\partial_{(j,J)}(B, (0, \xi))(z_0), \alpha_{z_0})_{z_0} = 0. \tag{2.19}$$

It remains to show that $\alpha_{z_0} = 0$. For this, we have only to show that the image of the evaluation map

$$\xi \mapsto D_{J,(j,u)}\partial_{(j,J)}(0, (0, \xi))(z_0) = \partial\xi(z_0)$$

is surjective onto $H_{(J,(j,u),z_0)}^{(1,0)} = \Lambda_{(j,J)}^{(1,0)}(T_{u(z_0)}M)$. The equality comes from (2.15).

To show this surjectivity, we need to prove the existence of ξ satisfying

$$\partial\xi(z_0) = \zeta_0 \tag{2.20}$$

at the given point z_0 for any given ζ_0 . We can multiply a cut-off function χ to ζ_0 with $\chi \equiv 1$ to make $\zeta(z) := \chi(z)\zeta_0$ supported in a sufficient small neighborhood around z_0 , and apply Cauchy integral formula in coordinates to solve

$$\partial\xi = \zeta$$

in some neighborhood around z_0 . This finishes the existence of a solution to (2.10) and hence (2.8) and so the proof of the claim that the image of (2.6) with $v = 0$ is dense in

$$\Omega_{1,p}^{(0,1)}(u^*TM) \times H_{(J,(j,u),z_0)}^{(1,0)}.$$

We recall that for any fixed (J, j) , the image of $D_u\bar{\partial}_{(j,J)}$ on $T_J\mathcal{J}_\omega \times \Omega_{2,p}^0(u^*TM)$ is closed in $\Omega_{(1,p)}^{(0,1)}(u^*TM)$ and that $H_{(J,(j,u),z_0)}^{(1,0)}$ is a finite dimensional vector space. Therefore the image of the linearization (2.6) is also closed. Hence (2.6) is surjective $\Omega_{(1,p)}^{(0,1)}(u^*TM) \times H_{(J,(j,u),z_0)}^{(1,0)}$ as a map from $T_J\mathcal{J}_\omega \times T_{((j,u),z)}\mathcal{F}_1^{2,p}(\Sigma, M; \beta)$.

Now finally suppose γ is in the subspace $\Omega_{k-1,p}^{(0,1)}(u^*TM) \subset \Omega_{1,p}^{(0,1)}(u^*TM)$ of higher regularity. We recall $k \geq 3$. By the above analysis of the map (2.6) for $b = 0 = v$, we can find a solution $(B, (0, \xi), 0)$ of (2.7)-(2.8) with $B \in T_J\mathcal{J}_\omega$ and with ξ as an element

in $\Omega_{2,p}^0(u^*TM) = W^{2,p}(u^*TM)$ for any $\zeta_0 \in H_{(J,(j,u),z_0)}^{(1,0)}$. By elliptic regularity of (2.7), ξ indeed lies in $W^{k,p}$ if $\gamma \in W^{k-1,p}$, and hence in $\Omega_{k,p}^0(u^*TM)$. Therefore the map (2.6) is onto, i.e., Υ is transverse to the submanifold $o_{\mathcal{H}''} \times o_{H^{(1,0)}} \subset \mathcal{H}'' \times H^{(1,0)}$. \square

Finally we have the natural projection

$$\pi : \widetilde{\mathcal{M}}_1(M; \beta) := \bigcup_{J \in \mathcal{J}_\omega} \widetilde{\mathcal{M}}_1(M, J; \beta) \rightarrow \mathcal{J}_\omega.$$

The projection has index $2(c_1(\beta) + n(1 - g)) + 2$, so for any regular value J , the moduli space

$$\widetilde{\mathcal{M}}_1^{crit}(M, J; \beta) := \Upsilon^{-1}(o_{\mathcal{H}''} \times o_{H^{(1,0)}}) \cap \pi^{-1}(J)$$

is of dimension

$$2(c_1(\beta) + (3 - n)(g - 1)) + 2 - 2n$$

for $g \geq 2$. For the case $g = 0, 1$, the dimension can be similarly calculated from (1.1).

