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PROJECTIVE EQUIVALENCE OF PLANE CURVE SINGULARITIES DEFINED BY THE HOMOGENIZATION OF WEIGHTED HOMOGENEOUS POLYNOMIALS IN $\mathbb{C}[Y, Z]$ AND ITS DIFFERENCE FROM THEIR ANALYTIC EQUIVALENCE*

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Dedicated to my mother's 80-th birthday

Abstract. The aim in this paper is to solve the following two problems completely:

The First Problem. Given any two weighted homogeneous polynomials $f(y, z)$ and $g(y, z)$ in $\mathbb{C}[y, z]$ with isolated singularity at the origin in $\mathbb{C}^2$ and with $\deg(f) = \deg(g)$, which are not homogeneous, we find the necessary and sufficient condition for C_1 and C_2 to be projectively equivalent in the complex projective plane $\mathbb{P}^2(\mathbb{C})$ in an elementary way where $F(x, y, z)$ and $G(x, y, z)$ are the homogenization of $f(y, z)$ and $g(y, z)$ in $\mathbb{C}[x, y, z]$, respectively, and C_1 and C_2 are the zero sets of $F(x, y, z)$ and $G(x, y, z)$ in $\mathbb{P}^2(\mathbb{C})$, respectively.

The Second Problem. Under the the same assumption as in the first problem, we find an exact difference between the analytic equivalence of plane curve singularities defined by $f(y, z)$ and $g(y, z)$, and the projective equivalence of C_1 and C_2 where C_1 and C_2 are the zero sets of $F(x, y, z)$ and $G(x, y, z)$ in $\mathbb{P}^2(\mathbb{C})$, respectively.

Key words. isolated singularities, topological and analytic equivalence of plane curve singularities defined by weighted homogeneous polynomials in $\mathbb{C}[y, z]$, the projective equivalence of plane curve singularities in $\mathbb{P}^2(\mathbb{C})$ defined by the homogenization in $\mathbb{C}[x, y, z]$ of weighted homogeneous polynomials in $\mathbb{C}[y, z]$.

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1. Introduction.

Let ${}_n\mathcal{O}$ or $\mathbb{C}\{x_1, x_2 \dots, x_n\}$ be the ring of convergent power series at the origin in $\mathbb{C}^n$, and $\mathbb{C}[x_1, \dots, x_n]$ be the ring of polynomials in n variables with coefficients in $\mathbb{C}$ where $\mathbb{C}$ denotes the field of the complex numbers. Let $f(x_1, \dots, x_n) = \Sigma a_{\alpha_1 \cdots \alpha_n} x_1^{\alpha_1} \cdots x_1^{\alpha_1}$ be a polynomial in $\mathbb{C}[x_1, \dots, x_n]$. The degree of $f(x_1, \dots, x_n)$, denoted by $\deg f$, is defined to be the largest integer $\alpha_1 + \cdots + \alpha_n$ such that $a_{\alpha_1 \cdots \alpha_n} \neq 0$.

Throughout this paper, we assume the followings:

(i) By definition, let $f = f(y, z)$ and $g = g(y, z)$ be weighted homogeneous polynomials in $\mathbb{C}[y, z]$, having rational number weights (w_1, w_2) and (v_1, v_2), respectively where every monomial $z^\alpha y^\beta$ of f satisfies the equality $\frac{\alpha}{w_1} + \frac{\beta}{w_2} = 1$ and every monomial $z^\gamma y^\delta$ of g satisfies the equality $\frac{\gamma}{v_1} + \frac{\delta}{v_2} = 1$.

(ii) Let $f(y, z)$ and $g(y, z)$ be polynomials in $\mathbb{C}[y, z]$ with isolated singularity at the origin in $\mathbb{C}^2$, which are not homogeneous, and let $\deg f = p$ and $\deg g = q$. The homogeneous polynomial $F(x, y, z) \in \mathbb{C}[x, y, z]$ defining the homogenization of $f(y, z)$ and the homogeneous polynomial $G(x, y, z) \in \mathbb{C}[x, y, z]$ defining the homogenization of $g(y, z)$ can be written as follows:

$$(*) \qquad \begin{aligned} F(x, y, z) &= x^p f(y/x, z/x) \quad \text{and} \\ G(x, y, z) &= x^q g(y/x, z/x). \end{aligned}$$

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Under the same assumption as above, our aim in this paper is to solve the following two problems completely:

The First Problem: Given any two weighted homogeneous polynomials $f(y,z)$ and $g(y,z)$ in $\mathbb{C}[y,z]$ with isolated singularity at the origin in $\mathbb{C}^2$ and with $\deg(f) = \deg(g)$, which are not homogeneous, compute the necessary and sufficient condition for C_1 and C_2 to be projectively equivalent in the complex projective plane $\mathbb{P}^2(\mathbb{C})$ in an elementary way where $F(x,y,z)$ and $G(x,y,z)$ are the homogenization of $f(y,z)$ and $g(y,z)$ in $\mathbb{C}[x,y,z]$, respectively, and C_1 and C_2 are the zero sets of $F(x,y,z)$ and $G(x,y,z)$ in $\mathbb{P}^2(\mathbb{C})$, respectively.

The Second Problem: Under the the same assumption as in the first problem, compute the difference between the analytic equivalence of plane curve singularities defined by $f(y,z)$ and $g(y,z)$, and the projective equivalence of C_1 and C_2 where C_1 and C_2 are the zero sets of $F(x,y,z)$ and $G(x,y,z)$ in $\mathbb{P}^2(\mathbb{C})$, respectively. As a corollary, assuming that $f(y,z) = 0$ and $g(y,z) = 0$ have the same topological type of singularity at the origin in $\mathbb{C}^2$ and that $\deg(f) = \deg(g)$, then prove that the projectively equivalence of C_1 and C_2 implies the analytic equivalence of $f(y,z) = 0$ and $g(y,z) = 0$ at the origin in $\mathbb{C}^2$, but the converse may not be true.

(1) To find a complete solution of the first problem in Theorem 6.1, it suffices to use Theorem 2.7 and The Fundamental Theorem, which says that any given projective curve has a unique decomposition into irreducible curves in $\mathbb{P}^2(\mathbb{C})$, only.

(2) To find a solution of the second problem in Theorem 6.2, it suffices to find the difference between a solution for the first problem and the consequence for Theorem 2.9.

In preparation for the representation of the solutions of two problems, for simplicity of notations, let f and g be holomorphic functions near the origin in $\mathbb{C}^2$ with isolated singularity at the origin. If f and g have the same topological type of singularity at the origin in the sense of Definition 2.1, we denote this relation by $f \sim g$. Otherwise, we write $f \not\sim g$. Also, if f and g have the same analytic type of singularity at the origin in the sense of Definition 2.1, then we write $f \approx g$. Otherwise, we write $f \not\approx g$. Observe that $f = y(z + y^k)$ for any positive integer $k \geq 2$ is weighted homogeneous, which is not homogeneous, but $f \approx z^2 + y^2$. It was proved by Lemma 2.6 that f is not homogeneous with $f \not\sim z^2 + y^2$ if and only if $f \not\sim h$ for any homogeneous polynomial $h \in \mathbb{C}[y,z]$ where $f \in \mathbb{C}[y,z]$ is a weighted homogeneous polynomial with isolated singularity at the origin in $\mathbb{C}^2$.

Moreover, let $F = F(x,y,z)$ and $G = G(x,y,z)$ be homogeneous polynomials in $\mathbb{C}[x,y,z]$ where F and G are square-free. Then, $F = 0$ and $G = 0$ may be viewed as defining equations for two projective curves C_1 and C_2 in $\mathbb{P}^2(\mathbb{C})$, respectively. If $F = 0$ and $G = 0$ are projectively equivalent in $\mathbb{P}^2(\mathbb{C})$ in the sense of Definition 3.2, then we write sometimes $F(x,y,z) \sim_{proj} G(x,y,z)$ in $\mathbb{P}^2(\mathbb{C})$ for brevity of notation. Otherwise, we write $F(x,y,z) \not\sim_{proj} G(x,y,z)$ in $\mathbb{P}^2(\mathbb{C})$.

In order to find a solution in the above two problems, first of all, we are going to study the following three questions:

Assume that $f(y,z)$ is a weighted homogeneous polynomial in $\mathbb{C}[y,z]$, which is not homogeneous, with an isolated singular point at the origin in $\mathbb{C}^2$ and that $F(x,y,z)$ is the homogeneous polynomial in $\mathbb{C}[x,y,z]$ defining the homogenization of $f(y,z)$. Then,

Question 1. What kind of topological type of isolated singularity does $f(y,z)$ have?

Question 2. What is the difference between $\deg(f)$ and $\deg(g)$, whenever $f \sim g$ for any such weighted homogeneous polynomials f and g?

Question 3. How many distinct lines and tangent lines does the projective curve C have in $\mathbb{P}^2(\mathbb{C})$ when C has a unique decomposition into irreducible curves in $\mathbb{P}^2(\mathbb{C})$? Note that C is the projective curve defined by $F(x,y,z) = 0$.

Note by Theorem 2.7([K2]) that Question 1 was already solved. Now, in order to solve the remaining two questions as above, it is very interesting and important for us to have the following theorem (Theorem 1.1), denoted by Theorem 3.6 of §3 later, which will be shown by an application of Theorem 2.7 and The Fundamental Theorem(Any given projective curve has a unique decomposition into irreducible curves in $\mathbb{P}^2(\mathbb{C})$).

Theorem 1.1 (The topological types of plane curve singularities defined by the homogenization of weighted homogeneous polynomials in $\mathbb{C}[y,z]$).

<u>Assumption</u> *Let $1 \le n < k$. Assume that $f(y,z)$ is a weighted homogeneous polynomial in $\mathbb{C}[y,z]$, which is not homogeneous, with isolated singularity at the origin in $\mathbb{C}^2$. Let $F(x,y,z)$ be the homogeneous polynomial in $\mathbb{C}[x,y,z]$ defining the homogenization of the above $f(y,z)$.*

<u>Conclusion</u> *Using a nonsingular linear change of coordinates $T : \mathbb{C}^3 \to \mathbb{C}^3$ with $T(0) = 0$, we may assume without loss of generality that $F(x,y,z)$ with $F(1,y,z) = f(y,z)$ can be written as follows:*

$$f(y,z) = y^{\varepsilon_1} z^{\varepsilon_2} f_1(y,z) \quad \textit{with} \tag{1.1.1}$$

$$f_1(y,z) = \prod_{i=1}^{d} (z^{n_1} + s_i y^{k_1}),$$

$$F(x,y,z) = y^{\varepsilon_1} z^{\varepsilon_2} F_1(x,y,z) \quad \textit{with} \tag{1.1.2}$$

$$F_1(x,y,z) = \prod_{i=1}^{d} (x^{k_1 - n_1} z^{n_1} + s_i y^{k_1}),$$

where

(a) $1 \le n < k$, $d = \gcd(n,k)$ *with* $n = dn_1$ *and* $k = dk_1$,
(b) $\varepsilon_1, \varepsilon_2$ *are either* 1 *or* 0, *respectively,*
(c) *if* $\varepsilon_1 = \varepsilon_2 = 0$, *then* $n \ge 2$,
(d) *the* s_i *are nonzero distinct complex numbers for* $1 \le i \le d$.

Let C be the plane curve defined by $F(x,y,z) = 0$ in $\mathbb{P}^2(\mathbb{C})$. Then ε_1, ε_2 and k are invariant under projective equivalence of plane curve singularities in $\mathbb{P}^2(\mathbb{C})$ from the following four cases: Let $\mathbb{N}$ be the set of positive integers, and $\mathbb{Q}$ be the set of rational numbers.

Case(I) *Let $\varepsilon_1 = \varepsilon_2 = 0$ with $n \ge 2$. Then, $f \sim z^n + y^k$ with weights $(n,k) \in \mathbb{N} \times \mathbb{N}$ and $\deg(f) = k$. So, the projective curve C has no line when C has a decomposition into irreducible curves, and two distinct tangent lines in $\mathbb{P}^2(\mathbb{C})$.*

Case(II) *Let* $\varepsilon_1 = 0$ *and* $\varepsilon_2 = 1$ *with* $n \geq 1$. *Then,* $f \sim z(z^n + y^k)$ *with weights* $(n+1, k+\frac{k}{n}) \in \mathbb{N} \times \mathbb{Q}$ *and* $\deg(f) = k+1 < k+\frac{k}{n}$. *So, the projective curve* C *has one distinct line when* C *has a decomposition into irreducible curves, and two distinct tangent lines in* $\mathbb{P}^2(\mathbb{C})$.

Case(III) *Let* $\varepsilon_1 = 1$ *and* $\varepsilon_2 = 0$ *with* $n \geq 1$. *Then,* $f \sim y(z^n + y^k)$ *with weights* $(n+\frac{n}{k}, k+1) \in (\mathbb{Q} - \mathbb{N}) \times \mathbb{N}$ *and* $\deg(f) = k+1$. *So, the projective curve* C *has one distinct line when* C *has a decomposition into irreducible curves, and three distinct tangent lines in* $\mathbb{P}^2(\mathbb{C})$.

Case(IV) *Let* $\varepsilon_1 = 1$ *and* $\varepsilon_2 = 1$ *with* $n \geq 1$. *Then,* $f \sim yz(z^n + y^k)$ *with weights* $(n+1+\frac{n}{k},\ k+1+\frac{k}{n}) \in (\mathbb{Q} - \mathbb{N}) \times \mathbb{Q}$ *and* $\deg(f) = k+2 < k+1+\frac{k}{n}$. *So, the projective curve* C *has two distinct lines when* C *has a decomposition into irreducible curves, and three distinct tangent lines in* $\mathbb{P}^2(\mathbb{C})$.

Moreover, $f(y,z)$ *of* (1.1.1) *and* $F(x,y,z)$ *of* (1.1.2) *can be rewritten in the following form:*

$$
\begin{aligned}
(1.1.3) \qquad f(y,z) &= y^{\varepsilon_1} z^{\varepsilon_2} f_1(y,z) \quad \textit{with} \\
f_1(y,z) &= z^n + y^k + \sum_{i=1}^{d-1} A_i y^{ik_1} z^{(d-i)n_1} \quad \textit{and} \\
F(x,y,z) &= y^{\varepsilon_1} z^{\varepsilon_2} F_1(x,y,z) \quad \textit{with} \\
F_1(x,y,z) &= x^{k-n} z^n + y^k + \sum_{i=1}^{d-1} A_i x^{k-ik_1-(d-i)n_1} y^{ik_1} z^{(d-i)n_1},
\end{aligned}
$$

where the A_i *are complex numbers for* $1 \leq i \leq d-1$.

Remark 1.1.1. As a consequence of Theorem 1.1, let C_1 and C_2 be two projective curves defined by $F = 0$ and $G = 0$, respectively, each of which satisfies the same kind of properties and notations as we have seen in Theorem 1.1. Then, $C_1 \sim_{proj} C_2$ in $\mathbb{P}^2(\mathbb{C})$ if and only if C_1 and C_2 belongs to the same one and only one of the four cases in Theorem 1.1, so that there is a nonsingular linear change of coordinates $T : \mathbb{C}^3 \to \mathbb{C}^3$ with $T(0) = 0$ and $F \circ T = G$. In order to find a complete solution of two problems in the beginning, it is enough to compute $F \circ T = G$ directly, using Theorem 1.1.

Now, we are going to consider the following computations:

(i) We study Case(I) of Theorem 1.1 in terms of Theorem 4.1 and Theorem 4.2 in §4, and study Case(II) of Theorem 1.1 in terms of Theorem 4.4 and Theorem 4.5 in §4.

(ii) We study Case(III) of Theorem 1.1 in terms of Theorem 5.1 and Theorem 5.2 in §5, and study Case(IV) of Theorem 1.1 in terms of Theorem 5.4 and Theorem 5.5 in §5.

From (i) and (ii), summing up the results of Theorem 4.1, Theorem 4.4, Theorem 5.1 and Theorem 5.4, we will get a solution of the first problem in terms of Theorem 6.1(The projective equivalence of plane curve singularities defined by the homogenization of weighted homogeneous polynomials in $\mathbb{C}[y,z]$) in §6.

From (i) and (ii), summing up the results of Theorem 4.2, Theorem 4.5, Theorem 5.2 and Theorem 5.5, we will get a solution of the second problem in terms of Theorem 6.2(The difference between analytic equivalence for weighted homogeneous polynomials in $\mathbb{C}[y,z]$ and projective equivalence for their homogenization in $\mathbb{C}[x,y,z]$)

in §6.

2. Known preliminaries.

DEFINITION 2.1. Let $V = \{z \in \mathbb{C}^{n+1} : f(z) = 0\}$ and $W = \{z \in \mathbb{C}^{n+1} : g(z) = 0\}$ be germs of complex analytic hypersurfaces with isolated singularity at the origin.

(i) f and g are said to have the same topological type of singularity at the origin(equivalently, to be topologically equivalent at the origin) if there is a germ at the origin of homeomorphisms $\phi : (U_1, 0) \to (U_2, 0)$ such that $\phi(V) = W$ and $\phi(0) = 0$ where U_1 and U_2 are open subsets in $\mathbb{C}^{n+1}$. In this case, denote this relation by $f \sim g$. Otherwise, we write $f \not\sim g$.

(ii) f and g are said to have the same analytic type of singularity at the origin(equivalently, to be analytically equivalent at the origin) if there is a germ at the origin of biholomorphisms $\psi : (U_1, 0) \to (U_2, 0)$ such that $\psi(V) = W$ and $\psi(0) = 0$ where U_1 and U_2 are open subsets of $\mathbb{C}^{n+1}$, that is, $f \circ \psi = ug$ where u is a unit in ${}_{n+1}\mathcal{O}$, the ring of germs of holomorphic functions at the origin in $\mathbb{C}^{n+1}$. Then we write $f \approx g$. If not, we write $f \not\approx g$.

THEOREM 2.2 ([BR], [BU], [Z1]). *Let $f(y, z)$ be irreducible in ${}_2\mathcal{O}$ with an isolated singularity at the origin in $\mathbb{C}^2$. Then the curve defined by f at the origin can be described topologically by $y = t^n$ and $z = t^{\alpha_1} + \cdots + t^{\alpha_p}$ where $n < \alpha_1 < \cdots < \alpha_p$ and $n > (n, \alpha_1) > \cdots > (n, \alpha_1, \ldots, \alpha_p) = 1$. If for a given f there is another homeomorphic parametrization defined by $y = t^m$ and $z = t^{\beta_1} + \cdots + t^{\beta_q}$ where $m < \beta_1 < \cdots < \beta_q$ and $m > (m, \beta_1) > \cdots > (m, \beta_1, \cdots \beta_q) = 1$, then $n = m$, $p = q$ and $\alpha_i = \beta_i$ for $1 \le i \le p$. Conversely, the curve defined by the parameter with the same kind of inequality as above must be irreducible at the origin.*

THEOREM 2.3 ([L], [Z2]). *Let $f(y, z)$ be in ${}_2\mathcal{O}$ with isolated singularity at the origin in $\mathbb{C}^2$. Then the topological type of the plane curve singularity defined by f is determined by the topological type of every irreducible component of f at O and all the pairs of intersection multiplicity of these two components.*

THEOREM 2.4 ([K1]). *Let $f = z^n + y^n + \sum_{i=1}^{k} a_i y^{n-i} z^i$ and $g = z^n + y^n + \sum_{j=1}^{l} b_j y^{n-j} z^j$ be homogeneous polynomials with isolated singularity at the origin in $\mathbb{C}^2$ where $n \ge 2k + 3$, $n \ge 2l + 3$ and $n \ge 5$. Then $f \approx g$ if and only if there is a complex number ρ with $\rho^n = 1$ such that $b_i = a_i \rho^i$ for $i = 1, 2, \ldots, k = l$. Moreover, if $f = z^4 + ay^3 z + y^4$ and $g = z^4 + by^3 z + y^4$ have an isolated singularity at the origin, then $f \approx g$ if and only if $a^4 = b^4$.*

THEOREM 2.5 [YO AND SU]. *The topology of a quasihomogeneous singularity in $\mathbb{C}^2$ determines the weights of the polynomial defining the singularity.*

LEMMA 2.6 ([K2]). *Let $f \in {}_2\mathcal{O}$ and f be weighted homogeneous with an isolated singular point at the origin. Then f is not homogeneous with $f \not\sim z^2 + y^2$ if and only if $f \not\sim h$ for any homogeneous polynomial $h \in {}_2\mathcal{O}$.*

THEOREM 2.7 ([K2]). *Let $2 \le n < k$. Assume that $f(y, z)$ is a weighted homogeneous polynomial with an isolated singular point at the origin in $\mathbb{C}^2$ which is not homogeneous. If $f \not\sim z^2 + y^2$, then f is topologically equivalent to the one and only one of the followings: Let $d = gcd(n, k)$.*
(I) $f \sim z^n + y^k$ *with weights (n, k).*

(i) $d < n$,
(ii) $d = n$.
(II) $f \sim z(z^n + y^k)$ *with weights* $(n+1, k+\frac{k}{n})$.
(i) $d < n$.
(III) $f \sim y(z^n + y^k)$ *with weights* $(n+\frac{n}{k}, k+1)$.
(i) $d < n$,
(ii) $d = n$.
(IV) $f \sim yz(z^n + y^k)$ *with weights* $(n+1+\frac{n}{k},\ k+1+\frac{k}{n})$.
(i) $d < n$.

In general, suppose that either f satisfies the above assumption with $f \not\sim z^2 + y^2$, or f is homogeneous. Then the weights of f determine the topological type of f and conversely.

Lemma 2.8 [K2]. *Let f be a weighted homogeneous polynomial with isolated singularity at the origin in $\mathbb{C}^2$ where f is not a homogeneous polynomial. If $f \not\sim z^2+y^2$, then f can be written analytically without loss of generality as follows:*

$$f(y,z) = y^{\varepsilon_1} z^{\varepsilon_2} f_1(y,z) \quad \text{with}$$
$$f_1(y,z) = z^n + y^k + \sum_{i=1}^{d-1} A_i y^{ik_1} z^{(d-i)n_1}$$

where $2 \le n < k$ and $d = gcd(n,k)$ with $n = n_1 d$ and $k = k_1 d$, each ε_i is either 1 or 0 for $i = 1, 2$, and all A_i are complex numbers for $1 \le i \le d-1$, satisfying the following property (2.8.1):

$$\text{(2.8.1)} \qquad \text{if} \quad gcd(n,k) = n, \ \text{i.e.,}\ n_1 = 1, \ \text{then } A_1 \text{ is zero.}$$

Theorem 2.9 [K2]. *Let f and g be weighted homogeneous polynomials, which are not homogeneous, with isolated singularity at the origin in $\mathbb{C}^2$ such that $f \not\sim z^2+y^2$ and $g \not\sim z^2 + y^2$. Then we may assume without loss of generality that analytically,*

$$f = y^{\varepsilon_1} z^{\varepsilon_2} f_1 \quad \text{with}$$
$$f_1 = z^n + y^k + \sum_{i=1}^{d-1} A_i y^{ik_1} z^{(d-i)n_1}$$
$$g = y^{\delta_1} z^{\delta_2} g_1 \quad \text{with}$$
$$g_1 = z^m + y^l + \sum_{j=1}^{e-1} B_j y^{jl_1} z^{(e-j)m_1}$$

where
(a) $2 \le n < k$, $d = \gcd(n,k)$ *with* $n = dn_1$ *and* $k = dk_1$,
(b) $2 \le m < l$, $e = \gcd(m,l)$ *with* $m = em_1$ *and* $l = el_1$,
(c) $\varepsilon_1, \varepsilon_2, \delta_1, \delta_2$ *are either* 1 *or* 0, *respectively, and*
(d) A_i *and* B_j *are complex numbers for* $1 \le i \le d-1$ *and* $1 \le j \le e-1$.
Also, we need to assume without loss of generality that

$$\text{if } \gcd(n,k) = n, \quad \text{i.e.,} \quad n_1 = 1, \ \text{then } A_1 = 0 \text{ and}$$
$$\text{if } \gcd(m,l) = m, \quad \text{i.e.,} \quad m_1 = 1, \ \text{then } B_1 = 0.$$

As a conclusion, we get the following:

(i) $f \approx g$ *if and only if* $\varepsilon_i = \delta_i$ *for* $i = 1, 2$ *and* $f_1 \approx g_1$.

