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RIEMANNIAN EXPONENTIAL MAPS OF THE DIFFEOMORPHISM GROUP OF $\mathbb{T}^2$*

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Abstract. We study the exponential maps induced by right-invariant weak Riemannian metrics of Sobolev type of order $k \geq 0$ on the Lie group of smooth, orientation preserving diffeomorphisms of the two dimensional torus. We prove that for $k \geq 1$, but not for $k = 0$, each of them defines a smooth Fréchet chart of the identity.

Key words. Geodesic exponential maps, diffeomorphism group of the torus

AMS subject classifications. 58D05, 58E10

1. Introduction. The aim of this paper is to contribute towards a theory of Riemannian geometry on infinite dimensional Lie groups. These groups have attracted a lot of attention since Arnold's seminal paper [1] on hydrodynamics – e.g. [12], [19], [24], [25]. As a case study we consider the Lie group $\mathcal{D}_+ = \mathcal{D}_+(\mathbb{T}^2)$ of orientation preserving C^∞-diffeomorphisms of the two dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. The Lie algebra $T_{\mathtt{id}}\mathcal{D}_+$ of $\mathcal{D}_+$ is the space $C^\infty(\mathbb{T}^2, \mathbb{R}^2)$ of smooth vector fields on $\mathbb{T}^2$. We remark that $\mathcal{D}_+$ and its Lie algebra come up in hydrodynamics playing the role of configuration spaces for compressible and inviscid fluids on $\mathbb{T}^2$.

For any given $k \geq 0$, consider the scalar product $\langle \cdot, \cdot \rangle_k : C^\infty(\mathbb{T}^2, \mathbb{R}^2) \times C^\infty(\mathbb{T}^2, \mathbb{R}^2) \to \mathbb{R}$,

$$\langle u, v \rangle_k := \sum_{0 \leq j \leq k} \int_{\mathbb{T}^2} \langle (-\Delta)^j u, v \rangle dx$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^2$. It induces a C^1_F-smooth weak right-invariant Riemannian metric $\nu^{(k)}$ on $\mathcal{D}_+$,

$$\nu^{(k)}_\varphi(\xi, \eta) := \langle (d_{\mathtt{id}} R_\varphi)^{-1} \xi, (d_{\mathtt{id}} R_\varphi)^{-1} \eta \rangle_k, \quad \forall \varphi \in \mathcal{D}_+, \quad \text{and} \quad \forall \xi, \eta \in T_\varphi \mathcal{D}_+$$

where $R_\varphi : \mathcal{D}_+ \to \mathcal{D}_+$, $\psi \mapsto \psi \circ \varphi$ denotes the right translation by φ. The subscript F in C^1_F, refers to the calculus in Fréchet spaces – see Appendix A where we collect some definitions and notions of the calculus in Fréchet spaces. The metric $\nu^{(k)}$ being weak means that the topology induced by $\nu^{(k)}$ on the tangent space $T_\varphi \mathcal{D}_+$ at an arbitrary point φ in $\mathcal{D}_+$, is weaker than the Fréchet topology on $T_\varphi \mathcal{D}_+ \cong C^\infty(\mathbb{T}^2, \mathbb{R}^2)$ – see e.g. [12].

DEFINITION 1.1. *For any given $T > 0$, a C^2_F-smooth curve $\varphi : [0, T] \to \mathcal{D}_+$, is called a geodesic for $\nu^{(k)}$, or $\nu^{(k)}$-geodesic for short, if it is a critical point of the*

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action functional within the class of C^2_F-smooth variations γ constrained to keep the end points fixed. More precisely, for any C^2_F-smooth function

$$\gamma : (-\varepsilon, \varepsilon) \times [0, T] \to \mathcal{D}_+, \quad (s,t) \mapsto \gamma(s,t)$$

satisfying $\gamma(0,t) = \varphi(t)$, for every $0 \le t \le T$, and $\gamma(s,0) = \varphi(0)$ and $\gamma(s,T) = \varphi(T)$ for any $-\varepsilon < s < \varepsilon$, one has

$$\frac{d}{ds}\Big|_{s=0} \mathcal{E}_k^T(\gamma(s,\cdot)) = 0, \tag{1.1a}$$

where $\mathcal{E}_k^T$ denotes the action functional

$$\mathcal{E}_k^T(\gamma(s,\cdot)) := \frac{1}{2}\int_0^T \nu^{(k)}_{\gamma(s,t)}(\dot\gamma(s,t), \dot\gamma(s,t))dt\,, \tag{1.1b}$$

and $\dot\gamma(s,t) = \partial\gamma(s,t)/\partial t$.

The Euler-Lagrange equations for the action functional $\mathcal{E}_k^T$ defined in (1.1b) lead to the following initial value problem

$$\begin{cases} \dot\varphi = v, & \dot v = F_k(\varphi, v) \\ \varphi(0) = \mathtt{id}, & v(0) = v_0 \end{cases} \tag{1.2}$$

where

$$F_k(\varphi, v) := R_\varphi \circ A_k^{-1} \circ B_k(v \circ \varphi^{-1}) \tag{1.3}$$

in which

$$A_k := \mathrm{Id} + \sum_{i=1}^k (-\Delta)^i \tag{1.4}$$

and, for any smooth function u in $C^\infty(\mathbb{T}^2, \mathbb{R}^2)$,

$$B_k(u) := A_k((du)u) - (dA_k u)u - \big(\operatorname{div} u \cdot \mathrm{Id} + (du)^\dagger\big) A_k u. \tag{1.5}$$

Here $(\cdot)^{\cdot}$ stands for d/dt, $\Delta = \partial^2_{x_1} + \partial^2_{x_2}$ is the Laplacian, $((du)^\dagger)_{ij} = (du)_{ji}$ the transpose of du, and Id the 2×2 identity matrix. We remark that one can write $(dA_k u)u = (u\cdot\nabla)A_k u$, and $(du)u = (u\cdot\nabla)u$, where $u\cdot\nabla$ is the vector field $u_1\partial_{x_1} + u_2\partial_{x_2}$. The operators $u\cdot\nabla$, Δ, and A_k, act componentwise on functions in $C^\infty(\mathbb{T}^2, \mathbb{R}^2)$. Note that $t \mapsto \varphi(t)$ evolves in $\mathcal{D}_+$ whereas $t \mapsto \dot\varphi(t) = v(t)$ is a vector field along φ i.e., a section of $\varphi^* T\mathcal{D}_+$, and $v(t)$ in $T_{\varphi(t)}\mathcal{D}_+$. It is easy to check that (1.2) is equivalent to the two initial value problems

$$\begin{cases} \dot\varphi = u \circ \varphi \\ \varphi(0) = \mathtt{id} \end{cases} \tag{1.6}$$

and

$$\begin{cases} (A_k u)^{\cdot} + (dA_k u)u + \big(\operatorname{div} u \cdot \mathrm{Id} + (du)^\dagger\big) A_k u = 0 \\ u(0) = v_0 \end{cases} \tag{1.7}$$

where $t \mapsto u(t) = (d_{\mathtt{id}} R_{\varphi(t)})^{-1}\dot{\varphi}(t)$ is a curve in $T_{\mathtt{id}}\mathcal{D}_+$. The initial value problems (1.2) and (1.7) are, via (1.6), two alternative descriptions of the geodesic flow. The first corresponds to the Lagrangian description i.e., tracking the flow as a section of $\varphi^* T\mathcal{D}_+$ while the latter describes it in $T_{\mathtt{id}}\mathcal{D}_+$ from the Eulerian point of view of a fixed observer. For the convenience of the reader the derivation of (1.6)-(1.7) is reviewed in Appendix C.

For $k = 0$, equation (1.7) can be viewed as a generalization (from one to two space dimensions) of the inviscid Burgers' equation. For $k = 1$, the geodesic flow (1.2) is the analogue of the Camassa-Holm equation (1.7) in two space dimensions – see e.g. [19],[20], and [25].

Our first result concerns the existence of the geodesic flow associated with the weak right-invariant Riemannian metric $\nu^{(k)}$ for any $k \geq 1$.

THEOREM 1.2. *Let k be an arbitrary integer in $\mathbb{Z}_{\geq 1}$. Then there exists an open neighborhood $V^{(k)}$ of 0 in $C^\infty(\mathbb{T}^2, \mathbb{R}^2)$ such that for any v_0 in $V^{(k)}$ there is a unique $\nu^{(k)}$-geodesic $(-2, 2) \to \mathcal{D}_+, t \mapsto \varphi(t)$ issuing from the identity in the direction v_0 which depends C^1_F-smoothly on the initial data i.e., the map*

$$(-2, 2) \times V^{(k)} \to \mathcal{D}_+, \quad (t, v_0) \mapsto \varphi(t; v_0)$$

is C^1_F-smooth.

Theorem 1.2 allows to define, for any given $k \geq 1$, the Riemannian exponential map

$$\mathrm{Exp}_k \big|_{V^{(k)}} : V^{(k)} \to \mathcal{D}_+, \qquad v_0 \mapsto \varphi(1; v_0).$$

THEOREM 1.3. *For any integer $k \geq 1$, there exist an open neighborhood $\tilde{V}^{(k)} \subseteq V^{(k)}$ of 0 in $C^\infty(\mathbb{T}^2, \mathbb{R}^2)$ and an open neighborhood $U^{(k)}$ of id in $\mathcal{D}_+$ so that*

$$\mathrm{Exp}_k \big|_{\tilde{V}^{(k)}} : \tilde{V}^{(k)} \to U^{(k)}, \quad v_0 \mapsto \varphi(1; v_0)$$

is a C^1_F-diffeomorphism.

Most likely, Theorem 1.2 holds also for $k = 0$ (even though, for this case, our method of proof does not work). However, Theorem 1.3 is false for $k = 0$. Indeed, we construct a counter-example along the lines of [7, 8].

REMARK 1.4. (*i*) *In [8], it was shown that an analogous result to the one stated in Theorem 1.3 holds for a family of Riemannian exponential maps of the Lie group of orientation preserving C^∞-diffeomorphisms of the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. In [16] we have improved this result showing that the exponential maps are Fréchet bianalytic diffeomorphisms near* 0.

(*ii*) *While the proofs of Theorem 1.2 and Theorem 1.3 follow the same approach pioneered in [8], the integrals of the flow stemming from Noether's theorem are of no use in this higher dimensional situation. Instead, the proofs of these theorems are based on a novel interplay of the structure of the equations describing the geodesic flow from a Lagrangian perspective and the structure of the Euler equation.*

THEOREM 1.5. *Assume that the Riemannian exponential map Exp_k for $k = 0$ can be uniquely defined near 0 as a C^1_F map in $C^\infty(\mathbb{T}^2, \mathbb{R}^2)$. Then, there is no neighborhood*

$V^{(0)}$ *of* 0 *in* $C^\infty(\mathbb{T}^2, \mathbb{R}^2)$ *so that* Exp_0 *is a* C^1_F*-diffeomorphism from* $V^{(0)}$ *onto a neighborhood of the identity map in* $\mathcal{D}_+$.

The paper is organized as follows: Sections 2 and 3 are preliminary. Theorem 1.2 is proved in Section 4 and Theorem 1.3 in Section 5. To prove Theorem 1.3 we use a version of the inverse function theorem in a set-up with Fréchet spaces, discussed in Appendix A (Theorem A.5). The assumptions of Theorem A.5 are verified in Section 5. The key result, stated in Proposition 5.1, is that assumption (c) of Theorem A.5 holds. It says that for $k \geq 1$, a $\nu^{(k)}$-geodesic in $\mathcal{D}^\ell_+$ (with $\ell \geq 2k+5$) issuing from the identity (with sufficiently small speed) and such that at $t = 1$ it passes through an element in $\mathcal{D}^{\ell+1}_+$, actually evolves in $\mathcal{D}^{\ell+1}_+$ – a quite astonishing property for solutions of an evolution equation. Proposition 5.1 and Proposition 5.2, corresponding to assumption (c) and, respectively, (d) of Theorem A.5, are proved in Section 6. For the convenience of the reader, some elementary auxiliary results used to prove these propositions are collected in Appendix B. Theorem 1.5 is proved in Section 7.

We use standard notation. In particular, $H^\ell = H^\ell(\mathbb{T}^2, \mathbb{R}^2)$ denotes the space of $\mathbb{R}^2$-valued functions on $\mathbb{T}^2$ of Sobolev class H^ℓ. Depending on the context, we will also use H^ℓ to denote the Sobolev space $H^\ell(\mathbb{T}^2, \mathbb{R})$ or $H^\ell(\mathbb{T}^2, \mathrm{Mat}_{2\times 2})$ where $\mathrm{Mat}_{2\times 2}$ denotes the linear space of 2×2 real valued matrices. Further, for $\ell \geq 3$, $\mathcal{D}^\ell_+ = \mathcal{D}^\ell_+(\mathbb{T}^2)$ denotes the set of orientation preserving C^1-diffeomorphisms $\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ of class H^ℓ. It is a Hilbert manifold modeled on H^ℓ.

In the remainder of the paper, we will always identify a C^1-diffeomorphism $\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ with a lift $\mathbb{R}^2 \to \mathbb{R}^2$ of the form $T + f$ where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is periodic i.e., $f(x + \xi) = f(x)$ for any $x \in \mathbb{R}^2$ and $\xi \in \mathbb{Z}^2$, and $T \in SL(2;\mathbb{Z})$.

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2. Group of orientation preserving diffeomorphisms. In this section we introduce some more notations and review some properties of spaces of maps from $\mathbb{T}^2$ onto itself – see [12] as well as [11], [22], [28], [29].

