Asian J. Math.
Vol. 13, No. 2, pp. 151–180, June 2009

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ON LEVEL-RAISING CONGRUENCES*

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Abstract. In this paper we rewrite a work of Sorensen to include nontrivial types at the infinite places. This work extends results of K. Ribet and R. Taylor on level-raising for algebraic modular forms on $D^\times$, where D is a definite quaternion algebra over a totally real field F. We do this for any automorphic representations π of an arbitrary reductive group G over F which is compact at infinity. We do not assume π_∞ is trivial. If λ is a finite place of $\bar{\mathbb{Q}}$, and w is a place where π_w is unramified and $\pi_w \equiv \mathbf{1} \pmod{\lambda}$, then under some mild additional assumptions (we relax requirements on the relation between w and ℓ which appear in previous works) we prove the existence of a $\tilde{\pi} \equiv \pi \pmod{\lambda}$ such that $\tilde{\pi}_w$ has more parahoric fixed vectors than π_w. In the case where G_w has semisimple rank one, we sharpen results of Clozel, Bellaiche and Graftieaux according to which $\tilde{\pi}_w$ is Steinberg. To provide applications of the main theorem we consider two examples over F of rank greater than one. In the first example we take G to be a unitary group in three variables and a split place w. In the second we take G to be an inner form of GSp(2). In both cases, we obtain precise satisfiable conditions on a split prime w guaranteeing the existence of a $\tilde{\pi} \equiv \pi \pmod{\lambda}$ such that the component $\tilde{\pi}_w$ is generic and Iwahori spherical. For symplectic G, to conclude that $\tilde{\pi}_w$ is generic, we use computations of R. Schmidt. In particular, if π is of Saito-Kurokawa type, it is congruent to a $\tilde{\pi}$ which is not of Saito-Kurokawa type.

Key words. Level-Raising, Congruences, Algebraic Modular Forms.

AMS subject classifications. 11F33, 22E55, 11F70, 11F85, 11F46, 20G25, 22E35

Introduction. This paper stems from the following result of Ribet [R]:

Theorem 0.1. *Let $f \in S_2(\Gamma_0(N))$ be an eigenform. Let $\lambda|\ell$ be a finite place of $\bar{\mathbb{Q}}$ with $\ell \geq 5$ and f not congruent to an Eisenstein series modulo λ. Let q be a prime number with $(q, N\ell) = 1$ and $a_q(f)^2 \equiv (1+q)^2 \pmod{\lambda}$. Then there exists a q-new eigenform $\tilde{f} \in S_2(\Gamma_0(Nq))$ congruent to f mod λ.*

Two eigenforms f and $\tilde{f}$ are said to be *congruent modulo* λ if their Hecke eigenvalues are algebraic integers congruent for almost all primes, that is, if $a_p(f) \equiv a_p(\tilde{f}) \pmod{\lambda}$ for almost all p. The proof of this theorem can be reduced, via the correspondence from an inner form to GL(2) (see [F] for a simple proof), to the corresponding statement for $D^\times$ where D is a definite quaternion algebra over $\mathbb{Q}$.

A goal of this paper is to prove that an automorphic form of Saito-Kurokawa type is congruent to an automorphic form which is *not* of Saito-Kurokawa type. By functoriality ([F1]) the statement can be reduced to that for an inner form G of $\mathrm{PGSp}(2)/F$ such that $G(\mathbb{R})$ is compact. Indeed, the set of packets of automorphic representations of $G(\mathbb{A})$ can be identified with a subset of the set of such objects on $\mathrm{PGSp}(2,\mathbb{A})$, where almost all local components are the same. By a form on $G \simeq \mathrm{SO}(5)$ of Saito-Kurokawa type we mean the lift of $\mathbf{1} \times \rho$ from the endoscopic group $\mathrm{PGL}(2,\mathbb{A}) \times \mathrm{PGL}(2,\mathbb{A})$ to $\mathrm{PGSp}(2,\mathbb{A})$, where ρ is cuspidal and $\mathbf{1}$ is trivial on $\mathrm{PGL}(2,\mathbb{A})$. It is nontempered at almost all places. We achieve this goal in Theorem 0.6, proven in Section 12.

We apply ideas and methods of R. Taylor [T1] and [T2]. The level-raising part of Taylor's proof carries over to the following more general setup. Let F denote a totally real number field with ring $\mathbb{A} = F_\infty \times \mathbb{A}^\infty$ of adèles. We denote the set of real places

*Received June 11, 2008; accepted for publication November 11, 2008.
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of F by ∞. Let G be a connected reductive F-group such that $G^1_\infty := G_\infty \cap G(\mathbb{A})^1$ (see Sect. 3) is compact and the derived group $G^{\rm der}$ is simple and simply connected; here $G_\infty = G(F_\infty)$. When $F \neq \mathbb{Q}$, this just means that G_∞ is compact, see Prop. 3.1 below. However, when $F = \mathbb{Q}$ and the $\mathbb{Q}$- and $\mathbb{R}$-ranks of $G^{\rm ab}$ are equal, it suffices that $G^{\rm der}_\infty$ be compact. Here $G^{\rm ab}$ denotes the biggest quotient group of G which is a torus, thus $G^{\rm ab} = G/G^{\rm der}$. There are plenty of such groups G. In fact, any split simple F-group not of type A_n $(n \geq 2)$, D_{2n+1} or E_6 has infinitely many inner forms which are compact at infinity (and quasi-split at all but at most one finite place).

Fix a Haar measure $\mu = \otimes\mu_v$ on $G(\mathbb{A}^\infty)$. We state the results using the following notion of congruence. As K varies over the compact open subgroups of $G(\mathbb{A}^\infty)$, the centers $Z(\mathcal{H}_{K,\mathbb{Z}})$ of the Hecke algebras form an inverse system. To an automorphic representation π of $G(\mathbb{A})$ we associate the character $\eta_\pi : \varprojlim Z(\mathcal{H}_{K,\mathbb{Z}}) \to \mathbb{C}$ such that $\eta_\pi = \eta_{\pi^K} \circ \mathrm{pr}_K$ for every compact open subgroup K such that $\pi^K \neq 0$. The character η_π takes its values on $Z(\mathcal{H}_{K,\mathbb{Z}})$ in the ring of integers of some number field, depending on F, G and K. If λ is a finite place of $\bar{\mathbb{Q}}$, we say that $\tilde\pi$ and π are *congruent modulo* λ if their characters are. Write $\tilde\pi \equiv \pi \pmod{\lambda}$ in this case. A similar notion makes sense locally, and then $\tilde\pi \equiv \pi \pmod{\lambda}$ if and only if $\tilde\pi_v \equiv \pi_v \pmod{\lambda}$ for all finite v. Moreover, when both $\tilde\pi_v$ and π_v are unramified, $\tilde\pi_v \equiv \pi_v \pmod{\lambda}$ simply means that the Satake parameters are congruent.

Definition 0.2. *Let π be an automorphic representation of $G(\mathbb{A})$. Let λ be a finite place of $\bar{\mathbb{Q}}$. We say that π is abelian modulo λ relative to K if $\pi^K \neq 0$ and there exists an automorphic character χ of $G(\mathbb{A})$, trivial on K, such that $\eta_{\pi^K}(\phi) \equiv \eta_\chi(\phi)$ $(\mathrm{mod}\,\lambda)$, $\forall \phi \in Z(\mathcal{H}_{K,\mathbb{Z}})$. We say that π is abelian modulo λ if it is abelian modulo λ relative to some K, thus $\pi \equiv \chi$ $(\mathrm{mod}\,\lambda)$ for some χ.*

This is the analogue of the notion Eisenstein modulo λ in [Cl, p. 1269]. Since $G^{\rm der}$ is anisotropic in our setup, there are neither cusps nor Eisenstein series. Thus the terminology abelian modulo λ seems more suitable.

The following theorem is in some sense the main result of this paper.

Let F be a totally real number field, Σ a finite set of finite places of F. Fix a compact open subgroup K_v of $G_v = G(F_v)$ for each $v \notin \infty$, hyperspecial for almost all v. Fix an irreducible representation ρ_Σ of $K_\Sigma = \prod_{v\in\Sigma} K_v$ and an irreducible smooth unitary representation ρ_∞ of $G_\infty = \prod_{v\in\infty} G_v$. Then $K = \prod_{v\notin\Sigma} K_v$ is a compact open subgroup of $G(\mathbb{A}^\Sigma)$. Denote by e_K the constant measure supported on K of volume 1. It is the unit element in the Hecke algebra $\mathcal{H}_{K,\mathbb{Z}}$.

Theorem 0.3. *Let $\lambda|\ell$ be a finite place of $\bar{\mathbb{Q}}$ such that there exists at least two finite places v where $\ell \nmid |K_v|$ (this is automatic if there is an F-embedding $G \hookrightarrow \mathrm{GL}(n)$ and $\ell > [F:\mathbb{Q}]n+1$). Let $\pi = \otimes\pi_v$ be an automorphic representation of $G(\mathbb{A})$ such that $\pi_\infty = \rho_\infty$, $\pi_\Sigma \supset \rho_\Sigma$, and $\pi^K \neq 0$. Assume π is nonabelian modulo λ relative to K. Let w be a finite place of F such that K_w is hyperspecial. Let $q = q_w$ denote the residual cardinality of w. Let $J_w = K_w \cap K'_w$ be a parahoric subgroup, where $K'_w \neq K_w$ is maximal compact. Let $J = J_w K^w$ and $K' = K'_w K^w$. Put*

$$[K'_w : J_w]_{K_w} = [K'_w : J_w]/([K'_w : J_w], [K_w : J_w])$$

and

$$e_{K,K'} = [K_w : J_w][K'_w : J_w]_{K_w}(e_K * e_{K'} * e_K) \in Z(\mathcal{H}_{K,\mathbb{Z}}).$$

Assume $\ell \nmid q_w[K'_w : J_w]_{K_w}$ and $(\star)$ $\eta_{\pi^K}(e_{K,K'}) \equiv \eta_{\mathbf{1}}(e_{K,K'})$ $(\mathrm{mod}\,\lambda)$. Then there exists an automorphic representation $\widetilde\pi = \otimes\widetilde\pi_v$ of $G(\mathbb{A})$ such that $\widetilde\pi_\infty = \rho_\infty$, $\widetilde\pi_\Sigma \supset \rho_\Sigma$ and

$\widetilde{\pi}^{K^w} \neq 0$ *satisfying* $\widetilde{\pi}_w^{J_w} \neq \widetilde{\pi}_w^{K_w} + \widetilde{\pi}_w^{K'_w}$, *and* $\eta_{\widetilde{\pi}^J}(\phi) \equiv \eta_{\pi^K}(e_K * \phi) \pmod{\lambda}$ *for all* $\phi \in Z(\mathcal{H}_{J,\mathbb{Z}})$.

This theorem claims nothing unless $\pi_w^{J_w} = \pi_w^{K_w} + \pi_w^{K'_w}$. The assumption ($\star$) is implied by the stronger assumption: ($\star\star$) π_w is congruent to the trivial representation $\mathbf{1}$ modulo λ, namely $\pi(\phi) \equiv \mathbf{1}(\phi) \pmod{\lambda}$, $\forall \phi \in Z(\mathcal{H}_{K,\mathbb{Z}})$. But ($\star$) is strictly weaker than ($\star\star$). Our final conclusion is slightly more precise than $\widetilde{\pi} \equiv \pi \pmod{\lambda}$. If G_w^{der} has rank one, J_w is an Iwahori subgroup I_w and we show in Lemma 11.1 that $\widetilde{\pi}_w^{K_w} = 0$ (and $\widetilde{\pi}_w^{K'_w} = 0$) but $\widetilde{\pi}_w^{I_w} \neq 0$. The eigensystem of a modular form mod ℓ comes from an algebraic modular form mod ℓ on $D^\times$, where $D/\mathbb{Q}$ is the quaternion algebra with ramification locus $\{\infty, \ell\}$, see Serre [S]. Using the transfer of automorphic forms from $D(\mathbb{A})^\times$ to the split form $\mathrm{GL}(2,\mathbb{A})$ (see [F] for a simple proof) we get the result of Ribet after stripping powers of ℓ from the level. Note that $[K'_w : J_w]_{K_w} = 1$ when K'_w is conjugate to K_w. The condition $\ell \nmid [K'_w : J_w]$ which appears in [So] introduces the requirement $(\ell, 1+q) = 1$ in the formulation of Ribet's theorem in [So].

There is another proof of Ribet's theorem relying on the so-called Ihara lemma. It states that for $q \nmid N\ell$, the degeneracy maps $X_0(Nq) \rightrightarrows X_0(N)$ induce an injection $H^1(X_0(N), \mathbb{Z}_\ell)^{\oplus 2} \to H^1(X_0(Nq), \mathbb{Z}_\ell)$ with torsion-free cokernel. The proof of this lemma reduces to the congruence subgroup property of the group $\mathrm{SL}(2, \mathbb{Z}[1/q])$. In our case we are looking at functions on a finite set, and the analogue of the Ihara lemma can be proved by imitating the combinatorial argument of Taylor [T1, p. 274] in the diagonal weight 2 case. See section 7.3 below.

Here are a few applications of Theorem 0.3.

THEOREM 0.4. *Let F be a totally real number field. Let π be as in Theorem* 0.3. *Let w be a finite place of F such that K_w is hyperspecial and the F_w-rank of G_w^{der} is one. Let $I_w = K_w \cap K'_w$ be an Iwahori subgroup, where $K'_w \neq K_w$ is maximal compact. Put $I = I_w K^w$ and $K' = K'_w K^w$. Suppose $\ell \nmid q_w [K'_w : I_w]_{K_w}$ and $\eta_{\pi^K}(e_{K,K'}) \equiv \eta_{\mathbf{1}}(e_{K,K'}) (\mathrm{mod}\, \lambda)$, with $e_{K,K'}$ as in Theorem* 0.3. *Then there exists an automorphic representation $\widetilde{\pi} = \otimes \widetilde{\pi}_v$ of $G(\mathbb{A})$ such that $\widetilde{\pi}_\infty = \rho_\infty$, $\widetilde{\pi}^{K^w} \neq 0$, $\widetilde{\pi}_w^{I_w} \neq 0$, $\widetilde{\pi}_w^{K_w} = 0 = \widetilde{\pi}_w^{K'_w}$, satisfying $\eta_{\widetilde{\pi}^I}(\phi) \equiv \eta_{\pi^K}(e_K * \phi) (\mathrm{mod}\, \lambda)$ for all $\phi \in Z(\mathcal{H}_{I,\mathbb{Z}})$. If $\pi_\Sigma \supset \rho_\Sigma$ then $\widetilde{\pi}$ can be chosen to satisfy $\widetilde{\pi}_\Sigma \supset \rho_\Sigma$.*

This theorem is a variant of Bellaiche's Theorem 1.4.6, [Bel, p. 215]: It gives results modulo arbitrary $\lambda | \ell$ prime to $q_w [K'_w : I_w]_{K_w}$, independently of π, the level-raising condition is weaker, and we get information about the action of the center of the Iwahori-Hecke algebra on $\widetilde{\pi}_w^{I_w}$. Bellaiche's proof is different. He uses results of Lazarus and Vigneras from modular representation theory, such as the computation of the composition series of universal modules. His ℓ is prime to q_w times the number of neighbors of the vertex in the Bruhat-Tits building fixed by K_w, times the number of neighbors of the vertex fixed by K'_w, and has to lie outside a finite set depending on π, but his π is not required to be nonabelian mod λ. His level-raising condition: $\eta_{\pi^K}(\phi) \equiv \eta_{\mathbf{1}}(\phi)$ for *all* $\phi \in \mathcal{H}_{K_w}$, is stronger, and he can conclude that $\widetilde{\pi}_w$ is the actual Steinberg representation of G_w. We show this too, using the analysis of section 11. In [So] it is only shown that $\widetilde{\pi}_w$ is ramified. See also [BG] where general ρ_∞ are considered, and the only condition on $\ell \nmid q_w$ is that it lies outside an unknown finite set depending on π, but π is not assumed to be nonabelian mod λ.

Consider the special case where E/F is a totally imaginary quadratic extension of a totally real number field F, $G^{\mathrm{qs}} = \mathrm{U}(2,1)$ is the quasi-split unitary F-group in 3 variables split over E, and $G = \mathrm{U}(3)$ is an inner form of G^{qs} such that G_∞ is compact.

For F-primes w inert in E, the semisimple rank of $G(F_w)$ is one. In this case Theorem 0.4 strengthens (to $\ell \nmid q_w$) Clozel [Cl] (where $F = \mathbb{Q}$ and $\ell \nmid q_w(q_w^3+1)(q_w-1)$), [Bel] Theorem 1.4.6, where $\ell \nmid q_w(q_w^3+1)$ and – as in [BG] – ℓ is outside a finite set depending on π. Indeed, $[K'_w : I_w] = q_w+1$ divides $[K_w : I_w] = q_w^3+1$, hence $[K'_w : I_w]_{K_w} = 1$, so our condition on ℓ is only that it be prime to q_w. From $\widetilde{\pi}_w^{I_w} \neq 0$ and $\widetilde{\pi}_w^{K_w} = 0 = \widetilde{\pi}_w^{K'_w}$ we conclude that $\widetilde{\pi}_w$ is Steinberg (as $\pi^{\times K} \neq 0$ and $\pi^{+K'} \neq 0$). We recall the classification of reducible unramified induced representations, in particular in the case of $G = \mathrm{U}(3)$, in section 11.3.

If π is a representation of $G(\mathbb{A}) = \mathrm{U}(3,\mathbb{A})$ such that π_v is the nontempered $\pi_v^\times$ for almost all v (in the notations of [F2]), and $\pi_w = \pi_w^\times$, then $\widetilde{\pi}_w$ is not the cuspidal π_w^- (since $\widetilde{\pi}_w^{I_w} \neq 0$) and not $\pi_w^\times$ (as $\widetilde{\pi}_w^{K_w} = 0$), so $\widetilde{\pi}$ has no component $\pi_v^\times$, by the results of [F2]. Alternatively, this follows from $\widetilde{\pi}_w$ being Steinberg.

When $G = \mathrm{U}(3)$ and the F-prime w splits in E, thus $G(F_w) = \mathrm{GL}(3, F_w)$, we obtain the following as a corollary.

Theorem 0.5. *Let $\pi = \otimes \pi_v$ be an automorphic representation of $G(\mathbb{A})$ with $\pi_\infty = \rho_\infty$. Choose a compact open subgroup $K = \prod K_v \subset G(\mathbb{A}^\infty)$ such that $\pi^K \neq 0$. Let $\lambda|\ell$ be a finite place of $\bar{\mathbb{Q}}$ such that π is nonabelian modulo λ relative to K. If $\ell \leq 3$, or if $E = \mathbb{Q}(\sqrt{-7})$ and $\ell = 7$, assume $\ell \nmid |K_v|$ for at least two primes v. Let $w \nmid \ell$ be a prime in F, split in E, such that K_w is hyperspecial. Suppose there is a $w_E|w$ such that the Satake parameter is $\mathfrak{t}_{\pi_{w_E}} \equiv \mathrm{diag}(q_w, 1, q_w^{-1}) (\mathrm{mod}\, \lambda)$. Then there exists an automorphic representation $\widetilde{\pi} = \otimes \widetilde{\pi}_v$ of $G(\mathbb{A})$ with $\widetilde{\pi}_\infty = \rho_\infty$ and $\widetilde{\pi}^{K^w} \neq 0$ satisfying (1) $\widetilde{\pi}_w$ is either an irreducible unramified principal series or induced from a Steinberg representation (in particular $\widetilde{\pi}_w$ is generic, not square integrable), and $\widetilde{\pi}_w^{J_w} \neq 0$ for any maximal proper parahoric subgroup J_w, and (2) $\eta_{\widetilde{\pi}^J}(\phi) \equiv \eta_{\pi^K}(e_K * \phi) (\mathrm{mod}\, \lambda)$ for all $\phi \in Z(\mathcal{H}_{J,\mathbb{Z}})$, where $J = J_w K^w$, hence $\tilde{\pi} \equiv \pi \,(\mathrm{mod}\, \lambda)$.*

Theorem 0.5 claims nothing unless π_w is induced from the determinant (type IIb of Tables A, B in Section 11), that is, unramified and non-generic (and not 1-dimensional), which is the case for the endoscopic lifts from $\mathrm{U}(2) \times \mathrm{U}(1)$ considered in [Bel, p. 250]. Since we deal with any $\pi_\infty = \rho_\infty$, it follows that if π is endoscopic abelian (that is, a lift of a character of a proper endoscopic group), then it is congruent to a $\tilde{\pi}$ which is not endoscopic abelian. This is true even for $\mathrm{U}(n)$, for all $n \geq 2$. For $n = 3$ this result has been applied to the Bloch-Kato conjecture for certain Hecke characters of E [Bel]. In fact, the results one can get for $\mathrm{U}(n)$ indicate that an endoscopic abelian lift π is congruent to a $\widetilde{\pi}$ which is not endoscopic abelian. We cannot prove by our methods that $\tilde{\pi}_w$ is ramified. In his thesis [Bel, p. 218], Bellaiche also has a result in the split case. His ℓ is prime to $q_w(q_w^3-1)(q_w+1)$, and lies outside a finite set depending on π. If π occurs with multiplicity one (the multiplicity one theorem for $\mathrm{U}(3)$ is currently proven – in [F2] – only for representations satisfying some mild condition at the dyadic places; it is not yet proven in general, contrary to the assertion of [Cl]), he obtains a $\tilde{\pi}$ with $\widetilde{\pi}_w$ ramified. We classify the Iwahori-spherical representations of $\mathrm{GL}(3)$ and compute the dimensions of their parahoric fixed spaces. This allows us to conclude that $\widetilde{\pi}_w$ is either a full unramified principal series or induced from a Steinberg representation. Hence, from our analysis, $\widetilde{\pi}_w$ is induced from Steinberg. Theorem 0.5 seems related to the $n = 3$ case of conjecture 5.3 in [T2, p. 35], providing an analogue of Ihara's lemma. Automorphic representations of unitary groups with a generic component at a split prime come up naturally in the proof of the local Langlands correspondence for $\mathrm{GL}(n)$ [HT].