(ii) $f_1 \approx g_1$ *if and only if* $n = m$ *and* $k = l$ *and there is a complex number* ρ *with* $\rho^d = 1$ *such that* $A_i\rho^i = B_i$ *for* $i = 1, ..., d-1$.

REMARK 2.9.1. If $2 \leq n < k$ and $\gcd(n, k) = n$, then we proved by Theorem 2.9 that $z(z^n + y^k) \not\approx z^{n+1} + y^{k+\frac{k}{n}}$ and $yz(z^n + y^k) \not\approx y(z^{n+1} + y^{k+\frac{k}{n}})$. But note that $z(z^n + y^k) \sim z^{n+1} + y^{k+\frac{k}{n}}$ and $yz(z^n + y^k) \sim y(z^{n+1} + y^{k+\frac{k}{n}})$, because $\gcd(n, k) = n$ is a positive integer.

3. How to find the topological types of plane curve singularities defined by the homogenization of weighted homogeneous polynomials in $\mathbb{C}[y, z]$. Let $\mathbb{C}[x_1, \ldots, x_n]$ be the ring of polynomials in n variables with coefficients in $\mathbb{C}$ where $\mathbb{C}$ is the field of the complex numbers.

DEFINITION 3.1. Let $\mathbb{N}$ be the set of positive integers, and $\mathbb{N}^2$ be its two-dimensional copy. Let $f = f(y, z) \in \mathbb{C}[y, z]$ be a weighted homogeneous polynomial with weights (w_1, w_2), that is, any monomial $z^\alpha y^\beta$ of f satisfies the equality $\frac{\alpha}{w_1} + \frac{\beta}{w_2} = 1$.

(i) It is said that the weights (w_1, w_2) of f belong to $\mathbb{N}^2$, denoted by $(w_1, w_2) \in \mathbb{N}^2$, if the weights (w_1, w_2) of f are positive integers.

(ii) Also, it is said that the weights (w_1, w_2) of f do not belong to $\mathbb{N}^2$, denoted by $(w_1, w_2) \notin \mathbb{N}^2$, if the weights (w_1, w_2) of f are not positive integers. For example, if w_1 is not a positive integer and w_2 is a rational number for the weights (w_1, w_2) of f, then we write $(w_1, w_2) \in (\mathbb{Q} - \mathbb{N}) \times \mathbb{Q}$, if necessary.

DEFINITION 3.2. Let $F = F(x, y, z)$ and $G = G(x, y, z)$ be homogeneous polynomials in $\mathbb{C}[x, y, z]$ where F and G are square-free. Then F and G may be viewed as defining equations for two projective curves C_1 and C_2 in $\mathbb{P}^2(\mathbb{C})$, respectively. Then, we say that C_1 and C_2, i.e., $F(x, y, z) = 0$ and $G(x, y, z) = 0$ either have the same projective type of the singularity or are projectively equivalent in $\mathbb{P}^2(\mathbb{C})$ if there is a nonsingular linear change of coordinates $T : \mathbb{C}^3 \to \mathbb{C}^3$ with $T(0) = 0$ such that $F \circ T = G$. In this case, we write $F(x, y, z) \sim_{proj} G(x, y, z)$ in $\mathbb{P}^2(\mathbb{C})$ for brevity of notation. Otherwise, we write $F(x, y, z) \not\sim_{proj} G(x, y, z)$ in $\mathbb{P}^2(\mathbb{C})$.

REMARK 3.2.1. (1) A projective curve or plane curve is the zero set of a nonzero constant homogeneous polynomial in the complex projective plane $\mathbb{P}^2(\mathbb{C})$.

(2) A projective curve of degree 1, 2, 3, 4, 5, 6 is called a line, quadric, cubic, quartic, quintic, sextic, respectively.

(3) A projective curve is irreducible if it is not the union of two distinct plane curves.

(4) Let C be the plane curve and let F be a homogeneous polynomial with zero set C. Then C is irreducible just in case F is a power of an irreducible polynomial.

(5) The Fundamental Theorem Let C be any plane curve, the zero set of the homogeneous polynomial F such that $F = {F_1}^{k_1} \cdots {F_r}^{k_r}$ where the F_i are irreducible homogeneous polynomials for $1 \leq i \leq r$. Then,

(5a) C has a unique decomposition into irreducible curves, $C = C_1 \cup \cdots \cup C_r$.

(5b) With suitable numbering, the irreducible curve C_i is the zero set of F_i. The polynomial F_i are determined, up to a constant factor, by C_i(and so by C).

Finding a complete solution of two problems in the beginning, it is enough to consider two cases, respectively.

Case(A): The weights (w_1, w_2) of a weighted homogeneous polynomial $f(y,z) \in \mathbb{C}[y,z]$ belong to $\mathbb{N}^2$ in the sense of Definition 3.1.

Case(B): The weights (w_1, w_2) of a weighted homogeneous polynomial $f(y,z) \in \mathbb{C}[y,z]$ do not belong to $\mathbb{N}^2$ in the sense of Definition 3.1.

First of all, we will study Case(A) in Lemma 3.4 of this section and after then, we will study Case(B) in Lemma 3.5 of this section, too. In preparation for the study of Case(A) and Case(B) in this section, first of all, consider two examples with weights (w_1, w_2) such that either $(w_1, w_2) \in \mathbb{N}^2$ or $(w_1, w_2) \notin \mathbb{N}^2$.

Example 3.2.2. Let $f_t = f_t(y,z) = z^2 + ty^2z + y^4$ for any complex number t with $t^2 \neq 4$. Then, $f_t = 0$ has an isolated singularity at the origin in $\mathbb{C}^2$ for each t. For any complex number t with $t^2 \neq 4$, let $F_t(x,y,z)$ be the homogenization of $f_t(y,z)$ defined by $F_t(x,y,z) = x^2z^2 + txy^2z + y^4$ in $\mathbb{C}[x,y,z]$.

<u>Instead of a nonsingular linear change of coordinates from $\mathbb{C}^2$ to itself,</u>

<u>using a nonsingular holomorphic map ϕ from an open subset $U \subseteq \mathbb{C}^2$ to $\mathbb{C}^2$,</u>

it is easily proved that $f_t \approx z^2 + y^4$ for any complex number t with $t^2 \neq 4$.

But, it will proved by Theorem 4.1 that $F_t(x,y,z) \sim_{proj} F_s(x,y,z)$ in $\mathbb{P}^2(\mathbb{C})$ if and only if $\frac{t^2}{t^2-4} = \frac{s^2}{s^2-4}$. Observe that $F_t(x,y,z)$ and $F_s(x,y,z)$ are locally analytically equivalent at any singular point in $\mathbb{C}^3 - \{0\}$ for any s, t with $s^2 \neq 4$ and $t^2 \neq 4$. Thus, an analytic equivalence at any singular point does not give a projective equivalence, but the projective equivalence implies an analytic equivalence for this case.

Example 3.2.3. Let $f(y,z) = y^\varepsilon z(z^2 - y^4)$ and $g(y,z) = y^\varepsilon(z^3 + y^6)$ where ε is either 1 or 0. By Theorem 2.7 and Theorem 2.9, $f \sim g$ with the same weights, but $f \not\approx g$ and $\deg(f) = 5 + \varepsilon < \deg(g) = 6 + \varepsilon$. If $\varepsilon = 0$, then the weights of f are positive integers, but if $\varepsilon = 1$, then the weights of f are not positive integers. By a nonsingular nonlinear change of coordinates $T : (y,z) \to (y, z + y^2)$, which is not linear, we get that $f(y,z) = y^\varepsilon z(z - y^2)(z + y^2) \approx y^\varepsilon z(z + y^2)(z + 2y^2) = y^\varepsilon z(z^2 + 3y^2z + 2y^4) = h(y,z)$. Then, $h(y,z) \approx y^\varepsilon z(z^2 + \frac{3}{2^{1/2}}y^2z + y^4) = \ell(y,z)$ with $\deg(\ell) = 5 + \varepsilon$. Let $F(x,y,z) = x^{5+\varepsilon}f(y/x, z/x)$, $G(x,y,z) = x^{6+\varepsilon}g(y/x, z/x)$, and $H(x,y,z) = x^{5+\varepsilon}h(y/x, z/x)$ and $L(x,y,z) = x^{5+\varepsilon}\ell(y/x, z/x)$ be the homogenization of $f(y,z)$, $g(y,z)$, $h(y,z)$ and $\ell(y,z)$ in $\mathbb{C}[x,y,z]$, respectively. Then, any two of $F(x,y,z) = 0$, $G(x,y,z) = 0$ and $L(x,y,z) = 0$ are not projectively equivalent in $\mathbb{P}^2(\mathbb{C})$, but $H(x,y,z) \sim_{proj} L(x,y,z)$ in $\mathbb{P}^2(\mathbb{C})$, which will be proved by Theorem 5.4. Note that $F(1,y,z) \approx L(1,y,z)$ at the origin in $\mathbb{C}^2$ and also $F(x,y,1) \approx L(x,y,1)$ at the origin in $\mathbb{C}^2$, but $F(x,y,z) \not\sim_{proj} L(x,y,z)$ in $\mathbb{P}^2(\mathbb{C})$. So, locally analytic equivalence at any singular point does not give the projective equivalence.

Lemma 3.3. <u>*Assumption*</u> *Let $1 \leq n < k$. Let f be a weighted homogeneous polynomial in $\mathbb{C}[y,z]$, which is not homogeneous, with isolated singularity at the origin in $\mathbb{C}^2$. Let $F(x,y,z) \in \mathbb{C}[x,y,z]$ be the homogenization of the above $f(y,z)$.*

<u>*Conclusion*</u>

(I) *Then, $f(y,z)$ can be written without loss of generality as follows:*

$$f(y,z) = y^{\varepsilon_1} z^{\varepsilon_2} f_1(y,z) \quad \textit{with} \tag{3.3.1}$$

$$f_1(y,z) = A_0 z^n + A_d y^k + \sum_{i=1}^{d-1} A_i y^{ik_1} z^{(d-i)n_1}$$

$$= \prod_{i=1}^{d} (s_0 z^{n_1} + s_i y^{k_1}),$$

where

(a) $1 \le n < k$, $d = \gcd(n,k)$ *with* $n = dn_1$ *and* $k = dk_1$,
(b) $\varepsilon_1, \varepsilon_2$ *are either* 1 *or* 0, *respectively, and*
(c) *the* A_i *are complex numbers for* $0 \le i \le d$ *and* $A_0 A_d \ne 0$,
(d) *the* s_i *are nonzero distinct complex numbers for* $0 \le i \le d$,
(e) *if* $\varepsilon_1 = \varepsilon_2 = 0$, *then we may assume additionally that* $2 \le n < k$,

satisfying the following property:

$$\underline{\textit{if } \gcd(n,k) = n, \quad \textit{i.e.,} \quad n_1 = 1, \textit{ then either } A_1 = 0 \textit{ or } A_1 \ne 0.} \tag{3.3.2}$$

(II) *With the same notation and property as in* (I), *the homogenization $F(x,y,z)$ of the above $f(y,z)$ can be written as follows:*

$$F(x,y,z) = y^{\varepsilon_1} z^{\varepsilon_2} F_1(x,y,z) \quad \textit{with} \tag{3.3.3}$$

$$F_1(x,y,z) = A_0 x^{k-n} z^n + A_d y^k + \sum_{i=1}^{d-1} A_i x^{(k_1-n_1)(d-i)} y^{ik_1} z^{(d-i)n_1}$$

$$= \prod_{i=1}^{d} (s_0 x^{k_1-n_1} z^{n_1} + s_i y^{k_1}),$$

where $F_1(x,y,z)$ is the homogeneous polynomial in $\mathbb{C}[x,y,z]$ defining the homogenization of the above $f_1(y,z)$ in $\mathbb{C}[y,z]$, that is, satisfying

$$F(x,y,z) = y^{\varepsilon_1} z^{\varepsilon_2} F_1(x,y,z) \quad \textit{with } \deg(f) = k + \varepsilon_1 + \varepsilon_2, \tag{3.3.4}$$

$$F_1(x,y,z) = x^k f_1(y/x, z/x).$$

(III) *In order to find the difference between an analytic equivalence of weighted homogeneous polynomials with isolated singularity in* [I] *and the projective equivalence of projective curves defined by their homogenization in* [II] *at the same time, we need to assume without loss of generality that*

<u>instead of a nonsingular holomorphic map from an open subset $U \subseteq \mathbb{C}^2$ to $\mathbb{C}^2$</u>,
<u>just using a nonsingular linear change of coordinates from $\mathbb{C}^3$ to itself</u>,
$F(x,y,z)$ with $f(y,z) = F(1,y,z)$ satisfies the following property:

$$A_0 = A_d = s_0 = 1, \textit{ and also} \tag{3.3.5}$$

$$\textit{if } \gcd(n,k) = n, \quad \textit{i.e.,} \quad n_1 = 1, \textit{ then either } A_1 = 0 \textit{ or } A_1 \ne 0. \tag{3.3.6}$$

Remark 3.3.1. (a) *Note that a condition in* (3.3.5) *is invariant under projective equivalence of curves, but projective equivalence of curves depends on the condition in* (3.3.2) *or* (3.3.6), *which is going to prove by Theorem* 6.1 *of this paper.*

(b) *Assume that a projective curve $F = 0$ has a line, when $F = 0$ has a decomposition into irreducible curves in $\mathbb{P}^2(\mathbb{C})$. Then an equation for the line is always given by either $y^{\varepsilon_1} = 0$ or $z^{\varepsilon_2} = 0$, if exists because* $0 < n_1 < k_1$ *in* (3.3.3).

Proof of Lemma 3.3. [I] By assumption, we may put f as follows:

$$f(y,z) = y^{\varepsilon_1} z^{\varepsilon_2} \ell(y,z) \quad \text{with} \tag{3.3.7}$$

$$\ell(y,z) = B_0 z^n + B_n y^k + \sum_{i=1}^{n-1} B_i y^{\beta_i} z^{n-i},$$

where $1 \le n < k$, each ε_i is either 1 or 0 for $i = 1, 2$, and the B_i are complex numbers for $0 \le i \le n$ with $B_0 B_n \neq 0$ and $1 < \beta_1 < \cdots < \beta_{n-1} < k$. Observe that $f(y,z)$ is weighted homogeneous if and only if $\ell(y,z)$ is weighted homogeneous. Let $d = \gcd(n,k)$ with $n = n_1 d$ and $k = k_1 d$. Then $\ell(y,z)$ can be rewritten in the form

$$\ell(y,z) = B_0 z^n + B_n y^k + \sum_{i=1}^{d-1} B_i y^{ik_1} z^{(d-i)n_1}. \tag{3.3.8}$$

Then, we put $A_0 = B_0$, $A_d = B_n$ and $A_i = B_i$ for $1 \le i \le d-1$. Thus, it is clear that

$$f(y,z) = y^{\varepsilon_1} z^{\varepsilon_2} f_1(y,z) \quad \text{with} \tag{3.3.9}$$

$$\begin{aligned} f_1(y,z) &= A_0 z^n + A_d y^k + \sum_{i=1}^{d-1} A_i y^{ik_1} z^{(d-i)n_1} \\ &= \prod_{i=1}^{d} (s_0 z^{n_1} + s_i y^{k_1}), \end{aligned}$$

where the A_i are complex numbers for $0 \le i \le d$ and $A_0 A_d \neq 0$, and the s_i are nonzero distinct complex numbers for $0 \le i \le d$. Moreover, if $\varepsilon_1 = \varepsilon_2 = 0$, then we must assume additionally that $2 \le n < k$ because $f = 0$ has an isolated singular point at the origin in $\mathbb{C}^2$. Then, the proof of (I) is done.

(II) Let $F(x,y,z)$ be the homogeneous polynomial in $\mathbb{C}[x,y,z]$ defining the homogenization of the above $f(y,z)$ in $\mathbb{C}[y,z]$. Then, $F(x,y,z)$ can be written as follows:

$$F(x,y,z) = y^{\varepsilon_1} z^{\varepsilon_2} F_1(x,y,z) \quad \text{with} \tag{3.3.10}$$

$$\begin{aligned} F_1(x,y,z) &= A_0 x^{k-n} z^n + A_d y^k + \sum_{i=1}^{d-1} A_i x^{(k_1-n_1)(d-i)} y^{ik_1} z^{(d-i)n_1} \\ &= \prod_{i=1}^{d} (s_0 x^{k_1-n_1} z^{n_1} + s_i y^{k_1}), \end{aligned}$$

where $F_1(x,y,z)$ is the homogeneous polynomial in $\mathbb{C}[x,y,z]$ defining the homogenization of the above $f_1(y,z)$ in $\mathbb{C}[y,z]$. Then, the proof of (II) is done.

(III) Consider $F(x, ay, bz)$ for some nonzero numbers a, b such that $F(1, ay, bz) = f(ay, bz)$. That is, just using a nonsingular linear change of coordinates from $\mathbb{C}^3$ to itself, at the same time we may write without loss of generality that

$$A_0 = A_d = s_0 = 1\ . \tag{3.3.11}$$

It is clear that a condition in (3.3.5) is invariant under projective equivalence of curves, but another condition in (3.3.2) or (3.3.6) is not invariant under projective equivalence of curves, which need to prove later by Theorem 6.1. Then, the proof of (III) is done. Thus, the proof of lemma is done.

LEMMA 3.4 (THE TOPOLOGICAL TYPE OF PLANE CURVE SINGULARITY DEFINED BY THE HOMOGENIZATION OF THE WEIGHTED HOMOGENEOUS POLYNOMIAL $\in \mathbb{C}[y,z]$ WITH POSITIVE INTEGER WEIGHTS).

Assumption *Let* $2 \le n \le k$. *Let* $f(y,z)$ *be a weighted homogeneous polynomial with isolated singularity at the origin in* $\mathbb{C}^2$ *and with positive integer weights* (n,k), *which is not homogeneous. Let* $F(x,y,z) \in \mathbb{C}[x,y,z]$ *be the homogenization of the above* $f(y,z)$.

Since the weights of $f(y,z)$ *are positive integers, then by Theorem* 2.7 *and Lemma* 3.3, *we may put* $f(y,z)$ *and* $F(x,y,z)$ *as follows:*

$$\begin{aligned} f(y,z) &= B_0 z^n + B_n y^k + \sum_{i=1}^{n-1} B_i y^{\beta_i} z^{n-i}, \\ F(x,y,z) &= x^p f(y/x, z/x) \qquad \text{with } p = \deg(f), \end{aligned} \tag{3.4.1}$$

where $2 \le n < k$, *and all* B_i *are complex numbers for* $0 \le i \le n$, *and* $B_0 \ne 0$ *because* f *is a weighted homogeneous polynomial with isolated singularity at the origin in* $\mathbb{C}^2$ *and* $2 \le n$ *and* $1 < \beta_1 < \cdots < \beta_{n-1} < k$.

Conclusion *Using a nonsingular linear change of coordinate* $T : \mathbb{C}^3 \to \mathbb{C}^3$ *with* $T(0) = 0$, *instead of a nonsingular holomorphic map from an open subset* $U \subseteq \mathbb{C}^2$ *to* $\mathbb{C}^2$, *then* $f(y,z)$ *with* $\deg f$ *and* $F(x,y,z)$ *are simultaneously equivalent to one and only one of the following: Note first that either* $B_n \ne 0$ *or* $B_n = 0$.

(I) *Let* $B_n \ne 0$. *Then, we may put* $B_0 = B_n = 1$ *by Lemma* 3.3. *Let* $d = \gcd(n,k)$, *and then either* $d < n$ *or* $d = n$.

(I-1) *Then,* $f(y,z)$ *and* $F(x,y,z)$ *of* (3.4.1) *can be rewritten in the form*

$$\begin{aligned} f(y,z) &= z^n + y^k + \sum_{i=1}^{d-1} A_i y^{ik_1} z^{(d-i)n_1} \qquad \text{with } \deg f = k \\ &= \prod_{i=1}^{d} (z^{n_1} + s_i y^{k_1}) \qquad\qquad \text{with } 1 \le n_1 < k_1, \\ F(x,y,z) &= x^{k-n} z^n + y^k + \sum_{i=1}^{d-1} A_i x^{(k_1-n_1)(d-i)} y^{ik_1} z^{(d-i)n_1} \\ &= \prod_{i=1}^{d} (x^{k_1-n_1} z^{n_1} + s_i y^{k_1}), \end{aligned} \tag{3.4.2}$$

where

(a) $2 \le n < k$, $d = \gcd(n,k)$ *with* $n = dn_1$ *and* $k = dk_1$, *and*
(b) *the* A_i *are complex numbers for* $1 \le i \le d-1$,

(c) *the* s_i *are nonzero distinct complex numbers for* $1 \le i \le d$.

(I-2) *The projective curve* $F(x,y,z) = 0$ *has no line when* $F(x,y,z) = 0$ *has a decomposition into irreducible curves, and two distinct tangent lines in* $\mathbb{P}^2(\mathbb{C})$.

(II) *Let* $B_n = 0$. *Then,* $B_{n-1} \neq 0$, *and we may put* $B_0 = B_{n-1} = 1$ *by Lemma 3.3. Let* $d = \gcd(n,k)$, *and then* $d = n$.

(II-1) *Then,* $f(y,z)$ *and* $F(x,y,z)$ *of* (3.4.1) *can be rewritten in the form*

$$\begin{aligned}
(3.4.3)\quad f(y,z) &= zf_1(y,z) \qquad\qquad \textit{with } \deg f = k - k_1 + 1 < k,\\
f_1(y,z) &= z^{n-1} + y^{k_1(n-1)} + \sum_{i=1}^{n-2} A_i y^{ik_1} z^{(n-1-i)}\\
&= \prod_{i=1}^{n-1}(z + s_i y^{k_1}), \qquad\qquad \textit{with } 1 < k_1,\\
F(x,y,z) &= zF_1(x,y,z) \quad \textit{with}\\
F_1(x,y,z) &= x^{k-k_1-(n-1)}z^{n-1} + y^{k-k_1} + \sum_{i=1}^{n-2} A_i x^{k-k_1-ik_1-(n-1-i)} y^{ik_1} z^{n-1-i}\\
&= \prod_{i=1}^{n-1}(x^{k_1-1}z + s_i y^{k_1}),
\end{aligned}$$

where

(a) $2 \le n < k$, $n = \gcd(n,k)$ *with* $k = nk_1$ *and* $\gcd(n-1, k_1(n-1)) = n-1$,

(b) *the* A_i *are complex numbers for* $1 \le i \le n-2$,

(c) *the* s_i *are nonzero distinct complex numbers for* $1 \le i \le n-1$.