Let the moduli space

$$\mathcal{M}_1^{crit}(M, J; \beta) := \widetilde{\mathcal{M}}_1^{crit}(M, J; \beta) / \text{Aut}(\Sigma),$$

where $\text{Aut}(\Sigma)$ acts on marked Riemann surfaces $((\Sigma, j), z)$ by conformal equivalence then on the maps from them. Geometrically $\mathcal{M}_1^{crit}(M, J; \beta)$ consists of J -holomorphic curves in class β with at least one critical point. As a smooth orbifold, we have

$$\dim \mathcal{M}_1^{crit}(M, J; \beta) = 2(c_1(\beta) + (3 - n)(g - 1) + 1 - n)$$

for all g as stated in the introduction. Therefore, $\mathcal{M}_1^{crit}(M, J; \beta)$ is empty whenever this dimension is negative, i.e.,

$$c_1(\beta) + (3 - n)(g - 1) < n - 1.$$

We just set

$$\mathcal{J}_\omega^{imm} = \text{the set of regular values of } \pi$$

which finishes the proof of the following theorem.

THEOREM 2.6. *There exists a subset $\mathcal{J}_\omega^{imm} \subset \mathcal{J}_\omega$ of second category such that for $J \in \mathcal{J}_\omega^{imm}$ all somewhere injective (j, J) -holomorphic maps $u : \Sigma \rightarrow M$ are immersed for any $j \in \mathcal{M}(\Sigma)$, provided*

$$c_1(\beta) + (3 - n)(g - 1) < n - 1. \tag{2.21}$$

3. Proof of Lemma 2.5. In this section, we prove Lemma 2.5. Our primary goal is to prove

$$\langle D_u \bar{\partial}_{(j,J)} \xi, \eta \rangle = 0 \tag{3.1}$$

for all smooth $\xi \in \Omega^0(u^*TM)$, i.e., η is a distributional solution of $(D_u \bar{\partial}_{(j,J)})^\dagger \eta = 0$ on the whole Σ , not just on $\Sigma \setminus \{z_0\}$ which was shown in section 2.

We start with (2.17)

$$\langle D_u \bar{\partial}_{(j,J)} \xi, \eta \rangle + \langle \partial \xi, \delta_{z_0} \alpha_{z_0} \rangle = 0 \quad \text{for all } \xi \text{ of } C^\infty. \tag{3.2}$$

We first simplify the expression of the pairing $\langle D_u \bar{\partial}_{(j,J)} \xi, \eta \rangle$ knowing that $\text{supp } \eta \subset \{z_0\}$.

Let z be a complex coordinate centered at a fixed marked point z_0 and (w_1, \dots, w_n) be the complex coordinates on $T_{u(z_0)}M$ regarded as coordinates on a neighborhood of $u(z_0)$. We consider the standard metric

$$h = \frac{\sqrt{-1}}{2} dz d\bar{z}$$

on a neighborhood U of z_0 and with respect to the coordinates (w_1, \dots, w_n) we fix any Hermitian metric on \mathbb{C}^n .

The following lemma will be crucial in our proof.

LEMMA 3.1. *For any smooth section ξ of $u^*(TM)$ and η of $(\Omega_{1,p}^{(0,1)}(u^*TM))^*$*

$$\langle D_u \bar{\partial}_{(j,J)} \xi, \eta \rangle = \langle \bar{\partial} \xi, \eta \rangle,$$

where $\bar{\partial}$ is the standard Cauchy-Riemann operators on \mathbb{C}^n in the above coordinate.

Proof. We have already shown that η is a distribution with $\text{supp } \eta \subset \{z_0\}$. By the structure theorem on the distribution supported at a point z_0 (see section 4.5, especially p. 119, of [GS], for example), we have

$$\eta = P \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right) (\delta_{z_0})$$

where $z = s + it$ is the given complex coordinates at z_0 and $P \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right)$ is a differential operator associated by the polynomial P of two variables with coefficients in $(\Lambda_{(j_{z_0}, J_{u(z_0)})}^{(0,1)}(u^*TM))^*$.