Next, let G be an inner form of $\mathrm{GSp}(2)$ such that $G^{\mathrm{der}}(\mathbb{R})$ is compact. Concretely,

$G = \mathrm{GSpin}(f)$ for some definite quadratic form f in 5 variables over a totally real F. In this situation, Theorem 0.3 yields:

THEOREM 0.6. *Let $\pi = \otimes\pi_v$ be an automorphic representation of $G(\mathbb{A})$ with $\pi_\infty = \rho_\infty$. Choose a compact open subgroup $K = \prod K_v$ such that $\pi^K \neq 0$. Let $\lambda|\ell$ be a finite place of $\bar{\mathbb{Q}}$ such that π is nonabelian modulo λ relative to K. If $\ell \leq 5$ assume $\ell \nmid |K_v|$ for at least two primes v. Let $w \nmid \ell$ be a prime such that K_w is hyperspecial. Suppose $\mathfrak{t}_{\pi_w \otimes \nu_w^{-3/2}} \equiv \mathrm{diag}(1, q_w, q_w^2, q_w^3) (\bmod \lambda)$. Then there exists an automorphic representation $\widetilde{\pi} = \otimes\widetilde{\pi}_v$ of $G(\mathbb{A})$ with $\widetilde{\pi}_\infty = \rho_\infty$ and $\widetilde{\pi}^{K^w} \neq 0$ satisfying (1) $\widetilde{\pi}_w$ is generic and Heisenberg-spherical, and (2) $\eta_{\widetilde{\pi}^J}(\phi) \equiv \eta_{\pi^K}(e_K * \phi) (\bmod \lambda)$ for all $\phi \in Z(\mathcal{H}_{J,\mathbb{Z}})$, where $J = J_w K^w$.*

If in addition $q^4 \neq 1 (\bmod \ell)$, then $\widetilde{\pi}_w$ must be of type I, IIa or IIIa of Tables C, D in Appendix 2.

By the Heisenberg parahoric we mean the inverse image, under the reduction map, of the standard maximal parabolic in $\mathrm{GSp}(2, \mathbb{F}_w)$ whose unipotent radical is a Heisenberg group. The proof relies on computations of R. Schmidt [Sch]. If $m(\pi) = 1$, Bellaiche's methods seem to apply and one can probably show that $\tilde{\pi}_w$ is induced from a twisted Steinberg representation on the standard Heisenberg-Levi. It is known (see, e.g., [F1]) that Saito-Kurokawa lifts (that is, lifts of $\mathbf{1}\times$cuspidal from $\mathrm{PGL}(2, \mathbb{A}) \times \mathrm{PGL}(2, \mathbb{A})$ to $\mathrm{PGSp}(2, \mathbb{A})$) are locally non-generic everywhere. Therefore, if π is of Saito-Kurokawa type, it is congruent to a $\tilde{\pi}$ which is not of Saito-Kurokawa type. The interest in it stems from hoped for applications to the Bloch-Kato conjecture for the motives attached to classical modular forms. In particular, one hopes to establish a mod ℓ analogue of a result of Skinner and Urban [SU], which is valid for *all* (not necessarily ordinary) modular forms of classical weight at least 4.

There exists q with $q^4 \neq 1$ (mod ℓ) precisely when $\ell \geq 7$. In this case $\widetilde{\pi}_w$ is an unramified principal series (type I) or induced from a twisted Steinberg representation $\chi \,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \chi'$ or $\chi \rtimes \chi' \,\mathrm{St}_{\mathrm{GL}(2)}$ (type IIa and IIIa respectively). If one can show that $\widetilde{\pi}_w$ is para-ramified, meaning that $\widetilde{\pi}_w$ has no nonzero K_w'-fixed vectors, one can conclude that it is of type IIIa and therefore induced from a twisted Steinberg representation on the Heisenberg-Levi, since $m(\pi) = 1$ (see [F1]), using the methods of [Bel] and [Cl]. The result above only gives nontrivial congruences if π_w is nongeneric. If π is of Saito-Kurokawa type, it is locally nongeneric, and we get a $\widetilde{\pi}$ congruent to π which is not of Saito-Kurokawa type. If we know that $\widetilde{\pi}_w$ is of type IIIa, we can apply this strategy to the Bloch-Kato conjecture for the motives attached to classical modular forms of weight (at least) 4, using the methods of [Bel]. We should note that if we choose to work with the Siegel-parahoric J_w', we can only conclude that $\widetilde{\pi}_w$ is generic *or* a Saito-Kurokawa lift.

This work is simply an attempt to complete the beautiful paper [So] by extending it from the special case $\pi_\infty = \mathbf{1}$ to permit π_∞ to be any irreducible continuous representation ρ_∞ of G_∞. Further we optimize the constraint on ℓ and determine $\widetilde{\pi}_w$ to be Steinberg in the case of U(3). Except for these minor changes, we follow [So] very closely, attempting to expand some of the arguments there.

1. The abstract setup. In this section, we fix a ring $\mathcal{O}$ of characteristic zero which is a finite product of domains. Denote by L the associated product of fields of fractions. There are two cases of interest for us. The first is where $\mathcal{O}$ is the ring of integers in a number field $L \subset \mathbb{C}$. The case that we shall actually use in this paper is as follows. Let $L_1 \subset \mathbb{C}$ be a number field such that $[L_1 : L_0] = 2$ where $L_0 = L_1 \cap \mathbb{R}$. Let λ be a finite place of L_1, and $\lambda_0 = \lambda \cap L_0$ the place of L_0 under λ.

Let $L_{1\lambda}$ be the completion of L_1 at λ, and $(L_0)_{\lambda_0}$ the completion of L_0 at λ_0. Then $L_{1\lambda_0} = L_1 \otimes_{L_0} (L_0)_{\lambda_0}$ is $L_{1\lambda}$ if λ_0 stays prime in L_1, and it is $L_{1\lambda} \oplus L_{1\overline{\lambda}}$ if λ_0 splits as $\lambda\overline{\lambda}$ in L_1. Note that $L_{1\overline{\lambda}} \simeq L_{1\lambda}$. The ring of integers R_{λ_0} in $L_{1\lambda_0}$ is the ring of integers R_λ in $L_{1\lambda}$ if λ_0 stays prime, and $R_\lambda \oplus R_{\overline{\lambda}}$ if λ_0 splits. Then the case we shall actually use is that where $\mathcal{O} = R_{\lambda_0}$ and $L = L_{1\lambda_0}$.

Let H be a commutative L-algebra. We do not require H to be of finite dimension. However, we assume H comes equipped with an involution $\phi \mapsto \phi^\vee$. An *involution* is an L-linear anti-automorphism (thus $(\phi_1\phi_2)^\vee = \phi_2^\vee\phi_1^\vee$) of order two. Moreover, we fix an $\mathcal{O}$-order $H_\mathcal{O} \subset H$ preserved by $\vee$. An $\mathcal{O}$-*order* is an $\mathcal{O}$-subalgebra which is the $\mathcal{O}$-span of an L-basis for H. Then we look at a triple $(V, \langle -,-\rangle_V, V_\mathcal{O})$ consisting of:
(1) V is a finite-dimensional L-space with an action $r_V : H \to \mathrm{End}_L(V)$;
(2) $\langle -,-\rangle_V$ is a nondegenerate, symmetric, L-bilinear form $V \times V \to L$;
(3) $V_\mathcal{O} \subset V$ is an $\mathcal{O}$-*lattice* (that is, the $\mathcal{O}$-span of an L-basis).

We impose the following compatibility conditions on these data:
(1) $r_V(\phi^\vee)$ is the adjoint of $r_V(\phi)$ with respect to $\langle -,-\rangle_V$;
(2) $V_\mathcal{O} \subset V$ is preserved by the order $H_\mathcal{O} \subset H$;
(3) $V_\mathcal{O}/(V_\mathcal{O} \cap V_\mathcal{O}^\vee)$ and $V_\mathcal{O}^\vee/(V_\mathcal{O} \cap V_\mathcal{O}^\vee)$ are torsion $\mathcal{O}$-modules.

Here $V_\mathcal{O}^\vee = \{v \in V : \langle v, V_\mathcal{O}\rangle_V \subset \mathcal{O}\}$ is the *dual lattice* of $V_\mathcal{O}$ in V.

Choose nonzero annihilators A_V and B_V in $\mathcal{O}$ of the torsion modules above, that is, such that $A_V V_\mathcal{O} \subset V_\mathcal{O}^\vee$ and $B_V V_\mathcal{O}^\vee \subset V_\mathcal{O}$, thus

$$A_V\langle V_\mathcal{O}, V_\mathcal{O}\rangle_V \subset \mathcal{O} \quad \text{and} \quad \langle v, V_\mathcal{O}\rangle_V \subset \mathcal{O} \Rightarrow B_V v \in V_\mathcal{O}.$$

Let $(U, \langle -,-\rangle_U, U_\mathcal{O})$ be another such triple. Choose annihilators A_U and B_U for it too. Suppose we are given an H-linear map $\delta : U \to V$ satisfying:
(1) $U = \ker\delta \oplus (\ker\delta)^\perp$; $V = \mathrm{im}\,\delta \oplus (\mathrm{im}\,\delta)^\perp$;
(2) $\delta(U_\mathcal{O}) \subset V_\mathcal{O} \cap \delta(U)$, and the quotient is killed by $C \in \mathcal{O} - \{0\}$.

Put $V^{\mathrm{old}} = \mathrm{im}\,\delta$ and $V^{\mathrm{new}} = (\mathrm{im}\,\delta)^\perp$. These are H-stable subspaces of V. By assumption we have an orthogonal decomposition $V = V^{\mathrm{old}} \oplus V^{\mathrm{new}}$. The adjoint map $\delta^\vee : V \to U$ is defined by $\langle u, \delta^\vee v\rangle_U = \langle \delta u, v\rangle_V$. Then $\delta^\vee$ maps V^{new} to 0, and $\delta^\vee : V^{\mathrm{old}} \to (\ker\delta)^\perp$ is injective, with inverse δ.

Definition 1.1. *Put* $V_\mathcal{O}^{\mathrm{old}} = V_\mathcal{O} \cap V^{\mathrm{old}}$ *and* $V_\mathcal{O}^{\mathrm{new}} = V_\mathcal{O} \cap V^{\mathrm{new}}$.

These $H_\mathcal{O}$-stable submodules of $V_\mathcal{O}$ span V^{old} and V^{new} respectively. They are orthogonal, but their sum is not always all of $V_\mathcal{O}$. Note that $\delta(U_\mathcal{O}) \subset V_\mathcal{O}^{\mathrm{old}}$ and $CV_\mathcal{O}^{\mathrm{old}} \subset \delta(U_\mathcal{O})$ by assumption.

Definition 1.2. *Let* $\mathbb{T}_\mathcal{O}$ *be the image of* $H_\mathcal{O}$ *in* $\mathrm{End}_\mathcal{O}(V_\mathcal{O})$. *Let*

$$\mathbb{T}_\mathcal{O}^{\mathrm{old}} \subset \mathrm{End}_\mathcal{O}(V_\mathcal{O}^{\mathrm{old}}) \quad \textit{and} \quad \mathbb{T}_\mathcal{O}^{\mathrm{new}} \subset \mathrm{End}_\mathcal{O}(V_\mathcal{O}^{\mathrm{new}})$$

denote the images of $H_\mathcal{O}$ *defined by these submodules.*

Clearly we have natural surjective maps $\mathbb{T}_\mathcal{O} \twoheadrightarrow \mathbb{T}_\mathcal{O}^{\mathrm{old}}$ and $\mathbb{T}_\mathcal{O} \twoheadrightarrow \mathbb{T}_\mathcal{O}^{\mathrm{new}}$ given by restriction, and $\mathbb{T}_\mathcal{O}$ acts faithfully on $V_\mathcal{O}$.

2. Taylor's lemma. By a *congruence module* we mean a $\mathbb{T}_\mathcal{O}$-module, such that the action factors through both $\mathbb{T}_\mathcal{O}^{\mathrm{old}}$ and $\mathbb{T}_\mathcal{O}^{\mathrm{new}}$. The following lemma was stated for $\mathcal{O} = \mathbb{Z}$, trivial annihilators, and injective δ in [T2, p. 331]

Lemma 2.1. *Put* $E = A_U B_V C^2$ *and* $U'_\mathcal{O} = U_\mathcal{O} \cap (\ker\delta)^\perp$. *Then* $U''_\mathcal{O} = U'_\mathcal{O}/(U'_\mathcal{O} \cap E^{-1}\delta^\vee\delta(U_\mathcal{O}))$ *is a congruence module.*

Proof. Since $U_{\mathcal{O}}$ is preserved by $H_{\mathcal{O}}$ and δ is H-linear, the algebra $\mathbb{T}_{\mathcal{O}}$ acts naturally on $U'_{\mathcal{O}}$ via the embedding of $U'_{\mathcal{O}}$ into $V^{\mathrm{old}}_{\mathcal{O}}$ defined by δ. This action factors through $\mathbb{T}^{\mathrm{old}}_{\mathcal{O}}$.

It remains to show that $\mathbb{T}^{\mathrm{new}}_{\mathcal{O}}$ acts on $U''_{\mathcal{O}}$, namely that the action is well defined. So suppose that $\phi \in H_{\mathcal{O}}$ acts as zero on $V^{\mathrm{new}}_{\mathcal{O}}$. We must show that $E\phi$ maps $U'_{\mathcal{O}}$ into $\delta^\vee\delta(U_{\mathcal{O}})$, for then ϕ maps $U'_{\mathcal{O}}$ into $U'_{\mathcal{O}} \cap E^{-1}\delta^\vee\delta(U_{\mathcal{O}})$. In other words, if ϕ is zero in $\mathbb{T}^{\mathrm{new}}_{\mathcal{O}}$, then it is zero on $U''_{\mathcal{O}}$.

Note first that $\phi^\vee$ maps $V_{\mathcal{O}}$ into $V^{\mathrm{old}}_{\mathcal{O}}$. Indeed, for any $v \in V$, $v_n \in V^{\mathrm{new}}$ we have $\langle v_n, \phi^\vee v\rangle_V = \langle \phi v_n, v\rangle_V = 0$, thus $\phi^\vee v \in V^{\mathrm{old}}$, so $\phi^\vee$ maps V to V^{old}. Moreover $\phi \in H_{\mathcal{O}}$ implies $\phi^\vee \in H_{\mathcal{O}}$, thus $\phi^\vee$ maps $V_{\mathcal{O}}$ to itself, so to $V^{\mathrm{old}}_{\mathcal{O}}$.

Since $\phi^\vee$ also maps $V^{\mathrm{new}}_{\mathcal{O}}$ to itself, it maps $V^{\mathrm{new}}_{\mathcal{O}}$ to zero.

Now let $u = \delta^\vee(v) \in U_{\mathcal{O}}$ for some $v \in V^{\mathrm{old}}$. Note that $\delta^\vee V \subset (\ker\delta)^\perp$, thus $u \in U'_{\mathcal{O}}$. We have

$$A_U C\langle v, V^{\mathrm{old}}_{\mathcal{O}}\rangle_V \subset A_U\langle v, \delta(U_{\mathcal{O}})\rangle_V \subset A_U\langle u, U_{\mathcal{O}}\rangle_U \subset \mathcal{O},$$

from $CV^{\mathrm{old}}_{\mathcal{O}} \subset \delta(U_{\mathcal{O}})$, $u = \delta^\vee(v)$, $A_U U_{\mathcal{O}} \subset U^\vee_{\mathcal{O}}$. Since $\phi^\vee V_{\mathcal{O}} \subset V^{\mathrm{old}}_{\mathcal{O}}$, we have $A_U C\langle \phi v, V_{\mathcal{O}}\rangle_V \subset \mathcal{O}$. By definition of B_V, we deduce that $A_U B_V C(\phi v) \in V_{\mathcal{O}}$, hence it is in $V^{\mathrm{old}}_{\mathcal{O}}$, as $v \in V^{\mathrm{old}}$ and thus $\phi v \in V^{\mathrm{old}}$. We conclude from the definition of C that

$$A_U B_V C^2(\phi v) \in \delta(U_{\mathcal{O}}).$$

We get the result by applying $\delta^\vee$ to this: $E(\phi u) \in \delta^\vee\delta(U_{\mathcal{O}})$, as $u = \delta^\vee v$. Thus ϕ takes $U'_{\mathcal{O}}$ to $E^{-1}\delta^\vee\delta(U_{\mathcal{O}}) \cap U'_{\mathcal{O}}$. ◻

As in [T2, p. 331], we have the following useful corollary:

COROLLARY 2.2. *Let $\mathcal{O} = \mathcal{O}_L$ be the ring of integers of a number field $L \subset \mathbb{C}$, or $\mathcal{O} = R_{\lambda_0}$ and $L = L_{1\lambda_0}$. Suppose $u \in U_{\mathcal{O}} - \{0\}$ is an eigenvector for $H_{\mathcal{O}}$, with character $\eta : H_{\mathcal{O}} \to \mathcal{O}$. Define $\mathcal{E} = \{x \in \mathcal{O}; x(Lu \cap (U_{\mathcal{O}} + \ker\delta)) \subset \mathcal{O}u\}$; it is an ideal in $\mathcal{O}$. Suppose it is nonzero, and that $\delta^\vee\delta(u) \in mU_{\mathcal{O}}$, for some nonzero $m \in \mathcal{O}$. Then η induces a homomorphism $\mathbb{T}^{\mathrm{new}}_{\mathcal{O}} \to \mathcal{O}/(\mathcal{O} \cap mE^{-1}\mathcal{E}^{-1})$.*

Proof. Consider the action of $H_{\mathcal{O}}$ on $u + Lu \cap U'_{\mathcal{O}} \cap E^{-1}\delta^\vee\delta U_{\mathcal{O}}$. If $u_1^\perp \in U_{\mathcal{O}}$ and $u_1^\perp \perp u$, and $\delta^\vee\delta u_1^\perp \in Lu$, then $0 = \langle u_1^\perp, \delta^\vee\delta u_1^\perp\rangle_V = \langle \delta u_1^\perp, \delta u_1^\perp\rangle_V$. Hence $\delta u_1^\perp = 0$, thus $u_1^\perp \in \ker\delta$, and $Lu \cap \delta^\vee\delta U_{\mathcal{O}} = Lu \cap \delta^\vee\delta(U_{\mathcal{O}} \cap Lu)$. Also $U'_{\mathcal{O}} = U_{\mathcal{O}} \cap (\ker\delta)^\perp$, and $\operatorname{im}\delta^\vee \subset (\ker\delta)^\perp$. Thus $Lu \cap U'_{\mathcal{O}} \cap E^{-1}\delta^\vee\delta U_{\mathcal{O}} = \mathcal{E}^{-1}u \cap E^{-1}\delta^\vee\delta(U_{\mathcal{O}} \cap Lu) = \mathcal{E}^{-1}u \cap \mathcal{E}^{-1}E^{-1}mU_{\mathcal{O}}$. Now $\eta : H_{\mathcal{O}} \to \mathcal{O}$ is defined by $h \cdot u = \eta(h)u$, $h \in H_{\mathcal{O}}$, so the action of $h \in H_{\mathcal{O}}$ on a vector in $\mathcal{E}^{-1}u \cap \mathcal{E}^{-1}E^{-1}mU_{\mathcal{O}}$ is by multiplication by $\eta(h) \in \mathcal{O} \cap \mathcal{E}^{-1}E^{-1}m$, thus η induces a homomorphism as desired. ◻

We remark that $m = 0$ implies that $\delta^\vee\delta u = 0$, thus $\delta u = 0$ since $\delta^\vee$ is injective on V^{old}, so $u \in \ker\delta$. When $\mathcal{O} = R_{\lambda_0}$ and $L = L_{1\lambda_0}$ we get $\mathbb{T}^{\mathrm{new}}_{\mathcal{O}} \to \mathcal{O}/\lambda_0^n$ for every positive $n \le n_0 = v_{\lambda_0}(m) - v_{\lambda_0}(E) - v_{\lambda_0}(\mathcal{E})$. If $\mathcal{O}$ is the ring of integers in a number field L, and we factor the fractional ideal $\mathcal{O} \cap mE^{-1}\mathcal{E}^{-1}$ into prime powers and project further, we get the following. For every (nonzero) prime ideal $\lambda \subset \mathcal{O}$ there is a homomorphism

$$\mathbb{T}^{\mathrm{new}}_{\mathcal{O}} \to \mathcal{O}/\lambda^n$$

induced by η, where n is a nonnegative integer $\le n_0 = v_\lambda(m) - v_\lambda(E) - v_\lambda(\mathcal{E})$. Here we should think of $v_\lambda(m)$ as the main term, and the other two as controllable error terms. In our applications $\mathcal{O} = R_{\lambda_0}$ and $L = L_{1\lambda_0}$. We want to show that n_0 is positive.