(II-2) *The projective curve* $F(x,y,z) = 0$ *has one distinct line when* $F(x,y,z) = 0$ *has decomposition into irreducible curves, and two distinct tangent lines in* $\mathbb{P}^2(\mathbb{C})$.

Remark 3.4.1. (i) If $k = n+1$, then f and F have one singular point at $(y,z) = (0,0)$.

(ii) Hereafter, for simplicity of notations in solving the problems, $f(y,z)$ and $F(x,y,z)$ of (II) in Lemma 3.4 may be rewritten in the following form: $(n-1) \to m$ and $(k-k_1) \to \ell$. Then, $m = n-1 = \gcd(n-1, k-k_1) = \gcd(m,\ell)$.

$$\begin{aligned}
(3.4.4)\qquad f(y,z) &= zf_1(y,z) \qquad\qquad \text{with } \deg f = \ell + 1 < k,\\
f_1(y,z) &= z^m + y^\ell + \sum_{j=1}^{m-1} A_j y^{j\ell_1} z^{(m-j)}\\
&= \prod_{j=1}^{m}(z + s_i y^{\ell_1}), \qquad\qquad \text{with } 1 < \ell_1,\\
F(x,y,z) &= zF_1(x,y,z) \quad \text{with}\\
F_1(x,y,z) &= x^{\ell-m}z^m + y^\ell + \sum_{j=1}^{m-1} A_j x^{\ell-j\ell_1-(m-j)} y^{j\ell_1} z^{m-j}\\
&= \prod_{j=1}^{m}(x^{\ell_1-1}z + s_j y^{\ell_1}),
\end{aligned}$$

where
(a) $1 \leq m < \ell$ and $e = \gcd(m, \ell) = m$ with $m = m_1 e$ and $\ell = \ell_1 e$,
(b) the A_j are complex numbers for $1 \leq j \leq m-1$,
(c) the s_j are nonzero distinct complex numbers for $1 \leq j \leq m$.
Proof of Lemma 3.4. The proof is clear.

Lemma 3.5 (The topological type of plane curve singularity defined by the homogenization of the weighted homogeneous polynomial $\in \mathbb{C}[y,z]$ with no positive integer weights).

Assumption *Let $1 \leq n < k$. Let $f(y,z)$ be a weighted homogeneous polynomial with isolated singularity at the origin in $\mathbb{C}^2$ and with no positive integer weights, which f is not homogeneous. Let $F(x,y,z) \in \mathbb{C}[x,y,z]$ be the homogenization of the above $f(y,z)$.*

Since the weights of $f(y,z)$ are not positive integers, then by Theorem 2.7*, Lemma* 3.3 *and Lemma* 3.4*, then we may put $F(x,y,z)$ with $F(1,y,z) = f(y,z)$ as follows:*

$$f(y,z) = y^{\varepsilon_1} z^{\varepsilon_2} f_1(y,z) \quad \textit{with} \tag{3.5.1}$$

$$f_1(y,z) = z^n + y^k + \sum_{i=1}^{d-1} A_i y^{ik_1} z^{(d-i)n_1} = \prod_{i=1}^{d} (z^{n_1} + s_i y^{k_1}),$$

$$F(x,y,z) = y^{\varepsilon_1} z^{\varepsilon_2} F_1(x,y,z) \quad \textit{with} \tag{3.5.2}$$

$$F_1(x,y,z) = x^{k-n} z^n + y^k + \sum_{i=1}^{d-1} A_i x^{(k_1-n_1)(d-i)} y^{ik_1} z^{(d-i)n_1} = \prod_{i=1}^{d} (x^{k_1-n_1} z^{n_1} + s_i y^{k_1}),$$

where
(a) *$1 \leq n < k$, $d = \gcd(n,k)$ with $n = dn_1$ and $k = dk_1$,*
(b) *$\varepsilon_1, \varepsilon_2$ are either* 1 *or* 0*, respectively, and*
(c) *the A_i are complex numbers for $1 \leq i \leq d-1$,*
(d) *the s_i are nonzero distinct complex numbers for $1 \leq i \leq d$,*
(e) *if $\varepsilon_1 = 0$, then we may assume additionally that $\varepsilon_2 = 1$ and $\gcd(n,k) < n$,*
(f) *if $\varepsilon_2 = 0$, then we may assume additionally that $\varepsilon_1 = 1$,*
satisfying the following property:

$$\text{if } \gcd(n,k) = n, \text{ i.e., } n_1 = 1, \quad \text{then either } A_1 = 0 \text{ or } A_1 \neq 0. \tag{3.5.3}$$

Conclusion *Then, $f(y,z)$ with $\deg f$ and $F(x,y,z)$ are simultaneously equivalent to one and only one one of the following: Let C be the projective curve defined by $F(x,y,z) = 0$ in $\mathbb{P}^2(\mathbb{C})$.*

(I) *Let $\varepsilon_1 = \varepsilon_2 = 1$. Note that $n \geq 1$. Then, $f(y,z)$ and $F(x,y,z)$ have the same representation as in* (3.5.1) *and* (3.5.2)*, respectively and the projective curve C has two distinct lines when C has a decomposition into irreducible curves, and three distinct tangent lines in $\mathbb{P}^2(\mathbb{C})$.*

(II) *Let* $\varepsilon_1 = 0$, $\varepsilon_2 = 1$ *and* $\gcd(n,k) < n$. *Note that* $n \geq 1$. *Then,* $f(y,z)$ *and* $F(x,y,z)$ *have the same representation as in* (3.5.1) *and* (3.5.2), *respectively and the projective curve* C *has one distinct line when* C *has a decomposition into irreducible curves, and two distinct tangent line in* $\mathbb{P}^2(\mathbb{C})$.

(III) *Let* $\varepsilon_1 = 1$ *and* $\varepsilon_2 = 0$. *Note that* $n \geq 1$. *Then,* $f(y,z)$ *and* $F(x,y,z)$ *have the same representation as in* (3.5.1) *and* (3.5.2), *respectively and the projective curve* C *has one distinct line when* C *has a decomposition into irreducible curves, and three distinct tangent lines in* $\mathbb{P}^2(\mathbb{C})$.

Remark 3.5.1. Note the following observations:

(a) If $\varepsilon_1 = \varepsilon_2 = 0$ then f has the positive integer weights.

(b) If $\varepsilon_1 = 0$ and $\varepsilon_2 = 1$ then $\gcd(n,k) < n$, because otherwise f has the positive integer weights by Theorem 2.7 and Lemma 3.4.

Proof of Lemma 3.5. The proof is clear.

Let C be an arbitrary projective curve in $\mathbb{P}^2(\mathbb{C})$ defined by $F(x,y,z) = 0$ where $F(x,y,z) = 0$ is a homogeneous polynomial in $\mathbb{C}[x,y,z]$. Then, it is a well-known fact that whenever a projective curve C has a unique decomposition into irreducible curves in $\mathbb{P}^2(\mathbb{C})$ then the number of distinct lines of C and also the number of distinct tangent lines for C are projectively invariant under a nonsingular linear change of coordinates $T : \mathbb{C}^3 \to \mathbb{C}^3$ with $T(0) = 0$, respectively. In preparation for the complete solution of two problems in the beginning, first of all, we need to prove the following theorem by Lemma 3.4 and Lemma 3.5.

Theorem 3.6 (The topological types of plane curve singularities defined by the homogenization of weighted homogeneous polynomials in $\mathbb{C}[y,z]$).

<u>*Assumption*</u> *Let* $1 \leq n < k$. *Assume that* $f(y,z)$ *is a weighted homogeneous polynomial in* $\mathbb{C}[y,z]$, *which is not homogeneous, with isolated singularity at the origin in* $\mathbb{C}^2$. *Let* $F(x,y,z) \in \mathbb{C}[x,y,z]$ *be the homogenization of the above* $f(y,z)$. *Note that* $\mathbb{Q}$ *is the set of rational numbers.*

<u>*Conclusion*</u>

Fact(1): *Whether or not* $f \not\sim z^2 + y^2$, f *is topologically written in a unique way:*

$$f \sim y^{\varepsilon_1} z^{\varepsilon_2}(z^n + y^k) \quad \text{with } \deg(f) = k + \varepsilon_1 + \varepsilon_2 \text{ and}$$
$$\text{with weights } \left(n + \varepsilon_2 + \frac{n}{k}\varepsilon_1, k + \varepsilon_1 + \frac{k}{n}\varepsilon_2\right) \in \mathbb{Q} \times \mathbb{Q}, \tag{3.6.1}$$

where

(a) $1 \leq n < k$, $d = \gcd(n,k)$ *with* $n = dn_1$ *and* $k = dk_1$,

(b) $\varepsilon_1, \varepsilon_2$ *are either* 1 *or* 0, *respectively,*

(c) *if* $\varepsilon_1 = \varepsilon_2 = 0$, *then* $n \geq 2$.

Fact(2): *Using a nonsingular linear change of coordinate* $T : \mathbb{C}^3 \to \mathbb{C}^3$ *with* $T(0) = 0$, *we may assume without loss of generality that* $F(x,y,z)$ *with* $F(1,y,z) =$

$f(y,z)$ can be written as follows:

$$f(y,z) = y^{\varepsilon_1} z^{\varepsilon_2} f_1(y,z) \quad \textit{with} \tag{3.6.2}$$
$$f_1(y,z) = \prod_{i=1}^{d} (z^{n_1} + s_i y^{k_1}),$$
$$F(x,y,z) = y^{\varepsilon_1} z^{\varepsilon_2} F_1(x,y,z) \quad \textit{with} \tag{3.6.3}$$
$$F_1(x,y,z) = \prod_{i=1}^{d} (x^{k_1 - n_1} z^{n_1} + s_i y^{k_1}),$$

where the s_i are nonzero distinct complex numbers for $1 \le i \le d$ and $0 < n_1 < k_1$.

Fact(3): *For brevity of notation, let C be the projective curve defined by $F(x,y,z) = 0$ in $\mathbb{P}^2(\mathbb{C})$. Then ε_1, ε_2 and k are invariant under projective equivalence of plane curves in $\mathbb{P}^2(\mathbb{C})$, using the following four cases:*

Case(I) *Let $\varepsilon_1 = \varepsilon_2 = 0$ with $n \ge 2$. Then, $f \sim z^n + y^k$ with weights $(n,k) \in \mathbb{N} \times \mathbb{N}$ and $\deg(f) = k$. So, the projective curve C has no line when C has a decomposition into irreducible curves, and two distinct tangent lines in $\mathbb{P}^2(\mathbb{C})$.*

Case(II) *Let $\varepsilon_1 = 0$ and $\varepsilon_2 = 1$ with $n \ge 1$. Then, $f \sim z(z^n + y^k)$ with weights $(n+1, k+\frac{k}{n}) \in \mathbb{N} \times \mathbb{Q}$ and $\deg(f) = k+1 < k + \frac{k}{n}$. So, the projective curve C has one distinct line when C has a decomposition into irreducible curves, and two distinct tangent lines in $\mathbb{P}^2(\mathbb{C})$.*

Case(III) *Let $\varepsilon_1 = 1$ and $\varepsilon_2 = 0$ with $n \ge 1$. Then, $f \sim y(z^n + y^k)$ with weights $(n+\frac{n}{k}, k+1) \in (\mathbb{Q} - \mathbb{N}) \times \mathbb{N}$ and $\deg(f) = k+1$. So, the projective curve C has one distinct line when C has a decomposition into irreducible curves, and three distinct tangent lines in $\mathbb{P}^2(\mathbb{C})$.*

Case(IV) *Let $\varepsilon_1 = 1$ and $\varepsilon_2 = 1$ with $n \ge 1$. Then, $f \sim yz(z^n + y^k)$ with weights $(n+1+\frac{n}{k},\ k+1+\frac{k}{n}) \in (\mathbb{Q} - \mathbb{N}) \times \mathbb{Q}$ and $\deg(f) = k+2 < k+1+\frac{k}{n}$. So, the projective curve C has two distinct lines when C has a decomposition into irreducible curves, and three distinct tangent lines in $\mathbb{P}^2(\mathbb{C})$.*

Moreover, $f(y,z)$ of (3.6.2) and $F(x,y,z)$ of (3.6.3) can be rewritten in the following form:

$$f(y,z) = y^{\varepsilon_1} z^{\varepsilon_2} f_1(y,z) \quad \textit{with} \tag{3.6.4}$$
$$f_1(y,z) = z^n + y^k + \sum_{i=1}^{d-1} A_i y^{ik_1} z^{(d-i)n_1} \quad \textit{and}$$
$$F(x,y,z) = y^{\varepsilon_1} z^{\varepsilon_2} F_1(x,y,z) \quad \textit{with}$$
$$F_1(x,y,z) = x^{k-n} z^n + y^k + \sum_{i=1}^{d-1} A_i x^{k - ik_1 - (d-i)n_1} y^{ik_1} z^{(d-i)n_1},$$

where the A_i are complex numbers for $1 \le i \le d-1$.

Proof of Theorem 3.6. The proof just follows from Theorem 2.7, Lemma 3.3, Lemma 3.4 and Lemma 3.5.

Remark 3.6.1. Suppose that the same assumptions and conclusions with notations as in Theorem 3.6 hold.

(1) If the projective curve C belongs to Case(I), then the C has <u>no line</u> and the weights of $f(y,z)$ belong to $\underline{\mathbb{N}\times\mathbb{N}}$.

(2) If the projective curve C belongs to Case(II), then the C has <u>one line</u> and the weights of $f(y,z)$ belong to $\underline{\mathbb{N}\times\mathbb{Q}}$.

(3) If the projective curve C belongs to Case(III), then the C has <u>one line</u> and the weights of $f(y,z)$ belong to $\underline{(\mathbb{Q}-\mathbb{N})\times\mathbb{N}}$.

(4) If the projective curve C belongs to Case(IV), then the C has <u>two distinct lines</u> and the weights of $f(y,z)$ belong to $\underline{(\mathbb{Q}-\mathbb{N})\times\mathbb{Q}}$.

4. The projective equivalence of plane curves defined by the homogenization of weighted homogeneous polynomials with weights $\in\mathbb{N}\times\mathbb{Q}$ and its applications. Throughout this section, we study Case(I) of Theorem 3.6 in terms of Theorem 4.1 and Theorem 4.2, and study Case(II) of Theorem 3.6 in terms of Theorem 4.4 and Theorem 4.5, respectively.

Theorem 4.1 (The projective equivalence for Case(I) of Theorem 3.6).

<u>*Assumption*</u> *Let $f(y,z)$ and $g(y,z)$ be weighted homogeneous polynomials with positive integer weights in $\mathbb{C}[y,z]$, which are not homogeneous, with isolated singularity at the origin in $\mathbb{C}^2$, assuming that $f(y,z)$ and $g(y,z)$ belong to* Case(I) *of Theorem* 3.6.

Let $f\sim z^n+y^k$ with weights $(n,k)\in\mathbb{N}\times\mathbb{N}$ and $\deg(f)=k$, and let $g\sim z^m+y^\ell$ with weights $(m,\ell)\in\mathbb{N}\times\mathbb{N}$ and $\deg(g)=\ell$. By Theorem 3.6*, we may assume without loss of generality that $f(y,z)$ and $g(y,z)$ can be represented as follows:*

$$f(y,z)=z^n+y^k+\sum_{i=1}^{d-1}A_iy^{ik_1}z^{(d-i)n_1}\quad\text{and}\tag{4.1.1}$$
$$g(y,z)=z^m+y^\ell+\sum_{j=1}^{e-1}B_jy^{j\ell_1}z^{(e-j)m_1},$$

where

(a) $2\le n<k$ *and* $d=\gcd(n,k)$ *with* $n=n_1d$ *and* $k=k_1d$,

(b) $2\le m<\ell$ *and* $e=\gcd(m,\ell)$ *with* $m=m_1e$ *and* $\ell=\ell_1e$,

(c) *all the A_i and B_j are complex numbers for $1\le i\le d-1$ and $1\le j\le e-1$, respectively.*

Now, homogenize f and g as follows:

$$F(x,y,z)=x^kf(y/x,z/x)\quad\text{with}\quad\deg(f)=k,\tag{4.1.2}$$
$$G(x,y,z)=x^\ell g(y/x,z/x)\quad\text{with}\quad\deg(g)=\ell.$$

<u>*Conclusion*</u>

Then $F(x,y,z)\sim_{\mathrm{proj}}G(x,y,z)$ in $\mathbb{P}^2(\mathbb{C})$

$\Longleftrightarrow$ *there is a complex number ρ with $\rho^d=1$ such that $A_i\rho^i=B_i$ for $i=1,\dots,d-1$ where either $\{m=n\quad\text{and}\quad k=\ell\}$ or $\{m+n=k\quad\text{and}\quad k=\ell\}$.*

In particular, if $d=1$, then $F(x,y,z)\sim_{\mathrm{proj}}G(x,y,z)$ in $\mathbb{P}^2(\mathbb{C})$
$\Longleftrightarrow$ *either $\{m=n\quad\text{and}\quad k=\ell\}$ or $\{m+n=k\quad\text{and}\quad k=\ell\}$.*

Proof of Theorem 4.1. In preparation for the proof of the theorem, by (4.1.1) and

(4.1.2), $F = F(x, y, z)$ and $G = G(x, y, z)$ can be written by the following:

$$F = x^{k-n}z^n + y^k + \sum_{i=1}^{d-1} A_i x^{k-ik_1-(d-i)n_1} y^{ik_1} z^{(d-i)n_1}, \tag{4.1.3}$$

$$G = x^{\ell-m}z^m + y^\ell + \sum_{j=1}^{e-1} B_j x^{\ell-j\ell_1-(e-j)m_1} y^{j\ell_1} z^{(e-j)m_1},$$

where the A_i and B_j are complex numbers for $1 \leq i \leq d-1$ and $1 \leq j \leq e-1$.

First, to prove the necessity of the condition, suppose that $F \sim_{proj} G$ in $\mathbb{P}^2(\mathbb{C})$. Then, $k = \deg(f) = \deg(g) = \ell$, and there is a nonsingular linear change of coordinates $T : \mathbb{C}^3 \to \mathbb{C}^3$ with $T(0) = 0$, satisfying the following conditions:

$$\begin{aligned} & F \circ T(x, y, z) = G(x, y, z) \quad \text{with} \\ & T(x, y, z) = (a_1x + a_2y + a_3z, b_1x + b_2y + b_3z, c_1x + c_2y + c_3z), \end{aligned} \tag{4.1.4}$$

where the a_i, b_i, and c_i are complex numbers for $1 \leq i \leq 3$.

Since the number of singular points for the projective curve is invariant by a nonsingular linear change of coordinates $T : \mathbb{C}^3 \to \mathbb{C}^3$ with $T(0) = 0$, for the proof of the necessity of the condition, it suffices to consider the following two cases, respectively:

Case(i): If $k = n+1$ and $\ell = m+1$, then it is clear that $F = 0$ and $G = 0$ have one and only one singular point in $\mathbb{P}^2(\mathbb{C})$, which is denoted by $(x, y, z) = (1, 0, 0)$, and conversely. Then, we may assume that $T(1, 0, 0) = \lambda(1, 0, 0)$ for some nonzero complex number λ.

Case(ii): If $k \geq n+2$ and $\ell \geq m+2$, then it is clear that $F = 0$ and $G = 0$ have two singular points in $\mathbb{P}^2(\mathbb{C})$, which are denoted by $(x, y, z) = (1, 0, 0)$ and $(x, y, z) = (0, 0, 1)$, and conversely. For the proof of this case, it is enough to consider the following two subcases, respectively:

Case(ii-a): $T(1, 0, 0) = \lambda(1, 0, 0)$ and $T(0, 0, 1) = \mu(0, 0, 1)$ for some nonzero complex numbers λ and μ.

Case(ii-b): $T(1, 0, 0) = \lambda(0, 0, 1)$ and $T(0, 0, 1) = \mu(1, 0, 0)$ for some nonzero complex numbers λ and μ.

For the proof of the necessity of the condition in these cases, first of all, observe the followings by (4.1.3) and (4.1.4):

$$\begin{aligned} & F \circ T(x, y, z) \\ & = (a_1x + a_2y + a_3z)^{k-n}(c_1x + c_2y + c_3z)^n + (b_1x + b_2y + b_3z)^k \\ & \quad + \sum_{i=1}^{d-1} A_i H_i(x, y, z) \\ & = x^{\ell-m}z^m + y^\ell + \sum_{j=1}^{e-1} B_j K_j(x, y, z) \\ & = G(x, y, z), \end{aligned} \tag{4.1.5}$$

where $H_i(x, y, z) = (a_1x + a_2y + a_3z)^{k-ik_1-(d-i)n_1}(b_1x + b_2y + b_3z)^{ik_1}(c_1x + c_2y + c_3z)^{(d-i)n_1}$ and $K_j(x, y, z) = x^{\ell-j\ell_1-(e-j)m_1} y^{j\ell_1} z^{(e-j)m_1}$.

Case(i): Let $k = n+1$ and $\ell = m+1$. Note that $d = \gcd(n,k) = 1$ and $e = \gcd(m,\ell) = 1$. Since $F \sim_{proj} G$ in $\mathbb{P}^2(\mathbb{C})$, then $k = \deg(f) = \deg(g) = \ell$, and so $n = m$. Therefore, it is clear that $f(y,z) = z^n + y^k$ and $g(y,z) = z^m + y^\ell$ are the same, and also $F(x,y,z) = x^{k-n}z^n + y^k$ and $G(x,y,z) = x^{\ell-m}z^m + y^\ell$ are the same. So, there is nothing to prove for the necessity of the condition in this case.

Case(ii): Let $k \geq n+2$ and $\ell \geq m+2$.

Case(ii-a): Suppose that $T(1,0,0) = \lambda(1,0,0)$ and $T(0,0,1) = \mu(0,0,1)$ for some nonzero complex numbers λ and μ. Now, we claim the following:

$$T(x,y,z) = (a_1x, b_2y, c_3z). \tag{4.1.6}$$

In preparation for the proof of the claim in (4.1.6), first of all, we will prove the following sublemma by using the equation in (4.1.5).

Sublemma 4.1.1. Let T be a nonsingular linear change of coordinates $T : \mathbb{C}^3 \to \mathbb{C}^3$ with $T(0) = 0$ satisfying the equation $T(x,y,z) = (a_1x+a_2y+a_3z, b_1x+b_2y+b_3z, c_1x+c_2y+c_3z)$ where the a_i, b_i, and c_i are complex numbers for $1 \leq i \leq 3$. Suppose that $T(1,0,0) = \lambda(1,0,0)$ and $T(0,0,1) = \mu(0,0,1)$ for some nonzero complex numbers λ and μ.

As a conclusion, $b_1 = c_1 = a_3 = b_3 = 0$, and so the Jacobian determinant of T is $a_1b_2c_3 \neq 0$.

Proof of Sublemma 4.1.1. By (4.1,4), $T(1,0,0) = (a_1,b_1,c_1) = \lambda_1(1,0,0)$ and $T(0,0,1) = (a_3,b_3,c_3) = \mu(0,0,1)$ for some nonzero complex numbers λ and μ. Then, $b_1 = c_1 = a_3 = b_3 = 0$, and so the Jacobian determinant of T is $a_1b_2c_3 \neq 0$ because T is nonsingular. Thus, the proof of Sublemma 4.1.1 is done.