Furthermore since $\eta \in (W^{1,p})^*$, the degree of P must be zero and so we obtain

$$\eta = \alpha_{z_0} \delta_{z_0} \tag{3.3}$$

for some constant vector α_{z_0} : This is because the ‘evaluation at a point of the derivative’ of $W^{1,p}$ map does not define a continuous functional on $W^{1,p}$.

We can write

$$D_u \bar{\partial}_{(j,J)} \xi = \bar{\partial} \xi + E \cdot \partial \xi + F \cdot \xi$$

near z_0 in coordinates similarly as we did in (2.14) for the operator $D_u \partial_{(j,J)}$, where E and F are zero-order matrix operators with $E(z_0) = 0 = F(z_0)$.

Therefore by (3.3), we derive

$$\langle E \cdot \partial \xi + F \cdot \xi, \eta \rangle = \langle E \cdot \partial \xi + F \cdot \xi, \alpha_{z_0} \delta_{z_0} \rangle = (E(z_0) \partial \xi(z_0) + F(z_0) \xi(z_0), \alpha_{z_0})_{z_0} = 0.$$

Therefore we obtain

$$\langle D_u \bar{\partial}_{(j,J)} \xi, \eta \rangle = \langle \bar{\partial} \xi + E \cdot \partial \xi + F \cdot \xi, \eta \rangle = \langle \bar{\partial} \xi, \eta \rangle$$

which finishes the proof. \square

REMARK 3.2. We note that (3.3) is where we needed to have made the reduction of the regularity requirement for the map ξ from $W^{k,p}$ to $W^{2,p}$ in the first half of the proof of Proposition 2.3 : if we had not made the reduction but required ξ to be in $W^{k,p}$, its derivative would have lied in $W^{k-1,p}$ and hence we could have only concluded

$$\eta = P \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right) (\delta_{z_0})$$

with $\deg P \leq k - 1$. This would then cause a problem in the argument for the rest of the proof of this lemma.

This lemma then implies that (3.2) becomes

$$\langle \bar{\partial}\xi, \eta \rangle + \langle \partial\xi, \delta_{z_0}\alpha_{z_0} \rangle = 0 \quad \text{for all } \xi. \tag{3.4}$$

We decompose ξ as

$$\xi(z) = \tilde{\xi}(z) + \chi(z)(z - z_0)a(z_0)$$

by defining $\tilde{\xi}$ by

$$\tilde{\xi}(z) = \xi(z) - \chi(z)(z - z_0)a(z_0)$$

where we express the one-form $\partial\xi$ as

$$\partial\xi(z) = a(z)dz \tag{3.5}$$

on U in coordinates with $a(z_0) \in \mathbb{C}^n$, and χ is a cut-off function with $\chi \equiv 1$ in a small neighborhood V of z_0 and satisfies $\text{supp } \chi \subset U$. The choice of this decomposition is dictated by the fact

$$\partial(\chi(z)(z - z_0)a(z_0))(z_0) = a(z_0)dz. \tag{3.6}$$

Then $\tilde{\xi}$ is a smooth section on Σ , and satisfies

$$\partial\tilde{\xi}(z_0) = 0, \quad \bar{\partial}\tilde{\xi} = \bar{\partial}\xi \quad \text{on } V.$$

Therefore applying (3.4) to $\tilde{\xi}$ instead of ξ , we have

$$\langle \bar{\partial}\tilde{\xi}, \eta \rangle + \langle \partial\tilde{\xi}, \delta_{z_0}\alpha_{z_0} \rangle = 0.$$