Throughout this section s denotes an integer satisfying $s \geq 3$. Denote by $\mathcal{D}^s_+$ the set of all orientation preserving C^1-diffeomorphisms $\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ such that $\partial_{x_j}\varphi$ $(j = 1, 2)$ are in H^{s-1} i.e.,

$$\mathcal{D}^s_+ := \{\varphi : \mathbb{T}^2 \to \mathbb{T}^2 \bigm| C^1 - \text{diffeomorphism; } \det(d\varphi) > 0;\ d\varphi \in H^{s-1}\}. \tag{2.1a}$$

$\mathcal{D}^s_+$ is in a natural way a Hilbert manifold modeled on the Hilbert space $H^s = H^s(\mathbb{T}^2, \mathbb{R}^2)$. An atlas of $\mathcal{D}^s_+$ can be described in terms of lifts of φ in $\mathcal{D}^s_+$ i.e.,

$$\mathbb{R}^2 \to \mathbb{R}^2, x \mapsto Tx + f(x) \text{ where } f = (f_1, f_2) \in H^s \text{ and } T \in SL(2;\mathbb{Z})\,.$$

For $j = 1, 2$, let $\mathsf{C}_1(f_j)$ and $\mathsf{C}_2(f_j)$ be short for the conditions on the function f_j, $|f_j(0)| < 1/2$, respectively $0 < f_j(0) < 1$. Then the following set of Hilbert charts $(1 \leq i, j \leq 2, T \in SL(2;\mathbb{Z}))$ form an atlas of $\mathcal{D}^s_+$,

$$\mathfrak{U}^s_{i,j,T} := \{\varphi = T + f \bigm| f \in H^s\,, \mathsf{C}_i(f_1)\,, \mathsf{C}_j(f_2)\,, \det(T + df) > 0\}. \tag{2.1b}$$

Note that for $s \geq 3$, H^s is a Banach algebra. Moreover generally, for any $s \geq 3$ and any $|m| \leq s$, the bilinear mapping

$$H^s \times H^m \to H^m,\ (f, g) \mapsto fg$$

is bounded. (For $0 \leq m \leq 3$ this follows from the Sobolev embedding theorem and for $-3 \leq m \leq -1$ one argues by duality.) The fact that H^s is an algebra can be used to show that for $s \geq 3$, the inverse φ^{-1} of an element φ in $\mathcal{D}^s_+$ is again in $\mathcal{D}^s_+$, $v \circ \varphi$ belongs to H^s for any v in H^s and φ in $\mathcal{D}^s_+$, and the right translation $R_\varphi : H^s \to H^s$ is a bounded linear map. The following results are well-known – see e.g. [11], [12].

Proposition 2.1. *For any $s \geq 3$, the inverse and composition maps are continuous. More precisely, for any integer $m \geq 0$,*
(i) $H^{s+m} \times \mathcal{D}^s_+ \to H^s, \quad (v, \varphi) \mapsto v \circ \varphi$ *is C^m-smooth;*
(ii) $\mathcal{D}^{s+m}_+ \to \mathcal{D}^s_+, \quad \varphi \mapsto \varphi^{-1}$ *is C^m-smooth.*

Remark 2.2. *Note that by Proposition 2.1, the composition and the inverse maps, $\mathcal{D}_+ \times \mathcal{D}_+ \to \mathcal{D}_+$, respectively $\mathcal{D}_+ \to \mathcal{D}_+$, are smooth, making $\mathcal{D}_+$ into a Lie group. Its tangent space at the identity, $T_{\mathtt{id}}\mathcal{D}_+$, is the Lie algebra of C^∞-smooth vector fields on $\mathbb{T}^2$.*

3. The vector field $\mathcal{F}_k$. In this section we assume that k and ℓ are integers satisfying $k \geq 1$ and $\ell \geq \ell_k := 2k + 5$.[1] For any (φ, v) in $\mathcal{D}^\ell \times H^\ell$, consider

$$\mathcal{F}_k(\varphi, v) := (v, F_k(\varphi, v)),$$

where $F_k = R_\varphi \circ A_k^{-1} \circ B_k(v \circ \varphi^{-1})$ is defined in (1.3). Note that each component of $B_k(u)$, defined in (1.5), is a polynomial of degree 2 in u_1, u_2, and its derivatives up to order $2k$. Indeed, in the expression (1.5), the terms of order $2k+1$ occurring in the first two terms on the r.h.s. of (1.5) cancel out as $A_k((u \cdot \nabla)u) - (u \cdot \nabla)A_k u$ is in $H^{\ell-2k}$, being the commutator $[A_k, (u \cdot \nabla)]$ acting on u. Now since $A_k : H^\ell \to H^{\ell-2k}$ as well as $R_\varphi : H^j \to H^j$ and its inverse R_φ^{-1} (cf. Proposition 2.1) are bounded linear operators for $3 \leq j \leq \ell$, the composition $R_\varphi \circ A_k \circ R_\varphi^{-1} : H^\ell \to H^{\ell-2k}$ is a bounded linear operator. Moreover, as $\ell \geq \ell_k$ and H^j is a Banach algebra for $j \geq 3$, $B_k : H^\ell \to H^{\ell-2k}$ and therefore $R_\varphi \circ B_k \circ R_\varphi^{-1} : H^\ell \to H^{\ell-2k}$ are continuous maps. In particular

$$\mathcal{A}_k : \mathcal{D}^\ell_+ \times H^\ell \to \mathcal{D}^\ell_+ \times H^{\ell-2k}, \ \ (\varphi, v) \mapsto (\varphi, R_\varphi \circ A_k \circ R_\varphi^{-1} v) \tag{3.1}$$

and

$$\mathcal{B}_k : \mathcal{D}^\ell_+ \times H^\ell \to \mathcal{D}^\ell_+ \times H^{\ell-2k}, \ \ (\varphi, v) \mapsto (\varphi, R_\varphi \circ B_k \circ R_\varphi^{-1} v). \tag{3.2}$$

are well-defined. Further let

$$\mathcal{P}roj_2 : \mathcal{D}^\ell_+ \times H^\ell \to H^\ell, \ (\varphi, v) \mapsto v$$

denote the projection onto the second component.

Proposition 3.1. *Let $k \geq 1$, and $\ell \geq \ell_k = 2k + 5$. Then*
(i) the map $\mathcal{A}_k : \mathcal{D}^\ell_+ \times H^\ell \to \mathcal{D}^\ell_+ \times H^{\ell-2k}$ defined in (3.1) is a C^1-diffeomorphism with inverse given by

$$\mathcal{A}_k^{-1} : \mathcal{D}^\ell_+ \times H^{\ell-2k} \to \mathcal{D}^\ell_+ \times H^\ell, \ \ (\varphi, v) \mapsto (\varphi, R_\varphi \circ A_k^{-1} \circ R_\varphi^{-1} v)\,. \tag{3.3}$$

(ii) The map $\mathcal{B}_k : \mathcal{D}^\ell_+ \times H^\ell \to \mathcal{D}^\ell_+ \times H^{\ell-2k}$ defined by (3.2) is C^1-smooth.

[1]Some of the statements below remain valid for ℓ_k smaller.

As a consequence, $F_k = \mathcal{P}roj_2 \circ \mathcal{A}_k^{-1} \circ \mathcal{B}_k$ *and*

(*iii*) *the vector field*

$$\mathcal{F}_k : \mathcal{D}_+^\ell \times H^\ell \to H^\ell \times H^\ell,\ (\varphi, v) \mapsto (v, F_k(\varphi, v)) \tag{3.4}$$

is C^1*-smooth.*

REMARK 3.2. *The arguments used in [16] show that the vector field* $\mathcal{F}_k$ *described in Proposition 3.1* (*iii*) *is in fact analytic.*

To prove Proposition 3.1 we need two auxiliary lemmas. First, let us introduce the standard multi-index notation for differential operators in two independent variables i.e., for $n = (n_1, n_2)$ in $\mathbb{Z}^2_{\geq 0}$, we define ∂_x^n to be the differential operator $\partial_{x_1}^{n_1} \partial_{x_2}^{n_2}$ of order $|n| = n_1 + n_2$.

LEMMA 3.3. *Let* $k \geq 1$*,* $\ell \geq \ell_k = 2k + 5$*, and* $n = (n_1, n_2) \in \mathbb{Z}^2_{\geq 0}$*. Then, for any* φ *in* $\mathcal{D}_+^\ell$ *and* v *in* H^ℓ*, and for for any* $1 \leq |n| \leq 2k$*,*

$$R_\varphi \circ \partial_x^n \circ R_\varphi^{-1} v = \sum_{1 \leq |j| \leq |n|} p_{n,j}(\varphi) \partial_x^j v$$

where the coefficients $p_{n,j}(\varphi)$ *are polynomials in* $(\det(d\varphi))^{-1}$ *and* $\partial_x^\alpha \varphi_m$ *(*$1 \leq |\alpha| \leq |n| + 1 - |j|$*,* $m = 1, 2$*) with integer coefficients.*

Proof. The proof of the claimed formula follows in a straightforward way using that the two components of $(\partial_x^n \varphi^{-1}) \circ \varphi$ are polynomials in $(\det(d\varphi))^{-1}$ and $\partial_x^\alpha \varphi_m$ ($1 \leq |\alpha| \leq |n|$, $m = 1, 2$) with integer coefficients. ◻

LEMMA 3.4. *For any integer* $s \geq 3$*, introduce the open subset* $W^s := \{f \in H^s : f(x) \neq 0\ \forall x \in \mathbb{T}^2\}$ *of* $H^s = H^s(\mathbb{T}^2, \mathbb{R})$*. Then, the map* $W^s \to H^s$*,* $f \mapsto 1/f$ *is* C^1*-smooth.*

Proof. Let f in W^s, and $f + U_\epsilon$ be the neighborhood of f in H^s where

$$U_\epsilon = \{g \in H^s \mid \|g\|_{H^s} < \epsilon\}$$

and $\epsilon > 0$ is so small that $\|g/f\|_{H^s} < 1$. Such a choice is possible since H^s is a Banach algebra for $s \geq 3$ and therefore $\|g/f\|_{H^s} \leq C\|g\|_{H^s}\|1/f\|_{H^s}$ so that it suffices to pick $0 < \epsilon < (C\|1/f\|_{H^s})^{-1}$. Then, $f + g$ is in W^s for any $g \in U_\epsilon$, and $1/(f + g)$ can be written in terms of a series

$$\frac{1}{f+g} = \frac{1}{f}\left(1 - \frac{g}{f} + \left(\frac{g}{f}\right)^2 - \dots\right)$$

which converges uniformly in $U_\epsilon(f)$ to an element in H^s. In particular, the map $f + U_\epsilon \to H^s$, $f + g \mapsto 1/(f + g)$ is C^1-smooth. ◻

COROLLARY 3.5. *For any* $s \geq 4$*, the map* $\mathcal{D}_+^s \to H^{s-1}$*,* $\varphi \mapsto (\det(d\varphi))^{-1}$ *is* C^1*-smooth.*

Proof. The map $\mathcal{D}_+^s \to H^{s-1}$, $\varphi \mapsto (\det(d\varphi))^{-1}$, is the composition of the map $\mathcal{D}_+^s \to H^{s-1}$, $\varphi \mapsto \det(d\varphi)$, with the C^1-map $W^{s-1} \to H^{s-1}$, $f \mapsto 1/f$. ◻

Proof of Proposition 3.1. (i) Clearly the map defined in (3.3) is the inverse of (3.1). In particular, this shows that $\mathcal{A}_k$ is bijective. By Lemma 3.3 and the definition of A_k we have that

$$\begin{aligned}R_\varphi \circ A_k \circ R_\varphi^{-1} v &= v + \sum_{1\le i\le k} (-1)^i R_\varphi \circ \Delta^i \circ R_\varphi^{-1} v \\ &= v + \sum_{1\le |j|\le 2k} q_{2k,j}(\varphi)\partial_x^j v \end{aligned}\tag{3.5}$$

where the coefficients of the matrices $q_{2k,j}(\varphi)$ are polynomials in $(\det(d\varphi))^{-1}$ and $\partial_x^\alpha \varphi_m$ ($|\alpha| \le 2k+1-|j|$, $m = 1,2$) with integer coefficients. By Corollary 3.5, the map $\mathcal{D}_+^\ell \times H^\ell \to (H^{\ell-2k})^{k^*}$,

$$(\varphi, v) \mapsto ((\det(d\varphi))^{-1}, ((\partial_x^\alpha \varphi_1), (\partial_x^\alpha \varphi_2))_{1\le|\alpha|\le 2k}, ((\partial_x^\alpha v_1), (\partial_x^\alpha v_2))_{0\le|\alpha|\le 2k})$$

is a C^1-map where $k^* = 4k(2k+3) - 1$. Since for $\ell \ge \ell_k = 2k+5$, $H^{\ell-2k}$ is a Banach algebra, we conclude that the r.h.s. of (3.5) is C^1-smooth and hence that $\mathcal{A}_k$ is C^1-smooth. Moreover, for any (φ_0, v_0) in $\mathcal{D}_+^\ell \times H^\ell$, the differential $d_{(\varphi_0,v_0)}\mathcal{A}_k :$ $H^\ell \times H^\ell \to H^\ell \times H^{\ell-2k}$ is of the form

$$d_{(\varphi_0,v_0)}\mathcal{A}_k(\delta\varphi, \delta v) = \begin{pmatrix} \delta\varphi & 0 \\ \Lambda(\delta\varphi) & R_{\varphi_0} \circ A_k \circ R_{\varphi_0}^{-1}\delta v \end{pmatrix}\tag{3.6}$$

where

$$\Lambda : H^\ell \to H^{\ell-2k}, \quad \text{and} \quad R_{\varphi_0} \circ A_k \circ R_{\varphi_0}^{-1} : H^\ell \to H^{\ell-2k}$$

are bounded linear maps. As the latter map has a bounded inverse, $d_{(\varphi_0,v_0)}\mathcal{A}_k$ is a linear isomorphism and, by the inverse function theorem, $\mathcal{A}_k$ is a local C^1-diffeomorphism. Since we have seen that $\mathcal{A}_k$ is bijective, assertion (i) then follows. The proof of item (ii) is similar to the proof of the C^1-smoothness of $\mathcal{A}_k$ in part (i). □

Proposition 3.1 allows to apply the existence and uniqueness theorems of ODE's in Banach spaces (see e.g. [22]) to the initial value problems (1.2) on $\mathcal{D}_+^\ell \times H^\ell$ with $\ell \ge \ell_k = 2k+5$. As for any $k \ge 1$, the element $(\mathtt{id}, 0)$ in $\mathcal{D}_+^\ell \times T_{\mathtt{id}}\mathcal{D}_+$ is a stationary solution of (1.2) one gets the following result.