3. Compactness at infinity. Let F be a totally real number field. Let ∞ be the set of archimedean places. Denote the ring of adèles by $\mathbb{A} = \mathbb{A}_F = F_\infty \times \mathbb{A}^\infty$. Consider a connected reductive F-group G. Each F-rational character $\chi \in X^*(G)_F$ gives a continuous homomorphism $G(\mathbb{A}) \to \mathbb{R}_+^\times$ by composing with the idèle norm. Define

$$G(\mathbb{A})^1 = \{g \in G(\mathbb{A}); |\chi(g)| = 1, \forall \chi \in X^*(G)_F\}.$$

This group is known to be unimodular. By the product formula, $G(F)$ is a discrete subgroup of $G(\mathbb{A})^1$. The quotient $G(F)\backslash G(\mathbb{A})^1$ has finite volume. This quotient is compact if and only if G^{ad} is anisotropic. We shall naturally be led to studying groups for which $G_\infty^1 = G_\infty \cap G(\mathbb{A})^1$ is compact. Let G^{ab} denote the biggest quotient group of G which is a torus, namely $G^{\mathrm{ab}} = G/G^{\mathrm{der}}$ (G/G^{der} is connected as it is the quotient of a connected group, G, it is reductive since G is, and it is abelian as it is the quotient by G^{der}, hence it is a torus).

Proposition 3.1. *The group G_∞^1 is compact if and only if G_∞ is compact, or: $F = \mathbb{Q}$, $\mathrm{rk}_{\mathbb{Q}}\, G^{\mathrm{ab}} = \mathrm{rk}_{\mathbb{R}}\, G^{\mathrm{ab}}$, and G_∞^{der} is compact.*

Proof. Suppose first that G_∞^1 is compact. Let A denote the biggest quotient group of G^{ab} which is a split torus. Then $X^*(G)_F = X^*(G^{\mathrm{ab}})_F = X^*(A)_F = X^*(A)$. Set $G' = \ker[G \to A]$. Since we have an exact sequence $1 \to G'_\infty \to G_\infty^1 \to A_\infty^1 \to 1$, we see that both G'_∞ and A_∞^1 are compact. We may assume that $A \neq 1$ (otherwise $G_\infty = G_\infty^1$, hence G_∞ is compact). Choosing a basis for $X^*(A)$, we see that (with $r = \dim A$)

$$A_\infty^1 \simeq \{x \in F_\infty^\times; \prod_{v\in\infty} |x_v|_v = 1\}^r.$$

Therefore $\{x \in F_\infty^\times; \prod_{v|\infty} |x_v|_v = 1\}$ is compact. We conclude that F has a unique infinite place. That is, $F = \mathbb{Q}$. If $\mathrm{rk}_{\mathbb{Q}}\, G^{\mathrm{ab}} < \mathrm{rk}_{\mathbb{R}}\, G^{\mathrm{ab}}$, the $\mathbb{Q}$-anisotropic component $N(= (G^{\mathrm{ab}})' = \ker[G^{\mathrm{ab}} \to A])$ of G^{ab} is not $\mathbb{R}$-anisotropic, hence N_∞ is not compact, so G'_∞ is not compact and thus G_∞^1 is not compact. The converse is clear. □

4. Hecke algebras. From now on we fix a totally real number field F, and a connected reductive F-group G, not a torus, such that G_∞^1 is compact. Consider the locally profinite group of finite adèles $G(\mathbb{A}^\infty)$. Let Σ be a finite set of finite places. Let $\mathbb{A}^\Sigma$ be the ring of adèles without component at ∞ and Σ. Consider the subgroup $G(\mathbb{A}^\Sigma)$ of $G(\mathbb{A}^\infty)$. Choose a Haar measure $\mu = \otimes\mu_v$ on $G(\mathbb{A}^\Sigma)$ once and for all. Consider the vector space of all locally constant compactly supported $\mathbb{C}$-valued functions on $G(\mathbb{A}^\Sigma)$:

$$\mathcal{H} = \mathcal{H}(G(\mathbb{A}^\Sigma)) = C_c^\infty(G(\mathbb{A}^\Sigma), \mathbb{C}).$$

This becomes an associative $\mathbb{C}$-algebra, without neutral element, under μ-convolution. There is a canonical involution (anti-automorphism) on $\mathcal{H}$ defined by $\phi^\vee(x) = \phi(x^{-1})$. If $K \subset G(\mathbb{A}^\Sigma)$ is a compact open subgroup, $e_K = \mu(K)^{-1}\chi_K \in \mathcal{H}$ is an idempotent. This is the neutral element in the subalgebra of K-biinvariant compactly supported functions:

$$\mathcal{H}_K = \mathcal{H}(G(\mathbb{A}^\Sigma), K) = C_c(G(\mathbb{A}^\Sigma)//K, \mathbb{C}) = e_K * \mathcal{H} * e_K.$$

Clearly $\vee$ preserves $\mathcal{H}_K$. In addition, there is a canonical $\mathbb{Z}$-order $\mathcal{H}_{K,\mathbb{Z}} \subset \mathcal{H}_K$ consisting of all $\mu(K)^{-1}\mathbb{Z}$-valued functions. As a ring, $\mathcal{H}_{K,\mathbb{Z}}$ is isomorphic to

$C_c(G(\mathbb{A}^\Sigma)//K, \mathbb{Z})$ endowed with the K-normalized convolution. If R is a commutative ring, with neutral element, we define

$$\mathcal{H}_{K,R} = \mathcal{H}_{K,\mathbb{Z}} \otimes_{\mathbb{Z}} R, \quad \text{thus} \quad \mathcal{H}_K = \mathcal{H}_{K,\mathbb{C}}.$$

The algebras $\mathcal{H}_K$ are not commutative when K is not maximal. However, by a result of Bernstein [B], $\mathcal{H}_K$ is a finite module over its center $Z(\mathcal{H}_K)$. Now, suppose $J \subset K$ is a (proper) compact open subgroup. Then obviously $\mathcal{H}_K \subset \mathcal{H}_J$. However, $\mathcal{H}_K$ is not a subring since $e_K \neq e_J$. There is a natural retraction $\mathcal{H}_J \twoheadrightarrow \mathcal{H}_K$ defined by $\phi \mapsto e_K * \phi * e_K$. It does map $e_J \mapsto e_K$, but does not preserve $*$ unless we restrict it to the centralizer $Z_{\mathcal{H}_J}(e_K)$. Clearly, $Z_{\mathcal{H}_J}(\mathcal{H}_K)$ maps to the center $Z(\mathcal{H}_K)$. In particular,

$$Z(\mathcal{H}_J) \to Z(\mathcal{H}_K), \quad \phi \mapsto \phi * e_K = e_K * \phi,$$

gives a canonical homomorphism of algebras. It maps $Z(\mathcal{H}_{J,\mathbb{Z}})$ into $Z(\mathcal{H}_{K,\mathbb{Z}})$.

5. Algebraic modular forms. In this section we define algebraic modular forms with weight and type, using the exposition of Bellaiche and Graftieaux [BG]. For each finite place v of F, let K_v be a compact open subgroup of $G_v = G(F_v)$, which is a maximal compact hyperspecial subgroup for almost all v. Let Σ be a finite set of finite places of F. Write K_Σ for $\prod_{v \in \Sigma} K_v$ and $K = \prod_{v \notin \Sigma} K_v$. Put $K'' = K \times K_\Sigma$. Then $K'' = \prod_v K_v$ ($v < \infty$) is a compact open subgroup of $G(\mathbb{A}^\infty)$. It is called the *level*. Let $\rho_\Sigma : K_\Sigma \to \mathrm{GL}(V_\Sigma)$ be a smooth complex irreducible representation, named the *type*. It can be viewed as a representation of K'' trivial on K.

Put $G_\infty = G(F_\infty) = \prod_v G(F_v)$, $F_\infty = \prod_v F_v$ (v archimedean). Let $\rho_\infty : G_\infty \to \mathrm{GL}(V_\infty)$ be an irreducible, complex, continuous unitary representation, named the *weight*. Denote by Z_∞ the center of G_∞, and by ω_∞ the central character of ρ_∞.

Denote by ρ_∞^* and ρ_Σ^* the contragredient representations.

Note that $G(F) \subset G(\mathbb{A})/Z_\infty$ is a discrete cocompact subgroup. Consider the Hilbert space $L^2(G(F)\backslash G(\mathbb{A}), \omega_\infty)$ of L^2-functions on the quotient $G(F)\backslash G(\mathbb{A})$ which transform under Z_∞ via the unitary character ω_∞. There is a unitary representation r of $G(\mathbb{A})$ on this space given by right translations. This space is a direct sum, with finite multiplicities $m(\pi)$, of irreducible $G(\mathbb{A})$-submodules π, called *automorphic representations*. An admissible irreducible representation π decomposes as a product $\otimes_v \pi_v$ over all places v of F. Put $\pi_\Sigma = \otimes_{v \in \Sigma} \pi_v$. We shall be interested only in the part which contains the representation $\rho = \rho_\Sigma \otimes \rho_\infty$ of $K_\Sigma \times G_\infty$. It is $\mathcal{A}(K_\Sigma, \rho, \mathbb{C}) = \cup_K \mathcal{A}(K, \rho, \mathbb{C})$

$$= \mathrm{Hom}_{K_\Sigma \times G_\infty}(\rho_\Sigma \otimes \rho_\infty, C^\infty(G(F)\backslash G(\mathbb{A}), \omega_\infty; \mathbb{C})).$$

Here K runs through all compact open subgroups of $G(\mathbb{A}^\Sigma)$, and

$$\mathcal{A}(K, \rho, \mathbb{C}) = \mathrm{Hom}_{K_\Sigma \times G_\infty}(\rho_\Sigma \otimes \rho_\infty, C^\infty(G(F)\backslash G(\mathbb{A})/K, \omega_\infty; \mathbb{C})).$$

The Hecke algebra $\mathcal{H} = \mathcal{H}(G(\mathbb{A}^\Sigma))$ acts on $\mathcal{A}(K_\Sigma, \rho, \mathbb{C})$ by convolution, and $\mathcal{A}(K, \rho, \mathbb{C})$ is the space $r(e_K)\mathcal{A}(K_\Sigma, \rho, \mathbb{C})$ of K-invariants. It is a finite dimensional space, as the double coset space $X_K = G(F)\backslash G(\mathbb{A}^\infty)/K''$ is finite. Recall that $K'' = K \times K_\Sigma$. Thus $\mathcal{A}(K_\Sigma, \rho, \mathbb{C}) = \oplus m(\pi)\pi$, sum over the irreducible π with $\pi_\infty = \rho_\infty$, $\pi_\Sigma \supset \rho_\Sigma$. Also $\mathcal{A}(K, \rho, \mathbb{C}) = \oplus m(\pi)\pi^K$, sum over the same π, but for which the space π^K of K-invariants in π is nonzero.

We have the following compatibility between this action and the inner product:

$$(r(\bar{\phi})f, g) = (f, r(\phi^\vee)g); \qquad \phi \in \mathcal{H}; \;\; f, g \in \mathcal{A}(K_\Sigma, \rho, \mathbb{C}).$$

Definition 5.1. *A complex valued automorphic form of level K'', type ρ_Σ, and weight ρ_∞, is a function f'' in $\mathcal{A}(K,\rho,\mathbb{C})$ where $\rho = (\rho_\Sigma, \rho_\infty)$.*

Using the relation $[f'(g)](v,w) = [f''(v,w)](g)$, the space $\mathcal{A}(K,\rho,\mathbb{C})$ is isomorphic to the space of functions $f' : G(F)\backslash G(\mathbb{A})/K \to V_\Sigma^* \otimes_{\mathbb{C}} V_\infty^*$ with

$$f'(gku_\infty) = [\rho_\Sigma^*(k)^{-1} \otimes \rho_\infty^*(u_\infty)^{-1}]f'(g) \quad (g \in G(\mathbb{A}),\, k \in K'',\, u_\infty \in G_\infty).$$

The restriction f of such f' to $G(\mathbb{A}^\infty)$ satisfies, where $u \in G(F)$, u_∞ is the image of u in G_∞, so that $uu_\infty^{-1} \in G(F) \cap G(\mathbb{A}^\infty)$, the relation

$$f(ugk)\ [= f'(ugku_\infty^{-1}) = f'(gku_\infty^{-1})] = (\rho_\Sigma^*(k)^{-1} \otimes \rho_\infty^*(u_\infty))f(g), \tag{5.1}$$

where $g \in G(\mathbb{A}^\infty)$, $k \in K''$; it determines f'.

Since ρ_Σ factorizes through a finite quotient of K'', there exists a *number field* L (in $\mathbb{C}$) and an L-model $(\rho_{\Sigma,L}, V_{\Sigma,L})$ of (ρ_Σ, V_Σ). Increasing L we may assume it is stable under complex conjugation. We can then talk about Hermitian products on $V_{\Sigma,L}$. Choose such an Hermitian product on $V_{\Sigma,L}$ which is stable under $\rho_{\Sigma,L}$ (using the finiteness of $\rho_{\Sigma,L}(K)$).

Extend the representation ρ_∞ of the algebraic group $G_\infty = G(\mathbb{R})^{\Sigma_\infty}$ on V_∞ to a representation $\rho_{\infty,\mathbb{C}}$ of $G(\mathbb{C})^{\Sigma_\infty}$ on the same complex space V_Σ. Since $\rho_{\infty,\mathbb{C}}$ is algebraic, it has a model over a number field, which can be assumed to be L (on increasing L, so that in particular it contains F). Thus $\rho_{\infty,L}$ is an algebraic representation of $G(L)^{\Sigma_\infty}$ on $V_{\infty,L}$, defined by a morphism $G^{\Sigma_\infty} \otimes_F L \to \mathrm{GL}(V_{\infty,L})$ of group schemes over L, which specializes to $\rho_{\infty,\mathbb{C}}$ over $\mathbb{C}$. As usual, $\otimes_F L$ abbreviates $\times_{\mathrm{Spec}\,F} \mathrm{Spec}\,L$.

Increasing L we may assume it is Galois over $\mathbb{Q}$. Then $\sigma(F) \subset L$ for each σ in Σ_∞. Embed $G(F)$ in $G(L)^{\Sigma_\infty}$ by $r : u \mapsto (\sigma(u); u \in \Sigma_\infty)$. Then by definition, for every $u \in G(F)$ we have

$$\rho_{\infty,L}(r(u)) \otimes 1 = \rho_\infty(u_\infty) \in \mathrm{GL}(V_\infty), \tag{5.2}$$

the tensor product is $V_{\infty,L} \otimes_L \mathbb{C} = V_\infty$.

Definition 5.2. *For any commutative unitary L-algebra A, put $V_{\Sigma,A} = V_{\Sigma,L} \otimes_L A$, $V_{\infty,A} = V_{\infty,L} \otimes_L A$. Let $\rho_{\Sigma,A}$, $\rho_{\infty,A}$ be the corresponding representations. Let $\mathcal{A}(K,\rho,A)$ be the A-module of smooth functions*

$$f : G(\mathbb{A}^\infty) \to V_{\Sigma,A}^* \otimes_A V_{\infty,A}^* \tag{5.3}$$

satisfying for $g \in G(\mathbb{A}^\infty)$, $u \in G(F)$, $k \in K''$,

$$f(ugk) = [\rho_{\Sigma,A}^*(k)^{-1} \otimes \rho_{\infty,A}^*(r(u))]f(g).$$

When $A = \mathbb{C}$, (5.3) coincides with (5.1) in view of (5.2).

Since G_∞ is compact, ρ_∞ preserves an Hermitian form on V_∞. The restriction of $\rho_{\infty,L}$ to $G(L_0)^{\Sigma_0}$ does too. Here $L_0 = L \cap \mathbb{R}$, and we assume $L \neq L_0$, thus $[L : L_0] = 2$. Choose such an Hermitian form on $V_{\infty,L}$. Given $u \in G(F)$, for any embedding σ of F in $\mathbb{C}$, we have $\sigma(u) \in G(L_0)$ since F is totally real. Hence $\rho_{\infty,L}(r(u))$ is a unitary element in $\mathrm{GL}(V_{\infty,L})$. The space $G(F)\backslash G(\mathbb{A}^\infty)$ is compact, thus there is a unique right invariant measure on it which assigns it volume one. We obtain an Hermitian form on $\mathcal{A}(K,\rho,L)$ on integrating over $G(F)\backslash G(\mathbb{A}^\infty)$ that on $V_{\Sigma,L}^* \otimes_L V_{\infty,L}^*$.

Definition 5.3. *Let λ be a finite place of L. Let $\lambda_0 = \lambda \cap L_0$ be the place of $L_0 = L \cap \mathbb{R}$ dividing λ. Let $(L_0)_{\lambda_0}$ be the completion of L_0 at λ_0. Put $L_{\lambda_0} = L \otimes_{L_0} (L_0)_{\lambda_0}$. Let R_{λ_0} be the ring of integers of L_{λ_0}. If λ_0 splits in L into $\lambda \cdot \overline{\lambda}$, then $R_{\lambda_0} = R_\lambda \times R_{\overline{\lambda}}$. If λ_0 stays prime, R_{λ_0} is R_λ ($[L_{\lambda_0} : L_{0,\lambda_0}] = 2$, then $L_{\lambda_0} = L_\lambda$ and $R_{\lambda_0} = R_\lambda$). Then R_{λ_0} has an involution extending that on the ring of integers R_L of L.*

Let $\mathbb{F}_\lambda = R_\lambda/\lambda$ be the residue field of R_λ ($\mathbb{F}_{\overline{\lambda}}$ of $R_{\overline{\lambda}}$). Put $\mathbb{F}_{\lambda_0} = R_{\lambda_0}/\lambda_0$ (it is $\mathbb{F}_\lambda \times \mathbb{F}_{\overline{\lambda}}$ if λ_0 splits, or $\mathbb{F}_\lambda$ if λ_0 stays prime). Note that $\mathbb{F}_{\overline{\lambda}} \simeq \mathbb{F}_\lambda$.