But we have

$$\langle \bar{\partial}\tilde{\xi}, \eta \rangle = \langle \bar{\partial}\xi, \eta \rangle \tag{3.7}$$

since $\bar{\partial}\tilde{\xi} = \bar{\partial}\xi$ on V and $\text{supp } \eta \subset \{z_0\}$. Again using the support property $\text{supp } \eta \subset \{z_0\}$ and (3.5), (3.6), we derive

$$\begin{aligned} \langle \partial\tilde{\xi}, \delta_{z_0}\alpha_{z_0} \rangle &= \langle \partial\xi, \delta_{z_0}\alpha_{z_0} \rangle - \langle \partial(\chi(z)(z - z_0)a(z_0)), \delta_{z_0}\alpha_{z_0} \rangle \\ &= (\partial\xi(z_0), \alpha_{z_0})_{z_0} - (a(z_0)dz, \alpha_{z_0})_{z_0} \\ &= (\partial\xi(z_0) - a(z_0)dz, \alpha_{z_0})_{z_0} = 0. \end{aligned} \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.4), we obtain

$$\langle \bar{\partial}\xi, \eta \rangle = 0$$

and so we have finished the proof of (3.1).

By the elliptic regularity, η must be smooth. Since we have already shown $\eta = 0$ on $\Sigma \setminus \{z_0\}$, continuity of η proves $\eta = 0$ on the whole Σ .

4. Generic one-one property. In this subsection, we will give the embedding part of the main theorem. As before we assume that (j, u) is a somewhere injective J -holomorphic curve in class $\beta \in H_2(M, \mathbb{Z})$.

We note that for any given self-intersection point $x \in \text{Im } u$ we have

$$u^{-1}(x) = \{z_1, \dots, z_k\}$$

for some integer $k \geq 2$. So we will consider the moduli space of J -holomorphic maps with 2 marked points and in the homology class β .

We consider the triples $(J, (j, u), (z_1, z_2))$ and the map

$$\Upsilon_2 : (J, (j, u), (z_1, z_2)) \mapsto (\bar{\partial}_{(j,J)}u, (u(z_1), u(z_2))). \tag{4.1}$$

We introduce the necessary framework for the Fredholm theory needed to prove Theorem 2.6. Similarly as $\mathcal{F}_1(\Sigma, M; \beta)$, we define

$$\mathcal{F}_2(\Sigma, M; \beta) = \{((j, u), (z_1, z_2)) \mid (j, u) \in \mathcal{F}(\Sigma, M; \beta), z_1, z_2 \in \Sigma\}$$

and by $\mathcal{M}_2(M, J; \beta) = \widetilde{\mathcal{M}}_2(M, J; \beta) / \text{Aut}(\Sigma)$ the corresponding moduli spaces of J -holomorphic curves where

$$\widetilde{\mathcal{M}}_2(M, J; \beta) = \{((j, u), (z_1, z_2)) \in \mathcal{F}_2(\Sigma, M; \beta) \mid \bar{\partial}_{j,J}u = 0\}.$$

We set

$$\widetilde{\mathcal{M}}_2(M; \beta) := \bigcup_{J \in \mathcal{J}_\omega} \widetilde{\mathcal{M}}_2(M, J; \beta).$$

We have the natural projection map $\pi : \widetilde{\mathcal{M}}_2(M; \beta) \rightarrow \mathcal{J}_\omega$.

We have the natural evaluation map

$$ev : \mathcal{F}_2(\Sigma, M; \beta) \rightarrow M \times M; \quad ev((j, u), (z_1, z_2)) = (u(z_1), u(z_2)).$$

Then the above map Υ_2 defines a map

$$\Upsilon_2 : \mathcal{J}_\omega \times \mathcal{F}_2(\Sigma, M; \beta) \rightarrow \mathcal{H}'' \times M \times M.$$

We now prove the following lemma by a standard argument via the linearization of Υ_2 .

PROPOSITION 4.1. *The map Υ_2 is transverse to the submanifold*

$$o_{\mathcal{H}''} \times \Delta \subset \mathcal{H}'' \times (M \times M).$$

In particular the set

$$\Upsilon_2^{-1}(o_{\mathcal{H}''} \times \Delta)$$

is a submanifold of

$$\widetilde{\mathcal{M}}_2(M; \beta)$$

of codimension $2n$.

Proof. It is easy to check the statement on the codimension and so we will focus on proving the submanifold property.