For the proof of the claim in (4.1.6), it remains to show by Sublemma 4.1.1 that $a_2 = c_2 = 0$. Now, applying Sublemma 4.1.1 to (4.1.5), then we have the following:

$$\begin{aligned} F \circ T(x,y,z) =& (a_1x + a_2y)^{k-n}(c_2y + c_3z)^n + (b_2y)^k \\ &+ \sum_{i=1}^{d-1} A_iH_i(x,y,z) \\ =& x^{\ell-m}z^m + y^\ell + \sum_{j=1}^{e-1} B_jK_j(x,y,z) \\ =& G(x,y,z), \end{aligned} \tag{4.1.7}$$

where $H_i(x,y,z) = (a_1x + a_2y)^{k-ik_1-(d-i)n_1}(b_2y)^{ik_1}(c_2y + c_3z)^{(d-i)n_1}$ and $K_j(x,y,z) = x^{\ell-j\ell_1-(e-j)m_1}y^{j\ell_1}z^{(e-j)m_1}$.

In order to prove the claim in (4.1.6), first of all, we need to use the following two facts (4.1.8) and (4.1.9), which can be easily proved from (4.1.7): Note that $0 < k_1 - n_1$ and $0 < \ell_1 - m_1$ by (4.1.1).

(4.1.8)

(i) $0 < k - ik_1 - (d-i)n_1 = (k_1 - n_1)(d-i) < k - n$ for $1 \leq i \leq d-1$.

(ii) $0 < ik_1 < k$ for $1 \leq i \leq d-1$.

(iii) $0 < (d-i)n_1 < n$ for $1 \leq i \leq d-1$.

(iv) $0 < \ell - j\ell_1 - (e-j)m_1 = (\ell_1 - m_1)(e-j) < \ell - m$ for $1 \leq j \leq e-1$.

(v) $0 < j\ell_1 < \ell$ for $1 \leq j \leq e-1$.

(vi) $0 < (e-j)m_1 < m$ for $1 \leq j \leq e-1$.

(4.1.9) Whenever any monomial $x^\alpha y^\beta z^\gamma$ belongs to $K_j(x,y,z)$ for all $j = 1, \ldots, e-1$, then α, β and γ are all positive integers by (iv), (v) and (vi) of (4.1.8).

Now, to prove that $a_2 = c_2 = 0$, it is enough to consider an existence of the coefficients of monomials $x^{k-n}y^n$ and $y^{k-n}z^n$ in $F \circ T = G$, respectively.

Then, it is easy to prove the following:

(a) By (4.1.7) and (4.1.9), two monomials $x^{k-n}y^n$ and $y^{k-n}z^n$ do not belong to $G(x,y,z)$.

(b) By (4.1.7) and (i) of (4.1.8), the monomial $x^{k-n}y^n$ has the coefficient $a_1^{k-n}c_2^n$ in $F \circ T$ because $x^{k-n}y^n \notin H_i(x,y,z)$ for any $i = 1, \ldots, d-1$, if exists.

(c) By (4.1.7) and (iii) of (4.1.8), the monomial $y^{k-n}z^n$ has the coefficient $a_2^{k-n}c_3^n$ in $F \circ T$ because $y^{k-n}z^n \notin H_i(x,y,z)$ for any $i = 1, \ldots, d-1$, if exists.

Because $F \circ T = G$ and the Jacobian determinant of T is $a_1b_2c_3 \neq 0$, then it is trivial by (a), (b) and (c) that $a_1^{k-n}c_2^n = a_2^{k-n}c_3^n = 0$, and therefore $c_2 = a_2 = 0$. Thus, we proved that $T(x,y,z) = (a_1x, b_2y, c_3z)$ in (4.1.6).

Using (4.1.6) and (4.1.7), we have the following:

$$(4.1.10) \qquad \begin{aligned} F \circ T(x,y,z) &= (a_1x)^{k-n}(c_3z)^n + (b_2y)^k + \sum_{i=1}^{d-1} A_iH_i(x,y,z) \\ &= x^{\ell-m}z^m + y^\ell + \sum_{j=1}^{e-1} B_jK_j(x,y,z) \\ &= G(x,y,z), \\ \text{where} \quad H_i(x,y,z) &= (a_1x)^{k-ik_1-(d-i)n_1}(b_2y)^{ik_1}(c_3z)^{(d-i)n_1}, \\ K_j(x,y,z) &= x^{\ell-j\ell_1-(e-j)m_1}y^{j\ell_1}z^{(e-j)m_1}. \end{aligned}$$

From (4.1.10) with (4.1.8) and (4.1.9), comparing the coefficients of monomials $x^{k-n}z^n$, y^k and $x^{k-ik_1-(d-i)n_1}y^{ik_1}z^{(d-i)n_1}$ in $F \circ T(x,y,z)$, with the coefficients of monomials $x^{\ell-m}z^m$, y^ℓ and $x^{\ell-j\ell_1-(e-j)m_1}y^{j\ell_1}z^{(e-j)m_1}$ in $G(x,y,z)$, respectively on both sides where $1 \leq i \leq d-1$ and $1 \leq j \leq e-1$, then we get the following equations: Note that $k = \ell$, $d = \gcd(n,k)$ and $e = \gcd(m,\ell)$.

$$(4.1.11) \qquad \begin{aligned} &x^{k-n}z^n = x^{\ell-m}z^m \text{ and } y^k = y^\ell \\ &\text{imply that } k = \ell \text{ and } n = m, \quad \text{and} \\ &a_1^{k-n}c_3^n = 1, \qquad b_2^k = 1 \qquad \text{and} \\ &A_ia_1^{k-ik_1-(d-i)n_1}b_2^{ik_1}c_3^{(d-i)n_1} = B_i \quad \text{for} \quad 1 \leq i \leq d-1 = e-1. \end{aligned}$$

Using (4.1.11) with $a_1^{k-n}c_3^n = 1$, then $A_ia_1^{k-ik_1-(d-i)n_1}b_2^{ik_1}c_3^{(d-i)n_1} = B_i$ can be rewritten as $A_ia_1^{-i(k_1-n_1)}b_2^{ik_1}c_3^{-in_1} = B_i$.

Let $\rho = a_1^{-(k_1-n_1)}b_2^{k_1}c_3^{-n_1}$. By (4.1.11), $\rho^d = 1$ and then $A_i\rho^i = B_i$ for each $i = 1, \cdots, d-1$. Thus, the proof for the necessity of the condition in Case(ii-a) is done.

Case(ii-b): Suppose that $T(1,0,0) = \lambda(0,0,1)$ and $T(0,0,1) = \mu(1,0,0)$ for some nonzero complex numbers λ and μ. Now, we claim the following:

$$T(x,y,z) = (a_3x, b_2y, c_1z). \tag{4.1.12}$$

In preparation for the proof of the claim in (4.1.12), first of all, we will prove the following sublemma by using the equation in (4.1.5).

Sublemma 4.1.2: Let T be a nonsingular linear change of coordinates $T : \mathbb{C}^3 \to \mathbb{C}^3$ with $T(0) = 0$ satisfying the equation $T(x,y,z) = (a_1x+a_2y+a_3z, b_1x+b_2y+b_3z, c_1x+c_2y+c_3z)$ where the a_i, b_i, and c_i are complex numbers for $1 \leq i \leq 3$. Suppose that $T(1,0,0) = \lambda(0,0,1)$ and $T(0,0,1) = \mu(1,0,0)$ for some nonzero complex numbers λ and μ.

As a conclusion, $a_1 = b_1 = b_3 = c_3 = 0$, and so the Jacobian determinant of T is $-a_3b_2c_1 \neq 0$.

Proof of Sublemma 4.1.2. By (4.1,4), $T(1,0,0) = (a_1,b_1,c_1) = \lambda(0,0,1)$ and $T(0,0,1) = (a_3,b_3,c_3) = \mu(1,0,0)$ for some nonzero complex numbers λ and μ. Then, $a_1 = b_1 = b_3 = c_3 = 0$, and so the Jacobian determinant of T is $-a_3b_2c_1 \neq 0$ because T is nonsingular. Thus, the proof of Sublemma 4.1.2 is done.

For the proof of the claim in (4.1.12), it remains to show by Sublemma 4.1.2 that $a_2 = c_2 = 0$. Now, applying Sublemma 4.1.2 to (4.1.5), then we have the following:

$$\begin{aligned} F \circ T(x,y,z) &= (a_2y + a_3z)^{k-n}(c_1x + c_2y)^n + (b_2y)^k \\ &\quad + \sum_{i=1}^{d-1} A_i H_i(x,y,z) \\ &= x^{\ell-m}z^m + y^\ell + \sum_{j=1}^{e-1} B_j K_j(x,y,z) \\ &= G(x,y,z), \end{aligned} \tag{4.1.13}$$

$$\begin{aligned} \text{where} \quad H_i(x,y,z) &= (a_2y + a_3z)^{k-ik_1-(d-i)n_1}(b_2y)^{ik_1}(c_1x + c_2y)^{(d-i)n_1}, \\ K_j(x,y,z) &= x^{\ell-j\ell_1-(e-j)m_1}y^{j\ell_1}z^{(e-j)m_1}. \end{aligned}$$

Now, to prove that $a_2 = c_2 = 0$, it is enough to compute an existence of the coefficients of monomials $y^{k-n}x^n$ and $z^{k-n}y^n$ in $F \circ T = G$ of (4.1.13), respectively. For such computations, we use the same notations and methods, that is, two facts (4.1.8) and (4.1.9) as we have done in the proof of Case(ii-a).

Then, it is easy to prove the following:

(a) By (4.1.13) and (4.1.9), two monomials $y^{k-n}x^n$ and $z^{k-n}y^n$ do not belong to $G(x,y,z)$.

(b) By (4.1.13) and (iii) of (4.1.8), the monomial $y^{k-n}x^n$ has the coefficient $a_2^{k-n}c_1^n$ in $F \circ T$ because $y^{k-n}x^n \notin H_i(x,y,z)$ for any $i = 1,\ldots,d-1$, if exists.

(c) By (4.1.13) and (i) of (4.1.8), the monomial $z^{k-n}y^n$ has the coefficient $a_3^{k-n}c_2^n$ in $F \circ T$ because $z^{k-n}y^n \notin H_i(x,y,z)$ for any $i = 1,\ldots,d-1$, if exists.

Because $F \circ T = G$ and the Jacobian determinant of T is $-a_3b_2c_1 \neq 0$, then it is trivial by (a),(b) and (c) that $a_2^{k-n}c_1^n = a_3^{k-n}c_2^n = 0$, and therefore $a_2 = c_2 = 0$. Thus, we proved that $T(x,y,z) = (a_3x, b_2y, c_1z)$ in (4.1.12).

Using (4.1.12) and (4.1.13), then we have the following:

$$
\begin{aligned}
(4.1.14) \qquad F\circ T(x,y,z) &= (a_3 z)^{k-n}(c_1 x)^n + (b_2 y)^k + \sum_{i=1}^{d-1} A_i H_i(x,y,z) \\
&= x^{\ell-m} z^m + y^\ell + \sum_{j=1}^{e-1} B_j x^{\ell - j\ell_1 - (e-j)m_1} y^{j\ell_1} z^{(e-j)m_1} \\
&= G(x,y,z), \\
\text{where} \quad H_i(x,y,z) &= (a_3 z)^{k-ik_1-(d-i)n_1} (b_2 y)^{ik_1} (c_1 x)^{(d-i)n_1}.
\end{aligned}
$$

From (4.1.14) with (4.1.8) and (4.1.9), comparing the coefficients of monomials $z^{k-n}x^n$, y^k and $z^{k-ik_1-(d-i)n_1}y^{ik_1}x^{(d-i)n_1}$ in $F\circ T(x,y,z)$, with the coefficients of monomials $x^{\ell-m}z^m$, y^ℓ and $x^{\ell-j\ell_1-(e-j)m_1}y^{j\ell_1}z^{(e-j)m_1}$ in $G(x,y,z)$, respectively on both sides where $1\le i\le d-1$ and $1\le j\le e-1$, then we get the following equations: Note that $k=\ell$, $d=\gcd(n,k)$ and $e=\gcd(m,\ell)$.

$$
\begin{aligned}
(4.1.15) \qquad & z^{k-n}x^n = x^{\ell-m}z^m \text{ and } y^k = y^\ell \text{ imply that } k=\ell=n+m \quad \text{and} \\
& a_3^{k-n}c_1^n = 1, \qquad b_2^k = 1 \qquad \text{and} \\
& A_i a_3^{k-ik_1-(d-i)n_1} b_2^{ik_1} c_1^{(d-i)n_1} = B_i \quad \text{for} \quad 1\le i\le d-1 = e-1,
\end{aligned}
$$

noting by (4.1.15) that $z^{k-ik_1-(d-i)n_1}y^{ik_1}x^{(d-i)n_1}$ and $x^{\ell-i\ell_1-(e-i)m_1}y^{i\ell_1}z^{(e-i)m_1}$ are the same monomial for each $i=1,2,\ldots,d-1=e-1$.

Using (4.1.15) with $a_3^{k-n}c_1^n=1$, then $A_i a_3^{k-ik_1-(d-i)n_1} b_2^{ik_1} c_1^{(d-i)n_1} = B_i$ can be rewritten as $A_i a_3^{-i(k_1-n_1)} b_2^{ik_1} c_1^{-in_1} = B_i$.

Let $\rho = a_3^{-(k_1-n_1)} b_2^{k_1} c_1^{-n_1}$. By (4.1.15), $\rho^d = 1$ and then $A_i\rho^i = B_i$ for each $i=1,\cdots,d-1$. Thus, the proof for the necessity of the condition in Case(ii-b) is done, and so we proved the necessity for Case(ii).

Therefore, we finished the proof for the necessity of the condition.

Next, to prove the sufficiency of the condition, since the number of singular points for the projective curve is invariant by a nonsingular linear change of coordinates $T:\mathbb{C}^3\to\mathbb{C}^3$ with $T(0)=0$, by the same method as we have used in the proof for the necessity of the condition, it is enough to consider the following two cases, respectively:

Case(i): Let $k=n+1$ and $\ell=m+1$. Since $k=\ell$ by assumption, then $n=m$. Note that $d=\gcd(n,k)=1$ and $e=\gcd(m,\ell)=1$. So, there is nothing to prove for Case(i), because $f(y,z)=g(y,z)$ and then $F(x,y,z)=G(x,y,z)$. Thus, the proof for the sufficiency of the condition in Case(i) is done.

Case(ii): Let $k\ge n+2$ and $\ell\ge m+2$. Suppose that there is a complex number ρ with $\rho^d=1$ such that $A_i\rho^i=B_i$ for $i=1,\ldots,d-1$ where either $\{m=n \quad \textit{and} \quad k=\ell\}$ or $\{m+n=k \quad \text{and} \quad k=\ell\}$, and $d=\gcd(n,k)$ and $e=\gcd(m,\ell)$. If either $\{m=n \quad \textit{and} \quad k=\ell\}$ or $\{m+n=k \quad \text{and} \quad k=\ell\}$, note that $d=e$. If $\{m=n \quad \textit{and} \quad k=\ell\}$, define $T:\mathbb{C}^3\to\mathbb{C}^3$ by $T(x,y,z)=(x,by,z)$ for some number b such that $b^{k/d}=\rho$. If $\{m+n=k \quad \text{and} \quad k=\ell\}$, define $T:\mathbb{C}^3\to\mathbb{C}^3$ by $T(x,y,z)=(z,by,x)$ for some number b such that $b^{k/d}=\rho$. Then it is clear that $F\circ T=G$ whether $\{m=n \quad \textit{and} \quad k=\ell\}$ or $\{m+n=k \quad \text{and} \quad k=\ell\}$. Thus, the proof for the sufficiency of the condition in Case(ii) is done.

So, we finished the proof for the sufficiency of the condition in Case(i) and Case(ii). Therefore, the proof for the projective equivalence is completely done.

THEOREM 4.2 (THE DIFFERENCE BETWEEN ANALYTIC EQUIVALENCE FOR WEIGHTED HOMOGENEOUS POLYNOMIALS IN $\mathbb{C}[y,z]$ AND PROJECTIVE EQUIVALENCE FOR THEIR HOMOGENIZATION IN $\mathbb{C}[x,y,z]$ FOR CASE(I) OF THEOREM 3.6).

Assumption *Let $f(y,z)$ and $g(y,z)$ be weighted homogeneous polynomials in $\mathbb{C}[y,z]$, and let $F(x,y,z)$ and $G(x,y,z)$ be the homogenization of $f(y,z)$ and $g(y,z)$ in $\mathbb{C}[x,y,z]$, respectively, satisfying the same assumptions and notations as in Theorem* 4.1.

Let $f \sim z^n + y^k$ with weights $(n,k) \in \mathbb{N}\times\mathbb{N}$ and $\deg(f) = k$, *and let $g \sim z^m + y^\ell$ with weights $(m,\ell) \in \mathbb{N}\times\mathbb{N}$ and* $\deg(g) = \ell$.

Conclusion *The difference between projective equivalence and analytic equivalence can be represented by three cases* (I), (II) *and* (III), *below:*

(I) *Let $gcd(n,k) < n$.*
(I-a) *Let $m = n$ and $k = \ell$. Then,*
$F(x,y,z) \sim_{proj} G(x,y,z)$ *in* $\mathbb{P}^2(\mathbb{C})$
$\iff f \approx g$ *at origin in* $\mathbb{C}^2$.

(I-b) *Let $m + n = k$ and $k = \ell$. Then,*
$F(x,y,z) \sim_{proj} G(x,y,z)$ *in* $\mathbb{P}^2(\mathbb{C})$
$\iff f \approx h$ *at origin in* $\mathbb{C}^2$.

(II) *Let $gcd(n,k) = n$. Suppose that $A_1 = B_1 = 0$ in* (4.1.1).
(II-a) *Let $m = n$ and $k = \ell$. Then,*
$F(x,y,z) \sim_{proj} G(x,y,z)$ *in* $\mathbb{P}^2(\mathbb{C})$
$\iff f \approx g$ *at origin in* $\mathbb{C}^2$.

(II-b) *Let $m + n = k$ and $k = \ell$. Then,*
$F(x,y,z) \sim_{proj} G(x,y,z)$ *in* $\mathbb{P}^2(\mathbb{C})$
$\iff f \approx h$ *at origin in* $\mathbb{C}^2$.

(III) *Let $gcd(n,k) = n$. Suppose that either $A_1 \neq 0$ or $B_1 \neq 0$ in* (4.1.1).
(III-a) *Let $m = n$ and $k = \ell$. Then,*
$F(x,y,z) \sim_{proj} G(x,y,z)$ *in* $\mathbb{P}^2(\mathbb{C})$
$\Longrightarrow f \approx g$ *at origin in* $\mathbb{C}^2$.

(III-b) *Let $m + n = k$ and $k = \ell$. Then,*
$F(x,y,z) \sim_{proj} G(x,y,z)$ *in* $\mathbb{P}^2(\mathbb{C})$
$\Longrightarrow f \approx h$ *at origin in* $\mathbb{C}^2$.

But the converse for (III) *does not hold, which will be proved by the next corollary, Corollary* 4.3.

REMARK 4.2.1. Under the same assumptions and conclusions as in Theorem 4.1, observe by Theorem 4.1 that if $F(x,y,z) \sim_{\text{proj}} G(x,y,z)$ in $\mathbb{P}^2(\mathbb{C})$ then either $\{n = m \quad \textit{and} \quad k = \ell\}$ or $\{m + n = k \quad \textit{and} \quad k = \ell\}$, and also by Theorem 3.6 that (i) and (ii) are true.

(i) $m = n$ and $k = \ell \iff f \sim g$ at the origin in $\mathbb{C}^2$ with $\deg(f) = \deg(g)$ where $f(y,z) = F(1,y,z)$ and $g(y,z) = G(1,y,z)$.

(ii) $m+n=k$ and $k=\ell \iff f\sim h$ at the origin in $\mathbb{C}^2$ with $\deg(f)=\deg(h)$ where $f(y,z)=F(1,y,z)$ and $h(x,y)=G(x,y,1)$.

Proof of Theorem 4.2. For the proof of the theorem, it is enough to consider the following two cases, respectively:
Case(A) $n=m$ and $k=\ell$.
Case(B) $n+m=k$ and $k=\ell$.

Case(A) Let $n=m$ and $k=\ell$. Suppose that $F\sim_{proj} G$ in $\mathbb{P}^2(\mathbb{C})$. To prove that $f\approx g$ at origin in $\mathbb{C}^2$, using the same notations and methods as we have seen in the proof of Case(i) and Case(ii) for the necessity of the condition of Theorem 4.1, then $G(x,y,z)=F\circ T(x,y,z)$ implies that $g(y,z)=f(by,z)$, and so there is nothing to prove. Conversely, assuming that $f\approx g$ at origin in $\mathbb{C}^2$ except for the case (III), to prove that $F(x,y,z)\sim_{proj} G(x,y,z)$ in $\mathbb{P}^2(\mathbb{C})$ just follows from Theorem 2.9 and Theorem 4.1.

Case(B) Let $n+m=k$ and $k=\ell$. Suppose that $F\sim_{proj} G$ in $\mathbb{P}^2(\mathbb{C})$. To prove that $f\approx h$ at origin in $\mathbb{C}^2$, using the same notations and methods as we have seen in the proof of Case (i) and Case(ii) for the necessity of the condition of Theorem 4.1, then $G(x,y,z)=F\circ T(x,y,z)$ implies that $h(y,z)=f(by,z)$, and so there is nothing to prove. Conversely, assuming that $f\approx h$ at origin in $\mathbb{C}^2$ except for the case (III), to prove that $F(x,y,z)\sim_{proj} G(x,y,z)$ in $\mathbb{P}^2(\mathbb{C})$ just follows from Theorem 2.9 and Theorem 4.1.

Moreover, it will be shown by Corollary 4.3 that the converse for the case (III) is not true. Therefore, the proof of theorem can be completely finished.

Corollary 4.3. *Let $F_t(x,y,z)=x^2z^2+txy^2z+y^4$ for any complex number t with $t^2\neq 4$. By Theorem 4.1, $F_t(x,y,z)\sim_{proj} F_s(x,y,z)$ in $\mathbb{P}^2(\mathbb{C})$ if and only if $\frac{t^2}{t^2-4}=\frac{s^2}{s^2-4}$. But, observe by Theorem 2.9 that $F_t(1,y,z)\approx F_s(1,y,z)$ at the origin in $\mathbb{C}^2$ and also $F_t(1,y,z)\approx F_s(x,y,1)$ at the origin in $\mathbb{C}^2$ for any s,t such that $\frac{t^2}{t^2-4}=\frac{s^2}{s^2-4}$ or $\frac{t^2}{t^2-4}\neq\frac{s^2}{s^2-4}$. Thus, an analytic equivalence at any singular point does not give a projective equivalence.*

Theorem 4.4 (The projective equivalence for Case(II) of Theorem 3.6).
Assumption Let $f(y,z)$ and $g(y,z)$ be weighted homogeneous polynomials in $\mathbb{C}[y,z]$ with weights in $\mathbb{N}\times\mathbb{Q}$, which are not homogeneous, with isolated singularity at the origin in $\mathbb{C}^2$, assuming that $f(y,z)$ and $g(y,z)$ belong to Case(II) *of Theorem* 3.6.