THEOREM 3.6. *Let $k \ge 1$ and $\ell \ge \ell_k = 2k+5$. Then there exists an open neighborhood $V_{k;\ell}$ of 0 in H^ℓ so that, for any v_0 in $V_{k;\ell}$, the initial value problem* (1.2) *has a unique C^1-solution*

$$(-2,2) \to \mathcal{D}_+^\ell \times H^\ell, \quad t \mapsto (\varphi(t), v(t)).$$

Moreover, the flow map is C^1,

$$(-2,2) \times V_{k;\ell} \to \mathcal{D}_+^\ell \times H^\ell, \quad (t, v_0) \mapsto (\varphi(t;v_0), v(t;v_0)).$$

REMARK 3.7. *Note that Theorem 3.6 does not exclude that $\bigcap_{\ell\ge\ell_k} V_{k;\ell} = \{0\}$. This possibility is ruled out by Theorem 4.1 of the next section.*

As discussed in the introduction, the initial value problem (1.2) on $\mathcal{D}_+^\ell \times H^\ell$ is closely related to the initial value problems (1.6)-(1.7) on $\mathcal{D}_+^\ell$ and H^ℓ respectively. A solution (φ, v) in $C^1((-2,2), \mathcal{D}^\ell_+ \times H^\ell)$ of (1.2) corresponds via (1.6) to a solution of (1.7) in $C^0((-2,2), H^\ell) \cap C^1((-2,2), H^{\ell-1})$. To check this, note that from the first equation of (1.2) and (1.6) we have that

$$u = v \circ \varphi^{-1}$$

so that $u(t)$ is in H^ℓ for any $-2 < t < 2$. Further, $t \mapsto \varphi(t)$ evolves in $\mathcal{D}^\ell_+$ and hence so does $t \mapsto \varphi^{-1}(t)$; they are continuous within the same time interval. Thus, by Proposition 2.1,

$$(-2,2) \to H^\ell, \quad t \mapsto u(t) = v(t) \circ \varphi^{-1}(t)$$

is a continuous curve in H^ℓ and a C^1-curve in $H^{\ell-1}$, emanating from $v(0) \circ \varphi^{-1}(0) = v_0$. Altogether, we have

$$u \in C^0((-2,2), H^\ell) \cap C^1((-2,2), H^{\ell-1}).$$

A direct computation shows that u is a solution of (1.7) on $(-2,2) \times \mathbb{T}^2$.

4. Riemannian exponential maps. Throughout this section we assume that k and ℓ are integers satisfying $k \geq 1$, and $\ell \geq \ell_k := 2k+5$. By Theorem 3.6, there exists a neighborhood $V^{(k)}_{\ell_k} := V_{k;\ell_k}$ of 0 in H^{ℓ_k} so that for any v_0 in $V^{(k)}_{\ell_k}$, the initial value problem (1.2) has a unique C^1-solution

$$(-2,2) \to \mathcal{D}^{\ell_k}_+ \times H^{\ell_k}, \quad t \mapsto (\varphi(t), v(t)).$$

We emphasize the dependence on the initial data by writing $t \mapsto (\varphi(t; v_0),\ v(t; v_0))$ instead of $t \mapsto (\varphi(t), v(t))$. Moreover, the map

$$(-2,2) \times V^{(k)}_{\ell_k} \to \mathcal{D}^{\ell_k}_+ \times H^{\ell_k},\ (t, v_0) \mapsto (\varphi(t; v_0), v(t; v_0))$$

is C^1. Hence, in particular, the exponential map

$$\mathrm{Exp}_{k,\ell_k} : V^{(k)}_{\ell_k} \to \mathcal{D}^{\ell_k}_+, \quad v_0 \mapsto \varphi(1; v_0) \tag{4.1}$$

is C^1. In this section we want to study the restriction of Exp_{k,ℓ_k} to $V^{(k)}_{\ell_k} \cap C^\infty(\mathbb{T}^2, \mathbb{R}^2)$. As explained at the end of the previous section, for any v_0 in $V^{(k)}_{\ell_k}$, the curve $t \mapsto u(t) = v(t) \circ \varphi(t)^{-1}$ is a solution of equation (1.7) in $C^0((-2,2), H^{\ell_k}) \cap C^1((-2,2), H^{\ell_k - 1})$.

We start by deriving transport equations for $(A_k u) \circ \varphi$ and $d\varphi$. This will be helpful to study the regularity properties of the exponential map as we will see shortly. As $C^1((-2,2), H^3) \hookrightarrow C^1((-2,2) \times \mathbb{T}^2)$ in view of the Sobolev embedding theorem, one obtains by pointwise differentiation

$$[(A_k u) \circ \varphi]^\cdot = (A_k \dot{u}) \circ \varphi + [(dA_k u) \circ \varphi]\, \dot{\varphi}.$$

Notice that the latter identity actually holds in $H^{\ell_k - 2k - 1}$. As $\dot{\varphi} = v = u \circ \varphi$, (1.7) leads to the following linear initial value problem in $H^{\ell_k - 2k - 1}$ for $w = w(t) := (A_k u(t)) \circ \varphi(t)$

$$\begin{cases} \dot{w} + \left[((\operatorname{div} u) \circ \varphi)\, \mathrm{Id} + (du)^\dagger \circ \varphi\right] w = 0 \\ w(0) = A_k v_0\,. \end{cases} \tag{4.2}$$

On the other hand, differentiating equation (1.6), considered in H^{ℓ_k}, with respect to the spatial variables one gets

$$\begin{cases} (d\varphi)^{\cdot} = [(du)\circ\varphi]d\varphi \\ d\varphi(0) = \mathrm{Id} \end{cases} \tag{4.3}$$

which we view as a linear system of ODE's in $H^{\ell_k - 1}$ for the two columns of the Jacobian $d\varphi$. We will use (4.2) and (4.3) to prove that any solution curve $t \mapsto (\varphi(t;v_0), v(t;v_0))$ in $C^1((-2,2), \mathcal{D}_+^{\ell_k} \times H^{\ell_k})$ emanating from $(\varphi(0), v(0)) = (\mathtt{id}, v_0)$, with v_0 in $V_{\ell_k}^{(k)} \cap H^\ell$ and $\ell \geq \ell_k$, actually evolves in $\mathcal{D}_+^\ell \times H^\ell$.

THEOREM 4.1. *Let $k \geq 1$ and $\ell \geq \ell_k = 2k+5$. If v_0 is in $V_\ell^{(k)} := V_{\ell_k}^{(k)} \cap H^\ell$, then there exists a unique solution (φ, v) of (1.2) in $C^1((-2,2), \mathcal{D}_+^\ell \times H^\ell)$. Moreover, $u := v \circ \varphi^{-1}$ is a solution of (1.7) in $C^0((-2,2), H^\ell) \cap C^1((-2,2), H^{\ell-1})$.*

REMARK 4.2. *The solution (φ, v) in Theorem 4.1 depends C^1-smoothly on the initial data $v_0 \in V_\ell^{(k)}$ so that $(\varphi, v) \in C^1((-2,2) \times V_\ell^{(k)}, \mathcal{D}_+^\ell \times H^\ell)$ (cf. e.g. [22], Chapter IV).*

Proof of Theorem 4.1. We argue by induction with respect to ℓ. For $\ell = \ell_k$, the statement follows from the definition of $V_{\ell_k}^{(k)}$, and the discussion at the end of Section 3. Now, assume that the proposition holds for a fixed $\ell \geq \ell_k$ i.e., assume that any given v_0 in $V_\ell^{(k)}$, the solution curve (φ, v) of (1.2) is in $C^1((-2,2), \mathcal{D}_+^\ell \times H^\ell)$ and the corresponding solution u of (1.7) lies in $C^0((-2,2), H^\ell) \cap C^1((-2,2), H^{\ell-1})$. Let $v_0 \in V_{\ell+1}^{(k)}$. As $V_{\ell+1}^{(k)} \subseteq V_\ell^{(k)}$ we get from the induction hypothesis that $w = (A_k u) \circ \varphi$ belongs to $C^0((-2,2), H^{\ell-2k})$ – see Propositon 2.1 (i). Moreover for any $1 \leq j \leq 2k$

$$[(\operatorname{div} u \cdot \mathrm{Id} + (du)^\dagger) \circ \varphi] \in C^0((-2,2), \mathcal{L}(H^{\ell-j}, H^{\ell-j})) \tag{4.4}$$

where $\mathcal{L}(H^s, H^s)$ denotes the space of bounded linear operators from H^s to H^s. At this point it is crucial that we assume $k \geq 1$. It garantees that $2k-1 \geq 1$ so that we can apply (4.4) for $j = 2k-1$ to conclude that (4.2) is a linear ODE in $H^{\ell+1-2k}$. As $w(0) = A_k v_0 \in H^{\ell+1-2k}$, it then follows that w is in $C^1((-2,2), H^{\ell+1-2k})$. On the other hand, since φ^{-1} is in $C^0((-2,2), H^\ell)$ (cf. Proposition 2.1 (ii)), $w \circ \varphi^{-1} = A_k u$ is in $C^0((-2,2), H^{\ell+1-2k})$ i.e., u belongs to $C^0((-2,2), H^{\ell+1})$. We now want to show that u lies in $C^1((-2,2), H^\ell)$. Recall that by the induction hypothesis, (1.7) holds in $H^{\ell-2k-1}$. Hence when integrating (1.7) with respect to t in $H^{\ell-2k-1}$, we get

$$A_k u(t) = v_0 - \int_0^t ((dA_k u)u + (\operatorname{div} u \cdot \mathrm{Id} + (du)^\dagger)A_k u) d\tau\,.$$

As by the considerations above, the integrand in the latter formula is in $C^0((-2,2), H^{\ell-2k})$, it follows that $A_k u \in C^1((-2,2), H^{\ell-2k})$. Altogether we have

$$u \in C^0((-2,2), H^{\ell+1}) \cap C^1((-2,2), H^\ell)\,.$$

Furthermore, as φ is in $C^1((-2,2), \mathcal{D}_+^\ell)$, and u is in $C^0((-2,2), H^{\ell+1})$, $(du)\circ\varphi$ is in $C^0((-2,2), \mathcal{L}(H^\ell, H^\ell))$ so that (4.3) is a linear system of ODEs in H^ℓ for the two columns of the Jacobian $d\varphi$. Integrating (4.3) with respect to t in H^ℓ, one argues as above to conclude that the columns of $d\varphi$ are in $C^1((-2,2), H^\ell)$, and thus that φ is in $C^1((-2,2), \mathcal{D}_+^{\ell+1})$. Moreover, $v = u \circ \varphi$ is in $C^0((-2,2), H^{\ell+1})$. By (1.2),

$\dot{v} = F_k(\varphi, v)$. By Proposition 3.1, $\dot{v}$ is in $C^0((-2,2), H^{\ell+1})$. Arguing as above one then concludes that (φ, v) is in $C^1((-2,2), \mathcal{D}_+^{\ell+1} \times H^{\ell+1})$. □

Proof of Theorem 1.2. Theorem 1.2 is an immediate consequence of Theorem 4.1 with $V^{(k)}$ given by $V^{(k)} := V_{\ell_k}^{(k)} \cap C^\infty(\mathbb{T}^2, \mathbb{R}^2)$. □

By shrinking $V_{\ell_k}^{(k)}$ if necessary, we can assume that the image of (4.1) is contained in the Hilbert chart

$$\mathfrak{U}_{1,1,\mathtt{id}}^{\ell_k} := \{\mathtt{id} + g | \, g \in H^\ell \, , |g_i(0)| < 1/2 \; (i = 1, 2) \, , \det(\mathrm{Id} + dg) > 0\}$$

defined in (2.1b). Note that $d_0 \mathrm{Exp}_{k,\ell_k} = \mathrm{Id}_{H^{\ell_k}}$. It then follows from the inverse function theorem that by shrinking $V_{\ell_k}^{(k)}$ further, if necessary, one can ensure that there exists an open neighborhood $U_{\ell_k}^{(k)}$ of $\mathtt{id}$ in $\mathcal{D}_+^{\ell_k}$ such that

$$\mathrm{Exp}_{k,\ell_k} : V_{\ell_k}^{(k)} \to U_{\ell_k}^{(k)}$$

is a C^1-diffeomorphism and $U_{\ell_k}^{(k)} \subseteq \mathfrak{U}_{1,1,\mathtt{id}}^{\ell_k}$. In what follows we will assume that the latter two conditions are satisfied.

Theorem 4.1 allows to define the exponential map on a neighborhood of 0 in $T_{\mathtt{id}}\mathcal{D}_+^\ell$ for any $\ell \geq \ell_k$. Recall that for any $k \geq 1$ and $\ell \geq \ell_k$, $V_\ell^{(k)} = V_{\ell_k}^{(k)} \cap H^\ell$, and

$$V^{(k)} = V_{\ell_k}^{(k)} \cap C^\infty(\mathbb{T}^2, \mathbb{R}^2) = \bigcap_{\ell \geq \ell_k} V_\ell^{(k)}.$$

Define for any $\ell \geq \ell_k$,

$$\mathrm{Exp}_{k,\ell} : V_\ell^{(k)} \to \mathcal{D}_+^\ell, \quad v_0 \mapsto \varphi(1; v_0)\,.$$

By Theorem 4.1, the restriction $\mathrm{Exp}_{k,\ell}$ of Exp_{k,ℓ_k} to $V_\ell^{(k)}$ takes values in $\mathcal{D}_+^\ell$ and by Remark 4.2, $\mathrm{Exp}_{k,\ell} : V_\ell^{(k)} \to \mathcal{D}_+^\ell$ is a C^1-map. But then, the restriction Exp_k of Exp_{k,ℓ_k} to $V^{(k)}$ takes values in $\mathcal{D}_+$,

$$\mathrm{Exp}_k : V^{(k)} \to \mathcal{D}_+\,.$$

5. C_F^1-smooth charts of $\mathtt{id}$ in $\mathcal{D}_+$. Theorem 1.3 states that, for any integer $k \geq 1$, the exponential map

$$\mathrm{Exp}_k \big|_{V^{(k)}} : V^{(k)} \to \mathcal{D}_+$$

can be used to define a C_F^1-smooth chart of the identity in $\mathcal{D}_+$. To prove Theorem 1.3 we will use the following two propositions which we state in a slightly stronger version than needed. Denote $U_\ell^{(k)} := U_{\ell_k}^{(k)} \cap H^\ell$.