The linearization of Υ_2 is given by

$$(B, (\xi, v_1, v_2)) \mapsto (D_{J,u}\bar{\partial}(B, \xi), \xi(z_1) + du(v_1), \xi(z_2) + du(v_2)). \quad (4.2)$$

We will focus on the problem of finite dimensional transversality of the linear map

$$(\xi, v_1, v_2) \mapsto (\xi(z_1) + du(v_1), \xi(z_2) + du(v_2))$$

to the subspace $T\Delta \subset T(M \times M)$. This diagonal transversality is well-known (e.g., see Proposition 3.4.2 [MS]). In fact an easier variation of our proof of the 1-jet evaluation transversality adapted to the usual 0-jet evaluation map gives rise to a simple proof of the well-known transversality result of the evaluation map e.g. of Proposition 3.4.2 [MS]. For the readers' convenience, we provide the details of this transversality result in the Appendix.

Now we consider the natural projection

$$\pi_{\Upsilon_2} : \Upsilon_2^{-1}(o_{\mathcal{H}''} \times \Delta) \rightarrow \mathcal{J}_\omega$$

which is the restriction of the projection map $\pi : \widetilde{\mathcal{M}}_2(M; \beta) \rightarrow \mathcal{J}_\omega$. Since π has the index $2(c_1(\beta) + (3 - n)(g - 1)) + 4$ (for $g \geq 2$), the Fredholm index of π_{Υ_2} is given by $2(c_1(\beta) + (3 - n)(g - 1)) + 4 - 2n$.

Therefore for any regular value J of π_{Υ_2} ,

$$\widetilde{\mathcal{M}}_2^{doub}(M, J; \beta) := \Upsilon_2^{-1}(o_{\mathcal{H}''} \times \Delta) \cap \pi^{-1}(J)$$

is a smooth manifold of dimension

$$2(c_1(\beta) + (3 - n)(g - 1)) + 4 - 2n$$

for $g \geq 2$. Again similar dimension formula can be derived from (1.1) for $g = 0, 1$. We just set

$$\mathcal{J}_\omega^{inj} = \text{the set of regular values of } \pi_{\Upsilon_2}.$$

Again we define

$$\mathcal{M}_2^{doub}(M, J; \beta) := \widetilde{\mathcal{M}}_2^{doub}(M, J; \beta) / \text{Aut}(\Sigma),$$

where $\text{Aut}(\Sigma)$ acts on marked Riemann surfaces $((\Sigma, j), (z_1, z_2))$ by conformal equivalence and then on maps from them. Geometrically, $\mathcal{M}_2^{doub}(M, J; \beta)$ consists of J -holomorphic curves in class β with self-intersections. It is a smooth orbifold of dimension

$$\dim \mathcal{M}_2^{doub}(M, J; \beta) = 2(c_1(\beta) + (3 - n)(g - 1) + 2 - n)$$

for all genus g . Therefore for any $J \in \mathcal{J}_\omega^{inj}$ $\mathcal{M}_2^{doub}(M, J; \beta)$ will be empty whenever

$$c_1(\beta) + (3 - n)(g - 1) < n - 2$$

and in particular when $n \geq 3$ and $c_1(\beta) \leq 0$. \square

We denote

$$\mathcal{J}_\omega^{emb} = \mathcal{J}_\omega^{imm} \cap \mathcal{J}_\omega^{inj}$$

which is again of second category of \mathcal{J}_ω since both \mathcal{J}_ω^{imm} and \mathcal{J}_ω^{inj} are of second category thereof.