Let $f\sim z(z^n+y^k)$ with weights $(n,k+\frac{k}{n})\in\mathbb{N}\times\mathbb{Q}$ and $\deg(f)=k+1$, and let $g\sim z(z^m+y^\ell)$ with weights $(m,\ell+\frac{\ell}{m})\in\mathbb{N}\times\mathbb{Q}$ and $\deg(g)=\ell+1$.

By Theorem 3.6*, we may assume without loss of generality that $f(y,z)$ and $g(y,z)$ can be represented as follows:*

$$(4.4.1)\qquad \begin{aligned} f(y,z)&=zf_1(y,z)\quad \textit{with}\\ f_1(y,z)&=z^n+y^k+\sum_{i=1}^{d-1}A_iy^{ik_1}z^{(d-i)n_1},\\ g(y,z)&=zg_1(y,z)\quad \textit{with}\\ g_1(y,z)&=z^m+y^\ell+\sum_{j=1}^{e-1}B_jy^{j\ell_1}z^{(e-j)m_1},\end{aligned}$$

where

(a) $1 \leq n < k$ *and* $d = \gcd(n, k)$ *with* $n = n_1 d$ *and* $k = k_1 d$,

(b) $1 \leq m < \ell$ *and* $e = \gcd(m, \ell)$ *with* $m = m_1 e$ *and* $\ell = \ell_1 e$,

(c) *all the* A_i *and* B_j *are complex numbers for* $1 \leq i \leq d-1$ *and* $1 \leq j \leq e-1$, *respectively.*

Now, homogenize f *and* g *as follows:*

$$F(x, y, z) = x^p f(y/x, z/x) \quad \text{with } p = k+1, \tag{4.4.2}$$
$$G(x, y, z) = x^q g(y/x, z/x) \quad \text{with } q = \ell+1.$$

Conclusion

Then $F(x, y, z) \sim_{\text{proj}} G(x, y, z)$ *in* $\mathbb{P}^2(\mathbb{C})$

$\Longleftrightarrow$ *there is a complex number* ρ *with* $\rho^d = 1$ *such that* $A_i \rho^i = B_i$ *for* $i = 1, \ldots, d-1 = e-1$ *where* $n = m$ *and* $k = \ell$.

In particular, if $d = 1$, *then* $F(x, y, z) \sim_{\text{proj}} G(x, y, z)$ *in* $\mathbb{P}^2(\mathbb{C})$ $\Longleftrightarrow$ $\{n = m \quad \text{and} \quad k = \ell\}$.

Proof of Theorem 4.4. In preparation for the proof of the theorem, by (4.4.1) and (4.4.2), $F = F(x, y, z)$ and $G = G(x, y, z)$ can be written by the following:

$$F = zF_1 \quad \text{with} \tag{4.4.3}$$
$$F_1 = x^{k-n} z^n + y^k + \sum_{i=1}^{d-1} A_i x^{k - ik_1 - (d-i)n_1} y^{ik_1} z^{(d-i)n_1},$$
$$G = zG_1 \quad \text{with}$$
$$G_1 = x^{\ell-m} z^m + y^\ell + \sum_{j=1}^{e-1} B_j x^{\ell - j\ell_1 - (e-j)m_1} y^{j\ell_1} z^{(e-j)m_1},$$

where the A_i and B_j are complex numbers for $1 \leq i \leq d-1$ and $1 \leq j \leq e-1$.

First, to prove the necessity of the condition, suppose that $F \sim_{proj} G$ in $\mathbb{P}^2(\mathbb{C})$. Then, $k+1 = \deg(f) = \deg(g) = \ell+1$, and there is a nonsingular linear change of coordinates $T : \mathbb{C}^3 \to \mathbb{C}^3$ with $T(0) = 0$, satisfying the following conditions:

$$F \circ T(x, y, z) = G(x, y, z) \quad \text{with} \tag{4.4.4}$$
$$T(x, y, z) = (a_1 x + a_2 y + a_3 z, b_1 x + b_2 y + b_3 z, c_1 x + c_2 y + c_3 z),$$

where the a_i, b_i, and c_i are complex numbers for $1 \leq i \leq 3$.

Since the number of singular points for the projective curve is invariant by a nonsingular linear change of coordinates $T : \mathbb{C}^3 \to \mathbb{C}^3$ with $T(0) = 0$, for the proof of the necessity of the condition, it suffices to consider the following two cases, respectively:

Case(i): If $k = n+1$ and $\ell = m+1$, then it is clear that $F = 0$ and $G = 0$ have one and only one singular point in $\mathbb{P}^2(\mathbb{C})$, which is denoted by $(x, y, z) = (1, 0, 0)$, and conversely. Then, we may assume that $T(1, 0, 0) = \lambda(1, 0, 0)$ for some nonzero complex number λ.

Case(ii): If $k \geq n+2$ and $\ell \geq m+2$, then it is clear that $F = 0$ and $G = 0$ have two singular points in $\mathbb{P}^2(\mathbb{C})$, which are denoted by $(x, y, z) = (1, 0, 0)$ and $(x, y, z) = (0, 0, 1)$, and conversely. For the proof of this case, it is enough to consider the following two subcases, respectively:

Case(ii-a): $T(1,0,0) = \lambda(1,0,0)$ and $T(0,0,1) = \mu(0,0,1)$ for some nonzero complex numbers λ and μ.

Case(ii-b): $T(1,0,0) = \lambda(0,0,1)$ and $T(0,0,1) = \mu(1,0,0)$ for some nonzero complex numbers λ and μ.

For the proof of the necessity of the condition in these cases, by (4.4.3) and (4.4.4), observe the followings:

$$\begin{aligned} &F \circ T(x,y,z) \\ &= (c_1x + c_2y + c_3z)\{(a_1x + a_2y + a_3z)^{k-n}(c_1x + c_2y + c_3z)^n \\ &\quad + (b_1x + b_2y + b_3z)^k + \sum_{i=1}^{d-1} A_i H_i(x,y,z)\} \\ &= z\{x^{\ell-m}z^m + y^\ell + \sum_{j=1}^{e-1} B_j K_j(x,y,z)\} \\ &= G(x,y,z), \end{aligned} \tag{4.4.5}$$

where $H_i(x,y,z) = (a_1x + a_2y + a_3z)^{k-ik_1-(d-i)n_1}(b_1x + b_2y + b_3z)^{ik_1}(c_1x + c_2y + c_3z)^{(d-i)n_1}$ and $K_j(x,y,z) = x^{\ell-j\ell_1-(e-j)m_1}y^{j\ell_1}z^{(e-j)m_1}$.

Case(i): Let $k = n+1$ and $\ell = m+1$. Note that $d = \gcd(n,k) = 1$ and $e = \gcd(m,\ell) = 1$. Since $F \sim_{proj} G$ in $\mathbb{P}^2(\mathbb{C})$, then $k+1 = \deg(f) = \deg(g) = \ell+1$, and so $k = \ell$ and $n = m$. Therefore, it is clear that $f(y,z) = z(z^n + y^k) = g(y,z)$ and $F(x,y,z) = z(x^{k-n}z^n + y^k) = G(x,y,z)$. So, there is nothing to prove for the necessity of the condition in this case.

Case(ii): Let $k \geq n+2$ and $\ell \geq m+2$.

Case(ii-a): Suppose that $T(1,0,0) = \lambda(1,0,0)$ and $T(0,0,1) = \mu(0,0,1)$ for some nonzero complex numbers λ and μ. Now, we claim the following:

$$T(x,y,z) = (a_1x, b_2y, c_3z). \tag{4.4.6}$$

From Sublemma 4.1.1 in the proof of Theorem 4.1 and the assumption of Case(ii-a), it is clear that $b_1 = c_1 = a_3 = b_3 = 0$ and the Jacobian determinant of T is $a_1b_2c_3 \neq 0$.

For the proof of the claim in (4.4.6), it remains to show by (4.4.5) that $a_2 = c_2 = 0$. Using (4.4.4) and (4.4.5) with $b_1 = c_1 = a_3 = b_3 = 0$, then we have the following:

$$\begin{aligned} F \circ T(x,y,z) &= (c_2y + c_3z)\{(a_1x + a_2y)^{k-n}(c_2y + c_3z)^n \\ &\quad + (b_2y)^k + \sum_{i=1}^{d-1} A_i H_i(x,y,z)\} \\ &= z\{x^{\ell-m}z^m + y^\ell + \sum_{j=1}^{e-1} B_j K_j(x,y,z)\} \\ &= G(x,y,z), \end{aligned} \tag{4.4.7}$$

$$\begin{aligned} \text{where}\quad H_i(x,y,z) &= (a_1x + a_2y)^{k-ik_1-(d-i)n_1}(b_2y)^{ik_1}(c_2y + c_3z)^{(d-i)n_1}, \\ K_j(x,y,z) &= x^{\ell-j\ell_1-(e-j)m_1}y^{j\ell_1}z^{(e-j)m_1}. \end{aligned}$$

In order to prove the claim in (4.4.6), by the same notations and methods as we have seen in Case (ii-a) of the proof of Theorem 4.1, then we can use the following two facts (4.4.8) and (4.4.9) from (4.4.7):

(4.4.8)

(i) $0 < k - ik_1 - (d-i)n_1 = (k_1 - n_1)(d-i) < k - n$ for $1 \le i \le d-1$.
(ii) $0 < ik_1 < k$ for $1 \le i \le d-1$.
(iii) $0 < (d-i)n_1 < n$ for $1 \le i \le d-1$.
(iv) $0 < \ell - j\ell_1 - (e-j)m_1 = (\ell_1 - m_1)(e-j) < \ell - m$ for $1 \le j \le e-1$.
(v) $0 < j\ell_1 < \ell$ for $1 \le j \le e-1$.
(vi) $0 < (e-j)m_1 < m$ for $1 \le j \le e-1$.

(4.4.9) Whenever any monomial $x^\alpha y^\beta z^\gamma$ belongs to $K_j(x,y,z)$ for all $j = 1, \ldots, e-1$, then α, β and γ are all positive integers by (iv), (v) and (vi) of (4.4.8).

Now, to prove that $a_2 = c_2 = 0$, it is enough to consider an existence of the coefficients of monomials $yx^{k-n}y^n$ and $zy^{k-n}z^n$ in $F \circ T = G$, respectively.

Then, it is easy to prove the following:

(a) By (4.4.7) and (4.4.9), these two monomials $yx^{k-n}y^n$ and $zy^{k-n}z^n$ do not belong to $G(x,y,z)$.

(b) By (4.4.7) and (i) of (4.4.8), the monomial $yx^{k-n}y^n$ has the coefficient $c_2a_1^{k-n}c_2^n$ in $F \circ T$ because $x^{k-n}y^n \notin H_i(x,y,z)$ for any $i = 1, \ldots, d-1$, if exists.

(c) By (4.4.7) and (iii) of (4.4.8), the monomial $zy^{k-n}z^n$ has the coefficient $c_3a_2^{k-n}c_3^n$ in $F \circ T$ because $y^{k-n}z^n \notin H_i(x,y,z)$ for any $i = 1, \ldots, d-1$, if exists.

Because $F \circ T = G$ and the Jacobian determinant of T is $a_1b_2c_3 \neq 0$, then it is trivial by (a), (b) and (c) that $c_2a_1^{k-n}c_2^n = c_3a_2^{k-n}c_3^n = 0$, and therefore $c_2 = a_2 = 0$. Thus, we proved that $T(x,y,z) = (a_1x, b_2y, c_3z)$ in (4.4.6).

Using (4.4.6) and (4.4.7), then we have the following:

$$
\begin{aligned}
(4.4.10)\quad F \circ T(x,y,z) &= c_3z\{(a_1x)^{k-n}(c_3z)^n + (b_2y)^k + \sum_{i=1}^{d-1} A_iH_i(x,y,z)\} \\
&= z\{x^{\ell-m}z^m + y^\ell + \sum_{j=1}^{e-1} B_jK_j(x,y,z)\} \\
&= G(x,y,z), \\
\text{where}\quad H_i(x,y,z) &= (a_1x)^{k-ik_1-(d-i)n_1}(b_2y)^{ik_1}(c_3z)^{(d-i)n_1}, \\
K_j(x,y,z) &= x^{\ell-j\ell_1-(e-j)m_1}y^{j\ell_1}z^{(e-j)m_1}.
\end{aligned}
$$

From (4.4.10) with (4.4.8) and (4.4.9), comparing the coefficients of monomials $zx^{k-n}z^n$ and zy^k and $zx^{k-ik_1-(d-i)n_1}y^{ik_1}z^{(d-i)n_1}$ in $F \circ T(x,y,z)$, with the coefficients of monomials $zx^{\ell-m}z^m$ and zy^ℓ and $zx^{\ell-j\ell_1-(e-j)m_1}y^{j\ell_1}z^{(e-j)m_1}$ in $G(x,y,z)$, respectively on both sides where $1 \le i \le d-1$ and $1 \le j \le e-1$, then we get the following equations: Note that $d = \gcd(n, k)$ and $e = \gcd(m, \ell)$.

$$
\begin{aligned}
(4.4.11) \qquad & zx^{k-n}z^n = zx^{\ell-m}z^m \text{ and } zy^k = zy^\ell \\
& \text{imply that } n = m \text{ and } k = \ell \text{ ,} \quad \text{and} \\
& c_3a_1^{k-n}c_3^n = 1, \qquad c_3b_2^k = 1 \qquad \text{and} \\
& c_3A_ia_1^{k-ik_1-(d-i)n_1}b_2^{ik_1}c_3^{(d-i)n_1} = B_i \quad \text{for} \quad 1 \le i \le d-1 = e-1.
\end{aligned}
$$

Using (4.4.11) with $c_3a_1^{k-n}c_3^n = 1$, then $c_3A_ia_1^{k-ik_1-(d-i)n_1}b_2^{ik_1}c_3^{(d-i)n_1} = B_i$ can be rewritten as $A_ia_1^{-i(k_1-n_1)}b_2^{ik_1}c_3^{-in_1} = B_i$.

Let $\rho = a_1^{-(k_1-n_1)}b_2^{k_1}c_3^{-n_1}$. By (4.4.11), $\rho^d = 1$ and then $A_i\rho^i = B_i$ for each $i = 1, \cdots, d-1$. Thus, the proof for the necessity of the condition in Case (ii-a) is done.

Case(ii-b): Suppose that $T(1, 0, 0) = \lambda(0, 0, 1)$ and $T(0, 0, 1) = \mu(1, 0, 0)$ for some nonzero complex numbers λ and μ.

We claim that there is no such case.

Assume the contrary. From Sublemma 4.1.2 in the proof of Theorem 4.1 and the assumption of Case(ii-b), it is clear that $a_1 = b_1 = b_3 = c_3 = 0$, and also the Jacobian determinant of T is $-a_3b_2c_1 \neq 0$.

Using (4.4.3) and (4.4.5) with $a_1 = b_1 = b_3 = c_3 = 0$, then we have the following:

$$
\begin{aligned}
(4.4.12) \qquad F \circ T(x, y, z) &= (c_1x + c_2y)F_1 \circ T(x, y, z) \quad \text{with} \\
F_1 \circ T(x, y, z) &= (a_2y + a_3z)^{k-n}(c_1x + c_2y)^n + (b_2y)^k \\
&\quad + \sum_{i=1}^{d-1} A_iH_i(x, y, z), \\
G(x, y, z) &= zG_1(x, y, z) \quad \text{with} \\
G_1(x, y, z) &= x^{\ell-m}z^m + y^\ell + \sum_{j=1}^{e-1} B_jK_j(x, y, z), \\
\text{where} \quad H_i(x, y, z) &= (a_2y + a_3z)^{k-ik_1-(d-i)n_1}(b_2y)^{ik_1}(c_1x + c_2y)^{(d-i)n_1}, \\
K_j(x, y, z) &= x^{\ell-j\ell_1-(e-j)m_1}y^{j\ell_1}z^{(e-j)m_1}.
\end{aligned}
$$

For an easy proof, by the defining equations of $F = F(x, y, z)$ and $G = G(x, y, z)$ in (4.4.3), F and G can be rewritten in the following form:

$$
\begin{aligned}
(4.4.13) \qquad F(x, y, z) &= zF_1(x, y, z) \quad \text{with} \\
F_1(x, y, z) &= \prod_{i=1}^{d}(x^{k_1-n_1}z^{n_1} + s_iy^{k_1}), \\
G(x, y, z) &= zG_1(x, y, z) \quad \text{with} \\
G_1(x, y, z) &= \prod_{j=1}^{e}(x^{\ell_1-m_1}z^{m_1} + t_iy^{\ell_1}),
\end{aligned}
$$

where

(a) all the s_i are nonzero distinct complex numbers for $1 \le i \le d$,

(b) all the t_j are nonzero distinct complex numbers for $1 \leq j \leq e$,
(c) $0 < k_1 - n_1 < k_1$ and $0 < \ell_1 - m_1 < \ell_1$.

From (4.4.12) and (4.4.13), if $F \circ T = G$, then $c_1x + c_2y = \alpha z$ for a nonzero constant α, because $0 < k_1 - n_1 < k_1$ and $0 < \ell_1 - m_1 < \ell_1$ imply that two plane curves $F = 0$ and $G = 0$ have one and only one line at the same time, when $F = 0$ and $G = 0$ have a unique decomposition into irreducible curves in $\mathbb{P}^2(\mathbb{C})$, respectively. This would be impossible.

Thus, the claim for this case is proved, and then the proof for the necessity of the condition in Case(ii-b) is done. So we proved the necessity for Case(ii).

Therefore, we finished the proof for the necessity of the condition.

Next, to prove the sufficiency of the condition, since the number of singular points for the projective curve is invariant by a nonsingular linear change of coordinates $T : \mathbb{C}^3 \to \mathbb{C}^3$ with $T(0) = 0$, then by the same method as we have seen in the proof for the necessity of the condition, it is enough to consider the following two cases, respectively:

Case(i): Let $k = n + 1$ and $\ell = m + 1$. Since $k = \ell$ by assumption, then $n = m$. Note that $d = \gcd(n, k) = 1$ and $e = \gcd(m, \ell) = 1$. So, there is nothing to prove for Case(i), because $f(y, z) = g(y, z)$ and then $F(x, y, z) = G(x, y, z)$. Thus, the proof for the sufficiency of the condition in Case(i) is done.

Case(ii): Let $k \geq n+2$ and $\ell \geq m+2$. Suppose that there is a complex number ρ with $\rho^d = 1$ such that $A_i\rho^i = B_i$ for $i = 1, 2, \ldots, d-1$ where $m = n$ and $k = \ell$, and $d = \gcd(n, k)$. Define $T : \mathbb{C}^3 \to \mathbb{C}^3$ by $T(x, y, z) = (x, by, z)$ for some number b such that $b^{k/d} = \rho$, and then it is clear that $F \circ T = G$.

So, the proof of sufficiency is done in any case.

Therefore, the proof for the projective equivalence is completely finished.

Theorem 4.5 (The difference between analytic equivalence for weighted homogeneous polynomials in $\mathbb{C}[y, z]$ and projective equivalence for their homogenization in $\mathbb{C}[x, y, z]$ for Case(II) of Theorem 3.6).

Assumption *Let $f(y, z)$ and $g(y, z)$ be weighted homogeneous polynomials in $\mathbb{C}[y, z]$, and let $F(x, y, z)$ and $G(x, y, z)$ be the homogenization of $f(y, z)$ and $g(y, z)$ in $\mathbb{C}[x, y, z]$, respectively, satisfying the same assumptions and notations as in Theorem 4.4.*

Let $f \sim z(z^n + y^k)$ with weights $(n, k + \frac{k}{n}) \in \mathbb{N} \times \mathbb{Q}$ and $\deg(f) = k + 1$, and let $g \sim z(z^m + y^\ell)$ with weights $(m, \ell + \frac{\ell}{m}) \in \mathbb{N} \times \mathbb{Q}$ and $\deg(g) = \ell + 1$.

Conclusion *The difference between projective equivalence and analytic equivalence can be represented by three cases* (I), (II) *and* (III), *below:*

(I) *Let $gcd(n, k) < n$. Then,*
$F(x, y, z) \sim_{proj} G(x, y, z)$ *in* $\mathbb{P}^2(\mathbb{C})$
$\iff$ $f \approx g$ *at origin in* $\mathbb{C}^2$.

(II) *Let $gcd(n, k) = n$. Suppose that $A_1 = B_1 = 0$ in* (4.4.1). *Then,*
$F(x, y, z) \sim_{proj} G(x, y, z)$ *in* $\mathbb{P}^2(\mathbb{C})$
$\iff$ $f \approx g$ *at origin in* $\mathbb{C}^2$.

(III) *Let $gcd(n, k) = n$. Suppose that either $A_1 \neq 0$ or $B_1 \neq 0$ in* (4.4.1). *Then,*
$F(x, y, z) \sim_{proj} G(x, y, z)$ *in* $\mathbb{P}^2(\mathbb{C})$

$\Longrightarrow$ *$f \approx g$ at origin in $\mathbb{C}^2$.*

But the converse for (III) *does not hold, which will be proved by the next corollary, Corollary* 4.6.

REMARK 4.5.1. Under the same assumptions and conclusions as in Theorem 4.4, observe by Theorem 4.4 that if $F(x,y,z) \sim_{\text{proj}} G(x,y,z)$ in $\mathbb{P}^2(\mathbb{C})$ then $n=m$ and $k=\ell$, and also by Theorem 3.6 that the followings $(*)$ are true.
$(*)$ $m=n$ and $k=\ell \iff f \sim g$ at the origin in $\mathbb{C}^2$ with $\deg(f)=\deg(g)$ where $f(y,z)=F(1,y,z)$ and $g(y,z)=G(1,y,z)$.

Proof of Theorem 4.5. The proof of the theorem can be done by the same way as we have seen in the proof of Theorem 4.2.

COROLLARY 4.6. *For example, let $f(y,z)=z(z^2+y^4)$ and $g(y,z)=z(z^2+\frac{3}{2^{1/2}}y^2z+y^4)$. Put $F(x,y,z)=x^5f(y/x,z/x)$ and $G(x,y,z)=x^5g(y/x,z/x)$. Then, $F(x,y,z)\not\sim_{\text{proj}} G(x,y,z)$ in $\mathbb{P}^2(\mathbb{C})$, but $F(1,y,z)\approx G(1,y,z)$ at the origin in $\mathbb{C}^2$ and also $F(x,y,1)\approx G(x,y,1)$ at the origin in $\mathbb{C}^2$, too. Thus, locally analytic equivalence at any singular point does not give a projective equivalence.*

Proof of Corollary 4.6. Let $f(y,z)=z(z^2+y^4)$. Then, $f(y,z)\approx z(z^2-y^4)=z(z-y^2)(z+y^2)\approx z(z+y^2)(z+2y^2)=z(z^2+3y^2z+2y^4)$. Define $h(y,z)=z(z^2+3y^2z+2y^4)$. Then, $h(y,z)\approx z(z^2+\frac{3}{2^{1/2}}y^2z+y^4)=g(y,z)$. So, $f(y,z)\approx g(y,z)$, but it is clear by by Theorem 4.4 that $F(x,y,z)=x^5f(y/x,z/x)$ and $G(x,y,z)=x^5g(y/x,z/x)$ are not projectively equivalent in $\mathbb{P}^2(\mathbb{C})$. Note that $F(x,y,1)\approx G(x,y,1)$ at the origin in $\mathbb{C}^2$. Thus, the proof is done.