PROPOSITION 5.1. *For any $k \geq 1$ and any $\ell \geq \ell_k := 2k + 5$, $\mathrm{Exp}_{k,\ell}$ maps $V_\ell^{(k)}$ onto $U_\ell^{(k)}$.*

PROPOSITION 5.2. *Let $k \geq 1$, $\ell \geq \ell_k = 2k + 5$, and assume that v_0 belongs to $V_{\ell+2}^{(k)}$. Then*

$$(d_{v_0} \mathrm{Exp}_{k,\ell})(H^\ell \backslash H^{\ell+1}) \subseteq H^\ell \backslash H^{\ell+1}.$$

We will prove these two propositions in the next section.

Proof of Theorem 1.3. We want to apply the inverse function theorem in Fréchet spaces as stated in Theorem A.5 of Appendix A. Fix $k \geq 1$ and let $\ell_k = 2k + 5$. To match the notation of this theorem, we write $\ell = \ell_k + n$, $n \geq 0$ and define for any integer $n \geq 0$

$$X_n := H^\ell, \quad Y_n := H^\ell, \qquad \text{and} \quad V_n := V_\ell^{(k)}, \quad U_n := \mathrm{Exp}_{k,\ell}(V_\ell^{(k)}),$$

where $V_\ell^{(k)}$ and $\mathrm{Exp}_{k,\ell}$ were introduced in Section 4. Further, define the map f of Theorem A.5 by $f := \mathrm{Exp}_{k,\ell_k} : V_0 \to U_0$. By the choice of $V_{\ell_k}^{(k)}$, for any $n \geq 0$, U_n is contained in $\mathfrak{U}_{1,1}^\ell$ and therefore can be identified with an open neighborhood of 0 in $X_n = H^\ell$. As by construction $\mathrm{Exp}_{k,\ell_k} : V_{\ell_k}^{(k)} \to U_{\ell_k}^{(k)}$ is a C^1-diffeomorphism, item (a) of Theorem A.5 is verified. Assumption (b) of the latter theorem holds in view of Theorem 4.1 and Remark 4.2, whereas items (c) and (d) hold by Proposition 5.1, and Proposition 5.2, respectively. Finally, upon setting $\tilde{V}^{(k)} = V^{(k)} := \cap_{n \geq 0} V_n$ and $U^{(k)} := \cap_{n\geq 0} U_n$, Theorem 1.3 then follows from Theorem A.5. □

6. Proof of Propositions 5.1 and 5.2. In this section we prove Propositions 5.1 and 5.2 which were used in the proof of Theorem 1.3. Assume that $\ell \geq 2k+5$ and let $\varphi \in C^1((-2,2), \mathcal{D}_+^\ell)$ and $u \in C^0((-2,2), H^\ell) \cap C^1((-2,2), H^{\ell-1})$ be solutions of (1.6) and (1.7) respectively. For any given x in $\mathbb{T}^2$ and $-2 < t < 2$ consider the linear system of ODE's

$$\begin{cases} \dot{\Phi} - [(du) \circ \varphi]\, \Phi = 0 \\ \Phi|_{t=0} = \mathrm{Id}. \end{cases} \tag{6.1}$$

Note that by the Sobolev embedding theorem for any given $x \in \mathbb{T}^2$ the elements of the 2×2 matrix $(du) \circ \varphi$ are continuous real-valued functions of t on $(-2, 2)$. Denote by $\Phi^t \equiv \Phi^t(x)$ the fundamental matrix of (6.1). Arguing as in §4 one sees that

$$\Phi^t(x) = d_x\varphi(t) \quad \forall x \in \mathbb{T}^2, \quad \text{and} \quad \forall t \in (-2, 2). \tag{6.2}$$

Further, for any x in $\mathbb{T}^2$ and $-2 < t < 2$ denote by $\Psi^t = \Psi^t(x)$ the 2×2 fundamental matrix of the linear system

$$\begin{cases} \dot{\Psi} + \big[\big(\mathrm{div}\, u \cdot \mathrm{Id} + (du)^\dagger\big) \circ \varphi\big]\, \Psi = 0 \\ \Psi|_{t=0} = \mathrm{Id}\,. \end{cases} \tag{6.3}$$

Regarding (6.3) as a linear ODE for Ψ in $H^{\ell-1}$ one concludes that $\Psi \in C^1((-2,2), H^{\ell-1})$. Moreover, the arguments in §4 show that

$$(A_k u) \circ \varphi = \Psi^t A_k u(0)\,. \tag{6.4}$$

Proof of Proposition 5.1. We argue by induction. For $\ell = \ell_k$, $\mathrm{Exp}_{k,\ell_k} : V_{\ell_k}^{(k)} \to U_{\ell_k}^{(k)}$ is a C^1-diffeomorphism by the definition of $V_{\ell_k}^{(k)}$ and $U_{\ell_k}^{(k)}$. Now, given an arbitrary $\ell \geq \ell_k$, assume that any solution (φ, v) of (1.2) with $v_0 \in V_{\ell_k}^{(k)}$ for which $\varphi(1)$ is in $U_\ell^{(k)}$, has initial data (id, v_0) with v_0 in $V_\ell^{(k)}$. By Theorem 4.1, it is equivalent to assume that for the given data, the solution curve (φ, v) be in $C^1((-2,2), \mathcal{D}_+^\ell \times H^\ell)$. Then, $u = v \circ \varphi^{-1}$ is a solution of (1.7) in $C^0((-2,2), H^\ell) \cap C^1((-2,2), H^{\ell-1})$. We need to show that, if in addition $\mathrm{Exp}_{k,\ell}(v_0) = \varphi(1)$ is in $\mathcal{D}_+^{\ell+1}$, then v_0 must be in $H^{\ell+1}$, and hence that (φ, v) is in $C^1((-2,2), \mathcal{D}_+^{\ell+1} \times H^{\ell+1})$. To this end note that

$$-\Delta(u \circ \varphi) = [(du) \circ \varphi](-\Delta\varphi) - \sum_{1 \le i,j \le 2} [(d\varphi)(d\varphi)^\dagger]_{ij} [(\partial_{x_i} \partial_{x_j} u) \circ \varphi]. \tag{6.5}$$

Hence, applying $-\Delta$ to the identity $\dot\varphi = u \circ \varphi$ yields

$$(-\Delta\varphi)^{\cdot} - [(du) \circ \varphi](-\Delta\varphi) = (\mathcal{P}_\lambda u) \circ \varphi - \lambda u \circ \varphi \tag{6.6}$$

where $\mathcal{P}_\lambda$ is the second order elliptic differential operator

$$\mathcal{P}_\lambda = \lambda - \sum_{1 \le i,j \le 2} p_{ij} \partial_{x_i} \partial_{x_j} \tag{6.7}$$

where the coefficients $p_{ij} = p_{ij}(t,x)$, $1 \le i,j \le 2$,

$$p_{ij} := [\{(d\varphi)(d\varphi)^\dagger\} \circ \varphi^{-1}]_{ij}$$

are in $C^0((-2,2), H^{\ell-1})$ (cf. Proposition 2.1) . We will choose the parameter $\lambda \ge 1$ so that inequality (B.2) in Lemma B.2 in Appendix B holds uniformly in $0 \le t \le 1$. Our strategy is to find a formula relating $\varphi(1)$ and v_0 which will allow us to show that v_0 actually lies in $H^{\ell+1}$. To this end it turns out to be more convenient not to work with identity (6.6) but rather the one which arises by applying $-\Delta$ to (6.6) once more i.e.,

$$\begin{cases} (\Delta^2\varphi)^{\cdot} - [(du) \circ \varphi](\Delta^2\varphi) = (\mathcal{P}_\lambda^\dagger \mathcal{P}_\lambda u) \circ \varphi + g \\ \Delta^2\varphi(0) = 0 \end{cases} \tag{6.8}$$

where g is in $C^0((-2,2), H^{\ell-3})$. Here we have used the fact that $(\mathcal{P}_\lambda \mathcal{P}_\lambda u) \circ \varphi = (\mathcal{P}_\lambda^\dagger \mathcal{P}_\lambda u) \circ \varphi + \ldots$, where $\ldots$ stand for terms in $C^0((-2,2), H^{\ell-3})$ which are included in g. By the Sobolev embedding theorem $C^1((-2,2), H^{\ell-4}) \hookrightarrow C^1((-2,2) \times \mathbb{T}^2, \mathbb{R}^2)$ and hence for any fixed $x \in \mathbb{T}^2$ we can view (6.8) as an inhomogeneous linear ODE for $\Delta^2\varphi$. This implies that for any given $x \in \mathbb{T}^2$, $\Delta^2\varphi$ admits a representation of the form

$$\begin{aligned} \Delta^2\varphi(t) = \Phi^t \int_0^t (\Phi^s)^{-1} \circ R_{\varphi(s)} [\mathcal{P}_\lambda^\dagger(s) \mathcal{P}_\lambda(s) u(s)] \, ds \\ + \Phi^t \int_0^t (\Phi^s)^{-1} \, g \, ds \end{aligned} \tag{6.9}$$

where $\Phi^t = \Phi^t(x) = d_x\varphi(t)$ is the fundamental solution of (6.1). As $t \mapsto \Phi^t$ is in $C^1((-2,2), H^{\ell-1})$ and $g \in C^0((-2,2), H^{\ell-3})$, the integral in (6.9) is in $H^{\ell-3}$ and therefore represents an element in $H^{\ell-3}$. Hence

$$\Delta^2\varphi(t) = \Phi^t \int_0^t (\Phi^s)^{-1} \circ R_{\varphi(s)} [\mathcal{P}_\lambda^\dagger(s) \mathcal{P}_\lambda(s) u(s)] \, ds + \ldots \tag{6.10}$$

where $\ldots$ stand for terms in $C^1((-2,2), H^{\ell-3})$. Note that the integral in (6.10) converges in $H^{\ell-4}$. In particular, (6.10) can be considered not only as a pointwise equality but also as an equality in $H^{\ell-4}$.

To analyze (6.10) further, recall that for any given $-2 < t < 2$ and $x \in \mathbb{T}^2$,

$$(A_k u(t)) \circ \varphi(t) = \Psi^t A_k v_0$$

where Ψ^t is the fundamental matrix of (6.3). Hence, for any $-2 < t < 2$, the solution of (1.7) reads

$$u(t) = A_k^{-1} \circ R_{\varphi(t)}^{-1} \circ \Psi^t (A_k v_0).$$

Upon substituting the latter into the r.h.s. of equation (6.10), we get that $\Delta^2 \varphi(t)$ is equal to

$$\begin{aligned}
&\Phi^t \int_0^t (\Phi^s)^{-1} \circ R_{\varphi(s)} [\mathcal{P}_\lambda^\dagger(s) \mathcal{P}_\lambda(s) \{A_k^{-1} \circ R_{\varphi(s)}^{-1} \circ \Psi^s (A_k v_0)\}] ds + \dots \\
&= \Phi^t \int_0^t (\Phi^s)^{-1} \circ R_{\varphi(s)} \circ A_k^{-1/2} \circ \mathcal{P}_\lambda^\dagger(s) \circ \mathcal{P}_\lambda(s) \circ A_k^{-1/2} \circ R_{\varphi(s)}^{-1} \circ \Psi^s (A_k v_0)\, ds \\
&+ \Phi^t \int_0^t (\Phi^s)^{-1} \circ R_{\varphi(s)} \circ A_k^{-1/2} \circ [A_k^{1/2}, \mathcal{P}_\lambda^\dagger(s) \mathcal{P}_\lambda(s)] \circ A_k^{-1} \circ R_{\varphi(s)}^{-1} \circ \Psi^s (A_k v_0)\, ds \\
&+ \dots
\end{aligned} \tag{6.11}$$

where $\dots$ again stand for terms in $C^0((-2,2), H^{\ell-3})$, and $[A_k^{1/2}, \mathcal{P}_\lambda^\dagger \mathcal{P}_\lambda]$ denotes the commutator

$$[A_k^{1/2}, \mathcal{P}_\lambda^\dagger \mathcal{P}_\lambda] = A_k^{1/2} \circ \mathcal{P}_\lambda^\dagger \mathcal{P}_\lambda - \mathcal{P}_\lambda^\dagger \mathcal{P}_\lambda \circ A_k^{1/2}.$$

Note that, by the induction hypothesis, the coefficients of the differential operator $\mathcal{P}_\lambda^\dagger \mathcal{P}_\lambda$ are in $C^1((-2,2), H^{\ell-3})$. By Remark B.5 at the end of Appendix B below, the second term on the r.h.s. of (6.11) is in $C^0((-2,2), H^{\ell-3})$. Hence, at $t = 1$, we have

$$\Delta^2 \varphi(1) = \Phi^1 \mathcal{Q}_k (A_k v_0) + \dots \tag{6.12}$$

where $\dots$ stand for elements in $H^{\ell-3}$, and where for $f \in H^{\ell-2k}$, $\mathcal{Q}_k$ is defined by

$$\mathcal{Q}_k f := \int_0^1 (\Phi^s)^{-1} \circ R_{\varphi(s)} \circ A_k^{-1/2} \circ \mathcal{P}_\lambda^\dagger(s) \circ \mathcal{P}_\lambda(s) \circ A_k^{-1/2} \circ R_{\varphi(s)}^{-1} \circ \Psi^s f\, ds\,. \tag{6.13}$$

Here the integration is carried out in $H^{\ell-4}$. Recall that λ is chosen so that (B.2) in Lemma B.2 holds. We claim that $\mathcal{Q}_k$ extends to a bounded linear operator $\mathcal{Q}_k : H^{-(k-2)} \to H^{k-2}$. First note that for any $0 \le j \le \ell - 1$, the linear mapping $R_{\varphi(s)} : H^j \to H^j$ is bounded uniformly in $0 \le s \le 1$. Moreover, for every $0 \le s \le 1$, $R_{\varphi(s)}$ is well defined on the spaces of distributions H^{-j}, $1 \le j \le \ell - 1$. More precisely, for f in H^{-j} and g in H^j, we define (cf. (B.1))

$$\langle R_{\varphi(s)} f, g \rangle = \langle f, R_{\varphi(s)}^{-1} \left([\det(d\varphi(s))]^{-1} g \right) \rangle \tag{6.14}$$

where $\langle \cdot, \cdot \rangle$ denotes the extension of the L^2-inner product $\langle \cdot, \cdot \rangle_{L^2}$ as a dual pairing between H^{-j} and H^j. Indeed, as $\det(d\varphi(s) \circ \varphi^{-1}(s))$ is in $H^{\ell-1}$ where $\ell \ge 7$ and g belongs to H^j, it follows that $R_{\varphi(s)}^{-1}([\det(d\varphi(s))]^{-1} g)$ is in H^j and hence (6.14) is well-defined. Moreover, it follows from (6.14) that for any $|j| \le \ell - 1$ there exist $C_1, C_2 > 0$ such that

$$C_1 \|f\|_{H^j} \le \|R_\varphi(s) f\|_{H^j} \le C_2 \|f\|_{H^j}$$

uniformly for $0 \leq s \leq 1$. Finally, the boundedness of $\mathcal{Q}_k : H^{-(k-2)} \to H^{k-2}$ follows from the uniform boundedness of the operators appearing in (6.13). By similar arguments one sees that, more generally, for any m satisfying $2-k \leq m \leq \ell - 2k + 1$,

$$\mathcal{Q}_k|_{H^m} : H^m \to H^{m+2k-4}$$

is a bounded linear operator.