Write Σ_{λ_0} for the set of places of F of the same residual characteristic as that of λ_0.

For $\sigma \in \Sigma_\infty$, $\sigma : F \hookrightarrow L_0$, we have $\sigma^{-1}(\lambda_0) \in \Sigma_{\lambda_0}$ (as $\lambda_0 \subset R_0 \subset L_0$). The embedding σ extends by continuity to an embedding $\sigma : F_{\sigma^{-1}(\lambda_0)} \hookrightarrow L_{0\lambda_0} \hookrightarrow L_{\lambda_0}$. Then we have $r_\sigma : G(F_{\sigma^{-1}(\lambda_0)}) \hookrightarrow G(L_{\lambda_0})$ for each $\sigma \in \Sigma_\infty$, and the product $r_{L_{\lambda_0}} : G(F_{\Sigma_{\lambda_0}}) \hookrightarrow G(L_{\lambda_0})^{\Sigma_\infty}$,

$$(x_{\sigma^{-1}(\lambda_0)}; \sigma \in \Sigma_\infty) \mapsto (r_\sigma(x_{\sigma^{-1}(\lambda_0)}); \sigma \in \Sigma_\infty).$$

Let $p_{\Sigma_{\lambda_0}} : G(\mathbb{A}^\infty) \to G(F_{\Sigma_{\lambda_0}})$ be the natural projection. Put

$$\widetilde{\rho}_{\infty,L_{\lambda_0}} = \rho_{\infty,L_{\lambda_0}} \circ r_{L_{\lambda_0}} \circ p_{\Sigma_{\lambda_0}} : G(\mathbb{A}^\infty) \to \mathrm{GL}(V_{\infty,L_{\lambda_0}}).$$

It has, for $u \in G(F)$, that $\widetilde{\rho}_{\infty,L_{\lambda_0}}(u) \in \mathrm{GL}(V_{\infty,L_{\lambda_0}})$ is in fact in $\mathrm{GL}(V_{\infty,L})$, equal to

$$\widetilde{\rho}_{\infty,L_{\lambda_0}}(u) = \rho_{\infty,L}(r(u)) = \rho_\infty(u_\infty) \qquad (u \in G(F)).$$

Definition 5.4. *Fix any R_L-lattices V_{Σ,R_L} in $V_{\Sigma,L}$ and V_{∞,R_L} in $V_{\infty,L}$. Put $V_{\Sigma,R_{\lambda_0}} = V_{\Sigma,R_L} \otimes_{R_L} R_{\lambda_0}$, $V_{\infty,R_{\lambda_0}} = V_{\infty,R_L} \otimes_{R_L} R_{\lambda_0}$. Let $\mathcal{A}(K,\rho,R_{\lambda_0})$ be the sub-R_{λ_0}-module of $\mathcal{A}(K,\rho,L_{\lambda_0})$ consisting of the functions*

$$f : G(\mathbb{A}^\infty) \to V^*_{\Sigma,L_{\lambda_0}} \otimes_{L_{\lambda_0}} V^*_{\infty,L_{\lambda_0}}$$

with

$$\widetilde{f}(g) = \widetilde{\rho}^*_{\infty,L_{\lambda_0}}(g)^{-1} f(g) \in V^*_{\Sigma,R_{\lambda_0}} \otimes_{R_{\lambda_0}} V^*_{\infty,R_{\lambda_0}}$$

for all g in $G(\mathbb{A}^\infty)$. Put $K^{\lambda_0} = \prod_v K_v$ ($v \notin \Sigma \cup \Sigma_{\lambda_0} \cup \infty$).

Note that $\widetilde{\rho}_{\infty,L_{\lambda_0}}$ factorizes through $G(F_{\Sigma_{\lambda_0}})$, thus $\mathcal{A}(K,\rho,R_{\lambda_0})$ can be viewed as a set of K^{λ_0}-invariants of a space of functions on $G(\mathbb{A}^\infty)$ on which $G(\mathbb{A}^\infty)$ acts by right translation. We aim to show, in Lemma 5.6, that for almost all places λ, we have $\mathcal{A}(K,\rho,R_{\lambda_0}) \otimes_{R_{\lambda_0}} L_{\lambda_0} = \mathcal{A}(K,\rho,L_{\lambda_0})$.

The morphism $\rho_{\infty,L} : G^{\Sigma_\infty} \otimes_F L \to \mathrm{GL}(V_{\infty,L})$ extends – since G and $\mathrm{GL}(V_{\infty,L})$ are schemes of finite type – to a morphism

$$\mathbf{G}^{\Sigma_\infty} \otimes_{R_F} R_L[1/N] \to \mathrm{GL}(V_{\infty,R_L[1/N]})$$

over the open subset $\operatorname{Spec} R_L[1/N]$ of $\operatorname{Spec} R_L$, where N is a positive integer and $\mathbf{G}$ is a smooth affine group scheme of finite type over R_F with generic fiber G. For each λ_0 not dividing N there is then a model

$$\rho_{\infty,\lambda_0} : \mathbf{G}^{\Sigma_\infty} \otimes_{R_F} R_{\lambda_0} \to \mathrm{GL}(V_{\infty,R_{\lambda_0}})$$

over R_{λ_0} of $\rho_{\infty,L_{\lambda_0}}$. The lattice $V_{\infty,R_{\lambda_0}}$ of $V_{\infty,L_{\lambda_0}}$ is stable under the restriction of $\rho_{\infty,L_{\lambda_0}}$ to the subgroup $\mathbf{G}^{\Sigma_\infty}(R_{\lambda_0})$ of $\mathbf{G}^{\Sigma_\infty}(L_{\lambda_0})$. Increasing N we may also assume that $\mathbf{G}(R_v) = K_v$ for each F-place v prime to N. Then for each λ_0 prime to N the morphism $r_{L_{\lambda_0}} \circ p_{\Sigma_{\lambda_0}}$ maps K'' to $\mathbf{G}^{\Sigma_\infty}(R_{\lambda_0})$. Hence the restriction of $\widetilde{\rho}_{\infty,L_{\lambda_0}}$ to K'' leaves stable the lattice $V_{\infty,R_{\lambda_0}}$ of $V_{\infty,L_{\lambda_0}}$. For each R_{λ_0}-algebra A we can then define the representation $\widetilde{\rho}_{\infty,A}$ of the group K'' on $V_{\infty,A} = V_{\infty,R_{\lambda_0}} \otimes_{R_{\lambda_0}} A$. In particular one has a representation $\widetilde{\rho}_{\infty,\mathbb{F}_{\lambda_0}}$ of K'' on $V_{\infty,\mathbb{F}_{\lambda_0}}$.

Lemma 5.5. *Increasing N if necessary, for each place λ_0 of L_0 prime to N, the restriction of $\widetilde{\rho}_{\infty,\mathbb{F}_{\lambda_0}}$ to any subgroup H of K'' whose image under reduction to $\prod_{v\in\Sigma_{\lambda_0}} \mathbf{G}(\mathbb{F}_v)$ contains $\prod_{v\in\Sigma_{\lambda_0}} \mathbf{G}^{\mathrm{der}}(\mathbb{F}_v)$ is absolutely irreducible.*

Proof. The embedding $r : G(F) \to G^{\Sigma_\infty}(L)$ is the realization for $\mathbb{Q}$-points of a morphism $\mathrm{R}_{F/\mathbb{Q}}G \to \mathrm{R}_{L/\mathbb{Q}}G^{\Sigma_\infty}$ of algebraic $\mathbb{Q}$-groups. There exists a model of this morphism over $\operatorname{Spec}\mathbb{Z}[1/N]$ for a suitable N. For a prime ℓ prime to N one has the morphism

$$r_{\mathbb{Z}_\ell} : \prod_{v|\ell} \mathbf{G}(R_v) \to \prod_{\lambda_0|\ell} \mathbf{G}^{\Sigma_\infty}(R_{\lambda_0})$$

(v are F-places, λ_0 are L_0-places). There are analogous morphisms $r_{\mathbb{Q}_\ell}$ with R_v replaced by F_v and R_{λ_0} by L_{λ_0}, and $r_{\mathbb{F}_\ell}$ with $\mathbb{F}_v$ and $\mathbb{F}_{\lambda_0}$ (for R_v, R_{λ_0}). Note that $\{v; v|\ell\}$ is Σ_{λ_1} for any λ_1 dividing ℓ. The morphism $r_{L_{\lambda_1}}$ is then $r_{\mathbb{Q}_\ell} \circ \mathrm{pr}$, pr being the projection of $\prod_{\lambda_0|\ell} \mathbf{G}^{\Sigma_\infty}(L_{\lambda_0})$ to its factor $\mathbf{G}^{\Sigma_\infty}(L_{\lambda_1})$.

Since $G(F)$ is Zariski dense in $G(\mathbb{C})^{\Sigma_\infty}$, the morphism

$$\mathrm{R}_{F/\mathbb{Q}}G \to \mathrm{R}_{L/\mathbb{Q}}G^{\Sigma_\infty} \xrightarrow{\mathrm{R}_{L/\mathbb{Q}}\rho_{\infty,L}} \mathrm{R}_{L/\mathbb{Q}}\,\mathrm{GL}(V_{\infty,L})$$

is absolutely irreducible. The same holds with G replaced by G^{der}, since $G = G^{\mathrm{der}} \cdot Z$, Z being the center of G. The same holds for almost all ℓ and λ_0 dividing ℓ, for the morphism

$$\psi_{\lambda_0} : \prod_{v\in\Sigma_{\lambda_0}} \mathbf{G}(\mathbb{F}_v) \xrightarrow{r_{\mathbb{F}_\ell}} \prod_{\lambda_0|\ell} \mathbf{G}^{\Sigma_\infty}(\mathbb{F}_{\lambda_0}) \xrightarrow{\mathrm{pr}} \mathbf{G}^{\Sigma_\infty}(\mathbb{F}_{\lambda_1}) \xrightarrow{\rho_{\infty,\mathbb{F}_{\lambda_1}}} \mathrm{GL}(V_{\infty,\mathbb{F}_{\lambda_1}}),$$

and also with $\mathbf{G}^{\mathrm{der}}$ replacing $\mathbf{G}$.

The lemma follows from the commutativity of the lower triangle in the following diagram, where the square is clearly commutative:

$$\begin{array}{ccc} \prod_{v\in\Sigma_{\lambda_1}} K_v & \xrightarrow{\widetilde{\rho}_{\infty,R_{\lambda_1}}} & \mathrm{GL}(V_{\infty,R_{\lambda_1}}) \\ \downarrow \text{reduction} & \searrow \ \widetilde{\rho}_{\infty,\mathbb{F}_{\lambda_1}} & \downarrow \text{reduction} \\ \prod_{v\in\Sigma_{\lambda_1}} \mathbf{G}(\mathbb{F}_v) & \xrightarrow{\psi_{\lambda_1}} & \mathrm{GL}(V_{\infty,\mathbb{F}_{\lambda_1}}); \end{array}$$

the upper triangle is commutative by the definition of $\widetilde{\rho}_{\infty,\mathbb{F}_{\lambda_1}}$. □

Since $\rho_\Sigma(K'')$ is finite, increasing N we may assume that for λ_0 prime to N the lattice $V_{\Sigma,R_{\lambda_0}}$ is stable under $\rho_\Sigma(K)$, and the representation ρ_Σ of K'' on $V_{\Sigma,R_{\lambda_0}}/\lambda_0 V_{\Sigma,R_{\lambda_0}} = V_{\Sigma,\mathbb{F}_{\lambda_0}}$ is absolutely irreducible.

As $G(F)\backslash G(\mathbb{A}^\infty)$ is compact, there are $x_1, \ldots, x_h$ in $G(\mathbb{A}^\infty)$ with $G(\mathbb{A}^\infty) = \coprod_{1\le i\le h} G(F)x_iK''$. The group $\Delta_i = G(F)\cap x_iK''x_i^{-1}$ is finite ($1\le i\le h$), $G(F)\backslash G(\mathbb{A}^\infty) = \coprod_{1\le i\le h}\Delta_i\backslash x_iK''$. The map

$$f\mapsto (f(x_1),\ldots,f(x_h)),\quad \mathcal{A}(K,\rho,A)\to \oplus_{1\le i\le h}(V^*_{\Sigma,A}\otimes_A V^*_{\infty,A})^{\Delta_i},$$

where $\alpha = x_ikx_i^{-1}\in\Delta_i$ acts by $\rho^*_{\Sigma,A}(k)^{-1}\otimes_A \rho^*_{\infty,A}(r(\alpha))^{-1}$, is an isomorphism. The Hermitian product on $\mathcal{A}(K,\rho,A)$ is the sum over i ($1\le i\le h$) of $|\Delta_i|^{-1}$ times that on the spaces $(V^*_{\Sigma,L}\otimes_L V^*_{\infty,L})^{\Delta_i}$. Thus

$$\mu(K'')^{-1}(f,g) = \sum_{1\le i\le h}(f(x_i),g(x_i))|\Delta_i|^{-1}.$$

So $\mathcal{A}(K,\rho,A') = \mathcal{A}(K,\rho,A)\otimes_A A'$ for any $A\to A'$. As $Z(\mathcal{H}_K)$-modules, $\mathcal{A}(K,\rho,L)\otimes_L\mathbb{C} = \mathcal{A}(K,\rho,\mathbb{C})$. The Hermitian product on $\mathcal{A}(K,\rho,L)$ is nondegenerate: the adjoint of the action of an element of $Z(\mathcal{H}_K)$ on $\mathcal{A}(K,\rho,L)$ is still the action of an element of $Z(\mathcal{H}_K)$. This last algebra is commutative, thus the elements of $Z(\mathcal{H}_K)$ act as normal ($DD^* = D^*D$) operators, and $\mathcal{A}(K,\rho,L)$ is a semisimple $Z(\mathcal{H}_K)\otimes L$-module. The sub-$R_L$-module $(V^*_{\Sigma,R_L}\otimes_{R_L} V^*_{\infty,R_L})^{\Delta_i}$ of the L-vector space $(V^*_{\Sigma,L}\otimes_L V^*_{\infty,L})^{\Delta_i}$ is a lattice.

Increasing N we may assume that each $|\Delta_i|$ divides N, and for any λ_0 prime to N, the restriction of the Hermitian product of $V^*_{\Sigma,L_{\lambda_0}}\otimes_{L_{\lambda_0}} V^*_{\infty,L_{\lambda_0}}$ to $(V^*_{\Sigma,L_{\lambda_0}}\otimes_{L_{\lambda_0}} V^*_{\infty,L_{\lambda_0}})^{\Delta_i}$ is nondegenerate and R_{λ_0}-valued. Note: N depends only on (K'',ρ). Replacing K'' by a subgroup we need not change N.

Lemma 5.6. *For any place λ_0 of L_0 prime to N we have*

$$\mathcal{A}(K,\rho,R_{\lambda_0})\otimes_{R_{\lambda_0}} L_{\lambda_0} = \mathcal{A}(K,\rho,L_{\lambda_0})$$

as a $Z(\mathcal{H}_{K^{\lambda_0}})\otimes_{\mathbb{Z}} L_{\lambda_0}$-module. The Hermitian product on $\mathcal{A}(K,\rho,R_{\lambda_0})$ is nondegenerate.

Proof. To show that $\mathcal{A}(K,\rho,R_{\lambda_0})$ is a lattice in $\mathcal{A}(K,\rho,L_{\lambda_0})$, note that for $f\in\mathcal{A}(K,\rho,R_{\lambda_0})$, $g\in G(\mathbb{A}^\infty)$, $u\in G(F)$, $k\in K''$, we have

$$\widetilde{f}(ugk) = \widetilde{\rho}^*_{\infty,L_{\lambda_0}}(ugk)^{-1}f(ugk)$$

by definition of $\widetilde{f}$,

$$= (\rho^*_\Sigma(k)^{-1}\otimes_{L_{\lambda_0}}[\widetilde{\rho}^*_{\infty,L_{\lambda_0}}(k)^{-1}\widetilde{\rho}^*_{\infty,L_{\lambda_0}}(g)^{-1}\widetilde{\rho}^*_{\infty,L_{\lambda_0}}(u)^{-1}]\rho^*_\infty(u_\infty))f(g)$$

by definition of f,

$$= (\rho^*_\Sigma(k)^{-1}\otimes_{L_{\lambda_0}}\widetilde{\rho}^*_{\infty,L_{\lambda_0}}(k)^{-1})(\widetilde{\rho}^*_{\infty,L_{\lambda_0}}(g)^{-1}f(g))$$

since $\rho_\infty(u_\infty) = \widetilde{\rho}_{\infty,L_{\lambda_0}}(u)$,

$$= (\rho^*_\Sigma(k)^{-1}\otimes_{L_{\lambda_0}}\widetilde{\rho}^*_{\infty,L_{\lambda_0}}(k)^{-1})\widetilde{f}(g)$$

by definition of $\widetilde{f}$.

By definition of $\mathcal{A}(K,\rho,R_{\lambda_0})$, f lies in this space precisely when $\widetilde{f}(x_i) = \widetilde{\rho}^*_{\infty,L_{\lambda_0}}(x_i)^{-1}f(x_i)$ lies in $V^*_{\Sigma,R_{\lambda_0}} \otimes_{R_{\lambda_0}} V^*_{\infty,R_{\lambda_0}}$ for all i $(1 \le i \le h)$. Thus we have an isomorphism

$$\mathcal{A}(K,\rho,R_{\lambda_0}) = \oplus_{1\le i\le h}(\widetilde{\rho}^*_{\infty,R_{\lambda_0}}(x_i)[V^*_{\Sigma,R_{\lambda_0}} \otimes_{R_{\lambda_0}} V^*_{\infty,R_{\lambda_0}}])^{\Delta_i}.$$

Each of the R_{λ_0}-modules on the right is a lattice in the L_{λ_0}-vector space $(V^*_{\Sigma,L_{\lambda_0}} \otimes_{L_{\lambda_0}} V^*_{\infty,L_{\lambda_0}})^{\Delta_i}$. It remains to show that the restriction of the Hermitian product from $\mathcal{A}(K,\rho,L_{\lambda_0})$ to $\mathcal{A}(K,\rho,R_{\lambda_0})$ is R_{λ_0}-valued and nondegenerate. But this is explained in the paragraph before the lemma. Indeed, this Hermitian product is a direct sum, weighted by invertible elements of R_{λ_0}, of Hermitian products which are nondegenerate and R_{λ_0}-valued. □

For a place λ_0 of L_0 prime to N, and commutative R_{λ_0}-algebra A with a unit, put $V_{\Sigma,A} = V_{\Sigma,R_{\lambda_0}} \otimes_{R_{\lambda_0}} A$, $\rho_{\Sigma,A} = \rho_{\Sigma,R_{\lambda_0}} \otimes_{R_{\lambda_0}} A$, and similarly $V_{\infty,A}$, $\rho_{\infty,A}$. Define $\mathcal{A}(K,\rho,A)$ to be the A-module of smooth functions $\widetilde{f} : G(\mathbb{A}^\infty) \to V^*_{\Sigma,A} \otimes_A V^*_{\infty,A}$ satisfying, for $g \in G(\mathbb{A}^\infty)$, $u \in G(F)$, $k \in K''$,

$$\widetilde{f}(ugk) = [\rho^*_{\Sigma,A}(k)^{-1} \otimes_A \widetilde{\rho}^*_{\infty,A}(k)^{-1}]\widetilde{f}(g).$$

The $Z(\mathcal{H}_{K^{\lambda_0}})$-module $\mathcal{A}(K,\rho,A)$ commutes with base change $A \to A'$. For $A = R_{\lambda_0}$, $\mathcal{A}(K,\rho,A)$ is isomorphic to $\mathcal{A}(K,\rho,R_{\lambda_0})$ previously defined, by $\widetilde{f} \mapsto f$, $\widetilde{f}(g) = \widetilde{\rho}^*_{\infty,L_{\lambda_0}}(g)^{-1}f(g)$. Lemma 5.6 implies that any character of $Z(\mathcal{H}_{K^{\lambda_0}})$ on $\mathcal{A}(K,\rho,R_{\lambda_0})$ is R_{λ_0}-valued for almost all places λ of L.