5. The projective equivalence of plane curves defined by the homogenization of weighted homogeneous polynomials whose weights $\in(\mathbb{Q}-\mathbb{N})\times\mathbb{Q}$ and its applications. Throughout this section, we study Case(III) of Theorem 3.6 in terms of Theorem 5.1 and Theorem 5.2, and study Case(IV) of Theorem 3.6 in terms of Theorem 5.4 and Theorem 5.5, respectively.

THEOREM 5.1(THE PROJECTIVE EQUIVALENCE FOR CASE(III) OF THEOREM 3.6).

Assumption *Let $f(y,z)$ and $g(y,z)$ be weighted homogeneous polynomials with no positive integer weights in $\mathbb{C}[y,z]$, which are not homogeneous, with isolated singularity at the origin in $\mathbb{C}^2$, assuming that $f(y,z)$ and $g(y,z)$ belong to* Case(III) *of Theorem* 3.6.

Let $f\sim y(z^n+y^k)$ with weights $(n+\frac{n}{k},k+1)\in(\mathbb{Q}-\mathbb{N})\times\mathbb{N}$ and $\deg(f)=k+1$, and let $g\sim y(z^m+y^\ell)$ with weights $(m+\frac{m}{\ell},\ell+1)\in(\mathbb{Q}-\mathbb{N})\times\mathbb{N}$ and $\deg(g)=\ell+1$. By Theorem 3.6, *we may assume without loss of generality that $f(y,z)$ and $g(y,z)$ can be represented as follows:*

(5.1.1)
$$\begin{aligned} f(y,z)&=yf_1(y,z) \quad \textit{with}\\ f_1(y,z)&=z^n+y^k+\sum_{i=1}^{d-1}A_iy^{ik_1}z^{(d-i)n_1},\\ g(y,z)&=yg_1(y,z) \quad \textit{with}\\ g_1(y,z)&=z^m+y^\ell+\sum_{j=1}^{e-1}B_jy^{j\ell_1}z^{(e-j)m_1},\end{aligned}$$

where

(a) $1 \leq n < k$ *and* $d = \gcd(n, k)$ *with* $n = n_1 d$ *and* $k = k_1 d$,

(b) $1 \leq m < \ell$ *and* $e = \gcd(m, \ell)$ *with* $m = m_1 e$ *and* $\ell = \ell_1 e$,

(c) *all the* A_i *and* B_j *are complex numbers for* $1 \leq i \leq d-1$ *and* $1 \leq j \leq e-1$, *respectively.*

Now, homogenize f *and* g *as follows:*

$$\begin{aligned} F(x,y,z) &= x^p f(y/x, z/x) \quad \textit{with } p = k+1, \\ G(x,y,z) &= x^q g(y/x, z/x) \quad \textit{with } q = \ell+1. \end{aligned} \tag{5.1.2}$$

Conclusion

Then, $F(x,y,z) \sim_{proj} G(x,y,z)$ *in* $\mathbb{P}^2(\mathbb{C})$

$\Longleftrightarrow$ *there is a complex number* ρ *with* $\rho^d = 1$ *such that* $A_i \rho^i = B_i$ *for* $i = 1, \ldots, d-1$ *where either* $\{m = n \quad \textit{and} \quad k = \ell\}$ *or* $\{m + n = k \quad \textit{and} \quad k = \ell\}$.

In particular, if $d = 1$, *then* $F(x,y,z) \sim_{\mathrm{proj}} G(x,y,z)$ *in* $\mathbb{P}^2(\mathbb{C})$

$\Longleftrightarrow$ *either* $\{m = n \quad \textit{and} \quad k = \ell\}$ *or* $\{m + n = k \quad \textit{and} \quad k = \ell\}$.

Proof of Theorem 5.1. In preparation for the proof of the theorem, by (5.1.1) and (5.1.2), $F = F(x,y,z)$ and $G = G(x,y,z)$ can be written by the following:

$$\begin{aligned} &F = yF_1 \quad \text{with} \\ &F_1 = x^{k-n} z^n + y^k + \sum_{i=1}^{d-1} A_i x^{k - ik_1 - (d-i)n_1} y^{ik_1} z^{(d-i)n_1}, \\ &G = yG_1 \quad \text{with} \\ &G_1 = x^{\ell-m} z^m + y^\ell + \sum_{j=1}^{e-1} B_j x^{\ell - j\ell_1 - (e-j)m_1} y^{j\ell_1} z^{(e-j)m_1}, \end{aligned} \tag{5.1.3}$$

where the A_i and B_j are complex numbers for $1 \leq i \leq d-1$ and $1 \leq j \leq e-1$.

First, to prove the necessity of the condition, suppose that $F \sim_{proj} G$ in $\mathbb{P}^2(\mathbb{C})$. Then, $k+1 = \deg(f) = \deg(g) = \ell + 1$, and there is a nonsingular linear change of coordinates $T : \mathbb{C}^3 \to \mathbb{C}^3$ with $T(0) = 0$, satisfying the following conditions:

$$\begin{aligned} &F \circ T(x,y,z) = G(x,y,z) \quad \text{with} \\ &\quad T(x,y,z) = (a_1 x + a_2 y + a_3 z, b_1 x + b_2 y + b_3 z, c_1 x + c_2 y + c_3 z), \end{aligned} \tag{5.1.4}$$

where the a_i, b_i, and c_i are complex numbers for $1 \leq i \leq 3$.

Since both $F = 0$ and $G = 0$ have exactly two singular points in $\mathbb{P}^2(\mathbb{C})$, which are denoted by $(x,y,z) = (1,0,0)$ and $(x,y,z) = (0,0,1)$, for the proof of the necessity of the condition, it suffices to consider the following two subcases, respectively:

Case(i-a) $T(1,0,0) = \lambda(1,0,0)$ and $T(0,0,1) = \mu(0,0,1)$ for some nonzero complex numbers λ and μ.

Case(i-b) $T(1,0,0) = \lambda(0,0,1)$ and $T(0,0,1) = \mu(1,0,0)$ for some nonzero complex numbers λ and μ.

For the proof of the necessity of the condition in these cases, by (5.1.3) and (5.1.4), observe the followings:

(5.1.5)

$$
\begin{aligned}
&F \circ T(x,y,z)\\
&= (b_1x+b_2y+b_3z)\{(a_1x+a_2y+a_3z)^{k-n}(c_1x+c_2y+c_3z)^n + (b_1x+b_2y+b_3z)^k\\
&\quad + \sum_{i=1}^{d-1} A_i (a_1x+a_2y+a_3z)^{k-ik_1-(d-i)n_1}(b_1x+b_2y+b_3z)^{ik_1}(c_1x+c_2y+c_3z)^{(d-i)n_1}\}\\
&= y\{x^{\ell-m}z^m + y^\ell + \sum_{j=1}^{e-1} B_j x^{\ell-j\ell_1-(e-j)m_1}y^{j\ell_1}z^{(e-j)m_1}\}\\
&= G(x,y,z).
\end{aligned}
$$

Case(i-a): Suppose that $T(1,0,0)=\lambda(1,0,0)$ and $T(0,0,1)=\mu(0,0,1)$ for some nonzero complex numbers λ and μ. Now, we claim that

$$T(x,y,z) = (a_1x, b_2y, c_3z). \tag{5.1.6}$$

By Sublemma 4.1.1 in the proof of Theorem 4.1 and by the assumption of Case(i-a), it is clear that $b_1=c_1=a_3=b_3=0$ and the Jacobian determinant of T is $a_1b_2c_3 \neq 0$.

For the proof of the claim in (5.1.6), it remains to show by (5.1.5) that $a_2=c_2=0$.

Using (5.1.4) and (5.1.5) with $b_1=c_1=a_3=b_3=0$, then we have the following:

$$
\begin{aligned}
F \circ T(x,y,z) &= (b_2y)\{(a_1x+a_2y)^{k-n}(c_2y+c_3z)^n + (b_2y)^k\\
&\quad + \sum_{i=1}^{d-1} A_i H_i(x,y,z)\}\\
&= y\{x^{\ell-m}z^m + y^\ell + \sum_{j=1}^{e-1} B_j K_j(x,y,z)\}\\
&= G(x,y,z),
\end{aligned}
\tag{5.1.7}
$$

$$
\begin{aligned}
\text{where}\quad H_i(x,y,z) &= (a_1x+a_2y)^{k-ik_1-(d-i)n_1}(b_2y)^{ik_1}(c_2y+c_3z)^{(d-i)n_1},\\
K_j(x,y,z) &= x^{\ell-j\ell_1-(e-j)m_1}y^{j\ell_1}z^{(e-j)m_1}.
\end{aligned}
$$

For the proof of the claim in (5.1.6), using the same notations and methods as we have seen in Case (ii-a) for the proof of Theorem 4.1, then observe the following two facts (5.1.8) and (5.1.9) from (5.1.7):

(5.1.8)

(i) $0 < k-ik_1-(d-i)n_1 = (k_1-n_1)(d-i) < k-n$ for $1 \le i \le d-1$.

(ii) $0 < ik_1 < k$ for $1 \le i \le d-1$.

(iii) $0 < (d-i)n_1 < n$ for $1 \le i \le d-1$.

(iv) $0 < \ell-j\ell_1-(e-j)m_1 = (\ell_1-m_1)(e-j) < \ell-m$ for $1 \le j \le e-1$.

(v) $0 < j\ell_1 < \ell$ for $1 \le j \le e-1$.

(vi) $0 < (e-j)m_1 < m$ for $1 \le j \le e-1$.

(5.1.9) Whenever any monomial $x^\alpha y^\beta z^\gamma$ belongs to $K_j(x,y,z)$ for all $j=1,\ldots,e-1$, then α, β and γ are all positive integers by (iv), (v) and (vi) of (5.1.8).

Now, to prove that $a_2 = c_2 = 0$, it is enough to consider an existence of the coefficients of monomials $yx^{k-n}y^n$ and $yy^{k-n}z^n$ in $F \circ T = G$, respectively.

Then, it is easy to prove the following:

(a) By (5.1.7) and (5.1.9), two monomials $yx^{k-n}y^n$ and $yy^{k-n}z^n$ do not belong to $G(x, y, z)$.

(b) By (5.1.7) and (i) of (5.1.8), the monomial $yx^{k-n}y^n$ has the coefficient $b_2a_1^{k-n}c_2^n$ in $F \circ T$ because $x^{k-n}y^n \notin H_i(x, y, z)$ for any $i = 1, \dots, d-1$, if exists.

(c) By (5.1.7) and (iii) of (5.1.8), the monomial $yy^{k-n}z^n$ has the coefficient $b_2a_2^{k-n}c_3^n$ in $F \circ T$ because $y^{k-n}z^n \notin H_i(x, y, z)$ for any $i = 1, \dots, d-1$, if exists.

Because $F \circ T = G$ and the Jacobian determinant of T is $a_1b_2c_3 \neq 0$, then it is trivial by (a), (b) and (c) that $b_2a_1^{k-n}c_2^n = b_2a_2^{k-n}c_3^n = 0$, and therfore $c_2 = a_2 = 0$. Thus, we proved that $T(x, y, z) = (a_1x, b_2y, c_3z)$ in (5.1.6).

Using (5.1.6) and (5.1.7), we have the following:

$$
\begin{aligned}
(5.1.10) \qquad F \circ T(x, y, z) &= (b_2y)\{(a_1x)^{k-n}(c_3z)^n + (b_2y)^k + \sum_{i=1}^{d-1} A_iH_i(x, y, z)\} \\
&= y\{x^{\ell-m}z^m + y^\ell + \sum_{j=1}^{e-1} B_jK_j(x, y, z)\} \\
&= G(x, y, z),
\end{aligned}
$$

$$
\begin{aligned}
\text{where} \quad H_i(x, y, z) &= (a_1x)^{k-ik_1-(d-i)n_1}(b_2y)^{ik_1}(c_3z)^{(d-i)n_1}, \\
K_j(x, y, z) &= x^{\ell-j\ell_1-(e-j)m_1}y^{j\ell_1}z^{(e-j)m_1}.
\end{aligned}
$$

From (5.1.10) with (5.1.8) and (5.1.9), comparing the coefficients of monomials $yx^{k-n}z^n$ and yy^k and $yx^{k-ik_1-(d-i)n_1}y^{ik_1}z^{(d-i)n_1}$ in $F \circ T(x, y, z)$, with the coefficients of monomials $yx^{\ell-m}z^m$ and yy^ℓ and $yx^{\ell-j\ell_1-(e-j)m_1}y^{j\ell_1}z^{(e-j)m_1}$ in $G(x, y, z)$, respectively on both sides where $1 \le i \le d-1$ and $1 \le j \le e-1$, then we get the following equations: Note that $k = \ell$, $d = \gcd(n, k)$ and $e = \gcd(m, \ell)$.

$$
\begin{aligned}
(5.1.11) \qquad & yx^{k-n}z^n = yx^{\ell-m}z^m \text{ and } yy^k = yy^\ell \\
& \text{imply that } n = m \text{ and } k = \ell, \quad \text{and} \\
& b_2a_1^{k-n}c_3^n = 1, \qquad b_2b_2^k = 1 \qquad \text{and} \\
& b_2A_ia_1^{k-ik_1-(d-i)n_1}b_2^{ik_1}c_3^{(d-i)n_1} = B_i \quad \text{for} \quad 1 \le i \le d-1 = e-1.
\end{aligned}
$$

Using (5.1.11) with $b_2a_1^{k-n}c_3^n = 1$, then $b_2A_ia_1^{k-ik_1-(d-i)n_1}b_2^{ik_1}c_3^{(d-i)n_1} = B_i$ can be rewritten as $A_ia_1^{-i(k_1-n_1)}b_2^{ik_1}c_3^{-in_1} = B_i$.

Let $\rho = a_1^{-(k_1-n_1)}b_2^{k_1}c_3^{-n_1}$. By (5.1.11), $\rho^d = 1$, and then $A_i\rho^i = B_i$ for each $i = 1, \cdots, d-1$. Thus, the proof for the necessity of the condition in Case(i-a) is done.

Case(i-b): Suppose that $T(1, 0, 0) = \lambda(0, 0, 1)$ and $T(0, 0, 1) = \mu(1, 0, 0)$ for some nonzero complex numbers λ and μ. Now, we claim that

$$
(5.1.12) \qquad T(x, y, z) = (a_3x, b_2y, c_1z).
$$

By Sublemma 4.1.2 in the proof of Theorem 4.1 and by the assumption of Case(i-b), it is clear that $a_1 = b_1 = b_3 = c_3 = 0$, and also the Jacobian determinant of T is $-a_3b_2c_1 \neq 0$, if exists.

For the proof of the claim in (5.1.12), it remains to show by (5.1.4) that $a_2 = c_2 = 0$.

Using (5.1.4) and (5.1.5) with $a_1 = b_1 = b_3 = c_3 = 0$, then we have the following:

$$\begin{aligned}
F \circ T(x,y,z) &= (b_2y)\{(a_2y + a_3z)^{k-n}(c_1x + c_2y)^n + (b_2y)^k \\
&\quad + \sum_{i=1}^{d-1} A_iH_i(x,y,z)\} \\
&= y\{x^{\ell-m}z^m + y^\ell + \sum_{j=1}^{e-1} B_jK_j(x,y,z)\} \\
&= G(x,y,z),
\end{aligned} \tag{5.1.13}$$

$$\begin{aligned}
\text{where}\quad H_i(x,y,z) &= (a_2y + a_3z)^{k-ik_1-(d-i)n_1}(b_2y)^{ik_1}(c_1x + c_2y)^{(d-i)n_1}, \\
K_j(x,y,z) &= x^{\ell-j\ell_1-(e-j)m_1}y^{j\ell_1}z^{(e-j)m_1}.
\end{aligned}$$

Now, to prove that $a_2 = c_2 = 0$, it is enough to compute an existence of the coefficients of monomials $yy^{k-n}x^n$ and $yz^{k-n}y^n$ in $F \circ T = G$ of (5.1.13), respectively. For such computations, use two facts (5.1.8) and (5.1.9) by the same notations and methods as we have seen in the proof of Case (i-a) of this theorem.

Then, it is easy to prove the following:

(a) By (5.1.13) and (5.1.9), two monomials $yy^{k-n}x^n$ and $yz^{k-n}y^n$ do not belong to $G(x,y,z)$.

(b) By (5.1.13) and (iii) of (5.1.8), the monomial $yy^{k-n}x^n$ has the coefficient $b_2a_2^{k-n}c_1^n$ in $F \circ T$ because $y^{k-n}x^n \notin H_i(x,y,z)$ for any $i = 1, \ldots, d-1$, if exists.

(c) By (5.1.13) and (i) of (5.1.8), the monomial $yz^{k-n}y^n$ has the coefficient $b_2a_3^{k-n}c_2^n$ in $F \circ T$, because $z^{k-n}y^n \notin H_i(x,y,z)$ for any $i = 1, \ldots, d-1$, if exists.

Because $F \circ T = G$ and the Jacobian determinant of T is $-a_3b_2c_1 \neq 0$, then it is trivial by (a), (b) and (c) that $b_2a_2^{k-n}c_1^n = b_2a_3^{k-n}c_2^n = 0$, and therefore implies $a_2 = c_2 = 0$. Thus, we proved that $T(x,y,z) = (a_3x, b_2y, c_1z)$ in (5.1.12).

Using (5.1.12) and (5.1.13), we have the following:

$$\begin{aligned}
F \circ T(x,y,z) &= b_2y\{(a_3z)^{k-n}(c_1x)^n + (b_2y)^k + \sum_{i=1}^{d-1} A_iH(x,y,z)\} \\
&= y\{x^{\ell-m}z^m + y^\ell + \sum_{j=1}^{e-1} B_jK_j(x,y,z)\} \\
&= G(x,y,z),
\end{aligned} \tag{5.1.14}$$

$$\begin{aligned}
\text{where}\quad H(x,y,z) &= (a_3z)^{k-ik_1-(d-i)n_1}(b_2y)^{ik_1}(c_1x)^{(d-i)n_1}, \\
K_j(x,y,z) &= x^{\ell-j\ell_1-(e-j)m_1}y^{j\ell_1}z^{(e-j)m_1}.
\end{aligned}$$

From (5.1.14) with (5.1.8) and (5.1.9), comparing the coefficients of monomials $yz^{k-n}x^n$ and yy^k and $yz^{k-ik_1-(d-i)n_1}y^{ik_1}x^{(d-i)n_1}$ in $F \circ T(x,y,z)$, with the coefficients of monomials $yx^{\ell-m}z^m$ and yy^ℓ and $yx^{\ell-j\ell_1-(e-j)m_1}y^{j\ell_1}z^{(e-j)m_1}$ in $G(x,y,z)$, respectively on both sides where $1 \le i \le d-1$ and $1 \le j \le e-1$, then we get the following equations: Note that $k=\ell$, $d=\gcd(n,k)$ and $e=\gcd(m,\ell)$.

$$\begin{aligned}
&yz^{k-n}x^n = yx^{\ell-m}z^m \text{ and } yy^k = yy^\ell \\
&\text{imply that } k=\ell=n+m, \quad \text{and} \\
&b_2a_3^{k-n}c_1^n = 1, \qquad b_2b_2^k = 1 \qquad \text{and} \\
&b_2A_ia_3^{k-ik_1-(d-i)n_1}b_2^{ik_1}c_1^{(d-i)n_1} = B_i \quad \text{for} \quad 1 \le i \le d-1 = e-1,
\end{aligned} \tag{5.1.15}$$

noting by (5.1.15) that $yz^{k-ik_1-(d-i)n_1}y^{ik_1}x^{(d-i)n_1}$ and $yx^{\ell-i\ell_1-(e-i)m_1}y^{i\ell_1}z^{(e-i)m_1}$ are the same monomials.

Using (5.1.15) with $b_2a_3^{k-n}c_1^n = 1$, then $b_2A_ia_3^{k-ik_1-(d-i)n_1}b_2^{ik_1}c_1^{(d-i)n_1} = B_i$ can be rewritten as $A_ia_3^{-i(k_1-n_1)}b_2^{ik_1}c_1^{-in_1} = B_i$.

Let $\rho = a_3^{-(k_1-n_1)}b_2^{k_1}c_1^{-n_1}$. By (5.1.15) $\rho^d = 1$, and then $A_i\rho^i = B_i$ for each $i = 1, \cdots, d-1$.

Thus, the proof for the necessity of the condition in Case(i-b) is done.

Therefore, we finished the proof for the necessity of the condition.

<u>Next, to prove the sufficiency of the condition</u>, suppose that there is a complex number ρ with $\rho^d = 1$ such that $A_i\rho^i = B_i$ for $i = 1, \ldots, d-1$ where either $\{m = n \quad \textit{and} \quad k = \ell\}$ or $\{m+n = k \quad \text{and} \quad k = \ell\}$, and $d = \gcd(n,k)$ and $e = \gcd(m,\ell)$. If either $\{m = n \quad \textit{and} \quad k = \ell\}$ or $\{m+n = k \quad \text{and} \quad k = \ell\}$, note that $d = e$. If $\{m = n \quad \textit{and} \quad k = \ell\}$, define $T : \mathbb{C}^3 \to \mathbb{C}^3$ by $T(x,y,z) = (x, by, z)$ for some number b such that $b^{k/d} = \rho$. If $\{m+n = k \quad \text{and} \quad k = \ell\}$, define $T : \mathbb{C}^3 \to \mathbb{C}^3$ by $T(x,y,z) = (z, by, x)$ for some number b such that $b^{k/d} = \rho$. Then it is clear that $F \circ T = G$ whether $\{m = n \quad \textit{and} \quad k = \ell\}$ or $\{m+n = k \quad \text{and} \quad k = \ell\}$. So, the proof of the sufficiency is done.

Therefore, the proof for the projective equivalence is completely finished.

Theorem 5.2 (The difference between analytic equivalence for weighted homogeneous polynomials in $\mathbb{C}[y,z]$ and projective equivalence for their homogenization in $\mathbb{C}[x,y,z]$ for Case(III) of Theorem 3.6).

<u>*Assumption*</u> *Let $f(y,z)$ and $g(y,z)$ be weighted homogeneous polynomials in $\mathbb{C}[y,z]$, and let $F(x,y,z)$ and $G(x,y,z)$ be the homogenization of $f(y,z)$ and $g(y,z)$ in $\mathbb{C}[x,y,z]$, respectively, satisfying the same assumptions and notations as in Theorem* 5.1.

Let $f \sim y(z^n + y^k)$ with weights $(n + \frac{n}{k}, k+1) \in (\mathbb{Q} - \mathbb{N}) \times \mathbb{N}$ and $\deg(f) = k+1$, and let $g \sim y(z^m + y^\ell)$ with weights $(m + \frac{m}{\ell}, \ell+1) \in (\mathbb{Q} - \mathbb{N}) \times \mathbb{N}$ and $\deg(g) = \ell+1$.