Now we establish that $\mathcal{Q}_k : H^{-(k-2)} \to H^{k-2}$ is a linear isomorphism using the Lax-Milgram lemma – see e.g. [23, Chapter 6, Theorem 6]. Consider the bilinear form

$$\Lambda_k : H^{2-k} \times H^{2-k} \to \mathbb{R}, \quad (f,g) \mapsto \langle \mathcal{Q}_k f, g\rangle\,.$$

Note that by Lemma B.1 (ii)

$$\langle \mathcal{Q}_k f, g\rangle = \int_0^1 \langle \mathcal{P}_\lambda(s) \circ A_k^{-1/2} \circ R_{\varphi(s)}^{-1} \circ \Psi^s f, \mathcal{P}_\lambda(s) \circ A_k^{-1/2} \circ R_{\varphi(s)}^{-1} \circ \Psi^s g\rangle_{L^2} ds\,.$$

As $\mathcal{Q}_k : H^{-(k-2)} \to H^{k-2}$ is bounded, there exists a constant $C > 0$ such that for any $f, g \in H^{2-k}$,

$$|\langle \mathcal{Q}_k f, g\rangle| \leq C\|f\|_{H^{2-k}}\|g\|_{H^{2-k}}\,.$$

To see that Λ_k is positive definite, note that by Lemma B.2, there exists a constant $C > 0$ such that for any $0 \leq s \leq 1$ and f in H^2, $\|\mathcal{P}_\lambda(s)f\|_{L^2} \geq C\|f\|_{H^2}$. Hence, for any f in H^{2-k},

$$\begin{aligned}
\langle \mathcal{Q}_k f, f\rangle_{L^2} &= \int_0^1 \|\mathcal{P}_\lambda(s) \circ A_k^{-1/2} \circ R_{\varphi(s)}^{-1} \circ \Psi^s f\|_{L^2}^2 ds \\
&\geq C \int_0^1 \|A_k^{-1/2} \circ R_{\varphi(s)}^{-1} \circ \Psi^s f\|_{H^2}^2 ds \\
&\geq C_1 \int_0^1 \|R_{\varphi(s)}^{-1} \circ \Psi^s f\|_{H^{2-k}}^2 ds \\
&\geq C_2 \|f\|_{H^{2-k}}^2
\end{aligned}$$

for suitably chosen positive constants $C_1, C_2 > 0$. Hence, by the Lax-Milgram lemma, we conclude that $\mathcal{Q}_k : H^{2-k} \to H^{k-2}$ is a linear isomorphism.

Next, we use a bootstrapping argument to check that $\mathcal{Q}_k : H^m \to H^{m+2k-4}$ is a linear isomorphism for any $2-k \leq m \leq \ell - 2k + 1$. We already know that $\mathcal{Q}_k : H^m \to H^{m+2k-4}$ is a bounded linear operator and, by the previous step, that it is one-to-one. To show that it is onto, we argue by induction with respect to m. The case $m = 2-k$ has been treated above. Suppose that for an arbitrary m verifying $2-k \leq m \leq \ell - 2k$, we have that $\mathcal{Q}_k : H^m \to H^{m+2k-4}$ is onto (and hence a bijection). We have to show that $\mathcal{Q}_k : H^{m+1} \to H^{m+2k-3}$ is onto. By the induction hypothesis, for any given f in H^{m+2k-3}, there exists (a unique) q in H^m such that $\mathcal{Q}_k q = f$. Then, $q = \mathcal{Q}_k^{-1} f$, and for $j = 1, 2$, one has

$$\partial_{x_j} q = \partial_{x_j} \mathcal{Q}_k^{-1} f = \mathcal{Q}_k^{-1} \partial_{x_j} f + [\partial_{x_j}, \mathcal{Q}_k^{-1}] f\,. \tag{6.15}$$

As $[\partial_{x_j}, \mathcal{Q}_k^{-1}] = \mathcal{Q}_k^{-1}[\mathcal{Q}_k, \partial_{x_j}]\mathcal{Q}_k^{-1}$ we get

$$\partial_{x_j} q = \mathcal{Q}_k^{-1} \partial_{x_j} f + \mathcal{Q}_k^{-1}[\mathcal{Q}_k, \partial_{x_j}]\mathcal{Q}_k^{-1} f\,.$$

The first term on the r.h.s. of the latter identity is in H^m. To see that the second term also lies in H^m we write $\mathcal{Q}_k h$ for $h \in H^r$ with $2-k \le r \le \ell - 2k$ as

$$\mathcal{Q}_k h = \int_0^1 (\Phi^s)^{-1} \circ \mathcal{C}_s(A_k^{-1/2}) \circ \mathcal{C}_s(\mathcal{P}_\lambda^\dagger(s)) \circ \mathcal{C}_s(\mathcal{P}_\lambda(s)) \circ \mathcal{C}_s(A_k^{-1/2}) \circ \Psi^s h \, ds, \quad (6.16)$$

where $\mathcal{C}_s \equiv \mathcal{C}_{\varphi(s)}$ denotes conjugation of an operator by $R_{\varphi(s)}$,

$$\mathcal{C}_s(\cdot) = R_{\varphi(s)} \circ (\cdot) \circ R_{\varphi(s)}^{-1},$$

and the integration is performed in H^{r+2k-4}. Using (6.16) and the commutator identity $[a, bc] = [a, b]\, c + b\, [a, c]$ valid for any elements of a ring one sees that $[\mathcal{Q}_k, \partial_{x_j}]g$ with $g \in H^m$ can be represented as a sum of integrals each of which involves the commutator of ∂_{x_j} with one of the operators occuring in the integrand of (6.16). One then verifies that each of these summands is an element in $H^{m+(2k-4)}$ and hence

$$\mathcal{Q}_k^{-1}[\mathcal{Q}_k, \partial_{x_j}]\mathcal{Q}_k^{-1} f \in H^m \,. \quad (6.17)$$

To illustrate how this is done let us consider, for example, the commutator $[\mathcal{C}_s(A_k^{-1/2}), \partial_{x_j}]$. One has, with $\tilde{\mathcal{C}}_s = \mathcal{C}_{\varphi(s)^{-1}}$,

$$\begin{aligned}
&[\mathcal{C}_s(A_k^{-1/2}), \partial_{x_j}] = [\mathcal{C}_s(A_k^{-1/2}), \mathcal{C}_s(\tilde{\mathcal{C}}_s(\partial_{x_j}))] = \\
&= \mathcal{C}_s\Big([A_k^{-1/2}, \tilde{\mathcal{C}}_s(\partial_{x_j})]\Big) = \mathcal{C}_s\Big(A_k^{-1/2}[\tilde{\mathcal{C}}_s(\partial_{x_j}), A_k^{1/2}]A_k^{-1/2}\Big) \,. \quad (6.18)
\end{aligned}$$

A direct computation shows that $\tilde{\mathcal{C}}_s(\partial_{x_j}) = \sum_{1 \le l \le 2} r_{jl}(s)\, \partial_{x_l}$ where $r_{jl}(s) = \partial_{x_j}\varphi_l(s) \circ \varphi(s)^{-1}$ is in $C^0((-2,2), H^{\ell-1})$. Hence,

$$[\tilde{\mathcal{C}}_s(\partial_{x_j}), A_k^{1/2}] = \sum_l [A_k^{1/2}, r_{jl}]\, \partial_{x_l},$$

and by Lemma B.3, for any $k - \ell + 2 \le n \le \ell - 1$, $[\tilde{\mathcal{C}}_s(\partial_{x_j}), A_k^{1/2}] : H^n \to H^{n-k}$ is a bounded operator whose norm depends continuously on $s \in (-2, 2)$. In particular, (6.18) implies that for any $2 - \ell \le n \le \ell - k - 1$

$$[\mathcal{C}_s(A_k^{-1/2}), \partial_{x_j}] : H^n \to H^{n+k}$$

is a bounded operator depending strongly continuously on $s \in (-2, 2)$. Hence, for any $g \in H^m$ with $2 - \ell \le m \le \ell - k - 1$,

$$\int_0^1 (\Phi^s)^{-1} \circ \mathcal{C}_s(A_k^{-1/2}) \circ \mathcal{C}_s(\mathcal{P}_\lambda^\dagger(s) \circ \mathcal{P}_\lambda(s)) \circ [\mathcal{C}_s(A_k^{-1/2}), \partial_{x_j}] \circ \Psi^s\, g\, ds$$

and

$$\int_0^1 (\Phi^s)^{-1} \circ [\mathcal{C}_s(A_k^{-1/2}), \partial_{x_j}] \circ \mathcal{C}_s(\mathcal{P}_\lambda^\dagger(s) \circ \mathcal{P}_\lambda(s)) \circ \mathcal{C}_s(A_k^{-1/2}) \circ \Psi^s\, g\, ds$$

belong to H^{m+2k-4}. In a straightforward way one sees that for any $4 - \ell \le m \le \ell - 1$, $[\mathcal{C}_s(\mathcal{P}_\lambda(s)), \partial_{x_j}] : H^m \to H^{m-2}$ and $[\mathcal{C}_s(\mathcal{P}_\lambda^\dagger(s)), \partial_{x_j}] : H^m \to H^{m-2}$, whereas for any $2 - \ell \le m \le \ell - 2$, $[\Psi^s, \partial_{x_j}] : H^m \to H^m$ and $[(\Phi^s)^{-1}, \partial_{x_j}] : H^m \to H^m$, are bounded operators depending (strongly) continuously on $s \in (-2, 2)$. Hence all integrals appearing in the above mentioned representation of $[\mathcal{Q}_k, \partial_{x_j}]g$ are elements in H^{m+2k-4} and (6.17) holds as claimed.

Now (6.17) implies that, for $j = 1, 2$,

$$\partial_{x_j} q = \mathcal{Q}_k^{-1} \partial_{x_j} f + \mathcal{Q}_k^{-1} [\mathcal{Q}_k, \partial_{x_j}] \mathcal{Q}_k^{-1} f$$

is in H^m, and therefore q belongs to H^{m+1}. This shows that $\mathcal{Q}_k : H^{m+1} \to H^{m+2k-3}$ is onto and hence a linear isomorphism. This completes the induction argument.