Let $\mathbb{T}_{K^{\lambda_0},A}$ denote the image of the center $Z(\mathcal{H}_{K^{\lambda_0},A})$ in $\mathrm{End}_A\,\mathcal{A}(K,\rho,A)$. Hence $\mathbb{T}_{K^{\lambda_0},A}$ is a commutative A-algebra. Now, suppose $J \subset K$ is a (proper) compact open subgroup. Then $\mathcal{A}(K,\rho,A) \subset \mathcal{A}(J,\rho,A)$, and the canonical homomorphism $Z(\mathcal{H}_{J^{\lambda_0},A}) \to Z(\mathcal{H}_{K^{\lambda_0},A})$ descends to the restriction map $\mathbb{T}_{J^{\lambda_0},A} \to \mathbb{T}_{K^{\lambda_0},A}$.

6. Pairings. We review now the pairing on $\mathcal{A}(K,\rho,A)$. Here $(-,-)$ denotes the inner product on $V^*_{\Sigma,R_{\lambda_0}} \otimes V^*_{\infty,R_{\lambda_0}}$.

Definition 6.1. *For $f,g \in \mathcal{A}(K,\rho,A)$, define a symmetric bilinear form by*

$$\langle f,g\rangle_K = \mu(K'')^{-1}(f,\overline{g}) = \sum_{x\in X_K}(f(x),\overline{g}(x))|G(F)\cap {}^xK''|^{-1}.$$

Here ${}^xK'' = xK''x^{-1}$ and $X_K = G(F)\backslash G(\mathbb{A}^\infty)/K''$.

The factors $|G(F)\cap {}^xK''|^{-1}$ are missing in [T1] and [T2]. If K is sufficiently small, for example if $K = \prod_{v\notin\Sigma\cup\infty} K_v$ and K_v is torsion-free for some $v \notin \Sigma\cup\infty$ (this is the case if K_v is a sufficiently deep principal congruence subgroup), then indeed $G(F)\cap {}^xK'' = 1$. For $\phi \in \mathcal{H}_K$ and $f,g \in \mathcal{A}(K,\rho,A)$ we have the compatibility relation

$$\langle r(\phi)f,g\rangle_K = \langle f, r(\phi^\vee)g\rangle_K.$$

Next we have to show that the quotient $\mathcal{A}(K,\rho,R_{\lambda_0})/\mathcal{A}(K,\rho,R_{\lambda_0})^\vee$ is torsion and find a good annihilator A_K. The fact that it is torsion is immediate: it is killed by the positive integer

$$\prod_{x\in X_K}|G(F)\cap {}^xK''|.$$

This is 1 if K is sufficiently small in the sense above.

Lemma 6.2. *Let $K = \prod_{v \notin \Sigma} K_v \subset G(\mathbb{A}^\Sigma)$ be a decomposable compact open subgroup. Let ℓ be the residual characteristic of λ. Suppose $\ell \nmid |K_v|$ for some $v \notin \Sigma \cup \infty$. Then there exists a positive integer A_K, not divisible by ℓ, such that*

$$A_K \langle \mathcal{A}(K, \rho, R_{\lambda_0}), \mathcal{A}(K, \rho, R_{\lambda_0}) \rangle_K \subset R_{\lambda_0}.$$

Proof. Choose some torsion-free subgroup $\widetilde{K}_v \subset K_v$ and let $\widetilde{K} = \widetilde{K}_v K^v$. Then

$$\langle \mathcal{A}(\widetilde{K}, \rho, R_{\lambda_0}), \mathcal{A}(\widetilde{K}, \rho, R_{\lambda_0}) \rangle_{\widetilde{K}} \subset R_{\lambda_0}$$

as we have observed above. Thus, for $f, g \in \mathcal{A}(K, \rho, R_{\lambda_0}) \subset \mathcal{A}(\widetilde{K}, \rho, R_{\lambda_0})$, we have

$$[K_v : \widetilde{K}_v] \langle f, g \rangle_K = \langle f, g \rangle_{\widetilde{K}} \in R_{\lambda_0}.$$

We then take $A_K = [K_v : \widetilde{K}_v]$. This is not divisible by ℓ. □

Note that $\ell \nmid |K_v|$ if K_v is torsion-free and $v \nmid \ell$. For large ℓ this is automatic:

Lemma 6.3. *Given an F-embedding $G \hookrightarrow \mathrm{GL}(n)$, a compact open subgroup $K = \prod_{v \notin \Sigma} K_v$, and a prime number $\ell > [F : \mathbb{Q}]n + 1$, we have $\ell \nmid |K_v|$ for infinitely many places v.*

Proof. The group K_v embeds into a conjugate of $\mathrm{GL}(n, R_v)$. Therefore $|K_v|$ divides $|\mathrm{GL}(n, R_v)| = p^\infty \prod_{i=1}^n (q^i - 1)$. Assume ℓ divides $|K_v|$ for almost all v. Then p has order at most $[F : \mathbb{Q}]n$ in $(\mathbb{Z}/\ell)^\times$ for almost all primes p. Now, $(\mathbb{Z}/\ell)^\times$ is cyclic of order $\ell - 1$, so by Dirichlet's theorem on primes in arithmetic progressions we conclude that $\ell \leq [F : \mathbb{Q}]n + 1$. □

7. Ihara's lemma.

7.1. Parahoric subgroups. From now on we assume for simplicity that G^{der} is simple (that is, it has no nontrivial connected normal subgroups). Moreover, we fix a compact open subgroup $K = \prod_{v \notin \Sigma} K_v \subset G(\mathbb{A}^\Sigma)$. Then $K_v \subset G_v$ is a hyperspecial maximal compact subgroup for almost all places v, that is, $K_v = \mathbf{G}(R_v)$ for a smooth affine group scheme $\mathbf{G}$ of finite type over R_v with generic fiber G. Such $\mathbf{G}$ exists precisely when G_v is unramified. Let us look at a fixed finite place w of F where K_w is hyperspecial. Then write $K = K_w K^w$, where

$$K^w = \prod_{v \neq w} K_v \subset G(\mathbb{A}^{\Sigma, w}).$$

Let $\mathcal{B}_w$ denote the reduced Bruhat-Tits building of G_w (that is, the building of G_w^{ad}). We have assumed G^{der} is simple, so $\mathcal{B}_w$ is a simplicial complex. Let $x \in \mathcal{B}_w$ be the vertex fixed by K_w. Let (x, x') be an edge in the building. Consider the maximal compact subgroup $K'_w \subset G_w$ fixing the vertex x', and the parahoric subgroup $J_w = K_w \cap K'_w$ associated with the edge (x, x'). Let $K' = K'_w K^w$ and $J = J_w K^w$ be the corresponding subgroups of $G(\mathbb{A}^\Sigma)$.

Lemma 7.1. $\langle K_w, K'_w \rangle = G_w^0 := \{ g \in G_w : |\chi(g)| = 1, \forall \chi \in X^*(G)_{F_w} \}$.

Proof. This follows from Bruhat-Tits theory. □

Note that $G_w^{\mathrm{der}} \subset G_w^0 \subset G_w^1 = G_w \cap G(\mathbb{A})^1$.

7.2. The concrete setup. We apply the general results when L is L_{λ_0} and $\mathcal{O}$ is R_{λ_0}. Let $H = Z(\mathcal{H}_{J,L_{\lambda_0}})$. This is a commutative L_{λ_0}-algebra. It comes with the involution defined by $\phi^\vee(x) = \phi(x^{-1})$. The L_{λ_0}-space $V = \mathcal{A}(J,\rho,L_{\lambda_0})$ is finite-dimensional; $Z(\mathcal{H}_{J,L_{\lambda_0}})$ acts on it. The order $Z(\mathcal{H}_{J,R_{\lambda_0}})$ preserves the lattice $V_{R_{\lambda_0}} = \mathcal{A}(J,\rho,R_{\lambda_0})$. The space V comes with the bilinear form $\langle -,-\rangle_J$. The compatibility conditions between these data are satisfied. Let $U = \mathcal{A}(K,\rho,L_{\lambda_0}) \oplus \mathcal{A}(K',\rho,L_{\lambda_0})$. Then $Z(\mathcal{H}_{J,L_{\lambda_0}})$ acts on this space via the natural maps to $Z(\mathcal{H}_{K,L_{\lambda_0}})$ and $Z(\mathcal{H}_{K',L_{\lambda_0}})$. The lattice $U_{R_{\lambda_0}} = \mathcal{A}(K,\rho,R_{\lambda_0}) \oplus \mathcal{A}(K',\rho,R_{\lambda_0})$ is preserved by $Z(\mathcal{H}_{J,R_{\lambda_0}})$. The bilinear form on U is given by the sum $\langle -,-\rangle_K \oplus \langle -,-\rangle_{K'}$. The degeneracy map δ is given by

$$\delta : \mathcal{A}(K,\rho,L_{\lambda_0}) \oplus \mathcal{A}(K',\rho,L_{\lambda_0}) \stackrel{\text{sum}}{\to} \mathcal{A}(J,\rho,L_{\lambda_0}),$$

which is clearly $Z(\mathcal{H}_{J,L_{\lambda_0}})$-linear. Obviously, $\ker\delta$ consists of all pairs $(f,-f)$, where

$$f \in \mathcal{A}(K,\rho,L_{\lambda_0}) \cap \mathcal{A}(K',\rho,L_{\lambda_0}) = \{G_w^0 K^w\text{-invariant functions } f \in \mathcal{A}\}.$$

The decompositions $U = \ker\delta \oplus (\ker\delta)^\perp$ and $V = \operatorname{im}\delta \oplus (\operatorname{im}\delta)^\perp$ are immediate because of the relation between the pairings and the inner product.

7.3. Combinatorial Ihara lemma. The proof of the following lemma is a straightforward generalization of [T1, p. 274]. It asserts that the quotient $V_{R_{\lambda_0}} \cap \delta(U) \supset \delta(U_{R_{\lambda_0}})$ is killed by $C = 1$.

LEMMA 7.2. *We have that* $\mathcal{A}(J,\rho,R_{\lambda_0}) \cap \delta[\mathcal{A}(K,\rho,L_{\lambda_0}) \oplus \mathcal{A}(K',\rho,L_{\lambda_0})]$ *is equal to* $\delta[\mathcal{A}(K,\rho,R_{\lambda_0}) \oplus \mathcal{A}(K',\rho,R_{\lambda_0})]$.

Proof. Let us first set up some machinery for the proof. There are natural projections $\pi : X_J = G(F)\backslash G(\mathbb{A}^\infty)/J'' \to X_K$ and $\pi : X_J \to X_{K'}$. We define an equivalence relation on X_J by saying that $x, y \in X_J$ are equivalent ($x \sim y$) iff there exists a chain $x = x_0, \dots, x_d = y$ such that $\forall i$: $\pi(x_i) = \pi(x_{i+1})$ or $\pi'(x_i) = \pi'(x_{i+1})$. This gives a partition of X_J into equivalence classes X_J^j. For each j, we fix a representative $y^j \in X_J^j$. Correspondingly, we have a radius function $d : X_J \to \mathbb{Z}_{\geq 0}$ defined as follows. Given $x \in X_J$, there is a unique j such that $x \sim y^j$. Then $d(x)$ is the minimal length of a chain connecting x to y^j. Now, suppose $g = \delta(f,f') \in \mathcal{A}(J,\rho,R_{\lambda_0})$ for some $f \in \mathcal{A}(K,\rho,L_{\lambda_0})$ and $f' \in \mathcal{A}(K',\rho,L_{\lambda_0})$. We want to show $g \in \delta(\mathcal{A}(K,\rho,R_{\lambda_0}) \oplus \mathcal{A}(K',\rho,R_{\lambda_0}))$.

We claim that we may assume that $f(\pi(y^j)) = 0$ for all j. To see this, note that $X_K = \sqcup\pi(X_J^j)$ and $X_{K'} = \sqcup\pi'(X_J^j)$. We then define $\widetilde{f} \in \mathcal{A}(K,\rho,L_{\lambda_0})$ such that $\widetilde{f}|\pi(X_J^j) \equiv f(\pi(y^j))$, and $\widetilde{f}' \in \mathcal{A}(K',\rho,L_{\lambda_0})$ such that $\widetilde{f}'|\pi'(X_J^j) \equiv f(\pi(y^j))$. Then

$$g = \delta(f - \widetilde{f}, f' + \widetilde{f}')$$

and $(f - \widetilde{f})(\pi(y^j)) = 0$ for all j. This proves the claim.

From now on assume that $f(\pi(y^j)) = 0$ for all j. We claim, for every $m \geq 0$, that for every $x \in X_J$ with $d(x) = m$ we have that $f(\pi(x)) \in R_{\lambda_0}$ and $f'(\pi'(x)) \in R_{\lambda_0}$. We prove this by induction on $m \geq 0$. The case $m = 0$ is essentially just our assumption. Assume the statement is true for $m - 1 \geq 0$. Consider $x \in X_J$ with $d(x) = m$. Let $x = x_0, x_1, \dots, x_m = y^j$ be a chain of minimal length. Then $x' = x_1 \in X_J$ has $d(x') = m - 1$, so by induction $f(\pi(x')) \in R_{\lambda_0}$ and $f'(\pi'(x')) \in R_{\lambda_0}$. However, $\pi(x) = \pi(x')$ or $\pi'(x) = \pi'(x')$. In either case we get the statement for x. This proves the lemma, for then $f \in \mathcal{A}(K,\rho,R_{\lambda_0})$ and $f' \in \mathcal{A}(K',\rho,R_{\lambda_0})$. Note that $f(\pi(x)) \in R_{\lambda_0}$ if and only if $f'(\pi'(x)) \in R_{\lambda_0}$. $\square$

8. Applying the abstract theory.

8.1. Computing $\delta^\vee\delta$. To apply the abstract theory it is necessary to compute $\delta^\vee\delta$ explicitly.

Lemma 8.1. *The endomorphism $\delta^\vee\delta$ is given by the 2×2 matrix*

$$\delta^\vee\delta = \begin{pmatrix} [K:J] & [K:J]e_K \\ [K':J]e_{K'} & [K':J] \end{pmatrix}.$$

Proof. Put $k=[K:J]$ and $k'=[K':J]$. Recall that $\delta\begin{pmatrix}\phi_1^K\\ \phi_2^{K'}\end{pmatrix} = \phi_1^K+\phi_2^{K'} \in \mathcal{A}(K',\rho,L_{\lambda_0})$. Also, $\delta^\vee\phi^J = \begin{pmatrix} ke_K\phi^J + ke_Ke_{K'}\phi^J \\ k'e_{K'}e_K\phi^J + k'e_{K'}\phi^J\end{pmatrix}$.

Let us write the endomorphism $\delta^\vee\delta$ of $\begin{pmatrix}\mathcal{A}(K,\rho,L_{\lambda_0})\\ \mathcal{A}(K',\rho,L_{\lambda_0})\end{pmatrix}$ as $\delta^\vee\delta = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$, where $b:\mathcal{A}(K',\rho,L_{\lambda_0})\to\mathcal{A}(K,\rho,L_{\lambda_0})$ and so on. Then

$$\langle f_1+f_2, g_1+g_2\rangle_J = \left\langle \begin{pmatrix} f_1\\ f_2\end{pmatrix}, \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} g_1\\ g_2\end{pmatrix}\right\rangle_J$$

$$= \left\langle \begin{pmatrix} f_1\\ f_2\end{pmatrix}, \begin{pmatrix} ag_1+bg_2\\ cg_1+dg_2\rangle_J\end{pmatrix}\right\rangle_J = \langle f_1, ag_1+bg_2\rangle_K + \langle f_2, cg_1+dg_2\rangle_{K'}.$$

Taking $g_2=0=f_2$ we get $ag_1=kg_1$; with $g_1=0=f_2$, we get $bg_2=ke_Kg_2$; with $f_1=0=g_2$ we get $cg_1=k'e_{K'}g_1$; with $f_1=0=g_1$ we get $dg_2=k'g_2$. ∎

8.2. The main lemma. In our situation, Corollary 2.1 gives the following crucial lemma.

Lemma 8.2. *Let $f\in\mathcal{A}(K,\rho,R_{\lambda_0})$ be an eigenform for $Z(\mathcal{H}_{K,R_{\lambda_0}})$ with character $\eta_f:\mathbb{T}_{K,R_{\lambda_0}}\to R_{\lambda_0}$. Let ℓ be the residual characteristic of λ_0. Suppose there exist at least two places v such that $\ell\nmid |K_v|$. Assume that there exists a place w of F such that f modulo λ_0 is not G_w^0-invariant. Consider $e_{K,K'}=[K:J][K':J]_K(e_K*e_{K'}*e_K)\in Z(\mathcal{H}_{K,\mathbb{Z}})$ where $[K':J]_K = k'/(k',k)$ in the notation of the proof of Lemma 8.1. Suppose there is n with*

$$0<n\le v_{\lambda_0}(\eta_f(e_{K,K'})-[K:J][K':J]_K) - v_{\lambda_0}([K':J]_K).$$

Then the reduction of $\eta_f\circ *e_K$ modulo λ_0^n factors through $\mathbb{T}^{\mathrm{new}}_{J,R_{\lambda_0}}$.

Proof. (1) First we produce an eigenvector for $Z(\mathcal{H}_{J,R_{\lambda_0}})$ in $U_{R_{\lambda_0}}=\mathcal{A}(K,\rho,R_{\lambda_0})\oplus\mathcal{A}(K',\rho,R_{\lambda_0})$. For that we take

$$\vec{f} = [K':J]_K(f,-r(e_{K'})f)\in\mathcal{A}(K,\rho,R_{\lambda_0})\oplus\mathcal{A}(K',\rho,R_{\lambda_0}).$$

The factor $[K':J]_K$ is included since $r(e_{K'})f$ does not necessarily take values in R_{λ_0}: note that $e_{K'}=\chi_{K'}/\mu(K')=k\chi_{K'}/k'\mu(K)$, thus $k'e_{K'}/(k',k)\in\mathcal{H}_{K,\mathbb{Z}}$. Clearly, $\vec{f}$ is an eigenvector for $Z(\mathcal{H}_{J,R_{\lambda_0}})$. Its character is the composite

$$\eta_{\vec{f}}: Z(\mathcal{H}_{J,R_{\lambda_0}}) \xrightarrow{*e_K} Z(\mathcal{H}_{K,R_{\lambda_0}}) \xrightarrow{\eta_f} R_{\lambda_0}.$$

Indeed, for $h \in Z(\mathcal{H}_{J,R_{\lambda_0}})$, $f \in \mathcal{A}(K,\rho,R_{\lambda_0})$, we have $h(f, -e_{K'} \cdot f) = (he_K f, -e_{K'} he_K f) = \eta_f(he_K)(f, -e_{K'} f)$.
(2) Using the explicit formula for $\delta^\vee \delta$ in lemma 8.1 above, it follows that

$$\delta^\vee \delta(\vec{f}) = m(-f, 0), \qquad m = \eta_f(e_{K,K'}) - [K:J][K':J]_K.$$

Note that $(-f, 0) \in U_{R_{\lambda_0}}$. We claim that Corollary 2.1 applies with this $m \in R_{\lambda_0}$. Indeed, $m \neq 0$. If $m = 0$ then $\vec{f}$ must belong to the kernel of δ. Then f must be invariant under the group G_w^0 (say, on the right), contradicting our assumption.
(3) Define

$$\mathcal{F} = \{x \in L : xf \in \mathcal{A}(K,\rho,R_{\lambda_0}) + \mathcal{A}(K,\rho,L_{\lambda_0}) \cap \mathcal{A}(K',\rho,L_{\lambda_0})\}.$$

This is an R_{λ_0}-submodule of L containing R_{λ_0}. We have $\mathcal{F} = L$ if $f \in \mathcal{A}(K,\rho,L_{\lambda_0}) \cap \mathcal{A}(K',\rho,L_{\lambda_0})$. However, f is not G_w^0-invariant, hence $f \notin \mathcal{A}(K,\rho,L_{\lambda_0}) \cap \mathcal{A}(K',\rho,L_{\lambda_0})$.