<u>*Conclusion*</u> *The difference between projective equivalence and analytic equivalence can be represented by three cases* (I), (II) *and* (III), *below:*

(I) *Let $gcd(n,k) < n$.*

(I-a) *Let $m = n$ and $k = \ell$. Then,*

$$F(x,y,z) \sim_{proj} G(x,y,z) \text{ in } \mathbb{P}^2(\mathbb{C})$$
$$\iff f \approx g \text{ at origin in } \mathbb{C}^2.$$

(I-b) *Let $m+n = k$ and $k = \ell$. Then,*

$$F(x,y,z) \sim_{proj} G(x,y,z) \text{ in } \mathbb{P}^2(\mathbb{C})$$

$\iff$ *$f \approx h$ at origin in $\mathbb{C}^2$.*

(II) *Let $gcd(n,k) = n$. Suppose that $A_1 = B_1 = 0$ in* (5.1.1).
(II-a) *Let $m = n$ and $k = \ell$. Then,*
$F(x,y,z) \sim_{proj} G(x,y,z)$ *in* $\mathbb{P}^2(\mathbb{C})$
$\iff$ *$f \approx g$ at origin in $\mathbb{C}^2$.*

(II-b) *Let $m + n = k$ and $k = \ell$. Then,*
$F(x,y,z) \sim_{proj} G(x,y,z)$ *in* $\mathbb{P}^2(\mathbb{C})$
$\iff$ *$f \approx h$ at origin in $\mathbb{C}^2$.*

(III) *Let $gcd(n,k) = n$. Suppose that either $A_1 \neq 0$ or $B_1 \neq 0$ in* (5.1.1).
(III-a) *Let $m = n$ and $k = \ell$. Then,*
$F(x,y,z) \sim_{proj} G(x,y,z)$ *in* $\mathbb{P}^2(\mathbb{C})$
$\Longrightarrow$ *$f \approx g$ at origin in $\mathbb{C}^2$.*

(III-b) *Let $m + n = k$ and $k = \ell$. Then,*
$F(x,y,z) \sim_{proj} G(x,y,z)$ *in* $\mathbb{P}^2(\mathbb{C})$
$\Longrightarrow$ *$f \approx h$ at origin in $\mathbb{C}^2$.*

But the converse for (III) *does not hold, which will be proved by the next corollary, Corollary* 5.3.

Remark 5.2.1. Under the same assumptions and conclusions as in Theorem 5.1, observe by Theorem 5.1 that if $F(x,y,z) \sim_{\text{proj}} G(x,y,z)$ in $\mathbb{P}^2(\mathbb{C})$ then either $\{n = m \quad and \quad k = \ell\}$ or $\{m + n = k \quad and \quad k = \ell\}$, and also by Theorem 3.6 that the followings (i) and (ii) are true.

(i) $m = n$ and $k = \ell \iff f \sim g$ at the origin in $\mathbb{C}^2$ with $\deg(f) = \deg(g)$ where $f(y,z) = F(1,y,z)$ and $g(y,z) = G(1,y,z)$.

(ii) $\{m + n = k \quad \text{and} \quad k = \ell\} \iff f \sim h$ at the origin in $\mathbb{C}^2$ with $\deg(f) = \deg(h)$ where $f(y,z) = F(1,y,z)$ and $h(x,y) = G(x,y,1)$.

Proof of Theorem 5.2. The proof of the theorem can be done by the same way as we have seen in the proof of Theorem 4.2.

Corollary 5.3. *Let $F_t(x,y,z) = y(x^2z^2 + txy^2z + y^4)$ for any complex number t with $t^2 \neq 4$. By Theorem* 5.1, *$F_t(x,y,z) \sim_{proj} F_s(x,y,z)$ in $\mathbb{P}^2(\mathbb{C})$ if and only if $\frac{t^2}{t^2-4} = \frac{s^2}{s^2-4}$. But, observe by Theorem* 2.9 *that $F_t(1,y,z) \approx F_s(1,y,z)$ at $(y,z) = (0,0)$ in $\mathbb{C}^2$ and also $F_t(1,y,z) \approx F_s(x,y,1)$ at the origin in $\mathbb{C}^2$ for any s,t such that $\frac{t^2}{t^2-4} = \frac{s^2}{s^2-4}$ or $\frac{t^2}{t^2-4} \neq \frac{s^2}{s^2-4}$. Thus, an analytic equivalence at any singular point does not give a projective equivalence.*

Theorem 5.4 (The projective equivalence for Case(IV) of Theorem 3.6).

<u>Assumption</u> *Let $f(y,z)$ and $g(y,z)$ be weighted homogeneous polynomials in $\mathbb{C}[y,z]$, which are not homogeneous, with isolated singularity at the origin in $\mathbb{C}^2$, assuming that $f(y,z)$ and $g(y,z)$ belong to* Case(IV) *of Theorem* 3.6.

Let $f \sim yz(z^n + y^k)$ with weights $(n + 1 + \frac{n}{k}, k + 1 + \frac{k}{n}) \in (\mathbb{Q} - \mathbb{N}) \times \mathbb{Q}$ and $\deg(f) = k + 2$, *and let $g \sim yz(z^m + y^\ell)$ with weights $(m + 1 + \frac{m}{\ell}, \ell + 1 + \frac{\ell}{m}) \in (\mathbb{Q} - \mathbb{N}) \times \mathbb{Q}$ and* $\deg(g) = \ell + 2$.

By Theorem 3.6, *we may assume without loss of generality that* $f(y,z)$ *and* $g(y,z)$ *can be represented as follows:*

$$(5.4.1)\qquad \begin{aligned} f(y,z) &= yzf_1(y,z) \quad \textit{with} \\ f_1(y,z) &= z^n + y^k + \sum_{i=1}^{d-1} A_i y^{ik_1} z^{(d-i)n_1}, \\ g(y,z) &= yzg_1(y,z) \quad \textit{with} \\ g_1(y,z) &= z^m + y^\ell + \sum_{j=1}^{e-1} B_j y^{j\ell_1} z^{(e-j)m_1}, \end{aligned}$$

where

(a) $1 \le n < k$ *and* $d = \gcd(n,k)$ *with* $n = n_1 d$ *and* $k = k_1 d$,

(b) $1 \le m < \ell$ *and* $e = \gcd(m,\ell)$ *with* $m = m_1 e$ *and* $\ell = \ell_1 e$,

(c) *all the* A_i *and* B_j *are complex numbers for* $1 \le i \le d-1$ *and* $1 \le j \le e-1$, *respectively.*

Now, homogenize f *and* g *as follows:*

$$(5.4.2)\qquad \begin{aligned} F(x,y,z) &= x^p f(y/x, z/x) \quad \textit{with } p = k+2, \\ G(x,y,z) &= x^q g(y/x, z/x) \quad \textit{with } q = \ell+2. \end{aligned}$$

Conclusion

$F(x,y,z) \sim_{\text{proj}} G(x,y,z)$ *in* $\mathbb{P}^2(\mathbb{C})$

$\Longleftrightarrow$ *there is a complex number* ρ *with* $\rho^d = 1$ *such that* $A_i\rho^i = B_i$ *for* $i = 1,\dots,d-1 = e-1$ *where* $n = m$ *and* $k = \ell$.

In particular, if $d = 1$, *then* $F(x,y,z) \sim_{\text{proj}} G(x,y,z)$ *in* $\mathbb{P}^2(\mathbb{C})$

$\Longleftrightarrow$ $n = m$ *and* $k = \ell$.

Proof of Theorem 5.4. In preparation for the proof of the theorem, by (5.4.1) and (5.4.2), $F = F(x,y,z)$ and $G = G(x,y,z)$ can be written by the following:

$$(5.4.3)\qquad \begin{aligned} F &= yzF_1 \quad \text{with} \\ F_1 &= x^{k-n}z^n + y^k + \sum_{i=1}^{d-1} A_i x^{k-ik_1-(d-i)n_1} y^{ik_1} z^{(d-i)n_1}, \\ G &= yzG_1 \quad \text{with} \\ G_1 &= x^{\ell-m}z^m + y^\ell + \sum_{j=1}^{e-1} B_j x^{\ell-j\ell_1-(e-j)m_1} y^{j\ell_1} z^{(e-j)m_1}, \end{aligned}$$

where the A_i and B_j are complex numbers for $1 \le i \le d-1$ and $1 \le j \le e-1$.

First, to prove the necessity of the condition, suppose that $F \sim_{proj} G$ in $\mathbb{P}^2(\mathbb{C})$. Then, $k+2 = \deg(f) = \deg(g) = \ell+2$, and there is a nonsingular linear change of coordinates $T : \mathbb{C}^3 \to \mathbb{C}^3$ with $T(0) = 0$, satisfying the following conditions:

$$(5.4.4)\qquad \begin{aligned} &F \circ T(x,y,z) = G(x,y,z) \quad \text{with} \\ &T(x,y,z) = (a_1x + a_2y + a_3z, b_1x + b_2y + b_3z, c_1x + c_2y + c_3z), \end{aligned}$$

where the a_i, b_i, and c_i are complex numbers for $1 \le i \le 3$.

Since both $F = 0$ and $G = 0$ have exactly two singular points in $\mathbb{P}^2(\mathbb{C})$, which are denoted by $(x, y, z) = (1, 0, 0)$ and $(x, y, z) = (0, 0, 1)$, for the proof of the necessity of the condition, it suffices to consider the following two subcases, respectively:

Case(i-a) $T(1, 0, 0) = \lambda(1, 0, 0)$ and $T(0, 0, 1) = \mu(0, 0, 1)$ for some nonzero complex numbers λ and μ.

Case(i-b) $T(1, 0, 0) = \lambda(0, 0, 1)$ and $T(0, 0, 1) = \mu(1, 0, 0)$ for some nonzero complex numbers λ and μ.

For the proof of the necessity of the condition in these cases, by (5.4.3) and (5.4.4), observe the followings:

(5.4.5)

$$\begin{aligned}
&F \circ T(x, y, z)\\
&= (b_1x + b_2y + b_3z)(c_1x + c_2y + c_3z)\\
&\quad \times \{(a_1x + a_2y + a_3z)^{k-n}(c_1x + c_2y + c_3z)^n + (b_1x + b_2y + b_3z)^k\\
&\quad + \sum_{i=1}^{d-1} A_i(a_1x + a_2y + a_3z)^{k-ik_1-(d-i)n_1}(b_1x + b_2y + b_3z)^{ik_1}(c_1x + c_2y + c_3z)^{(d-i)n_1}\}\\
&= yz\{x^{\ell-m}z^m + y^\ell + \sum_{j=1}^{e-1} B_j x^{\ell-j\ell_1-(e-j)m_1}y^{j\ell_1}z^{(e-j)m_1}\}\\
&= G(x, y, z).
\end{aligned}$$

Case(i-a): Suppose that $T(1, 0, 0) = \lambda(1, 0, 0)$ and $T(0, 0, 1) = \mu(0, 0, 1)$ for some nonzero complex numbers λ and μ. Now, we claim that

$$T(x, y, z) = (a_1x, b_2y, c_3z). \tag{5.4.6}$$

By Sublemma 4.1.1 in the proof of Theorem 4.1 and by the assumption of Case(i-a), it is clear that $b_1 = c_1 = a_3 = b_3 = 0$ and the Jacobian determinant of T is $a_1b_2c_3 \neq 0$.

For the proof of the claim in (5.4.6), it remains to show by (5.4.5) that $a_2 = c_2 = 0$.

Using (5.4.4) and (5.4.5) with $b_1 = c_1 = a_3 = b_3 = 0$, then we have the following:

$$\begin{aligned}
F \circ T(x, y, z) &= (b_2y)(c_2x + c_3z)\{(a_1x + a_2y)^{k-n}(c_2y + c_3z)^n + (b_2y)^k\\
&\quad + \sum_{i=1}^{d-1} A_iH_i(x, y, z)\}\\
&= yz\{x^{\ell-m}z^m + y^\ell + \sum_{j=1}^{e-1} B_jK_j(x, y, z)\}\\
&= G(x, y, z),
\end{aligned} \tag{5.4.7}$$

$$\text{where}\quad H_i(x, y, z) = (a_1x + a_2y)^{k-ik_1-(d-i)n_1}(b_2y)^{ik_1}(c_2y + c_3z)^{(d-i)n_1},$$
$$K_j(x, y, z) = x^{\ell-j\ell_1-(e-j)m_1}y^{j\ell_1}z^{(e-j)m_1}.$$

For the proof of the claim in (5.4.6), using the same notations and methods as we have seen in Case (ii-a) for the proof of Theorem 4.1, then observe the following two facts (5.4.8) and (5.4.9) from (5.4.7):

(5.4.8)

(i) $0 < k - ik_1 - (d-i)n_1 = (k_1 - n_1)(d-i) < k - n$ for $1 \leq i \leq d-1$.
(ii) $0 < ik_1 < k$ for $1 \leq i \leq d-1$.
(iii) $0 < (d-i)n_1 < n$ for $1 \leq i \leq d-1$.
(iv) $0 < \ell - j\ell_1 - (e-j)m_1 = (\ell_1 - m_1)(e-j) < \ell - m$ for $1 \leq j \leq e-1$.
(v) $0 < j\ell_1 < \ell$ for $1 \leq j \leq e-1$.
(vi) $0 < (e-j)m_1 < m$ for $1 \leq j \leq e-1$.

(5.4.9) Whenever any monomial $x^\alpha y^\beta z^\gamma$ belongs to $K_j(x,y,z)$ for all $j = 1, \ldots, e-1$, then α, β and γ are all positive integers by (iv), (v) and (vi) of (5.4.8).

Now, to prove that $a_2 = c_2 = 0$, it is enough to consider an existence of the coefficients of monomials $(yz)x^{k-n}y^n$ and $(yz)y^{k-n}z^n$ in $F \circ T = G$, respectively.

Then, it is easy to prove the following:

(a) By (5.4.7) and (5.4.9), two monomials $(yz)x^{k-n}y^n$ and $(yz)y^{k-n}z^n$ do not belong to $G(x,y,z)$.

(b) By (5.4.7) and (i) of (5.4.8), the monomial $(yz)x^{k-n}y^n$ has the coefficient $(b_2c_3)a_1^{k-n}c_2^n$ in $F \circ T$ because $x^{k-n}y^n \notin H_i(x,y,z)$ for any $i = 1, \ldots, d-1$, if exists.

(c) By (5.4.7) and (iii) of (5.4.8), the monomial $(yz)y^{k-n}z^n$ has the coefficient $(b_2c_3)a_2^{k-n}c_3^n$ in $F \circ T$ because $y^{k-n}z^n \notin H_i(x,y,z)$ for any $i = 1, \ldots, d-1$, if exists.

Because $F \circ T = G$ and the Jacobian determinant of T is $a_1b_2c_3 \neq 0$, then it is trivial by (a), (b) and (c) that $(b_2c_3)a_1^{k-n}c_2^n = (b_2c_3)a_2^{k-n}c_3^n = 0$, and therefore $c_2 = a_2 = 0$. Thus, we proved that $T(x,y,z) = (a_1x, b_2y, c_3z)$ in (5.4.6).

Using (5.4.6) and (5.4.7), we have the following:

$$
\begin{aligned}
(5.4.10) \quad F \circ T(x,y,z) &= (b_2y)(c_3z)\{(a_1x)^{k-n}(c_3z)^n + (b_2y)^k + \sum_{i=1}^{d-1} A_iH_i(x,y,z)\} \\
&= yz\{x^{\ell-m}z^m + y^\ell + \sum_{j=1}^{e-1} B_jK_j(x,y,z)\} \\
&= G(x,y,z), \\
\text{where} \quad H_i(x,y,z) &= (a_1x)^{k-ik_1-(d-i)n_1}(b_2y)^{ik_1}(c_3z)^{(d-i)n_1}, \\
K_j(x,y,z) &= x^{\ell-j\ell_1-(e-j)m_1}y^{j\ell_1}z^{(e-j)m_1}.
\end{aligned}
$$

From (5.4.10) with (5.4.8) and (5.4.9), compare the coefficients of monomials $(yz)x^{k-n}z^n$, $(yz)y^k$, and $(yz)x^{k-ik_1-(d-i)n_1}y^{ik_1}z^{(d-i)n_1}$ in $F \circ T(x,y,z)$, with the coefficients of monomials $(yz)x^{\ell-m}z^m$, $(yz)y^\ell$ and $(yz)x^{\ell-j\ell_1-(e-j)m_1}y^{j\ell_1}z^{(e-j)m_1}$ in $G(x,y,z)$, respectively on both sides where $1 \leq i \leq d-1$ and $1 \leq j \leq e-1$. Then, we get the following equations: Note that $k+2 = \ell+2$, $d = \gcd(n,k)$ and $e = \gcd(m,\ell)$.

(5.4.11) $yzx^{k-n}z^n = yzx^{\ell-m}z^m$ and $yzy^k = yzy^\ell$ imply that $n = m$ and $k = \ell$, and

$(b_2c_3)a_1^{k-n}c_3^n = 1, \qquad (b_2c_3)b_2^k = 1 \qquad$ and

$(b_2c_3)A_ia_1^{k-ik_1-(d-i)n_1}b_2^{ik_1}c_3^{(d-i)n_1} = B_i \quad$ for $\quad 1 \leq i \leq d-1 = e-1$.

Using (5.4.11) with $(b_2c_3)a_1^{k-n}c_3^n = 1$, then $(b_2c_3)A_ia_1^{k-ik_1-(d-i)n_1}b_2^{ik_1}c_3^{(d-i)n_1} = B_i$ can be rewritten as $A_ia_1^{-i(k_1-n_1)}b_2^{ik_1}c_3^{-in_1} = B_i$.

Let $\rho = a_1^{-(k_1-n_1)}b_2^{k_1}c_3^{-n_1}$. By (5.4.11), $\rho^d = 1$, and then $A_i\rho^i = B_i$ for each $i = 1, \cdots, d-1$. Thus, the proof for the necessity of the condition in Case (i-a) is done.

Case(i-b): Suppose that $T(1,0,0) = \lambda(0,0,1)$ and $T(0,0,1) = \mu(1,0,0)$ for some nonzero complex numbers λ and μ.

We claim that there is no such case.

Assume the contrary. By Sublemma 4.1.2 in the proof of Theorem 4.1 and by the assumption of Case(i-b), it is clear that $a_1 = b_1 = b_3 = c_3 = 0$, and also the Jacobian determinant of T is $-a_3b_2c_1 \neq 0$.

Using (5.4.4) and (5.4.5) with $a_1 = b_1 = b_3 = c_3 = 0$, then we have the following:

$$\begin{aligned}
(5.4.12)\quad F\circ T(x,y,z) &= (b_2y)(c_1x+c_2y)F_1\circ T(x,y,z) \quad \text{with}\\
F_1\circ T(x,y,z) &= (a_2y+a_3z)^{k-n}(c_1x+c_2y)^n + (b_2y)^k\\
&\quad + \sum_{i=1}^{d-1} A_iH_i(x,y,z),\\
G(x,y,z) &= yzG_1(x,y,z) \quad \text{with}\\
G_1(x,y,z) &= x^{\ell-m}z^m + y^\ell + \sum_{j=1}^{e-1} B_jK_j(x,y,z),\\
\text{where}\quad H_i(x,y,z) &= (a_2y+a_3z)^{k-ik_1-(d-i)n_1}(b_2y)^{ik_1}(c_1x+c_2y)^{(d-i)n_1},\\
K_j(x,y,z) &= x^{\ell-j\ell_1-(e-j)m_1}y^{j\ell_1}z^{(e-j)m_1}.
\end{aligned}$$

To find an easy proof, from the defining equations of $F = F(x,y,z)$ and $G = G(x,y,z)$ in (5.4.3), F and G can be rewritten as follows:

$$\begin{aligned}
(5.4.13)\qquad F(x,y,z) &= yzF_1(x,y,z) \quad \text{with}\\
F_1(x,y,z) &= \prod_{i=1}^{d}(x^{k_1-n_1}z^{n_1} + s_iy^{k_1}),\\
G(x,y,z) &= yzG_1(x,y,z) \quad \text{with}\\
G_1(x,y,z) &= \prod_{j=1}^{e}(x^{\ell_1-m_1}z^{m_1} + t_iy^{\ell_1}),
\end{aligned}$$

where

(1) all the s_i are nonzero distinct complex numbers for $1 \leq i \leq d$,
(2) all the t_j are nonzero distinct complex numbers for $1 \leq j \leq e$,
(3) $0 < k_1 - n_1 < k_1$ and $0 < \ell_1 - m_1 < \ell_1$.

From (5.4.12) and (5.4.13), assuming that two plane curves $F(x,y,z) = 0$ and $G(x,y,z) = 0$ have a unique decomposition of irreducible curves in $\mathbb{P}^2(\mathbb{C})$, then $0 < k_1 - n_1 < k_1$ and $0 < \ell_1 - m_1 < \ell_1$ imply that $F_1(x,y,z) = 0$ and $G_1(x,y,z) = 0$ have no lines at all in $\mathbb{P}^2(\mathbb{C})$, and so two plane curves $F(x,y,z) = 0$ and $G(x,y,z) = 0$ have exactly two lines in $\mathbb{P}^2(\mathbb{C})$. Since $F\circ T = G$, then $b_2y(c_1x + c_2y) = \alpha yz$ for a nonzero constant α, which would be impossible. Thus, the claim for this case is proved. Thus, the proof for the necessity of the condition in Case(i-b) is done.

So, we finished the proof for the necessity of the condition.

Next, to prove the sufficiency of the condition , suppose that there is a complex number ρ with $\rho^d = 1$ such that $A_i\rho^i = B_i$ for $i = 1, 2, \dots, d-1$ where $m = n$ and $k = \ell$, and $d = \gcd(n, k)$. Define $T : \mathbb{C}^3 \to \mathbb{C}^3$ by $T(x, y, z) = (x, by, z)$ for some number b such that $b^{k/d} = \rho$. Then, it is clear that $F \circ T = G$, and so the proof of the sufficiency is done.

Thus, the proof for the projective equivalence is completely finished.

Theorem 5.5 (The difference between analytic equivalence for weighted homogeneous polynomials in $\mathbb{C}[y, z]$ and projective equivalence for their homogenization in $\mathbb{C}[x, y, z]$ for Case(IV) of Theorem 3.6).

Assumption *Let $f(y, z)$ and $g(y, z)$ be weighted homogeneous polynomials in $\mathbb{C}[y, z]$, and let $F(x, y, z)$ and $G(x, y, z)$ be the homogenization of $f(y, z)$ and $g(y, z)$ in $\mathbb{C}[x, y, z]$, respectively, satisfying the same assumptions and notations as in Theorem* 5.4.

Let $f \sim yz(z^n + y^k)$ with weights $(n + 1 + \frac{n}{k}, k + 1 + \frac{k}{n}) \in (\mathbb{Q} - \mathbb{N}) \times \mathbb{Q}$ and $\deg(f) = k + 2$, and let $g \sim yz(z^m + y^\ell)$ with weights $(m + 1 + \frac{m}{\ell}, \ell + 1 + \frac{\ell}{m}) \in (\mathbb{Q} - \mathbb{N}) \times \mathbb{Q}$ and $\deg(g) = \ell + 2$.

Conclusion *The difference between projective equivalence and analytic equivalence can be represented by three cases* (I), (II) *and* (III), *below:*

(I) *Let $gcd(n, k) < n$. Then,*
$F(x, y, z) \sim_{proj} G(x, y, z)$ *in* $\mathbb{P}^2(\mathbb{C})$
$\Longleftrightarrow$ $f \approx g$ *at origin in* $\mathbb{C}^2$.