In particular, $\mathcal{Q}_k : H^{\ell+1-2k} \to H^{\ell-3}$ is a linear isomorphism. As $\varphi(1)$ is assumed to be in $\mathcal{D}_+^{\ell+1}$, it then follows from (6.12) that

$$A_k v_0 = \mathcal{Q}_k^{-1} (\Phi^1)^{-1} (\Delta^2 \varphi(1) + \ldots)$$

is in $H^{\ell+1-2k}$ i.e., that v_0 is in $H^{\ell+1}$. $\square$

To prove Proposition 5.2 we need to make some preparations. The following arguments are valid for v_0 in $V_{\ell+2}^{(k)}$ where $\ell \geq \ell_k$. So let us assume that v_0 is in $V_{\ell+2}^{(k)}$. It follows by Theorem 4.1 that the solution (φ, v) of (1.2) issuing form (id, v_0) is in $C^1((-2,2), \mathcal{D}_+^{\ell+2} \times H^{\ell+2})$ and that the corresponding solution $u = v \circ \varphi^{-1}$ of (1.7) is in $C^0((-2,2), H^{\ell+2}) \cap C^1((-2,2), H^{\ell+1})$. Now, let us compute the derivative $\delta\varphi(t) := d_{v_0} \mathrm{Exp}_{k,\ell}(\delta v_0)$ of $\mathrm{Exp}_{k,\ell}$ at the point v_0 in the direction $\delta v_0 \in H^\ell$. For this purpose introduce $\delta v(t) := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} v(t; v_0 + \varepsilon \delta v_0)$ and $\delta u := \delta u(t) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} u(t; v_0 + \varepsilon \delta v_0)$. By Remark 4.2 one has $(\delta\varphi, \delta v) \in C^0((-2,2), H^\ell \times H^\ell)$. Moreover, the variation of (1.6) in $H^{\ell-1}$ leads to

$$\delta u = (\delta v) \circ \varphi^{-1} + [(dv) \circ \varphi^{-1}]\, \delta\varphi^{-1}\,,$$

where $\delta\varphi^{-1} \circ \varphi = -(d\varphi)^{-1} \delta\varphi$, so that

$$\delta u = [\delta v - (dv)(d\varphi)^{-1} \delta\varphi] \circ \varphi^{-1}\,.$$

Using that (φ, v) is in $C^1((-2,2), \mathcal{D}_+^{\ell+2} \times H^{\ell+2})$, one concludes that $\delta u \in C^0((-2,2), H^\ell)$. The variation of the integral analogue of (1.6),

$$\varphi(t) = \mathtt{id} + \int_0^t u \circ \varphi(s)\, ds\,,$$

leads to

$$(\delta\varphi)(t) = \int_0^1 [\delta u \circ \varphi + (du \circ \varphi)\, \delta\varphi]\, ds$$

where the integration is performed in $H^{\ell-1}$. As the integrand in the later formula lies in $C^0((-2,2), H^\ell)$, we see that $\delta\varphi \in C^1((-2,2), H^\ell)$ and satisfies the following inhomogeneous linear equation in H^ℓ,

$$\begin{cases} (\delta\varphi)^{\cdot} - [(du) \circ \varphi]\delta\varphi = \delta u \circ \varphi \\ \delta\varphi(0) = 0\,. \end{cases} \tag{6.19}$$

It follows from (6.19) and the method of the variation of parameters that

$$\delta\varphi(t) = \Phi^t \int_0^t (\Phi^s)^{-1} (\delta u(s) \circ \varphi(s))\, ds, \tag{6.20}$$

where, $\Phi^t = \Phi^t(x)$ is the 2×2 fundamental matrix solution of (6.1) and the integration is performed in H^ℓ. To express the r.h.s. of (6.20) in terms of the initial data, we need to investigate the inhomogeneous term on the r.h.s of (6.19). To this end, passing to the integral version of (1.7) and arguing as above we get, after composing with φ,

$$\begin{cases} [(A_k\delta u)\circ\varphi]^{\cdot} + \left\{[\text{div } u \cdot \text{Id} + (du)^\dagger]\circ\varphi\right\}[(A_k\delta u)\circ\varphi] = g_k \\ A_k\delta u(0) = A_k\delta v_0 \end{cases} \tag{6.21a}$$

where

$$g_k := -\left\{(d(A_k u))\delta u + \left[\text{div } (\delta u)\cdot \text{Id} + (d(\delta u))^\dagger\right] A_k u\right\}\circ\varphi. \tag{6.21b}$$

Note that g_k contains only derivatives of δu up to first order. Hence, by the regularity properties of φ, u, and δu discussed above, and the crucial assumption that $k \geq 1$, g_k is in $C^0((-2,2), H^{\ell+1-2k})$. Equation (6.21a) is a linear inhomogeneous ODE for $(A_k\delta u)\circ\varphi$. Hence, the solution of (6.21a) is given by

$$(A_k\delta u(s))\circ\varphi(s) = \Psi^s\left(A_k\delta v_0 + \int_0^s (\Psi^\tau)^{-1} g_k(\tau)d\tau\right), \tag{6.22}$$

where $\Psi^s = \Psi^s(x)$ is the 2×2 fundamental matrix solution of (6.3) and the integration is carried out in $H^{\ell+1-2k}$. Solving (6.22) for $\delta u(s)$ we get

$$\delta u(s)\circ\varphi(s) = R_{\varphi(s)}\circ A_k^{-1}\circ R_{\varphi(s)}^{-1}\left(\Psi^s\left(A_k\delta v_0 + \int_0^s (\Psi^\tau)^{-1} g_k(\tau)d\tau\right)\right). \tag{6.23}$$

We need to investigate the regularity of the r.h.s. of (6.23). As v_0 is assumed to be in $V_{\ell+2}^{(k)}$, the coefficients of the matrices $du\circ\varphi$ and $[\text{div } u\cdot \text{Id} + (du)^\dagger]\circ\varphi$ of (6.1) and (6.3) are in $C^0((-2,2), H^{\ell+1})$. In particular, $du\circ\varphi$ and $[\text{div } u\cdot\text{Id} + (du)^\dagger]\circ\varphi$ are in $C^0((-2,2), \mathcal{L}(H^{\ell+1}, H^{\ell+1}))$ and hence, by the ODE theory, the columns of the 2×2 matrices Φ and Ψ are in $C^1((-2,2), H^{\ell+1})$. It then follows from (6.23) that

$$\delta u(s)\circ\varphi(s) = R_{\varphi(s)}\circ A_k^{-1}\circ R_{\varphi(s)}^{-1}\circ\Psi^s(A_k\delta v_0) + \ldots \tag{6.24}$$

where $\ldots$ stand for terms in $C^0((-2,2), H^{\ell+1})$. Hence, by (6.20),

$$\begin{aligned} \delta\varphi(1) &= \Phi^1\int_0^1 (\Phi^s)^{-1}(\delta u(s)\circ\varphi(s))\,ds \\ &= \Phi^1\int_0^1 (\Phi^s)^{-1}\circ R_{\varphi(s)}\circ A_k^{-1}\circ R_{\varphi(s)}^{-1}\circ\Psi^s(A_k\delta v_0)\,ds + \ldots \\ &= \Phi^1\mathcal{R}_k(A_k\delta v_0) + \ldots \end{aligned} \tag{6.25}$$

where $\ldots$ stand for terms in $C^0((-2,2), H^{\ell+1})$ and for $f\in H^{\ell-2k}$

$$\mathcal{R}_k f := \int_0^1 (\Phi^s)^{-1}\circ R_{\varphi(s)}\circ A_k^{-1/2}\circ A_k^{-1/2}\circ R_{\varphi(s)}^{-1}\circ\Psi^s f\,ds. \tag{6.26}$$

Notice that the operator $\mathcal{R}_k$ is of the same form as the operator $\mathcal{Q}_k$ defined in (6.13). In particular, arguing as in the proof of Proposition 5.1, one sees that $\mathcal{R}_k$ extends to an operator $\mathcal{R}_k : H^{-k}\to H^k$ and for any $-k\leq m\leq \ell-2k+1$

$$\mathcal{R}_k|_{H^m} : H^m \to H^{m+2k} \tag{6.27}$$

is a linear isomorphism.

Proof of Proposition 5.2. Assume that $v_0 \in V^{(k)}_{\ell+2}$. Then since (6.27) is a linear isomorphism for $m = \ell - 2k$ and $\ell - 2k + 1$ and since the matrix Φ^1 is invertible with elements in $H^{\ell+1}$ we get from (6.25) that δv_0 is in $H^\ell \backslash H^{\ell+1}$ if and only if $\delta\varphi(1)$ is in $H^\ell \backslash H^{\ell+1}$. In other words, we have shown that

$$d_{v_0}\mathrm{Exp}_{k,\ell}\,(\delta v_0) \in H^\ell \backslash H^{\ell+1} \quad \text{if and only if} \quad \delta v_0 \in H^\ell \backslash H^{\ell+1}\,.$$

□

7. The exponential map for $k = 0$. In this section we prove Theorem 1.5. It says that even if we assume that the exponential map Exp_0 can be defined near zero it does *not* define a C^1_F-chart of id in $\mathcal{D}_+$.

Proof of Theorem 1.5. We follow the idea of the proof of the corresponding result for the circle in [7], [8]. For $k = 0$, (1.7) takes the form

$$\begin{cases} \dot u + (du + (du)^\dagger + \mathrm{div}\, u \cdot \mathrm{Id})u = 0 \\ u(0) = v_0 \end{cases} \tag{7.1}$$

Consider the initial data $v_0(x) = ce_1$ in which $e_1 = (1, 0)$, and c in $\mathbb{R}\backslash\{0\}$. Then $u(t, x) \equiv ce_1$ is a global (in time) solution of (7.1). The equation for the geodesic flow (1.6) then reads

$$\dot\varphi = ce_1,\ \varphi(0) = \mathrm{id}, \tag{7.2}$$

so that

$$\varphi(t, x) = x + cte_1\,.$$

By assumption, the exponential map Exp_0 is uniquely defined near 0; hence, it follows that for c sufficiently small

$$\mathrm{Exp}_0(ce_1)(x) = \varphi(1, x) = x + ce_1,\ \ \forall x \in \mathbb{T}^2.$$

In view of Remark A.4, Theorem 1.5 will follow if we can show that there exists c in $\mathbb{R}\backslash\{0\}$ arbitrarily close to 0 so that

$$d_{ce_1}\mathrm{Exp}_0 : C^\infty(\mathbb{T}^2, \mathbb{R}^2) \to T_{\mathrm{id}+ce_1}\mathcal{D}_+$$

is not one-to-one. To this end, we consider the variational equations for δu and $\delta\varphi$, obtained by linearizing equations (7.1) and (7.2) at the solutions $u(t, x) = ce_1$, and $\varphi(t, x) = x + cte_1$ respectively. One gets

$$\begin{cases} (\delta u)^{\cdot} + c[d(\delta u) + (d(\delta u))^\dagger + \mathrm{div}(\delta u) \cdot \mathrm{Id}]e_1 = 0 \\ \delta u(0) = \delta v_0 \end{cases} \tag{7.3a}$$

and

$$\begin{cases} (\delta\varphi)^{\cdot} = \delta u(t, x + cte_1) \\ \delta\varphi(0) = 0. \end{cases} \tag{7.3b}$$

In particular, for a variation of the form $\delta v_0 = he_1$, with $h(x) := h(x_1)$, we have that componentwise (7.3a) is given by

$$\begin{cases} (\delta u_1)^\cdot + c(3\partial_{x_1}(\delta u_1) + \partial_{x_2}(\delta u_2)) & = 0, \quad \delta u_1(0) = h \\ (\delta u_2)^\cdot + c(\partial_{x_1}(\delta u_2) + \partial_{x_2}(\delta u_1)) & = 0, \quad \delta u_2(0) = 0. \end{cases}$$

A solution is $\delta u(t,x) = h(x_1 - 3ct)e_1$. Hence the corresponding solution of (7.3b) is

$$\delta\varphi(t,x) = \int_0^t h(x_1 - 2cs)ds \cdot e_1,$$

and thus

$$(d_{ce_1}\mathrm{Exp}_0)(he_1)(x) = \int_0^1 h(x_1 - 2cs)ds \cdot e_1.$$

Now consider $c_\ell := \frac{1}{\ell}$, and let $h_\ell(x_1) := \cos(2\pi\ell x_1)$, with ℓ in $\mathbb{Z}_{\geq 1}$. Then

$$\int_0^1 h_\ell(x_1 - 2c_\ell s)ds = \int_0^1 \cos(2\pi(\ell x_1 - 2s))\, ds = 0,$$

i.e.,

$$(d_{\frac{1}{\ell}e_1}\mathrm{Exp}_0)(h_\ell e_1) = 0, \quad \forall \ell \in \mathbb{Z}_{\geq 1}.$$

Hence, by Remark A.4, the map Exp_0 cannot be a local C^1_F-diffeomorphism near the origin in $T_{\mathrm{id}}\mathcal{D}_+$. □

REMARK 7.1. *Note that the group $\mathcal{D}_+(\mathbb{T})$ of smooth orientation preserving diffeomorphisms of the circle $\mathbb{T}$ can be canonically embedded into $\mathcal{D}_+(\mathbb{T}^2)$,*

$$\mathcal{D}_+(\mathbb{T}) \to \mathcal{D}_+(\mathbb{T}^2), \ \varphi_1 \mapsto \varphi_1 \times \mathrm{id}\,.$$

It is straightforward to verify that $\mathcal{D}_+(\mathbb{T})$ is a totally geodesic Fréchet submanifold of $\mathcal{D}_+(\mathbb{T}^2)$ with respect to the right invariant metric $\nu^{(0)}$. In this way, Theorem 1.5 then follows from the corresponding result for $\mathcal{D}_+(\mathbb{T})$ (cf. [7], [8].)

Appendix A. Calculus on Fréchet spaces. For the convenience of the reader we recall in this appendix some definitions and notions from the calculus in Fréchet spaces put together in [6], and present an inverse function theorem valid in a set-up for Fréchet spaces which is suitable for our purposes. For more details we refer the reader to [14] and [21].

Fréchet spaces: Consider the pair $(X, \{\|\cdot\|_n\}_{n\geq 0})$ where X is a real vector space and $\{\|\cdot\|_n\}_{n\geq 0}$ is a countable collection of seminorms. A topology on X is defined in the usual way as follows: A basis of open neighborhoods of $0 \in X$ is given by the sets

$$U_{\epsilon,k_1,\ldots,k_s} := \{x \in X \ : \ \|x\|_{k_j} < \epsilon \ \forall 1 \leq j \leq s\}$$

where $s, k_1, \ldots k_s \in \mathbb{Z}_{\geq 0}$ and $\epsilon > 0$. Then the topology on X is defined as the collection of open sets generated by the sets $x + U_{\epsilon,k_1,\ldots,k_s}$, for arbitrary x in X and arbitrary

$s, k_1, ..., k_s \in \mathbb{Z}_{\geq 0}$ and $\epsilon > 0$. In this way X becomes a topological vector space. Note that a sequence x_k converges to x in X iff for any $n \geq 0$, $\|x_k - x\|_n \to 0$ as $k \to +\infty$.

Moreover, the topological vector space X described above is *Hausdorff* iff for any x in X, $\|x\|_n = 0$ for every n in $\mathbb{Z}_{\geq 0}$ implies $x = 0$. A sequence $(x_k)_{k\in\mathbb{N}}$ is called *Cauchy* iff it is a Cauchy sequence with respect to any of the seminorms. By definition, X is complete iff every Cauchy sequence converges in X.

Definition A.1. *A pair $(X, \{\|\cdot\|_n\}_{n\geq 0})$ consisting of a topological vector space X and a countable system of seminorms $\{\|\cdot\|_n\}_{n\geq 0}$ is called a* Fréchet space [2] *iff the topology of X is the one induced by $\{\|\cdot\|_n\}_{n\geq 0}$, and X is Hausdorff and complete.*

C^1_F-differentiability: Let $f : U \subseteq X \to Y$ be a map from an open set U of a Fréchet space X to a Fréchet space Y.