We summarize the discussion in this section into the following theorem

THEOREM 4.2. *There exists a subset $\mathcal{J}_\omega^{emb} \subset \mathcal{J}_\omega$ of second category such that for $J \in \mathcal{J}_\omega^{emb}$, all somewhere injective (j, J) -holomorphic maps $u : \Sigma \rightarrow M$ are embeddings for any $j \in \mathcal{M}_g$, provided*

$$c_1(\beta) + (3 - n)(g - 1) < n - 2. \tag{4.3}$$

Proof. [Proof of Theorem 1.1] Theorem 1.1 immediately follows from Theorem 2.6 and Theorem 4.2 by dimension counting. We have only to note that when $c_1(\beta) \leq 0$ and $n \geq 3$, both inequalities (2.21) and (4.3) are satisfied. \square

5. Compactification of moduli spaces in Calabi-Yau threefolds. In this section, we restrict our attention to the case $n = 3$, $c_1 = 0$ and let $J \in \mathcal{J}_\omega^{emb}$. By the dimension counting argument using the evaluation maps similar to the one in section 4, the following is easy to prove (See the proof of Theorem 6.3.1 [MS] or [OZ] for the set-up of the proof in a somewhat different context).

LEMMA 5.1. *There exists a subset $\mathcal{J}_\omega^{nodal} \subset \mathcal{J}_\omega^{emb}$ of \mathcal{J}_ω of second category such that any two somewhere injective curves do not intersect unless they have identical images.*

Proof. Recall (1.2) that the virtual dimension of $\mathcal{M}(M, J; \beta)$ is given by $(n - 3)(1 - g)$ for (M, ω) with $c_1 = 0$ and so is zero for $n = 3$. This result follows by the standard dimension counting argument for Fredholm regular *somewhere injective* curves. \square

This rules out nodal degeneration in the Gromov compactification and gives rise to the following compactification result.

THEOREM 5.2. *Let $J \in \mathcal{J}_\omega^{nodal}$. Fix $\beta \in H_2(M, \mathbb{Z})$ and consider a sequence of smooth J -holomorphic maps (u_i, Σ_i) with $u_i : \Sigma_i \rightarrow M$ in class $[u_i] = \beta$. Suppose that (u, Σ) is its stable limit. Then there exists an integer $d \geq 1$ a class $\gamma \in H_2(M, \mathbb{Z})$ with $\beta = d\gamma$ and an embedded curve $u' : \Sigma' \rightarrow M$ with $[u'] = \gamma$ such that (u, Σ) factors as the composition $u = u' \circ \phi$ where $\phi : \Sigma \rightarrow \Sigma'$ is a stable map into Σ' . Moreover, in the latter case, each non-constant irreducible component of Σ is a ramified covering of Σ' .*

Proof. If a J -holomorphic curve is somewhere injective and Fredholm regular, by dimension formula it is isolated and so there is no J -holomorphic curves in a sufficiently small C^∞ -neighborhood thereof. Furthermore from the above lemma, they do not intersect unless their images coincide whenever $J \in \mathcal{J}_\omega^{nodal}$.

Combining these, we derive that every irreducible component of (u, Σ) either is constant or has its image coinciding with, say, that of an embedded curve $u' : \Sigma' \rightarrow M$. Denote by C the image of u' . Then (u, Σ) defines a stable map into the curve C . Since $u' : \Sigma' \rightarrow C$ is a biholomorphic map, the stable map induces one into Σ' . Denoting this stable map u by $\phi : \Sigma \rightarrow \Sigma'$, we have finished the proof. \square

6. Appendix: A proof of 0-jet evaluation map transversality. In this appendix, we give a conceptually natural proof of the following well-known evaluation map transversality by adapting the proof of the 1-jet evaluation transversality given in the present paper. A detailed proof of this evaluation transversality is given in the proof of Proposition 3.4.2 [MS], with which our proof given here would like to be compared.

THEOREM 6.1 (0-jet evaluation transversality). *We consider the map*

$$\Upsilon_0(J, (j, u), z) = (\bar{\partial}(J, (j, u)), u(z))$$

as a map from $\mathcal{J}_\omega \times \mathcal{F}_1(\Sigma, M; \beta) \rightarrow \mathcal{H}'' \times M$. Then Υ_0 is transverse to the submanifold

$$o\mathcal{H}'' \times \{p\} \subset \mathcal{H}'' \times M$$

for any given point $p \in M$.