(II) *Let $gcd(n, k) = n$. Suppose that $A_1 = B_1 = 0$ in* (5.4.1). Then,
$F(x, y, z) \sim_{proj} G(x, y, z)$ in $\mathbb{P}^2(\mathbb{C})$
$\Longleftrightarrow$ $f \approx g$ at origin in $\mathbb{C}^2$.

(III) Let $gcd(n, k) = n$. Suppose that either $A_1 \neq 0$ or $B_1 \neq 0$ in (5.4.1). Then,
$F(x, y, z) \sim_{proj} G(x, y, z)$ in $\mathbb{P}^2(\mathbb{C})$
$\Longrightarrow$ $f \approx g$ at origin in $\mathbb{C}^2$.

But the converse for (III) does not hold, which will be proved by the next corollary, Corollary 5.6.

Remark 5.5.1. Under the same assumptions and conclusions as in Theorem 5.4, observe by Theorem 5.4 that if $F(x, y, z) \sim_{\text{proj}} G(x, y, z)$ in $\mathbb{P}^2(\mathbb{C})$ then $n = m$ and $k = \ell$, and also by Theorem 3.6 that $n = m$ and $k = \ell$ if and only if $f \sim g$ at origin in $\mathbb{C}^2$ with $\deg(f) = \deg(g)$.

Proof of Theorem 5.5. The proof of the theorem can be done by the same way as we have seen in the proof of Theorem 4.2.

Corollary 5.6. *For example, let $f(y, z) = yz(z^2 + y^4)$ and $g(y, z) = yz(z^2 + \frac{3}{2^{1/2}} y^2 z + y^4)$. Put $F(x, y, z) = x^6 f(y/x, z/x)$ and $G(x, y, z) = x^6 g(y/x, z/x)$. Then, $F(x, y, z) \not\sim_{\text{proj}} G(x, y, z)$ in $\mathbb{P}^2(\mathbb{C})$, but $F(1, y, z) \approx G(1, y, z)$ at the origin in $\mathbb{C}^2$ and* *also* $F(x,y,1) \approx G(x,y,1)$ *at the origin in* $\mathbb{C}^2$*, too. Thus, an analytic equivalence at any singular point does not give a projective equivalence.*

Proof of Corollary 5.6. Let $f(y,z) = yz(z^2+y^4)$. Then, $f(y,z) \approx yz(z^2 - y^4) = yz(z-y^2)(z+y^2) \approx yz(z+y^2)(z+2y^2) = yz(z^2+3y^2z+2y^4)$. Define $h(y,z) = yz(z^2+3y^2z+2y^4)$. Then, $h(y,z) \approx yz(z^2 + \frac{3}{2^{1/2}}y^2z + y^4) = g(y,z)$. So, $f(y,z) \approx g(y,z)$, but $F(x,y,z) = x^6 f(y/x, z/x)$ and $G(x,y,z) = x^6 g(y/x, z/x)$ are not projectively equivalent in $\mathbb{P}^2(\mathbb{C})$ by Theorem 5.4. Thus, the proof is done.

6. The summary for the projective equivalence of plane curve singularities defined by the homogenization of weighted homogeneous polynomials in $\mathbb{C}[y,z]$ and its applications to their analytic equivalence. Summing up the results of Theorem 4.1, Theorem 4.4, Theorem 5.1 and Theorem 5.4, we can find the solution of the first problem in terms of the following theorem, without any need of proof.

THEOREM 6.1 (THE PROJECTIVE EQUIVALENCE OF PLANE CURVE SINGULARITIES DEFINED BY THE HOMOGENIZATION OF WEIGHTED HOMOGENEOUS POLYNOMIALS IN $\mathbb{C}[y,z]$).

Assumption *Let* $f(y,z)$ *and* $g(y,z)$ *be weighted homogeneous polynomials in* $\mathbb{C}[y,z]$*, which are not homogeneous, with isolated singularity at the origin in* $\mathbb{C}^2$*.*

By Theorem 3.6*, whether or not* $f \not\sim z^2+y^2$ *and* $g \not\sim z^2+y^2$*,* f *and* g *can be topologically written in a unique way:*

(1) $f \sim y^{\varepsilon_1} z^{\varepsilon_2}(z^n + y^k)$ *with weights* $(n + \varepsilon_2 + \frac{n}{k}\varepsilon_1, k + \varepsilon_1 + \frac{k}{n}\varepsilon_2) \in \mathbb{Q} \times \mathbb{Q}$ *and with* $\deg(f) = k + \varepsilon_1 + \varepsilon_2$*,*

(2) $g \sim y^{\delta_1} z^{\delta_2}(z^m + y^\ell)$ *with weights* $(m + \delta_2 + \frac{m}{\ell}\delta_1, \ell + \delta_1 + \frac{\ell}{m}\delta_2) \in \mathbb{Q} \times \mathbb{Q}$ *and with* $\deg(g) = \ell + \delta_1 + \delta_2$*,*

where

(a) $1 \le n < k$*,* $d = \gcd(n,k)$ *with* $n = n_1 d$ *and* $k = k_1 d$*,*
(b) $\varepsilon_1, \varepsilon_2$ *are either* 1 *or* 0*, respectively,*
(c) $1 \le m < \ell$ *and* $e = \gcd(m,\ell)$ *with* $m = m_1 e$ *and* $\ell = \ell_1 e$*,*
(d) δ_1 *and* δ_2 *are either* 1 *or* 0*, respectively,*
(e) *if* $\varepsilon_1 = \varepsilon_2 = 0$*, then* $n \ge 2$*,*
(f) *if* $\delta_1 = \delta_2 = 0$*, then* $m \ge 2$*.*

By Theorem 3.6 *again, we may assume without loss of generality that*

$$
\begin{aligned}
(6.1.1)\qquad f(y,z) &= y^{\varepsilon_1} z^{\varepsilon_2} f_1(y,z) \quad \textit{with} \\
f_1(y,z) &= z^n + y^k + \sum_{i=1}^{d-1} A_i y^{ik_1} z^{(d-i)n_1}, \\
g(y,z) &= y^{\delta_1} z^{\delta_2} g_1(y,z) \quad \textit{with} \\
g_1(y,z) &= z^m + y^\ell + \sum_{j=1}^{e-1} B_j y^{j\ell_1} z^{(e-j)m_1},
\end{aligned}
$$

where

(a) *all the* A_i *are complex numbers for* $1 \le i \le d-1$*,*
(b) *all the* B_j *are complex numbers for* $1 \le j \le e-1$*,*

satisfying the following property:

$$\text{(6.1.2)} \qquad \underline{\textit{if } \gcd(n,k)=n, \quad \textit{i.e.,} \quad n_1=1, \textit{ then either } A_1=0 \textit{ or } A_1\neq 0,}$$
$$\underline{\textit{if } \gcd(m,\ell)=m, \quad \textit{i.e.,} \quad m_1=1, \textit{ then either } B_1=0 \textit{ or } B_1\neq 0.}$$

<u>Conclusion</u> *Now, homogenize f and g as follows:*

$$\text{(6.1.3)} \qquad \begin{aligned} F(x,y,z) &= x^p f(y/x, z/x) \quad \textit{with } p=k+\varepsilon_1+\varepsilon_2,\\ G(x,y,z) &= x^q g(y/x, z/x) \quad \textit{with } q=\ell+\delta_1+\delta_2. \end{aligned}$$

Then, $F(x,y,z) \sim_{proj} G(x,y,z)$ in $\mathbb{P}^2(\mathbb{C})$
$\iff$ *$k=\ell$, $\varepsilon_i=\delta_i$ for $i=1,2$, and there is a complex number ρ with $\rho^d=1$ such that $A_i\rho^i=B_i$ for $i=1,\ldots,d-1$ where either $\{n=m$ and $k=\ell\}$ or $\{n+m=k$ and $k=\ell\}$.*

Now, using Theorem 6.1 and Theorem 2.9, and also summing up the results of Theorem 4.2, Theorem 4.5, Theorem 5.2 and Theorem 5.5, then we can get the solution of the second problem in terms of the following theorem, without any need of proof.

Theorem 6.2 (The difference between analytic equivalence for weighted homogeneous polynomials in $\mathbb{C}[y,z]$ and projective equivalence for their homogenization in $\mathbb{C}[x,y,z]$).

<u>Assumption</u> *Let $f(y,z)$ and $g(y,z)$ be weighted homogeneous polynomials in $\mathbb{C}[y,z]$, and let $F(x,y,z)$ and $G(x,y,z)$ be the homogenization of $f(y,z)$ and $g(y,z)$ in $\mathbb{C}[x,y,z]$, respectively, satisfying the same assumptions and notations as in Theorem* 6.1*.*

For brevity of notation, observe by Theorem 6.1 *that if $F(x,y,z)\sim_{proj} G(x,y,z)$ in $\mathbb{P}^2(\mathbb{C})$, then either $\{n=m$ and $k=\ell\}$ or $\{n+m=k$ and $k=\ell\}$, and $\varepsilon_i=\delta_i$ for $i=1,2$, and that the followings* (i) *and* (ii) *are true.*

(i) *$n=m$, $k=\ell$ and $\varepsilon_i=\delta_i$ for $i=1,2$ if and only if $f\sim g$ at origin in $\mathbb{C}^2$ with* $\deg(f)=\deg(g)$ *where $f(y,z)=F(1,y,z)$ and $g(y,z)=G(1,y,z)$.*

(ii) *$n+m=k$, $k=\ell$ and $\varepsilon_1=\delta_1$ and $\varepsilon_2=\delta_2=0$ if and only if $f\sim h$ at origin in $\mathbb{C}^2$ with* $\deg(f)=\deg(h)$ *where $f(y,z)=F(1,y,z)$ and $h(x,y)=G(x,y,1)$.*

<u>Conclusion</u>

(I) *Let gcd$(n,k)<n$ with $d=\gcd(n,k)$.*

(I-a) *Assume that $n=m$ and $k=\ell$. Then,*
$F(x,y,z)\sim_{proj} G(x,y,z)$ *in* $\mathbb{P}^2(\mathbb{C})$
$\iff$ *$f\approx g$ at origin in $\mathbb{C}^2$.*

(I-b) *Assume that $n+m=k$ and $k=\ell$. Then,*
$F(x,y,z)\sim_{proj} G(x,y,z)$ *in* $\mathbb{P}^2(\mathbb{C})$
$\iff$ *$f\approx h$ at origin in $\mathbb{C}^2$.*

(II) *Let gcd$(n,k)=n$. Suppose that $A_1=B_1=0$ in* (6.1.1)*.*

(II-a) *Assume that $n=m$ and $k=\ell$. Then,*
$F(x,y,z)\sim_{proj} G(x,y,z)$ *in* $\mathbb{P}^2(\mathbb{C})$
$\iff$ *$f\approx g$ at origin in $\mathbb{C}^2$.*

(II-b) *Assume that $n+m=k$ and $k=\ell$. Then,*
$F(x,y,z)\sim_{proj} G(x,y,z)$ *in* $\mathbb{P}^2(\mathbb{C})$

$\Longleftrightarrow$ *$f \approx h$ at origin in $\mathbb{C}^2$.*

(III) *Let $gcd(n,k) = n$. Suppose that either $A_1 \neq 0$ or $B_1 \neq 0$ in* (6.1.1)*.*
(III-a) *Assume that $n = m$ and $k = \ell$. Then,*
$F(x,y,z) \sim_{proj} G(x,y,z)$ *in* $\mathbb{P}^2(\mathbb{C})$
$\Longrightarrow f \approx g$ *at origin in* $\mathbb{C}^2$*.*

(III-b) *Assume that $n + m = k$ and $k = \ell$. Then,*
$F(x,y,z) \sim_{proj} G(x,y,z)$ *in* $\mathbb{P}^2(\mathbb{C})$
$\Longrightarrow f \approx h$ *at origin in* $\mathbb{C}^2$*.*
But, the converse for (III) *does not hold, which will be proved by the next corollary, Corollary* 6.3*.*

In other words, summing up the results of (I)*,* (II) *and* (III)*, we have the followings:*
If $F(x,y,z) \sim_{proj} G(x,y,z)$ in $\mathbb{P}^2(\mathbb{C})$ and $f \sim g$ at origin in $\mathbb{C}^2$, then $f \approx g$ at origin in $\mathbb{C}^2$, and not conversely, and also if $F(x,y,z) \sim_{proj} G(x,y,z)$ in $\mathbb{P}^2(\mathbb{C})$ and $f \sim h$ at origin in $\mathbb{C}^2$, then $f \approx h$ at origin in $\mathbb{C}^2$, and not conversely.

Remark 6.2.1. On the assumption that $f \sim g$ with $\deg(f) = \deg(g)$ instead of assuming that $n = m$ and $k = \ell$, we may get the same statements (I-a), (II-a) and (III-a) as before, if necessary. Similarly, on the assumption that $f \sim h$ with $\deg(f) = \deg(h)$ instead of assuming that $n + m = \ell$ and $k = \ell$, we may get the same statements (I-b), (II-b) and (III-b) as before, if necessary.

Corollary 6.3. *Let $f(y,z) = y^{\varepsilon_1} z^{\varepsilon_2}(z^2 + y^4)$ and $g(y,z) = y^{\varepsilon_1} z^{\varepsilon_2}(z^2 + \frac{3}{2^{1/2}} y^2 z + y^4)$. Put $F(x,y,z) = x^{4+\varepsilon_1+\varepsilon_2} f(y/x, z/x)$ and $G(x,y,z) = x^{4+\varepsilon_1+\varepsilon_2} g(y/x, z/x)$. Then, $F(x,y,z) \not\sim_{\mathrm{proj}} G(x,y,z)$ in $\mathbb{P}^2(\mathbb{C})$, but $F(1,y,z) \approx G(1,y,z)$ at the origin in $\mathbb{C}^2$ and also $F(x,y,1) \approx G(x,y,1)$ at the origin in $\mathbb{C}^2$. Thus, an analytic equivalence at any singular point does not give a projective equivalence.*

Proof of Corollary 6.3. The proof just follows from Corollary 4.3, Corollary 4.6, Corollary 5.3 and Corollary 5.6.

As a generalization of Theorem 6.1, we have the following.

Corollary 6.4 (The projective equivalence of plane curve singularities defined by the homogenization of weighted homogeneous polynomials in $\mathbb{C}[y,z]$).

Assumption *Let $f(y,z)$ and $g(y,z)$ be weighted homogeneous polynomials in $\mathbb{C}[y,z]$, which are not homogeneous, with isolated singularity at the origin in $\mathbb{C}^2$.*

By Theorem 3.6*, we may assume without loss of generality that f and g can be written as follows:*

(1) *$f \sim y^{\varepsilon_1} z^{\varepsilon_2}(z^n + y^k)$ with weights $(n + \varepsilon_2 + \frac{n}{k}\varepsilon_1, k + \varepsilon_1 + \frac{k}{n}\varepsilon_2) \in \mathbb{Q} \times \mathbb{Q}$ and with* $\deg(f) = k + \varepsilon_1 + \varepsilon_2$*,*

(2) *$g \sim y^{\delta_1} z^{\delta_2}(z^m + y^\ell)$ with weights $(m + \delta_2 + \frac{m}{\ell}\delta_1, \ell + \delta_1 + \frac{\ell}{m}\delta_2) \in \mathbb{Q} \times \mathbb{Q}$ and with* $\deg(g) = \ell + \delta_1 + \delta_2$*,*

where

(a) *$1 \leq n < k$, $d = \gcd(n,k)$ with $n = n_1 d$ and $k = k_1 d$,*
(b) *$\varepsilon_1, \varepsilon_2$ are either* 1 *or* 0*, respectively,*
(c) *$1 \leq m < \ell$ and $e = \gcd(m,\ell)$ with $m = m_1 e$ and $\ell = \ell_1 e$,*
(d) *δ_1 and δ_2 are either* 1 *or* 0*, respectively,*

(e) *if* $\varepsilon_1 = \varepsilon_2 = 0$, *then* $n \geq 2$,
(f) *if* $\delta_1 = \delta_2 = 0$, *then* $m \geq 2$.

By Theorem 3.6, *we may assume without loss of generality that*

$$(6.4.1)\qquad \begin{aligned} f(y,z) &= y^{\varepsilon_1} z^{\varepsilon_2} f_1(y,z) \quad \textit{with} \\ f_1(y,z) &= A_0 z^n + A_d y^k + \sum_{i=1}^{d-1} A_i y^{ik_1} z^{(d-i)n_1}, \\ g(y,z) &= y^{\delta_1} z^{\delta_2} g_1(y,z) \quad \textit{with} \\ g_1(y,z) &= B_0 z^m + B_e y^\ell + \sum_{j=1}^{e-1} B_j y^{j\ell_1} z^{(e-j)m_1}, \end{aligned}$$

where

(a) *the* A_i *are complex numbers for* $0 \leq i \leq d$ *with* $A_0 A_d \neq 0$,
(b) *the* B_j *are complex numbers for* $0 \leq j \leq e$ *with* $B_0 B_e \neq 0$,

satisfying the following property:

$$(6.4.2)\qquad \begin{aligned} &\underline{\textit{if } \gcd(n,k) = n, \quad \textit{i.e.,} \quad n_1 = 1, \textit{ then either } A_1 = 0 \textit{ or } A_1 \neq 0,} \\ &\underline{\textit{if } \gcd(m,\ell) = m, \quad \textit{i.e.,} \quad m_1 = 1, \textit{ then either } B_1 = 0 \textit{ or } B_1 \neq 0.} \end{aligned}$$

Now, homogenize f *and* g *as follows:*

$$(6.4.3)\qquad \begin{aligned} F(x,y,z) &= x^p f(y/x, z/x) \quad \textit{with } p = k + \varepsilon_1 + \varepsilon_2, \\ G(x,y,z) &= x^q g(y/x, z/x) \quad \textit{with } q = \ell + \delta_1 + \delta_2. \end{aligned}$$

<u>Conclusion</u>

Then, $F(x,y,z) = 0$ *and* $G(x,y,z) = 0$ *are projectively equivalent in* $\mathbb{P}^2(\mathbb{C})$ $\iff$ $k = \ell$, $\varepsilon_i = \delta_i$ *for* $i = 1, 2$ *and there is a complex number* ρ *with* $\dfrac{A_d}{A_0}\rho^d = \dfrac{B_d}{B_0}$ *such that* $\dfrac{A_i}{A_0}\rho^i = \dfrac{B_i}{B_0}$ *for* $i = 1, \ldots, d-1$ *where either* $\{n = m$ *and* $k = \ell\}$ *or* $\{n + m = k$ *and* $k = \ell\}$.

Proof of Corollary 6.4. We may start to assume by (6.4.1) that $F = F(x,y,z)$ and $G = G(x,y,z)$ are written by the following:

$$(6.4.4)\qquad \begin{aligned} F &= y^{\varepsilon_1} z^{\varepsilon_2} (A_0 x^{k-n} z^n + A_d y^k + \sum_{i=1}^{d-1} A_i x^{k-ik_1-(d-i)n_1} y^{ik_1} z^{(d-i)n_1}), \\ G &= y^{\delta_1} z^{\delta_2} (B_0 x^{\ell-m} z^m + B_e y^\ell + \sum_{j=1}^{e-1} B_j x^{\ell-j\ell_1-(e-j)m_1} y^{j\ell_1} z^{(e-j)m_1}). \end{aligned}$$

From (6.4.4), define $F_1 = F_1(x,y,z)$ and $G_1 = G_1(x,y,z)$ by the following:

$$(6.4.5)\qquad \begin{aligned} F_1 &= y^{\varepsilon_1} z^{\varepsilon_2} \{x^{k-n} z^n + (\frac{A_d}{A_0}) y^k + \sum_{i=1}^{d-1} \frac{A_i}{A_0} x^{k-ik_1-(d-i)n_1} y^{ik_1} z^{(d-i)n_1}\}, \\ G_1 &= y^{\delta_1} z^{\delta_2} \{x^{\ell-m} z^m + (\frac{B_e}{B_0}) y^\ell + \sum_{j=1}^{e-1} \frac{B_j}{B_0} x^{\ell-j\ell_1-(e-j)m_1} y^{j\ell_1} z^{(e-j)m_1}\}. \end{aligned}$$

Note that $F(x,y,z) \sim_{proj} G(x,y,z)$ in $\mathbb{P}^2(\mathbb{C}) \iff F_1(x,y,z) \sim_{proj} G_1(x,y,z)$ in $\mathbb{P}^2(\mathbb{C})$.

Let $v = (\frac{A_d}{A_0})^{\frac{1}{k}} y$ and $w = (\frac{B_e}{B_0})^{\frac{1}{\ell}} y$. Then, $y^{ik_1} = (\frac{A_d}{A_0})^{\frac{-i}{d}} v^{ik_1}$ and $y^{j\ell_1} = (\frac{B_e}{B_0})^{\frac{-j}{e}} y^{j\ell_1}$.

From (6.4.5), define $F_2 = F_2(x,v,z)$ and $G_2 = G_2(x,w,z)$ by the following:

$$(6.4.6) \quad F_2 = v^{\varepsilon_1} z^{\varepsilon_2} \{x^{k-n} z^n + v^k + \sum_{i=1}^{d-1} \frac{A_i}{A_0} (\frac{A_d}{A_0})^{\frac{-i}{d}} x^{k-ik_1-(d-i)n_1} v^{ik_1} z^{(d-i)n_1}\},$$

$$G_2 = w^{\delta_1} z^{\delta_2} \{x^{\ell-m} z^m + w^\ell + \sum_{j=1}^{e-1} \frac{B_j}{B_0} (\frac{B_e}{B_0})^{\frac{-j}{e}} x^{\ell-j\ell_1-(e-j)m_1} w^{j\ell_1} z^{(e-j)m_1}\}.$$

Note that $F_1(x,y,z) \sim_{proj} G_1(x,y,z)$ in $\mathbb{P}^2(\mathbb{C}) \iff F_2(x,y,z) \sim_{proj} G_2(x,y,z)$ in $\mathbb{P}^2(\mathbb{C})$.

By Theorem 6.1, we have the following consequences:

$F_2(x,y,z) \sim_{proj} G_2(x,y,z)$ in $\mathbb{P}^2(\mathbb{C})$

$\iff k = \ell$, $\varepsilon_i = \delta_i$ for $i = 1,2$, and there is a complex number τ with $\tau^d = 1$ such that $P_i \tau^i = Q_i$ for $i = 1, \ldots, d-1 = e-1$ where either $\{n = m$ and $k = \ell\}$ or $\{n + m = k$ and $k = \ell\}$, noting that $P_i = \dfrac{A_i}{A_0}(\dfrac{A_d}{A_0})^{\frac{-i}{d}}$ and $Q_i = \dfrac{B_i}{B_0}(\dfrac{B_d}{B_0})^{\frac{-i}{d}}$.

Now, define ρ by $(\dfrac{A_d}{A_0})^{\frac{1}{d}} \rho = (\dfrac{B_d}{B_0})^{\frac{1}{d}} \tau$. Then $\rho^d = \dfrac{\frac{B_d}{B_0}}{\frac{A_d}{A_0}}$, and also $P_i \tau^i = Q_i$ implies that $\dfrac{A_i}{A_0} \rho^i = \dfrac{B_i}{B_0}$ for $i = 1, \ldots, d-1$, and conversely. Thus, the proof can be finished.

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