Definition A.2. *If the limit*

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}(f(x+\epsilon h) - f(x))$$

in Y exists with respect to the Fréchet topology of Y, we say that f is differentiable at x in the direction h. *The limit is declared to be the* directional derivative *of f at the point x in U in the direction h in X, and is denoted by $\delta_x f(h)$.*

Definition A.3. *If the directional derivative $\delta_x f(h)$ exists for any x in U and any h in X, and the map*

$$(x,h) \mapsto \delta_x f(h),\ U \times X \to Y$$

is continuous with respect to the Fréchet topology on $U \times X$ and Y, then f is called continuously differentiable on U *or* C^1_F-smooth. *The space of all such maps is denoted by $C^1_F(U, Y)$.*[3] *A map $f : U \to V$ from an open set $U \subseteq X$ onto an open set $V \subseteq Y$ is called* a C^1_F-diffeomorphism *if f is a homeomorphism and f as well as f^{-1} are C^1_F-smooth.*

Remark A.4. *Using the chain rule one easily obtains that for any x in U the directional derivative $\delta_x f : X \to Y$ of a C^1_F-diffeomorphism $f : U \to V$ is a linear isomorphism.*

In this paper we consider mainly the following spaces:

Fréchet space $C^\infty(\mathbb{T}^2, \mathbb{R}^2)$. The space $C^\infty(\mathbb{T}^2, \mathbb{R}^2)$ denotes the real vector space of C^∞-smooth, functions $u : \mathbb{T}^2 \to \mathbb{R}^2$. The topology on $C^\infty(\mathbb{T}^2, \mathbb{R}^2)$ is induced by the countable system of Sobolev norms:

$$\|u\|_n := \|u\|_{H^n} = \left(\sum_{j=0}^{n} \int_{\mathbb{T}^2} \langle (-\Delta)^j u, u \rangle \, dx \right)^{1/2}, \quad n \geq 0.$$

[2] Unlike for the standard notion of a Fréchet space, here the countable system of seminorms defining the topology of X is part of the structure of the space.

[3] Note that even in the case where X and Y are Banach spaces this definition of continuous differentiability is weaker than the usual one (cf. [14]). In order to distinguish it from the classical one we write C^1_F instead of C^1. We refer to [14] for a discussion of the reasons to introduce the notion of C^1_F-differentiability.

Fréchet manifold $\mathcal{D}_+(\mathbb{T}^2)$. By definition, $\mathcal{D}_+ = \mathcal{D}_+(\mathbb{T}^2)$ denotes the group of C^∞-smooth positively oriented diffeomorphisms of the 2-d torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$. A Fréchet manifold structure on $\mathcal{D}_+$ can be introduced as follows: Passing in domain and target to the universal cover $\mathbb{R}^2 \to \mathbb{T}^2$ of $\mathbb{T}^2$, any element φ of $\mathcal{D}_+$ gives rise to a smooth diffeomorphism of $\mathbb{R}^2$ in $C^\infty(\mathbb{R}^2, \mathbb{R}^2)$, again denoted by φ, where each of the components φ_j, $j = 1, 2$, satisfies the *normalization conditions*

$$\mathsf{C}_1(\varphi_j):\ -1/2 < \varphi_j(0) < 1/2 \qquad \text{or} \qquad \mathsf{C}_2(\varphi_j):\ 0 < \varphi_j(0) < 1. \tag{A.1}$$

Note that for any $k \in \mathbb{Z}^2$, $\varphi(x+k) - \varphi(x)$ is a continuous function of x with values in $\mathbb{Z}^2$. Hence, it is independent of $x \in \mathbb{R}^2$. Then, by a similar argument, the linear map

$$T_\varphi : \mathbb{R}^2 \to \mathbb{R}^2,\ \ (x_1, x_2) \mapsto x_1(\varphi(1,0) - \varphi(0,0)) + x_2(\varphi(0,1) - \varphi(0,0))$$

has the property that the function $f := \varphi - T_\varphi$ is 1-periodic in x_1 and x_2, and hence lies in $C^\infty(\mathbb{T}^2, \mathbb{R}^2)$. Note that $T_\varphi \in SL(2;\mathbb{Z})$ and $\det(T_\varphi + d_x f) > 0$ at any x in $\mathbb{R}^2$. The normalizations (A.1) give rise to the following atlas of charts $\{\mathfrak{U}_{i,j,T}\}_{1\le i,j\le 2, T\in SL(2;\mathbb{Z})}$ of $\mathcal{D}_+$, with $\bigcup_{i,j,T} \mathfrak{U}_{i,j,T} = \mathcal{D}_+$, defined by

$$\mathfrak{F}_{i,j,T} : \mathfrak{V}_{i,j,T} \to \mathfrak{U}_{i,j,T}, \quad f \mapsto \varphi := T + f$$

where $\mathfrak{U}_{i,j,T} = T + \mathfrak{V}_{i,j,T}$, and

$$\mathfrak{V}_{i,j,T} := \{f \in C^\infty(\mathbb{T}^2, \mathbb{R}^2) :\ \mathsf{C}_i(f_1); \mathsf{C}_j(f_2);\ \det(T + df) > 0\}. \tag{A.2}$$

As $\mathfrak{V}_{i,j,T}$ $(1 \le i,j \le 2, T \in SL(2;\mathbb{Z}))$ are open subsets in the Fréchet space $C^\infty(\mathbb{T}^2, \mathbb{R}^2)$, the construction above gives an atlas of Fréchet charts of $\mathcal{D}_+$. In this way, $\mathcal{D}_+$ is a Fréchet manifold modeled on $C^\infty(\mathbb{T}^2, \mathbb{R}^2)$.

Hilbert manifold $\mathcal{D}_+^\ell(\mathbb{T}^2)$ *($\ell \ge 3$).* $\mathcal{D}_+^\ell = \mathcal{D}_+^\ell(\mathbb{T}^2)$ denotes the group of positively oriented C^1-diffeomorphisms of $\mathbb{T}^2$ of class H^ℓ. By definition, a C^1-diffeomorphism φ of $\mathbb{T}^2$ is in H^ℓ iff any lift of φ is in $H^\ell_{loc}(\mathbb{R}^2, \mathbb{R}^2)$. As for $\mathcal{D}_+$, one can introduce an atlas for $\mathcal{D}_+^\ell$ making $\mathcal{D}_+^\ell$ into a Hilbert manifold modeled on H^ℓ.

Hilbert approximations: Assume that for a given Fréchet space X there is a sequence of Hilbert spaces $\{(X_n, \|\cdot\|_n)\}_{n\ge 0}$ such that

$$X_0 \supseteq X_1 \supseteq X_2 \supseteq ... \supseteq X \quad \text{and} \quad X = \bigcap_{n=0}^{\infty} X_n$$

where $\{\|\cdot\|_n\}_{n\ge0}$ is a sequence of norms inducing the topology on X so that $\|x\|_0 \le \|x\|_1 \le \|x\|_2 \le ...$ for any x in X. Such a sequence of Hilbert spaces $\{(X_n, \|\cdot\|_n)\}_{n\ge0}$ is called a *Hilbert approximation* of the Fréchet space X. For Fréchet spaces admitting Hilbert approximations one can prove the following version of the inverse function theorem.

THEOREM A.5. *Let X and Y be Fréchet spaces over $\mathbb{R}$ with Hilbert approximations $(X_n, \|\cdot\|_n)_{n\ge0}$, and, respectively, $(Y_n, |\cdot|_n)_{n\ge0}$. Let $f : V_0 \to U_0$ be a map between the open subsets $V_0 \subseteq X_0$ and $U_0 \subseteq Y_0$ of the Hilbert spaces X_0, respectively Y_0. Define, for any $n \ge 0$,*

$$V_n := V_0 \cap X_n, \qquad\qquad U_n := U_0 \cap Y_n.$$

Assume that, for any $n \ge 0$, the following properties are satisfied:

(a) *$f : V_0 \to U_0$ is a bijective C^1-map, and, for any x in $V := V_0 \cap X$, $d_x f : X_0 \to Y_0$ is a linear isomorphism;*
(b) *$f(V_n) \subseteq Y_n$, and the restriction $f\big|_{V_n} : V_n \to Y_n$ is a C^1-map;*
(c) *$f(V_n) \supseteq U_n$;*
(d) *for any x in V, $d_x f(X_n \backslash X_{n+1}) \subseteq Y_n \backslash Y_{n+1}$.*

Then for the open subsets $V := V_0 \cap X \subseteq X$ and $U := U_0 \cap Y \subseteq Y$, one has $f(V) \subseteq U$ and the map $f_\infty := f\big|_V : V \to U$ is a C^1_F-diffeomorphism.

Proof. By properties (a) and (b), $f_n := f\big|_{V_n} : V_n \to U_n$ is a well-defined, injective C^1-map. By (c), f_n is onto. Hence, $f_\infty := f\big|_V : V \to U$ is bijective. In order to prove that $f_\infty : V \to U$ is a C^1_F-diffeomorphism, consider, for any $n \geq 0$, and any x in V, the differential $d_x f_n : X_n \to Y_n$. As $(d_x f)\big|_{X_n} = d_x f_n$ and, by (a), $d_x f : X_0 \to Y_0$ is bijective, one concludes that $d_x f_n$ is one-to-one. We prove by induction (with respect to n) that, for any x in V, $d_x f_n : X_n \to Y_n$ is onto. For $n = 0$ ($V \subseteq V_0$), the statement is true by property (a). Next, assume that for arbitrary positive integer n, and arbitrary x in V, $d_x f_{n-1} : X_{n-1} \to Y_{n-1}$ is onto. Then, for any x in V, and η in $Y_n \subseteq Y_{n-1}$, there exists a (unique) ξ in X_{n-1} verifying $d_x f_{n-1}(\xi) = \eta$. By property (d), it follows that ξ belongs to X_n. In other words, for any given $n \geq 0$, and any x in V, we have that the map $d_x f_n : X_n \to Y_n$ is bijective, and thus, by open mappings theorem, the inverse $(d_x f_n)^{-1} : Y_n \to X_n$ is a bounded linear operator. As, for any $n \geq 0$, f_n is C^1-smooth, the map

$$V_n' \times X_n \to Y_n,\ (x, \xi) \mapsto d_x f_n(\xi) \tag{A.3}$$

is continuous and, by the inverse function theorem it follows that

$$U_n' \times Y_n \to X_n,\ (y, \eta) \mapsto d_y (f_n^{-1})(\eta) \tag{A.4}$$

is continuous as well. Here V_n' (and U_n') denotes the subset V (respectively U) with the topology induced by $|\cdot|_n$ (respectively $\|\cdot\|_n$). As for any x in V, and $n \geq 0$,

$$\delta_x f_\infty = d_x f_n\big|_X$$

one gets from (A.3) - (A.4) that

$$V \times X \to Y,\ (x, \xi) \to \delta_x f_\infty(\xi)$$

and

$$U \times Y \to X,\ (x, \eta) \mapsto \delta_y f_\infty^{-1}(\eta)$$

are continuous. In particular, one concludes (cf. Definition A.3) that

$$f_\infty : V \to U$$

is a C^1_F-diffeomorphism. □

Appendix B. Auxiliary results. In this appendix we collect some elementary auxiliary results used in the proofs of Propositions 5.1 and 5.2. We freely use the notations introduced in the main body of the paper.

LEMMA B.1. *Let (φ, v) and $u = v \circ \varphi^{-1}$ be the solutions of* (1.2)*, respectively,* (1.7) *given by Theorem 4.1. Then for any given $x \in \mathbb{T}^2$ and $-2 < t < 2$,*

(i) $(\Psi^t(x))^\dagger = [(\det \Phi^t(x))\Phi^t(x)]^{-1}$;

(ii) $\big((\Phi^t(x))^{-1} \circ R_{\varphi(t,x)}\big)^\dagger = R^{-1}_{\varphi(t,x)} \circ \Psi^t(x)$,

where $E^\dagger$ denotes the conjugation of a linear operator $E : H^m \to H^m$ with respect to the L^2-scalar product extended by continuity to a bilinear pairing $H^{-m} \times H^m \to \mathbb{R}$ $(0 \le m \le \ell - 1)$.