We claim that $\mathcal{F}$ is a fractional ideal. To see this note that if $x \in \mathcal{F}$, $xf \in \mathcal{A}(K,\rho,R_{\lambda_0}) + \mathcal{A}(K,\rho,L_{\lambda_0}) \cap \mathcal{A}(K',\rho,L_{\lambda_0})$. Then $A_K\langle xf, g\rangle_K \in R_{\lambda_0}$ by Lemma 6.2. Hence $A_K\langle f, g\rangle_K \mathcal{F} \subset R_{\lambda_0}$ for every $g \in \mathcal{A}(K,\rho,R_{\lambda_0}) \cap (\mathcal{A}(K,\rho,L_{\lambda_0}) \cap \mathcal{A}(K',\rho,L_{\lambda_0}))^\perp$. These g span $(\mathcal{A}(K,\rho,L_{\lambda_0}) \cap \mathcal{A}(K',\rho,L_{\lambda_0}))^\perp$ so f must belong to $\mathcal{A}(K,\rho,L_{\lambda_0}) \cap \mathcal{A}(K',\rho,L_{\lambda_0})$ if $\langle f, g\rangle_K = 0$ for all such g. Thus $\langle f, g\rangle_K$ is not identically zero, and $\mathcal{F}$ is a fractional ideal.
(4) Now, the nonzero ideal $\widetilde{\mathcal{E}} = \mathcal{F}^{-1}$ satisfies:

$$\widetilde{\mathcal{E}}(Lf \cap [\mathcal{A}(K,\rho,R_{\lambda_0}) + \mathcal{A}(K,\rho,L_{\lambda_0}) \cap \mathcal{A}(K',\rho,L_{\lambda_0})]) \subset R_{\lambda_0} f.$$

Therefore, $\mathcal{E} = [K':J]_K \widetilde{\mathcal{E}}$ satisfies the primitivity condition in corollary 2.1 (recall that $\vec{f} = [K':J]_K(f, -r(e_{K'})f)$):

$$\mathcal{E}(L\vec{f} \cap (\mathcal{A}(K,\rho,R_{\lambda_0}) \oplus \mathcal{A}(K',\rho,R_{\lambda_0}) + \ker \delta)) \subset R_{\lambda_0} \vec{f}.$$

Suppose that $v_{\lambda_0}(\widetilde{\mathcal{E}}) \neq 0$. Then $\mathcal{F}^{-1} \subset \lambda_0$, thus $\lambda_0^{-1} \subset \mathcal{F}$. It follows that $f \in \lambda_0(\mathcal{A}(K,\rho,R_{\lambda_0}) + \mathcal{A}(K,\rho,L_{\lambda_0}) \cap \mathcal{A}(K',\rho,L_{\lambda_0}))$. Hence $f = \lambda_0(g+h)$, thus $f - \lambda_0 g = \lambda_0 h \in \mathcal{A}(K,\rho,L_{\lambda_0}) \cap \mathcal{A}(K',\rho,L_{\lambda_0})$ is G_w^0-invariant. Hence the reduction $\bar{f} \in \mathcal{A}(K,\rho,\mathbb{F}_{\lambda_0})$ is G_w^0-invariant. Hence $v_{\lambda_0}(\widetilde{\mathcal{E}}) = 0$.

Since $\ell \nmid |K_v|$ holds for at least one $v \neq w$, by assumption, we can find A_K and $A_{K'}$ indivisible by ℓ according to Lemma 6.2. Also we can take $C = 1$ by Lemma 7.2. So if $0 < n \leq v_{\lambda_0}(m) - v_{\lambda_0}([K':J]_K) \leq v_{\lambda_0}(mE^{-1}\mathcal{E}^{-1})$, by Lemma 2.1, $\eta_{\vec{f}} \bmod \lambda_0^n$ factors through $\mathbb{T}^{\text{new}}_{J,R_{\lambda_0}} \to R_{\lambda_0}/\lambda_0^n$. □

9. Semisimplicity.

9.1. Semisimplicity in characteristic zero. Let π be an automorphic representation of $G(\mathbb{A})$ with $\pi_\infty = \rho_\infty$, $\pi_\Sigma \supset \rho_\Sigma$ and nonzero space π^K of K ($\subset G(\mathbb{A}^\Sigma)$)-fixed vectors. It is known that each π^K is a simple module over $\mathcal{H}_K$. Hence $\mathcal{A}(K,\rho,\mathbb{C})$ is semisimple. Moreover, by Schur's lemma, the center $Z(\mathcal{H}_K)$ acts on π^K by a $\mathbb{C}$-algebra homomorphism $\eta_{\pi^K} : Z(\mathcal{H}_K) \to \mathbb{C}$. For a character $\eta : Z(\mathcal{H}_K) \to \mathbb{C}$, we denote by $\mathcal{A}(K,\rho,\mathbb{C})(\eta)$ the η-isotypic component. That is, the eigenspace $\mathcal{A}(K,\rho,\mathbb{C})(\eta) = \{f \in \mathcal{A}(K,\rho,\mathbb{C}); r(\phi)f = \eta(\phi)f, \forall \phi \in Z(\mathcal{H}_K)\}$. Then there is a direct sum decomposition $\mathcal{A}(K,\rho,\mathbb{C}) = \bigoplus_\eta \mathcal{A}(K,\rho,\mathbb{C})(\eta)$. Clearly, $\mathcal{A}(K,\rho,\mathbb{C})(\eta) \neq 0$ if and only if $\eta = \eta_{\pi^K}$ for some π. The image $\mathbb{T}_K \subset \operatorname{End}_{\mathbb{C}} \mathcal{A}(K,\rho,\mathbb{C})$ of the center $Z(\mathcal{H}_K)$ is a commutative semisimple $\mathbb{C}$-algebra, that is, a direct product of copies of $\mathbb{C}$.

LEMMA 9.1. *The eigenspace $\mathcal{A}(K,\rho,\mathbb{C})(\eta)$ is nonzero if and only if η factors through $\mathbb{T}_K$.*

Proof. Obviously, η factors if $\mathcal{A}(K,\rho,\mathbb{C})(\eta) \neq 0$. Conversely, suppose η factors and look at its kernel $\mathfrak{m} = \ker(\eta) \subset \mathbb{T}_K$. This is a maximal ideal since $\mathrm{im}(\eta) = \mathbb{C}$ is a field. Since $\mathbb{T}_K$ acts faithfully on $\mathcal{A}(K,\rho,\mathbb{C})$, which is finite-dimensional, $\mathfrak{m}$ belongs to the support of $\mathcal{A}(K,\rho,\mathbb{C})$, namely the localization $\mathcal{A}(K,\rho,\mathbb{C})_{\mathfrak{m}}$ is nonzero. By the theory of associated primes, $\mathfrak{m}$ contains a prime ideal of the form $\mathrm{Ann}_{\mathbb{T}_K}(f)$ with $f \in \mathcal{A}(K,\rho,\mathbb{C})$ (Dummit and Foote, 3rd Ed., Sect. 15.4, Ex. 40, p. 730). All primes are maximal in $\mathbb{T}_K$, so in fact $\mathfrak{m} = \mathrm{Ann}_{\mathbb{T}_K}(f)$. Clearly $\mathfrak{m}$ contains $T - \eta(T)$ for every $T \in \mathbb{T}_K$, so $f \in \mathcal{A}(K,\rho,\mathbb{C})(\eta)$, and f must be nonzero as $\mathfrak{m} \neq \mathbb{T}_K$. ◻

Now, consider the $\mathcal{H}_{K,\mathbb{Q}}$-module $\mathcal{A}(K,\rho,L)$, and the image $\mathbb{T}_{K,\mathbb{Q}}$ of the center $Z(\mathcal{H}_{K,\mathbb{Q}})$ in the endomorphism algebra $\mathrm{End}_L\,\mathcal{A}(K,\rho,L)$. The algebra $\mathbb{T}_{K,\mathbb{Q}}$ can be viewed as a subring of $\mathbb{T}_K \simeq \mathbb{C} \otimes_{\mathbb{Q}} \mathbb{T}_{K,\mathbb{Q}}$. We deduce that $\mathbb{T}_{K,\mathbb{Q}}$ is a reduced commutative finite-dimensional $\mathbb{Q}$-algebra, that is, a product of number fields by Nakayama's lemma:

$$\mathbb{T}_{K,\mathbb{Q}} \simeq L_1 \times \cdots \times L_t.$$

Visibly, $\mathbb{T}_{K,\mathbb{Q}}$ is a semisimple $\mathbb{Q}$-algebra. (The L_i occurring in $\mathbb{T}_{K,\mathbb{Q}}$ are totally real or CM.)

9.2. Semisimplicity in positive characteristic. Now let R be a field of characteristic $p > 0$. We are interested in when $\mathcal{A}(K,\rho,R)$ is a semisimple module over $Z(\mathcal{H}_{K,R})$. As we have just seen, this means that $\mathbb{T}_{K,R}$ is a semisimple R-algebra. We have $\mathbb{T}_{K,R} \simeq R \otimes_{\mathbb{F}_p} \mathbb{T}_{K,\mathbb{F}_p}$, so equivalently, when is $\mathbb{T}_{K,\mathbb{F}_p}$ semisimple?

There is always a surjective homomorphism $\xi : \mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{T}_{K,\mathbb{Z}} \twoheadrightarrow \mathbb{T}_{K,\mathbb{F}_p}$. Indeed the image of $\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{T}_{K,\mathbb{Z}}$ in $\mathrm{End}_{\mathbb{F}_p}\,\mathcal{A}(K,\rho,\mathbb{F}_p)$ equals the image of $\mathbb{F}_p \otimes_{\mathbb{Z}} Z(\mathcal{H}_{K,\mathbb{Z}})$, and the natural map from $\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{T}_{K,\mathbb{Z}}$ to $Z(\mathcal{H}_{K,\mathbb{F}_p})$ is onto.

Put $\widetilde{\mathbb{T}}_{K,\mathbb{Z}} = \{T \in \mathbb{T}_{K,\mathbb{Q}}; T(\mathcal{A}(K,\rho,\mathbb{Z})) \subset \mathcal{A}(K,\rho,\mathbb{Z})\}$. This is a free finite $\mathbb{Z}$-module containing $\mathbb{T}_{K,\mathbb{Z}}$ as a subgroup of finite index.

LEMMA 9.2. *The kernel $\ker\xi$ is nilpotent. It is trivial iff $p \nmid [\widetilde{\mathbb{T}}_{K,\mathbb{Z}} : \mathbb{T}_{K,\mathbb{Z}}]$.*

Proof. For the first assertion it is enough to show that every element in $\ker(\xi)$ is nilpotent. Under the identification $\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{T}_{K,\mathbb{Z}} \simeq \mathbb{T}_{K,\mathbb{Z}}/p\mathbb{T}_{K,\mathbb{Z}}$, the kernel $\ker(\xi)$ corresponds to the ideal $(\mathbb{T}_{K,\mathbb{Z}} \cap p\widetilde{\mathbb{T}}_{K,\mathbb{Z}})/p\mathbb{T}_{K,\mathbb{Z}}$. Let $T \in \mathbb{T}_{K,\mathbb{Z}} \cap p\widetilde{\mathbb{T}}_{K,\mathbb{Z}}$. Obviously, $\widetilde{\mathbb{T}}_{K,\mathbb{Z}}$ is integral over $\mathbb{Z}$, so there is an equation

$$(p^{-1}T)^n + a_{n-1}(p^{-1}T)^{n-1} + \cdots + a_1(p^{-1}T) + a_0 = 0$$

for certain $a_i \in \mathbb{Z}$. Multiplying by p^n we see that $T^n \in p\mathbb{T}_{K,\mathbb{Z}}$.

For the last assertion, note that $\ker\xi = 0$ if and only if $\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{T}_{K,\mathbb{Z}} \to \mathbb{F}_p \otimes_{\mathbb{Z}} \widetilde{\mathbb{T}}_{K,\mathbb{Z}}$ is injective. ◻

In particular, $\ker\xi$ is contained in the Jacobson radical. We let $\bar{\mathbb{T}}_{K,\mathbb{Z}}$ denote the integral closure of $\mathbb{Z}$ in $\mathbb{T}_{K,\mathbb{Q}}$. It contains $\widetilde{\mathbb{T}}_{K,\mathbb{Z}}$ as a subgroup of finite index.

LEMMA 9.3. *If $p \nmid \Delta_K := [\bar{\mathbb{T}}_{K,\mathbb{Z}} : \widetilde{\mathbb{T}}_{K,\mathbb{Z}}] \cdot \prod_i \Delta_{L_i/\mathbb{Q}}$ then $\mathbb{T}_{K,\mathbb{F}_p}$ is semisimple.*

Proof. Note that $\mathbb{F}_p \otimes_{\mathbb{Z}} \widetilde{\mathbb{T}}_{K,\mathbb{Z}} \simeq \mathbb{F}_p \otimes_{\mathbb{Z}} \bar{\mathbb{T}}_{K,\mathbb{Z}}$ since $p \nmid [\bar{\mathbb{T}}_{K,\mathbb{Z}} : \widetilde{\mathbb{T}}_{K,\mathbb{Z}}]$.

As p does not divide the discriminant $\Delta_{L_i/\mathbb{Q}}$ for each i, p is unramified in every L_i occurring in $\mathbb{T}_{K,\mathbb{Q}}$. Hence $\mathbb{F}_p \otimes_{\mathbb{Z}} \bar{\mathbb{T}}_{K,\mathbb{Z}} \simeq \prod_i R_{L_i}/pR_{L_i} \simeq \prod_i \prod_{\mathfrak{p}|p} R_{L_i}/\mathfrak{p}$.

There is an embedding $\mathbb{T}_{K,\mathbb{F}_p} \simeq \mathbb{T}_{K,\mathbb{Z}}/\mathbb{T}_{K,\mathbb{Z}} \cap p\widetilde{\mathbb{T}}_{K,\mathbb{Z}} \hookrightarrow \widetilde{\mathbb{T}}_{K,\mathbb{Z}}/p\widetilde{\mathbb{T}}_{K,\mathbb{Z}} \simeq \mathbb{F}_p \otimes_{\mathbb{Z}} \widetilde{\mathbb{T}}_{K,\mathbb{Z}}$. It follows that $\mathbb{T}_{K,\mathbb{F}_p}$ is semisimple. □

The converse holds at least for $p \nmid [\widetilde{\mathbb{T}}_{K,\mathbb{Z}} : \mathbb{T}_{K,\mathbb{Z}}]$ (that is, when ξ is injective).

9.3. The simple modules. Let R be a perfect field of characteristic $p \geq 0$. Up to isomorphism, a simple $Z(\mathcal{H}_{K,R})$-module is given by an extension R'/R with an action given by a surjective R-algebra homomorphism $\eta : Z(\mathcal{H}_{K,R}) \twoheadrightarrow R'$. If (η, R') is such a submodule of $\mathcal{A}(K, \rho, R)$, the extension R'/R is finite and η factors through $\mathbb{T}_{K,R}$. If $p \nmid \Delta_K$, there exists a finite extension L/R such that we have a direct sum decomposition

$$\mathcal{A}(K, \rho, L) = \bigoplus_{\eta} \mathcal{A}(K, \rho, L)(\eta).$$

This is still true when $p|\Delta_K$, provided $\mathcal{A}(K, \rho, L)(\eta)$ denotes the generalized eigenspace:

$$\{f \in \mathcal{A}(K, \rho, L);\ \forall \phi \in Z(\mathcal{H}_{K,L}),\ (r(\phi) - \eta(\phi))^n f = 0\ \exists n \geq 1\}.$$

Observe the following:

Lemma 9.4. *Let R be a field. Choose a finite extension L/R as above. Let L'/L be an arbitrary extension. Suppose $\eta' : Z(\mathcal{H}_{K,L'}) \to L'$ occurs in $\mathcal{A}(K, \rho, L')$. Then $\eta' = 1 \otimes \eta$ for some character $\eta : Z(\mathcal{H}_{K,L}) \to L$ occurring in $\mathcal{A}(K, \rho, L)$. Moreover,*

$$\mathcal{A}(K, \rho, L')(1 \otimes \eta) \simeq L' \otimes_L \mathcal{A}(K, \rho, L)(\eta),$$

so η and $\eta' = 1 \otimes \eta$ occur with the same multiplicity.

Proof. Both $\mathcal{A}(K, \rho, L)$ and $\mathcal{A}(K, \rho, L') \simeq L' \otimes_L \mathcal{A}(K, \rho, L)$ have decompositions into direct sums of generalized eigenspaces. Under this isomorphism, $L' \otimes_L \mathcal{A}(K, \rho, L)(\eta) \hookrightarrow \mathcal{A}(K, \rho, L')(1 \otimes \eta)$. Therefore, every η' occurring in $\mathcal{A}(K, \rho, L')$ must come from an η, and the above injection must be an isomorphism. □

Let us apply these results to a number field $R = L'$. We conclude that there exists a number field L/L' such that $\mathcal{A}(K, \rho, L)$ is a direct sum of eigenspaces for characters $Z(\mathcal{H}_{K,L}) \to L$. Furthermore, if $\eta : Z(\mathcal{H}_K) \to \mathbb{C}$ is a character such that $\mathcal{A}(K, \rho, \mathbb{C})(\eta) \neq 0$, then η restricts to a $\mathbb{Q}$-algebra homomorphism $Z(\mathcal{H}_{K,\mathbb{Q}}) \to L$ occurring in $\mathcal{A}(K, \rho, L)$. In addition, since $Z(\mathcal{H}_{K,\mathbb{Z}})$ preserves $\mathcal{A}(K, \rho, R_L)$, η even restricts to a ring homomorphism $Z(\mathcal{H}_{K,\mathbb{Z}}) \to R_L$ occurring in $\mathcal{A}(K, \rho, R_L)$.

10. End of proof.

10.1. Invariance modulo λ. Denote by $\mathcal{A}^0(K, \rho, \mathbb{F}_\lambda)$ the space of the nonabelian modulo λ relative to K automorphic forms in $\mathcal{A}(K, \rho, \mathbb{F}_\lambda)$.