Proof. Its linearization $D\Upsilon_0(J, (j, u), z)$ is given by

$$(B, (b, \xi), v) \mapsto (D_{J,(j,u)}\bar{\partial}(B, (b, \xi)), \xi(u(z)) + du(z)(v))$$

for $B \in T_J\mathcal{J}_\omega$, $b \in T_j\mathcal{M}(\Sigma)$, $v \in T_z\Sigma$ and $\xi \in T_u\mathcal{F}(\Sigma, M; \beta)$. This defines a linear map

$$T_J\mathcal{J}_\omega \times T_j\mathcal{M}(\Sigma) \times T_u\mathcal{F}(\Sigma, M; \beta) \times T_z\Sigma \rightarrow \Omega_{(j,J)}^{(0,1)}(u^*TM) \times T_{u(z)}M.$$

We would like to prove that this linear map is surjective at every element $(u, z_0) \in \tilde{\mathcal{F}}_1(\Sigma, M; \beta)$ i.e., at the pair (u, z_0) satisfying

$$\bar{\partial}_{(j,J)}u = 0, \quad du(z_0) = 0.$$

For this purpose, we need to study solvability of the system of equations

$$D_{J,(j,u)}\bar{\partial}(B, (b, \xi)) = \gamma, \quad \xi(u(z_0)) + du(v) = X_0 \tag{6.1}$$

for given $\gamma \in \Omega_{(j,J)}^{(0,1)}(u^*TM)$ and $X_0 \in T_{u(z_0)}M$. Again it will be enough to consider the case $b = 0 = v$. Then this equation is reduced to

$$D_{J,u}\bar{\partial}(B, \xi) = \gamma, \quad \xi(u(z_0)) = X_0. \tag{6.2}$$

For the map Υ_0 to be differentiable, we need to choose the completion of $\mathcal{F}(\Sigma, M; \beta)$ in the $W^{k,p}$ -norm for $k \geq 2$.

Now we study (6.2) for $\xi \in W^{2,p}$ similarly as in sections 2 and 3 : This time we can use $W^{2,p}$ -norm instead of $W^{3,p}$ -norm since the 0-jet evaluation map does not involve taking a derivative of the map u unlike the 1-jet evaluation map. We regard

$$\Omega_{1,p}^{(0,1)}(u^*TM) \times T_{u(z_0)}M := \mathcal{B}_0$$

as a Banach space with the norm $\|\cdot\|_{1,p} + |\cdot|$ similarly as before with $H_{(J,(j,u),z_0)}^{(1,0)}$ replaced by $T_{u(z_0)}M$. By the same reasoning, we apply the Fredholm alternative and study those (η, X_0) that satisfy the equation

$$\langle D_u\bar{\partial}_{(j,J)}\xi + \frac{1}{2}B \circ du \circ j, \eta \rangle + \langle \xi, \delta_{z_0}X_0 \rangle = 0$$

for all B and ξ of C^∞ where δ_{z_0} is the Dirac-delta function supported at z_0 .

Now the rest of the proof will duplicate the proofs of Proposition 2.3 and Lemma 2.5 with $\partial\xi$ replaced by ξ . This finishes the proof of solvability of (6.2) for any given $\gamma \in W^{1,p}$ and $X_0 \in T_{u(z_0)}M$. As before if $\gamma \in W^{k-1,p}$, then $\xi \in W^{k,p}$ by the elliptic regularity. This finishes the proof of surjectivity of the map

$$(B, (0, \xi), 0) \mapsto (D_{J,u}\bar{\partial}(B, \xi), \xi(u(z))) : \\ T_J\mathcal{J}_\omega \times \Omega_{k,p}^0(u^*TM) \rightarrow \Omega_{k-1,p}^{(0,1)}(u^*TM) \times T_{u(z_0)}M$$

and hence proves the required transversality. \square

A proof of the diagonal transversality, which is the transversality of the map (4.2), can be given by an obvious modification of the above proof by considering the map

$$\Upsilon_2 : \mathcal{J}_\omega \times \mathcal{F}_2(\Sigma, M; \beta) \rightarrow \mathcal{H}'' \times M \times M$$

whose details we omit here.

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