Lemma 10.1. *Choose a number field $L/\mathbb{Q}$ such that $\mathcal{A}(K, \rho, L)$ is a direct sum of eigenspaces. Put $R = R_L$. Let π be an automorphic representation of $G(\mathbb{A})$ such that $\pi^K \neq 0$, $\pi_\Sigma \supset \rho_\Sigma$, and $\pi_\infty = \rho_\infty$. Denote by $\eta = \eta_{\pi^K} : Z(\mathcal{H}_{K,\mathbb{Z}}) \to R$ the character giving the action on π^K. Let w be a place such that K_w is hyperspecial, thus G_w is unramified. Suppose π is* non *abelian modulo λ relative to K, and $\overline{\eta} : Z(\mathcal{H}_{K,\mathbb{Z}}) \to \mathbb{F}_{\lambda_0} = R_{\lambda_0}/\lambda_0$ denotes the reduction of η. Then the eigenspace $\mathcal{A}^0(K, \rho, \mathbb{F}_\lambda)(\bar{\eta})$ contains no nonzero G_w^{der}-invariant functions.*

Proof. As observed above, η occurs in $\mathcal{A}(K, \rho, R_{\lambda_0})$, that is, there exists an eigenform $0 \neq f \in \mathcal{A}(K, \rho, R_{\lambda_0})$ with $\eta_f = \eta$. Let $\bar{f} = 1 \otimes f \in \mathcal{A}(K, \rho, \mathbb{F})$ be the reduction of f modulo λ, where $\mathbb{F} = R_\lambda/\lambda$ is a finite extension of $\mathbb{F}_\ell$. By scaling f, we can assume that $\bar{f} \neq 0$. Let us assume $\bar{f}$ is G_w^{der}-invariant. Now, G^{der} is simple, simply connected and G_w^{der} is noncompact. By the strong approximation theorem, $\bar{f}$ is in fact $G^{\mathrm{der}}(\mathbb{A}^\infty)$-invariant. In particular, $\dim \rho_\Sigma = 1$. As $H^1(F_v, G^{\mathrm{der}}) = 0$ for each finite place v, there is a short exact sequence

$$1 \to G^{\mathrm{der}}(\mathbb{A}^\infty) \to G(\mathbb{A}^\infty) \xrightarrow{\nu} G^{\mathrm{ab}}(\mathbb{A}^\infty) \to 1.$$

It follows that $\bar{f}$ lives on $G^{\mathrm{ab}}(\mathbb{A}^\infty)$. More precisely, there exists a unique function $\widetilde{f} : G^{\mathrm{ab}}(\mathbb{A}^\infty) \to \mathbb{F}$ such that $\bar{f} = \widetilde{f} \circ \nu$. It fits into the diagram

$$\begin{tikzcd} X_K = G(F)\backslash G(\mathbb{A}^\infty)/K'' \arrow[r, "\bar{f}"] \arrow[d, "\nu"'] & \mathbb{F} \\ Y_K = \nu(G(F))\backslash G^{\mathrm{ab}}(\mathbb{A}^\infty)/\nu(K'') \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

If R is a ring we denote by $\mathcal{A}(K,\rho,R)^{\mathrm{ab}}$ the module of R-valued functions on Y_K. Pulling back via ν, identifies $\mathcal{A}(K,\rho,R)^{\mathrm{ab}}$ with an $\mathcal{H}_{K,R}$-submodule of $\mathcal{A}(K,\rho,R)$. Then $0 \neq \widetilde{f} \in \mathcal{A}(K,\rho,\mathbb{F})^{\mathrm{ab},0}(\bar{\eta})$. By Lemme 6.11 of [DS, p. 522] we can lift $\bar{\eta}$ to characteristic zero: there exists an eigenform $0 \neq f' \in \mathcal{A}(K,\rho,L_\lambda)^{\mathrm{ab}}$ such that its character $\eta' : Z(\mathcal{H}_{K,\mathbb{Z}}) \to R_\lambda$ reduces to $\bar{\eta}$ modulo λ. From the results of the previous section we see that in fact η' maps into R, and it occurs in $\mathcal{A}(K,\rho,L)^{\mathrm{ab}}$ (and therefore in $\mathcal{A}(K,\rho,L_{\lambda_0})^{\mathrm{ab}}$). However, $\mathcal{A}(K,\rho,L_{\lambda_0})^{\mathrm{ab}}$ is just the space of L_{λ_0}-valued functions on the finite abelian group Y_K, so the characters form a basis. We conclude that there exists a character χ such that $\eta(\phi) \equiv \eta_\chi(\phi) \pmod{\lambda}$ for all $\phi \in Z(\mathcal{H}_{K,\mathbb{Z}})$. This contradicts the assumption that π is nonabelian mod λ relative to K. Hence $\mathcal{A}^0(K,\rho,\mathbb{F}_\lambda)(\bar{\eta})$ contains no nonzero G_w^{der}-invariant functions. ☐

10.2. Proof of Theorem 0.3. Note that $\pi \subset \mathcal{A}(K,\rho,L)$ for some number field L. The reduction $\bar{\eta}_{\pi^K}$ modulo $\lambda \cap R_L$ factors through $\mathbb{T}^{\mathrm{new}}_{J,R_{\lambda_0}}$, where R_{λ_0} is the completion of R_L at λ_0, by the main lemma (Lemma 8.2). That is, there exists a character $\eta' : Z(\mathcal{H}_{J,R_{\lambda_0}}) \to \mathbb{F}_\lambda$ factoring through $\mathbb{T}^{\mathrm{new}}_{J,R_{\lambda_0}}$ such that $\eta'(\phi) = \eta_{\pi^K}(e_K * \phi) \pmod{\lambda}$ for all $\phi \in Z(\mathcal{H}_{J,R_{\lambda_0}})$. As above, there is a surjective homomorphism with nilpotent kernel

$$\mathbb{F}_{\lambda_0} \otimes_{R_{\lambda_0}} \mathbb{T}^{\mathrm{new}}_{J,R_{\lambda_0}} \twoheadrightarrow \mathbb{T}^{\mathrm{new}}_{J,\mathbb{F}_{\lambda_0}}.$$

Thus η' gives rise to a character $\mathbb{T}^{\mathrm{new}}_{J,\mathbb{F}_{\lambda_0}} \to \mathbb{F}_{\lambda_0}$, also denoted by η'. By a standard argument (used above in section 8.2), there is an eigenform $f' \in \mathcal{A}(J,\rho,\mathbb{F}_{\lambda_0})^{\mathrm{new}}$ with character η'. Now we apply the Deligne-Serre lifting lemma, [DS, p. 522], to the finite free module $\mathcal{A}(J,\rho,R_{\lambda_0})^{\mathrm{new}}$. It gives the existence of a character $\widetilde{\eta} : \mathbb{T}^{\mathrm{new}}_{J,R_{\lambda_0}} \to \widetilde{R}_{\lambda_0}$ occurring in $\mathcal{A}(J,\rho,\widetilde{R}_{\lambda_0})^{\mathrm{new}}$ and reducing to η', where $\widetilde{R}_{\lambda_0}$ is the ring of integers in a finite extension of L_{λ_0}. Since $\mathbb{T}^{\mathrm{new}}_{J,R_{\lambda_0}}$ preserves the lattice $\mathcal{A}(J,\rho,R_{\lambda_0})^{\mathrm{new}}$, the values $\widetilde{\eta}(\phi)$ all lie in the ring of integers of some number field, $R_{\widetilde{L}}$. We deduce that there exists a character $\widetilde{\eta} : Z(\mathcal{H}_{J,R_L}) \to R_{\widetilde{L}}$, occurring in $\mathcal{A}(J,\rho,\widetilde{L})^{\mathrm{new}}$, such that

$$\widetilde{\eta}(\phi) \equiv \eta_{\pi^K}(e_K * \phi) (\mathrm{mod}\, \lambda)$$

for all $\phi \in Z(\mathcal{H}_{J,R})$. From the decomposition of $\mathcal{A}(J,\rho,L_{\lambda_0})$ in terms of automorphic representations, it follows that the new space $\mathcal{A}(J,\rho,L_{\lambda_0})^{\mathrm{new}}$ has the following description:

$$\mathcal{A}(J,\rho,L_{\lambda_0})^{\mathrm{new}} \simeq \bigoplus_{\{\pi\in\Pi_{\mathrm{unit}}(G(\mathbb{A}));\pi_\infty=\rho_\infty,\,\pi_\Sigma\supset\rho_\Sigma\}} m(\pi)\cdot\pi^J/(\pi^K+\pi^{K'}),$$

as $Z(\mathcal{H}_J)$-modules. The center $Z(\mathcal{H}_J)$ acts on the quotient $\pi^J/(\pi^K+\pi^{K'})$ by the character η_{π^J}. We conclude that there exists an automorphic representation $\widetilde{\pi}$ of $G(\mathbb{A})$ with $\widetilde{\pi}_\infty=\rho_\infty$, $\widetilde{\pi}_\Sigma\supset\rho_\Sigma$ and $\widetilde{\pi}^J\neq\widetilde{\pi}^K+\widetilde{\pi}^{K'}$, such that $\eta_{\widetilde{\pi}^J}=\widetilde{\eta}$. In particular,

$$\eta_{\widetilde{\pi}^J}(\phi)\equiv\eta_{\pi^K}(e_K*\phi)(\mathrm{mod}\,\lambda)$$

for all $\phi\in Z(\mathcal{H}_{J,R_L})$. This finishes the proof. □

11. Applications in rank one.

11.1. U(3) **- the nonsplit case.** Let F be a local nonarchimedean field. Suppose that the F-rank of G^{der} is one. In this rank one situation the parahoric $J=K\cap K'$ is an Iwahori subgroup, denoted I.

Lemma 11.1. *If* $\pi^I\neq\pi^K+\pi^{K'}$ *then* $\pi^K=\{0\}=\pi^{K'}$.

Proof. Suppose $\pi^I\neq\{0\}$. Then π is a constituent of a fully induced representation $\mathrm{Ind}(\chi)$, χ being an unramified character of the maximal torus A in the Borel subgroup B of G, by [Bo] or [B]. There are two cases.

If $\pi=\mathrm{Ind}(\chi)$ then $\dim_{\mathbb{C}}\pi^K=1=\dim_{\mathbb{C}}\pi^{K'}$. Indeed, the building of G is a tree and all vertices are special. Thus the maximal compact subgroup K' is special, so we have the Iwasawa decomposition $G=BK'$, and $B\cap K'=A(R)$ is the maximal compact subgroup in the maximal torus A in B. Then $f(g)=\chi(b)$, $g=bk'$, $b\in B$, $k'\in K'$, is well-defined, nonzero, fixed by K'.

Now $\dim_{\mathbb{C}}\pi^I=[W]$, and the number of elements $[W]$ in the Weyl group W of A in G is $2^{\mathrm{rk}(G^{\mathrm{der}})}$, namely 2. Our assumption is that $\pi^K+\pi^{K'}$ is not π^I, thus $\dim_{\mathbb{C}}(\pi^K+\pi^{K'})$ is 1. Hence $\pi^K=\pi^{K'}$ is a one-dimensional space fixed by K and K', hence by G^0 by Lemma 7.1, so that π is a character, contradicting our assumption that $\pi=\mathrm{Ind}(\chi)$.

The second case is where π is strictly contained in $\mathrm{Ind}(\chi)$. By [Bo] or [B], each constituent of $\mathrm{Ind}(\chi)$ has an Iwahori invariant vector. Hence $\dim_{\mathbb{C}}\pi^I=1$. But $\pi^K+\pi^{K'}$ is strictly contained in π^I. Hence $\pi^K=\pi^{K'}=\{0\}$. □

11.2. Proof of Theorem 0.4. This follows at once from Theorem 0.3, using Lemma 11.1.

In the case of $G=\mathrm{U}(3)$ where w stays prime in E, let $\widetilde{\pi}$ be the automorphic representation we get from Theorem 0.3. By Lemma 11.1 and [Bo] or [B], $\widetilde{\pi}_w$ is a ramified constituent of a reducible unramified induced representation. The constituents of the reducible unramified induced representations are the nontempered one-dimensional and $\pi^\times$, which are unramified, and the square integrable Steinberg and π^+. See 11.3 below. But $\pi^{+,K'}\neq 0$, hence $\widetilde{\pi}_w$ is Steinberg.

Finally, $[K_w:I_w]=q_w^3+1$, since K_w is the fixer of a hyperspecial vertex v in the Bruhat-Tits building, which has q_w^3+1 neighbors, and I_w is the fixer of an edge vv'. Thus $[K_w:I_w]$ counts the number of edges initiating from the vertex v. Similarly $[K'_w:I_w]=q_w+1$ as K'_w is the fixer of the special nonhyperspecial vertex v', which has q_w+1 neighbors. As q_w+1 divides q_w^3+1, $[K'_w:I_w]_{K_w}=1$. □

11.3. Reducibility of unramified representations. Let G be an unramified (split, or quasisplit and split over an unramified extension E) reductive group over a p-adic field F. An irreducible representation of G has a nonzero vector fixed by an Iwahori subgroup iff it is a constituent of a representation induced from an unramified character of a minimal parabolic subgroup ([Bo] or [B]). This induced representation is parametrized by the conjugacy class of a semisimple element s in the connected dual group $\widehat{G}$ if G is split, and in $\widehat{G}\sigma$ if G is quasisplit and splits over an unramified extension E/F, which we take to be minimal, and denote by σ a generator of the cyclic group $\mathrm{Gal}(E/F)$. Reducibility occurs precisely when there is a unipotent $u \neq 1$ in $\widehat{G}$ with $sus^{-1} = u^q$, where q is the residual cardinality of F (see e.g. [L]).

In the quasisplit case, if $s = s'\sigma$, the relation becomes $s'\sigma(u)s'^{-1} = u^q$. If $G = \mathrm{U}(3, E/F)$, E/F unramified quadratic extension, thus the residual cardinality of E is q^2, the representation $I(\eta)$ induced from the unramified character $\eta : t^n \mapsto a^n$, $t = \mathrm{diag}(\boldsymbol{\pi}^{-1}, 1, \boldsymbol{\pi})$ ($\boldsymbol{\pi}$ is a uniformizer in $F^\times$) is parametrized by $s = s'\sigma$ with $s' = \mathrm{diag}(a, 1, 1)$ (which is in $\mathrm{SL}(3, \mathbb{C})$ up to a scalar multiple; our representation has trivial central character so it can be viewed as one on the adjoint form of the group). Writing $u = [x, y, z]$ for the upper triangular unipotent matrix with top row $(1, x, y)$ and middle row $(0, 1, z)$, we check that $\sigma(u) = [z, xz - y, x]$, $s'\sigma(u)s'^{-1} = [az, axz - ay, x]$, $u^q = [qx, qy + q(q-1)xz/2, qz]$. Suppose $s'\sigma(u)s'^{-1} = u^q$ and $u \neq 1$. If $z \neq 0$, then $x = qz$, $a = q^2$, $y = qz^2/2$. If $z = 0$ then $x = 0$ and $-ya = qy$ implies $a = -q$. Thus reducibility occurs in two cases:

(1) $a = q^2$, the constituents are the nontempered trivial representation tr and the square integrable St;

(2) $a = -q$, the constituents are the nontempered representation which we denote by $\pi^\times$ and the square integrable π^+.

Put $r = \mathrm{antidiag}(1, -1, 1)$ and $r' = rt$ for the reflections in G with $K = I \cup IrI$ and $K' = I \cup Ir'I$. The Iwahori algebra H_I (of compactly supported I-biinvariant $\mathbb{C}$-valued functions on G) is generated over $\mathbb{C}$ by the characteristic functions T of IrI and T' of $Ir'I$, subject to the relations $(T + 1)(T - q^3) = 0$ and $(T' + 1)(T' - q) = 0$; see, e.g., [Bo], 3.2(2). The characteristic functions of K and K' are $T_K = 1 + T$ and $T_{K'} = 1 + T'$. The functor $V \mapsto V^I$ is an equivalence from the category of representations of G with a nonzero I-invariant vector to the category of H_I-modules. On the two dimensional H_I-module $I(\eta)^I$ the element TT' acts as $\delta^{1/2}(t)\,\mathrm{diag}(\eta(\boldsymbol{\pi}), \eta(\boldsymbol{\pi}^{-1}))$ for some basis, where $\delta(t) = |\det[\mathrm{Ad}(t)|\,\mathrm{Lie}\,N]| = q^4$, but T, T' are not diagonalizable with respect to a basis which diagonalizes TT'. When $I(\eta)$ is reducible, the constituents correspond to one dimensional representations of H_I. The possible images of T are -1 and q^3, of T' are -1 and q. Thus on the trivial representation $(T, T') \mapsto (q^3, q)$, and on the Steinberg $(T, T') \mapsto (-1, -1)$, so TT' acts on the corresponding induced $I(\eta)$ with eigenvalues $(q^4, 1) = q^2(q^2, q^{-2})$, and the induced is $I(\eta)$ with $\eta(t) = a$ equals q^2. On $\pi^\times$: $(T, T') \mapsto (q^3, -1)$, on π^+: $(T, T') \mapsto (-1, q)$, so TT' has eigenvalues $(-q^3, -q) = q^2(-q, -q^{-1})$ and the induced is $I(\eta)$ with $\eta(t) = -q$. Now the eigenvalues of $(T_K, T_{K'}) = (1 + T, 1 + T')$ are on tr: $(1 + q^3, 1 + q)$, on St: $(0, 0)$, on $\pi^\times$: $(1+q^3, 0)$, on π^+: $(0, 1+q)$. We conclude that the trivial representation has both (nonzero) K and K'-fixed vectors, the Steinberg has none, $\pi^\times$ has a K-fixed vector but no K'-fixed vector, and π^+ has a K'-fixed vector but no K-fixed vector, thus $\pi^{\times K} \neq 0 = \pi^{\times K'}$ and $\pi^{+K} = 0 \neq \pi^{+K'}$. Clearly each $I(\eta)$ has both K and K'-fixed vectors.

11.4. U(3) **- the split case.** Let E/F denote a totally imaginary quadratic extension E of a totally real number field F. Consider the quasi-split unitary F-group

$G^{\mathrm{qs}} = \mathrm{U}(2,1)$ in 3 variables, split over E. Let $G = \mathrm{U}(3)$ be an arbitrary inner form of G^{qs} such that G_∞ is compact. Such exist since E is CM. The rank is odd, so we may even assume G is quasi-split at all finite primes, but we do not need that here. Let w be a prime of F split in E. Denote by R_w the ring of integers in the completion F_w of F at w, and by $q = q_w$ the (residual) cardinality, of $\mathbb{F}_w = R_w/w$. Let w_E be a prime of E over w. Let us list the parahoric subgroups of $\mathrm{GL}(3, E_{w_E}) \simeq \mathrm{GL}(3, F_w)$. There is the hyperspecial maximal compact subgroup $K_w = \mathrm{GL}(3, R_w)$, and the Iwahori subgroup

$$I_w = \left\{ g \in K_w : g \equiv \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} (\operatorname{mod} w) \right\}.$$

There is only one $\mathrm{GL}(3, F_w)$-conjugacy class of maximal proper parahorics. Denote by $\boldsymbol{\pi}_w$ a generator of the maximal ideal w in the ring R_w of integers in F_w. Put $\mu_w = \mathrm{diag}(\boldsymbol{\pi}_w, \boldsymbol{\pi}_w, 1)$. Then

$$J_w = \left\{ g \in K_w : g \equiv \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} (\operatorname{mod} w) \right\} = K_w \cap \mu_w^{-1} K_w \mu_w.$$

is a representative.

11.5. Proof of Theorem 0.5. We first need to classify all the Iwahori-spherical representations of $\mathrm{GL}(3, F_w)$. It is a theorem of Borel [Bo] and Bernstein [B] that these are precisely the constituents of the unramified principal series. Let $\nu = |\cdot|$ be the absolute value character on F_w. Using the theory of Bernstein and Zelevinsky [BZ] we obtain the following table.

		constituent of	representation	unitary	tempered
I		$\chi_1 \times \chi_2 \times \chi_3$	$\chi_1 \times \chi_2 \times \chi_3$	below	$\lvert\chi_i\rvert = 1$
II	a	$\chi_1 \nu^{1/2} \times \chi_1 \nu^{-1/2} \times \chi_2$	$\chi_1 \mathrm{St}_{\mathrm{GL}(2)} \times \chi_2$	$\lvert\chi_i\rvert = 1$	$\lvert\chi_i\rvert = 1$
	b	$\chi_1 \chi_2^{-1} \neq \nu^{\pm 3/2}$	$\chi_1 \mathbf{1}_{\mathrm{GL}(2)} \times \chi_2$	$\lvert\chi_i\rvert = 1$	
III	a	$\chi\nu \times \chi \times \chi\nu^{-1}$	$\chi \, \mathrm{St}_{\mathrm{GL}(3)}$	$\lvert\chi\rvert = 1$	$\lvert\chi\rvert = 1$
	b		χV_P		
	c		χV_Q		
	d		$\chi \mathbf{1}_{\mathrm{GL}(3)}$	$\lvert\chi\rvert = 1$	

TABLE A. *Iwahori-spherical representations of* GL(3)

Only the representations of types I, IIa, IIIa are generic, and a representation in Table A is square integrable iff it is of type IIIa and $|\chi| = 1$.

The irreducible representation $\chi_1 \times \chi_2 \times \chi_2$ in case I is unitary if and only if either all the χ_i are unitary, or $\chi_1 \chi_2^{-1} = \nu^\alpha$ with $0 < \alpha < 1$ and χ_3 unitary (after a permutation). In the table, P and Q denote the parabolics of $G = \mathrm{GL}(3, F_w)$ of type $(2,1)$ and $(1,2)$ respectively. Moreover, $V_P = C^\infty(P \backslash G)/\mathbb{C}$ and $V_Q = C^\infty(Q \backslash G)/\mathbb{C}$. They are not unitary, and therefore irrelevant for the theory of automorphic forms. Next, we list the dimensions of their parahoric fixed spaces:

		representation	remarks	K	J	I
I		$\chi_1 \times \chi_2 \times \chi_3$		1	3	6
II	a	$\chi_1 \operatorname{St}_{\mathrm{GL}(2)} \times \chi_2$		0	1	3
	b	$\chi_1 \mathbf{1}_{\mathrm{GL}(2)} \times \chi_2$		1	2	3
III	a	$\chi \operatorname{St}_{\mathrm{GL}(3)}$		0	0	1
	b	χV_P	not unitary	0	1	2
	c	χV_Q	not unitary	0	1	2
	d	$\chi \mathbf{1}_{\mathrm{GL}(3)}$	irrelevant	1	1	1

Table B. *Dimensions of the parahoric fixed spaces*

To compute these dimensions, we use the following observation: If P is parabolic and J is parahoric, a choice of representatives $g \in P\backslash G/J$ determines an isomorphism

$$\operatorname{Ind}_P^G(\tau)^J \simeq \bigoplus\nolimits_{g \in P\backslash G/J} \tau^{P \cap gJg^{-1}},$$

for every representation τ of a Levi factor M_P. In particular, if $P = B$ is the Borel subgroup and τ is an unramified character, the dimension of $\operatorname{Ind}_B^G(\tau)^J$ equals the number of double cosets $|B\backslash G/J|$. With this information, the proof proceeds as follows. Theorem 0.3 gives an automorphic representation $\widetilde{\pi}$ congruent to π (modulo λ) such that $\widetilde{\pi}_w^{J_w} \neq \widetilde{\pi}_w^{K_w} + \widetilde{\pi}_w^{K'_w}$. Since $\widetilde{\pi}_w$ must be unitary, we see from table B that it is of type I or IIa. Then, from table A, we derive that $\widetilde{\pi}_w$ is generic and not L^2. Finally, note that there is a bijection $K/J \simeq \mathrm{GL}(3, \mathbb{F}_w)/\bar{P} \simeq \mathbb{P}_2(\mathbb{F}_{q_w})$, whose cardinality is $(q_w^3 - 1)/(q_w - 1)$, so $[K : J] = 1 + q_w + q_w^2$, $q_w = |R_w/w|$, but all maximal compact subgroups of $\mathrm{GL}(3, F_w)$ are conjugate, so $[K'_w : J_w]_{K_w} = 1$. $\square$

12. Applications for $\mathrm{GSp}(2)$**.** In this section we view the symplectic group $\mathrm{GSp}(2)$ of rank two as an algebraic F-subgroup of $\mathrm{GL}(4)$ by realizing it with respect to the standard skew-diagonal symplectic form. With this choice, the set of upper triangular matrices form a Borel subgroup $B = TU$. There are two maximal parabolic subgroups containing B. One is the Siegel parabolic

$$P = M_P \ltimes N_P = \left\{ \begin{pmatrix} g & \\ & \nu \cdot {}^\tau g^{-1} \end{pmatrix} \begin{pmatrix} 1 & & r & s \\ & 1 & t & r \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\},$$

where ${}^\tau g$ denotes the skew-transpose. The other is the Heisenberg parabolic

$$Q = M_Q \ltimes N_Q = \left\{ \begin{pmatrix} \nu & & \\ & g & \\ & & \nu^{-1} \cdot \det g \end{pmatrix} \begin{pmatrix} 1 & c & & \\ & 1 & & \\ & & 1 & -c \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & r & s \\ & 1 & & r \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\}.$$

We consider an inner form G of $\mathrm{GSp}(2)$ such that $G^{\mathrm{der}}(\mathbb{R})$ is compact. Concretely we have $G = \mathrm{GSpin}(f)$, where f is some definite quadratic form in 5 variables over F. Let us first describe the parahoric subgroups of $\mathrm{GSp}(2, F_w)$. There is the hyperspecial maximal compact subgroup $K_w = \mathrm{GSp}(2, R_w)$, and the Iwahori subgroup I_w consisting of elements in K_w with upper triangular reduction mod w. Similarly, P and Q define (non-conjugate) parahoric subgroups J'_w and J_w called the Siegel parahoric and the Heisenberg parahoric respectively. One can easily check that we have the identity,

$$J'_w = K_w \cap h_w K_w h_w^{-1}, \quad \text{where } h_w = \begin{pmatrix} & I \\ \boldsymbol{\pi}_w I & \end{pmatrix}, I = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}.$$

However, $J_w = K_w \cap K'_w$, where K'_w is the non-special *paramodular* (see [Sch], p. 267) maximal compact subgroup containing I_w. Since P and Q are not associated parabolics, the classification of the Iwahori-spherical representations of $\mathrm{GSp}(2, F_w)$ is more complicated than for $\mathrm{GL}(3, F_w)$. This is reproduced in Appendix 2 as Table C and Table D from Table 1 and Table 3 of R. Schmidt [Sch]. We use the notation from Appendix 2.

12.1. Proof of Theorem 0.6. We apply Theorem 0.3 to the Heisenberg parahoric J_w. An easy computation shows that $[K'_w : J_w] = q_w$ and $[K_w : J_w] = (q_w^4 - 1)/(q_w - 1)$, hence $[K'_w : J_w]_{K_w} = q_w$. We get an automorphic representation $\widetilde{\pi}$, congruent to π modulo λ, such that the component at w satisfies the identity:

$$\widetilde{\pi}_w^{J_w} \neq \widetilde{\pi}_w^{K_w} + \widetilde{\pi}_w^{K'_w}.$$

In particular, $\widetilde{\pi}_w^{J_w} \neq 0$. We must have that $\widetilde{\pi}_w^{K_w} \cap \widetilde{\pi}_w^{K'_w} = 0$, for otherwise $\dim \widetilde{\pi}_w = 1$ and therefore $\widetilde{\pi}$ is one-dimensional by the strong approximation theorem. However, π is assumed to be non-abelian modulo λ. Thus, equivalently we have

$$\dim \widetilde{\pi}_w^{J_w} > \dim \widetilde{\pi}_w^{K_w} + \dim \widetilde{\pi}_w^{K'_w}.$$

From table 3 of [Sch, p. 269] (copied as Table D in Appendix 2), we deduce that this inequality is satisfied precisely when $\widetilde{\pi}_w$ is of type I, IIa, IIIa, IVb, IVc, Va or VIa. However, the representations of type IVb and IVc are not unitary and can therefore be ruled out immediately. We are then left with the possible types I, IIa, IIIa, Va and VIa. Then from table 1 of [Sch, p. 264] (copied as Table C in Appendix 2), we read off that $\widetilde{\pi}_w$ is generic. Indeed all the representations of type Xa are generic, for X arbitrary.

Let us show that the types Va and VIa can also be ruled out if we assume $q^4 \neq 1 \pmod{\lambda}$. Suppose first that $\widetilde{\pi}_w$ is of type Va, that is, the unique subrepresentation of some $|\cdot|\xi_0 \times \xi_0 \rtimes |\cdot|^{-1/2}\sigma$ where ξ_0 has order two, in the notations of Sally-Tadic [ST]. By the main theorem, the center of the Heisenberg-Hecke algebra $Z(\mathcal{H}_{J_w,\mathbb{Z}})$ acts on $\widetilde{\pi}_w^{J_w}$ by a character $\eta_{\widetilde{\pi}_w^{J_w}}$ satisfying the congruence

$$\eta_{\widetilde{\pi}_w^{J_w}}(\phi) \equiv \eta_{\pi_w^{K_w}}(e_{K_w} * \phi) \equiv \eta_{\mathbf{1}}(e_{K_w} * \phi) \pmod{\lambda},$$

for all $\phi \in Z(\mathcal{H}_{J_w,\mathbb{Z}})$. We get immediately that the analogous statement is also true for the center of the Iwahori-Hecke algebra $Z(\mathcal{H}_{I_w,\mathbb{Z}})$. This, however, acts by a character on the Iwahori-fixed vectors in the principal series $|\cdot|\xi_0 \times \xi_0 \rtimes |\cdot|^{-1/2}\sigma$ (for it has an unramified Langlands quotient, so it is generated by any nonzero K_w-fixed vector). Hence $Z(\mathcal{H}_{I_w,\mathbb{Z}})$ acts on every constituent of this principal series by the same character $\eta_{\widetilde{\pi}_w^{I_w}}$. In particular, the action of the spherical Hecke algebra $\mathcal{H}_{K_w,\mathbb{Z}} \simeq Z(\mathcal{H}_{I_w,\mathbb{Z}})$ on the K_w-fixed vectors of the unramified quotient (type Vd) is given by a character congruent to $\eta_{\mathbf{1}}$. In terms of their Satake parameters we therefore must have (modulo the action of the Weyl group):

$$\mathrm{diag}(q_w^{-1/2}\sigma(\boldsymbol{\pi}_w),\, q_w^{-1/2}\xi_0\sigma(\boldsymbol{\pi}_w),\, q_w^{1/2}\xi_0\sigma(\boldsymbol{\pi}_w),\, q_w^{1/2}\sigma(\boldsymbol{\pi}_w))$$

$$\equiv \mathrm{diag}(q_w^{-3/2},\, q_w^{-1/2},\, q_w^{1/2},\, q_w^{3/2}) (\mathrm{mod}\, \lambda).$$

Since $\xi_0(\boldsymbol{\pi}_w) = -1$ we conclude that $q_w \equiv -1$ or $q_w^2 \equiv -1$ modulo λ.

Secondly, assume $\widetilde{\pi}_w$ is of type VIa, that is, the unique irreducible subrepresentation of some $|\cdot| \times \mathbf{1} \rtimes |\cdot|^{-1/2}\sigma$. Then, by the argument above, we conclude that the unramified quotient of this principal series must be congruent to $\mathbf{1}$. That is, in terms of their Satake parameters:

$$\operatorname{diag}(q_w^{-1/2}\sigma(\boldsymbol{\pi}_w),\, q_w^{-1/2}\sigma(\boldsymbol{\pi}_w),\, q_w^{1/2}\sigma(\boldsymbol{\pi}_w),\, q_w^{1/2}\sigma(\boldsymbol{\pi}_w))$$

$$\equiv \operatorname{diag}(q_w^{-3/2},\, q_w^{-1/2},\, q_w^{1/2},\, q_w^{3/2}) (\operatorname{mod} \lambda).$$

It follows that $q_w^2 \equiv 1$. The types I, IIa and IIIa cannot be excluded, even if π has trivial central character. □

Appendix 1. Congruent representations. The compact open subgroups $K \subset G(\mathbb{A}^\infty)$ form a directed set by opposite inclusion, that is $K \preccurlyeq J \Leftrightarrow K \supset J$. Let R be a commutative ring. As K varies over the compact open subgroups, the centers $Z(\mathcal{H}_{K,R})$ form an inverse system of R-algebras with respect to the canonical maps $Z(\mathcal{H}_{K,R}) \leftarrow Z(\mathcal{H}_{J,R})$ when $K \supset J$. Let

$$\mathcal{Z}_{G(\mathbb{A}^\infty),R} = \varprojlim Z(\mathcal{H}_{K,R}).$$

In this limit, it is enough to let K run through a neighborhood basis at the identity. Thus $\mathcal{Z}_{G(\mathbb{A}^\infty),R}$ is a commutative R-algebra, and it comes with projections ($K \supset J$)

$$\begin{tikzcd} & \mathcal{Z}_{G(\mathbb{A}^\infty),R} \arrow[dl, "\mathrm{pr}_K"'] \arrow[dr, "\mathrm{pr}_J"] & \\ Z(\mathcal{H}_{K,R}) & & Z(\mathcal{H}_{J,R}) \arrow[ll, "e_K * \phi \leftarrow \phi"] \end{tikzcd}$$

All we have said makes sense for any locally profinite group, so in particular we have local analogues $\mathcal{Z}_{G_v,R}$ for each finite place v. If $\mu = \otimes\mu_v$, it follows that

$$\mathcal{Z}_{G(\mathbb{A}^\infty),R} \simeq \bigotimes_{v<\infty} \mathcal{Z}_{G_v,R},$$

a restricted tensor product. Indeed the decomposable groups $K = \prod K_v$ form a cofinal system. It remains to determine the algebras $\mathcal{Z}_{G_v,R}$. By [B, 2.1], there exists a neighborhood basis at 1 consisting of compact open subgroups $K_v \subset G_v$ with Iwahori factorization with respect to a fixed minimal parabolic. If G_v is unramified, for such a K_v the canonical map $Z(\mathcal{H}_{K_v,R}) \to \mathcal{H}_{v,R}^{\mathrm{sph}}$ to the spherical Hecke algebra at v is an isomorphism [Bu]. This is a well-known result due to Bernstein when K_v is an actual Iwahori subgroup. Therefore,

$$G_v \text{ unramified} \Longrightarrow \mathcal{Z}_{G_v,R} \simeq \mathcal{H}_{v,R}^{\mathrm{sph}}.$$

The reason for introducing these objects is the following: Let $\pi = \otimes\pi_v$ be an irreducible admissible representation of $G(\mathbb{A})$. Then there exists a unique character

$$\eta_\pi : \mathcal{Z}_{G(\mathbb{A}^\infty),\mathbb{Z}} \to \mathbb{C},$$

such that $\eta_\pi = \eta_{\pi^K} \circ \mathrm{pr}_K$ for every K such that $\pi^K \neq 0$. Uniqueness is clear, and the existence reduces to showing that $\eta_{\pi^J}(\phi) = \eta_{\pi^K}(e_K * \phi)$ for $K \supset J$ when $\pi^K \neq 0$.

Similarly, we have characters η_{π_v} locally, and $\eta_\pi = \otimes\eta_{\pi_v}$ under the isomorphism above. If π is automorphic and $\pi_\infty = \rho_\infty$, the character η_π maps into the ring of integers of some number field. Our work suggests the following definition:

DEFINITION 12.1. *Let π and $\widetilde{\pi}$ be automorphic representations of $G(\mathbb{A})$, both ρ_∞ at infinity. Let λ be a finite place of $\bar{\mathbb{Q}}$. We say that π and $\widetilde{\pi}$ are congruent modulo λ, and we write $\widetilde{\pi} \equiv \pi \pmod{\lambda}$, if for all $\phi \in \mathcal{Z}_{G(\mathbb{A}^\infty),\mathbb{Z}}$ we have*

$$\eta_{\widetilde{\pi}}(\phi) \equiv \eta_\pi(\phi) \pmod{\lambda}.$$

Analogously, it makes sense to say that the local components $\widetilde{\pi}_v$ and π_v are congruent. Then $\widetilde{\pi} \equiv \pi$ (mod λ) if and only $\widetilde{\pi}_v \equiv \pi_v$ (mod λ) for all $v < \infty$. Note also that if $\widetilde{\pi}_v$ and π_v are both unramified, then $\widetilde{\pi}_v \equiv \pi_v$ (mod λ) means that the Satake parameters are congruent as it should. With these definitions, our results translate into those stated in the introduction.

Appendix 2. Iwahori-Spherical representations of GSp(4)**.** In this appendix we reproduce parts of Table 1 and Table 3 in [Sch]. The tables in [Sch] contain more information than what is listed here (such as Atkin-Lehner eigenvalues and signs of ε-factors). Below, we employ the notation of [ST]. Thus ν denotes the normalized absolute value of a non-archimedean local field. If χ_1, χ_2 and σ are unramified characters, $\chi_1 \times \chi_2 \rtimes \sigma$ denotes the principal series of GSp(2) obtained from

$$T \ni \mathrm{diag}(x, y, zy^{-1}, zx^{-1}) \mapsto \chi_1(x)\chi_2(y)\sigma(z) \in \mathbb{C}^\times$$

by normalized induction. Similarly, if π is a representation of GL(2), we denote by $\pi \rtimes \sigma$ and $\sigma \rtimes \pi$ the representations of GSp(2) induced from $\mathrm{diag}(X, z \cdot {}^\tau X^{-1}) \mapsto \pi(X)\sigma(z)$ and $\mathrm{diag}(z, X, z^{-1}\det X) \mapsto \sigma(z)\pi(X)$ respectively. By $L((-))$ we mean the unique irreducible quotient (the Langlands quotient) when it exists. The representations $\tau(S, \nu^{-1/2}\sigma)$ and $\tau(T, \nu^{-1/2}\sigma)$ are the constituents of $\mathbf{1} \rtimes \sigma\,\mathrm{St}_{\mathrm{GL}(2)}$. They can be called limits of discrete series. The nontrivial unramified quadratic character is denoted by ξ_0.

In the following Table C, a representation is generic iff it is of type I or Xa, and L^2 iff it is of type IVa or Va.

In table D below, our notation is different from [Sch]: K is hyperspecial, K' is paramodular, J is the Heisenberg parahoric, J' the Siegel parahoric and I is the Iwahori subgroup of GSp(4).

		constituent of	representation	tempered
I		$\chi_1 \times \chi_2 \rtimes \sigma$	$\chi_1 \times \chi_2 \rtimes \sigma$	$\lvert\chi_i\rvert = \lvert\sigma\rvert = 1$
II	a	$\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma$,	$\chi\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes\sigma$	$\lvert\chi\rvert = \lvert\sigma\rvert = 1$
	b	$\chi^2 \notin \{\nu^{\pm 1}, \nu^{\pm 3}\}$	$\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$	
III	a	$\chi \times \nu \rtimes \nu^{-1/2}\sigma$,	$\chi \rtimes \sigma\,\mathrm{St}_{\mathrm{GL}(2)}$	$\lvert\chi\rvert = \lvert\sigma\rvert = 1$
	b	$\chi \notin \{\mathbf{1}, \nu^{\pm 2}\}$	$\chi \rtimes \sigma\mathbf{1}_{\mathrm{GL}(2)}$	
IV	a	$\nu^2 \times \nu \rtimes \nu^{-3/2}\sigma$	$\sigma\,\mathrm{St}_{\mathrm{GSp}(4)}$	•
	b		$L((\nu^2, \nu^{-1}\sigma\,\mathrm{St}_{\mathrm{GL}(2)}))$	
	c		$L((\nu^{3/2}\,\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma))$	
	d		$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$	
V	a	$\nu\xi_0 \times \xi_0 \rtimes \nu^{-1/2}\sigma$,	$\delta([\xi_0, \nu\xi_0], \nu^{-1/2}\sigma)$	•
	b	$\xi_0^2 = \mathbf{1}$, $\xi_0 \neq \mathbf{1}$	$L((\nu^{1/2}\xi_0\,\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma))$	
	c		$L((\nu^{1/2}\xi_0\,\mathrm{St}_{\mathrm{GL}(2)}, \xi_0\nu^{-1/2}\sigma))$	
	d		$L((\nu\xi_0, \xi_0 \rtimes \nu^{-1/2}\sigma))$	
VI	a	$\nu \times \mathbf{1} \rtimes \nu^{-1/2}\sigma$	$\tau(S, \nu^{-1/2}\sigma)$	•
	b		$\tau(T, \nu^{-1/2}\sigma)$	•
	c		$L((\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma))$	
	d		$L((\nu, \mathbf{1} \rtimes \nu^{-1/2}\sigma))$	

TABLE C. *Iwahori-spherical representations of* GSp(4)

		representation	remarks	K	K'	J	J'	I
I		$\chi_1 \times \chi_2 \rtimes \sigma$		1	2	4	4	8
II	a	$\chi\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes\sigma$		0	1	2	1	4
	b	$\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$		1	1	2	3	4
III	a	$\chi \rtimes \sigma\,\mathrm{St}_{\mathrm{GL}(2)}$		0	0	1	2	4
	b	$\chi \rtimes \sigma\mathbf{1}_{\mathrm{GL}(2)}$		1	2	3	2	4
IV	a	$\sigma\,\mathrm{St}_{\mathrm{GSp}(4)}$		0	0	0	0	1
	b	$L((\nu^2, \nu^{-1}\sigma\,\mathrm{St}_{\mathrm{GL}(2)}))$	not unitary	0	0	1	2	3
	c	$L((\nu^{3/2}\,\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma))$	not unitary	0	1	2	1	3
	d	$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$	irrelevant	1	1	1	1	1
V	a	$\delta([\xi_0, \nu\xi_0], \nu^{-1/2}\sigma)$		0	0	1	0	2
	b	$L((\nu^{1/2}\xi_0\,\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma))$		0	1	1	1	2
	c	$L((\nu^{1/2}\xi_0\,\mathrm{St}_{\mathrm{GL}(2)}, \xi_0\nu^{-1/2}\sigma))$		0	1	1	1	2
	d	$L((\nu\xi_0, \xi_0 \rtimes \nu^{-1/2}\sigma))$		1	0	1	2	2
VI	a	$\tau(S, \nu^{-1/2}\sigma)$		0	0	1	1	3
	b	$\tau(T, \nu^{-1/2}\sigma)$		0	0	0	1	1
	c	$L((\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma))$		0	1	1	0	1
	d	$L((\nu, \mathbf{1} \rtimes \nu^{-1/2}\sigma))$		1	1	2	2	3

TABLE D. *Dimensions of the parahoric fixed spaces*

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