Proof. (i) From (6.1) and (6.3) it follows that, suppressing t in Ψ^t and Φ^t,

$$\Psi^\dagger \dot{\Phi} = \Psi^\dagger[(du) \circ \varphi]\Phi = -\Big((\dot{\Psi})^\dagger + [(\text{div } u) \circ \varphi]\Psi^\dagger\Big)\,\Phi,$$

so that

$$\begin{cases} (\Psi^\dagger\Phi)^{\cdot} = -[(\text{div } u) \circ \varphi]\Psi^\dagger\Phi \\ (\Psi^\dagger\Phi)^0 = \text{Id}. \end{cases}$$

Solving the latter we get, for any given $x \in \mathbb{T}^2$, and $-2 < t < 2$,

$$(\Psi^t(x))^\dagger\Phi^t(x) = e^{-\int_0^t (\text{div } u)(\tau,\varphi(\tau,x))\,d\tau}\,\text{Id}.$$

To prove item (i) it remains to show that the exponential factor in the latter identity is given by $(\det \Phi^t(x))^{-1}$. For any given $x \in \mathbb{T}^2$, this relation follows from the Liouville formula for the Wronskian of the linear equation (6.1),

$$\det \Phi^t(x) = e^{\int_0^t \text{Tr}(du(\tau,\varphi(\tau,x)))\,d\tau} = e^{\int_0^t (\text{div } u)(\tau,\varphi(\tau,x))\,d\tau}\,.$$

(ii) For any pair f, g of $\mathbb{R}^2$-valued L^2-functions on $\mathbb{T}^2$ we have

$$\begin{aligned} \langle f, R_{\varphi(t)}g\rangle_{L^2} &= \langle R^{-1}_{\varphi(t)}f, [\det(d\varphi^{-1}(t))]g\rangle_{L^2} \\ &= \langle R^{-1}_{\varphi(t)}\left([\det(d\varphi(t))]^{-1}f\right), g\rangle_{L^2}. \end{aligned} \tag{B.1a}$$

The first identity in the above display follows from an obvious change of variables. The second one follows from the fact that $(d\varphi^{-1}) \circ \varphi = (d\varphi)^{-1}$ which implies that $\det(d\varphi^{-1}) = [\det(d\varphi)]^{-1} \circ \varphi^{-1}$. Hence, since $d\varphi(t) = \Phi^t$, the L^2-transpose $R^\dagger_{\varphi(t)}$ of the right translation operator $R_{\varphi(t)}$ is given by the composition of multiplication by $[\det\Phi^t]^{-1}$ with $R^{-1}_{\varphi(t)}$ i.e.,

$$R^\dagger_{\varphi(t)} = R^{-1}_{\varphi(t)} \circ [\det\Phi^t]^{-1}. \tag{B.1b}$$

Combining this with item (i) we get

$$\begin{aligned} \big((\Phi^t)^{-1} \circ R_{\varphi(t)}\big)^\dagger &= \Big(R^{-1}_{\varphi(t)} \circ [\det\Phi^t]^{-1}\Big) \circ ((\Phi^t)^{-1})^\dagger \\ &= R^{-1}_{\varphi(t)} \circ \Big(\big[(\det\Phi^t)\Phi^t\big]^{-1}\Big)^\dagger \\ &= R^{-1}_{\varphi(t)} \circ \Psi^t. \end{aligned}$$

Clearly, (i) and (ii) extend by continuity to negative Sobolev spaces. □

Lemma B.2. *Assume that $\ell \ge 3$, $\varphi \in C^1((-2, 2), \mathcal{D}^\ell_+)$, and consider the one-parameter family of differential operators on the torus $\mathbb{T}^2$,*

$$\mathcal{P}_\lambda(s) := \lambda - \sum_{1 \le i,j \le 2} p_{ij}(s)\,\partial_{x_i}\partial_{x_j} \quad (-2 < s < 2)\,,$$

with coefficients $(p_{ij}(s,x))_{1\le i,j\le 2} := (d_{\varphi^{-1}(s,x)}\varphi(s))(d_{\varphi^{-1}(s,x)}\varphi(s))^\dagger$, $x\in\mathbb{T}^2$. *Then, there exists* $\lambda_0\ge 1$ *and* $C>0$ *so that for any* $\lambda\ge\lambda_0$, $0\le s\le 1$, *and* $u\in H^2(\mathbb{T}^2,\mathbb{R}^2)$,

$$\|\mathcal{P}_\lambda(s)u\|_{L^2}\ge C\|u\|_{H^2}\,. \tag{B.2}$$

Proof. Let $\lambda=1+\tilde\lambda$ and $\tilde\lambda\ge 0$. Then for any $u\in H^2(\mathbb{T}^2,\mathbb{R}^2)$ one has

$$\|\mathcal{P}_\lambda u\|^2_{L^2}=\|\mathcal{P}_1 u\|^2_{L^2}+2\tilde\lambda\int\limits_{\mathbb{T}^2}\mathcal{P}u\cdot u\,dx+(2\tilde\lambda+\tilde\lambda^2)\|u\|^2_{L^2} \tag{B.3}$$

where $\mathcal{P}:=\mathcal{P}_\lambda|_{\lambda=0}$, and the dot product denotes the Euclidean scalar product in $\mathbb{R}^2$. It follows from the formulas for the coefficients $p_{ij}(s,x)$ that there exists a constant $C_0>0$ so that for any $0\le s\le 1$, x in $\mathbb{T}^2$, and ξ in $\mathbb{R}^2$,

$$\sum_{1\le i,j\le 2}p_{ij}(s,x)\,\xi_i\,\xi_j\ge C_0\|\xi\|^2$$

and hence

$$\sum_{1\le i,j\le 2}p_{ij}(s)\,\partial_{x_i}u\cdot\partial_{x_j}u\ge C_0\Big(\sum_{i=1}^2|\partial_{x_i}u_1|^2+\sum_{i=1}^2|\partial_{x_i}u_2|^2\Big)$$

where u_1 and u_2 are the components of u. Using (B.3) in combination with the latter estimate, we get that there exists $C_1>0$ such that for any $\tilde\lambda\ge 0$, $0\le s\le 1$, and $u\in H^2(\mathbb{T}^2,\mathbb{R}^2)$,

$$\begin{aligned}\|\mathcal{P}_\lambda u\|^2_{L^2}&=\|\mathcal{P}_1 u\|^2_{L^2}+(2\tilde\lambda+\tilde\lambda^2)\|u\|^2_{L^2}\\&\quad+2\tilde\lambda\int_{\mathbb{T}^2}\sum_{1\le i,j\le 2}\left(p_{ij}(s)\,\partial_{x_i}u\cdot\partial_{x_j}u-\frac12\,(\partial_{x_i}\partial_{x_j}\,p_{ij}(s))\,u\cdot u\right)dx\\&\ge\|\mathcal{P}_1 u\|^2_{L^2}+(\tilde\lambda^2-C_1\tilde\lambda)\|u\|^2_{L^2}\,.\end{aligned}$$

Choosing $\tilde\lambda_0\ge 2\,C_1$ we obtain that for any $\tilde\lambda\ge\tilde\lambda_0$, $0\le s\le 1$, and $u\in H^2(\mathbb{T}^2,\mathbb{R}^2)$,

$$\|\mathcal{P}_\lambda u\|^2_{L^2}\ge\|\mathcal{P}_1 u\|^2_{L^2}+\tilde\lambda^2\|u\|^2_{L^2}/2\,. \tag{B.4}$$

Note that by (6.5),

$$\mathcal{P}_1 u=R_{\varphi^{-1}}\circ(1-\Delta)\circ R_\varphi u+du(R_{\varphi^{-1}}\circ\Delta(\varphi))\,.$$

Hence there exist positive constants so that for any $0\le s\le 1$ and $u\in H^2(\mathbb{T}^2,\mathbb{R}^2)$

$$\begin{aligned}\|\mathcal{P}_1 u\|^2_{L^2}&\ge\|R_{\varphi^{-1}}\circ(1-\Delta)\circ R_\varphi u\|^2_{L^2}\\&\quad+2\langle R_{\varphi^{-1}}\circ(1-\Delta)\circ R_\varphi u,du(R_{\varphi^{-1}}\circ\Delta(\varphi))\rangle_{L^2}\\&\ge C_2\|u\|^2_{H^2}-2\|R_{\varphi^{-1}}\circ(1-\Delta)\circ R_\varphi u\|_{L^2}\|du(R_{\varphi^{-1}}\circ\Delta(\varphi)\|_{L^2}\\&\ge C_2\|u\|^2_{H^2}-C_3\|u\|_{H^2}\|u\|_{H^1}\\&\ge\frac{C_2}{2}\,\|u\|^2_{H^2}-C_4\|u\|^2_{H^1}\\&\ge\frac{C_2}{4}\,\|u\|^2_{H^2}-C_5\|u\|^2_{L^2},\end{aligned}$$

where for the latter inequality we have used that, by interpolation, $\|u\|_{H^1} \leq C_6\|u\|_{H^2}^{1/2}\|u\|_{L^2}^{1/2}$, and hence $\|u\|_{H^1}^2 \leq \epsilon\,\|u\|_{H^2}^2 + C_6^4\|u\|_{L^2}^2/\epsilon$ for any $\epsilon > 0$. Thus

$$\|\mathcal{P}_1 u\|_{L^2}^2 + \tilde{\lambda}^2\|u\|_{L^2}^2/2 \geq \frac{C_2}{4}\,\|u\|_{H^2}^2 + \Big(\frac{\tilde{\lambda}^2}{2} - C_5\Big)\|u\|_{L^2}^2\,. \tag{B.5}$$

Together with (B.4) this implies that for $\tilde{\lambda}_0 = \max\{2\,C_1, \sqrt{2C_5}\}$ and $C = \sqrt{C_2}/2 > 0$ one has that for any $\tilde{\lambda} \geq \tilde{\lambda}_0$, $0 \leq s \leq 1$, and $u \in H^2(\mathbb{T}^2, \mathbb{R}^2)$,

$$\|\mathcal{P}_\lambda(s)\,u\|_{L^2} \geq C\|u\|_{H^2}\,. \tag{B.6}$$

□

The following result is well known. For the convenience of the reader we include an elementary proof for it.

Lemma B.3. *Let $k \geq 1$ and $s \geq k+2$. Then, for any $k - s \leq m \leq s-1$ and for any real-valued functions $f \in H^s$ and $g \in H^m$, $[A_k^{1/2}, f]\,g := A_k^{1/2}(fg) - fA_k^{1/2}g$ is in H^{m-k+1} and the bilinear mapping*

$$H^s \times H^m \to H^{m-k+1},\ (f,g) \mapsto [A_k^{1/2}, f]\,g$$

is continuous.

Proof. Consider the first order operator,

$$P_1 := (A_k)^{\frac{1}{2k}}\,.$$

Then the complete symbol of P_1 is the Fourier multiplier

$$\sigma_1(\xi) = \Big(\sum_{j=0}^{k}(2\pi)^{2j}(\xi_1^2 + \xi_2^2)^j\Big)^{\frac{1}{2k}}\,.$$

It means that for any $\mathbb{Z}^2$-periodic distribution $h \in \mathcal{D}'(\mathbb{T}^2)$ the Fourier coefficients $\widehat{(P_1 h)}(\xi)$, $\xi \in \mathbb{Z}^2$, of $P_1 h$ are given by $\widehat{(P_1 h)}(\xi) = \sigma_1(\xi)\hat{h}(\xi)$. Clearly, for any $l \in \mathbb{Z}$, $P_1 : H^l \to H^{l-1}$ is a bounded linear operator. Assume that $|m| \leq s-1$. For any $f \in H^s$ and $g \in H^m$ we have for $\xi \in \mathbb{Z}^2$,

$$\begin{aligned}
|(\widehat{[P_1, f]g})(\xi)| &= |\widehat{P_1(fg)}(\xi) - \widehat{fP_1 g}(\xi)|\\
&\leq \Big|\sigma_1(\xi)\sum_{\eta\in\mathbb{Z}^2}\hat{f}(\xi-\eta)\hat{g}(\eta) - \sum_{\eta\in\mathbb{Z}^2}\hat{f}(\xi-\eta)\sigma_1(\eta)\hat{g}(\eta)\Big|\\
&\leq \sum_{\eta\in\mathbb{Z}^2}|\sigma_1(\xi)-\sigma_1(\eta)||\hat{f}(\xi-\eta)||\hat{g}(\eta)|\\
&\leq C_k\sum_{\eta\in\mathbb{Z}^2}(1+|\xi-\eta|)|\hat{f}(\xi-\eta)||\hat{g}(\eta)|
\end{aligned} \tag{B.7}$$

where for the latter inequality we used that, for some constant $C_k > 0$,

$$|\sigma_1(\xi) - \sigma_1(\eta)| \leq C_k(1+|\xi-\eta|) \quad \forall \xi,\eta \in \mathbb{Z}^2\,.$$

It follows from (B.7) and Lemma B.4 below that there exists $C_k > 0$ such that for any $|m| \leq s-1$, and for any $f \in H^s$ and $g \in H^m$,

$$\|[P_1, f]g\|_{H^m} \leq C_k\|f\|_{H^s}\|g\|_{H^m}\,. \tag{B.8}$$

On the other side, as $A_k^{1/2} = P_1^k$, we have for any $f \in H^s$ and $g \in H^m$ with $k - s \leq m \leq s - 1$

$$[A_k^{1/2}, f]g = [P_1, f]P_1^{k-1}g + P_1[P_1, f]P_1^{k-2}g + ... + P_1^{k-1}[P_1, f]g \tag{B.9}$$

which together with (B.8) implies the statement of the lemma. □

In the proof of Lemma B.3 we used the following well-known fact. Denote by $h^l \equiv h^l(\mathbb{Z}^2, \mathbb{R})$ the Hilbert space

$$h^l = \{a = (a(\xi))_{\xi \in \mathbb{Z}^2} \mid \|a\|_{h^l} < \infty,\ a(-\xi) = a(\xi)\ \forall \xi \in \mathbb{Z}^2\}$$

where

$$\|a\|_{h^l} := \Big(\sum_{\xi \in \mathbb{Z}^2} (1 + \|\xi\|)^{2l} |a(\xi)|^2 \Big)^{1/2}.$$

LEMMA B.4. *Let $s \geq 3$ and $|m| \leq s$. Then the convolution*

$$h^s \times h^m \to h^m, \quad (a, b) \mapsto (a * b)(\xi) := \sum_{\eta \in \mathbb{Z}^2} a(\xi - \eta) b(\eta)$$

is continuous.

The proof of the lemma is straightforward and we omit it.

REMARK B.5. *In the proof of Proposition 5.1, we use Lemma B.3 as follows. First let us recall the set-up. Let $\ell \geq \ell_k = 2k+5$ be arbitrary and assume that the solution curve (φ, v) of (1.2) with initial data (id, v_0) and v_0 in H^ℓ, is in $C^1((-2,2), \mathcal{D}_+^\ell \times H^\ell)$. Then, $u = v \circ \varphi^{-1}$ is a solution of (1.7) in $C^0((-2,2), H^\ell) \cap C^1((-2,2), H^{\ell-1})$. Further, $\mathcal{P}_\lambda^\dagger \mathcal{P}_\lambda$, with $\mathcal{P}_\lambda$ given by (6.7), is a fourth order differential operator with coefficients in $C^0((-2,2), H^{\ell-3})$. Up to lower order, $\mathcal{P}_\lambda^\dagger \mathcal{P}_\lambda$ is equal to $\sum_{|\alpha|=4} f_\alpha \partial^\alpha$, where the coefficients f_α are in $C^0((-2,2), H^{\ell-3})$ (in fact they take values in $H^{\ell-1}$). This shows that for any h in H^ℓ,*

$$[A_k^{1/2}, \mathcal{P}_\lambda^\dagger \mathcal{P}_\lambda]h = A_k^{1/2} \sum_{|\alpha|=4} f_\alpha \partial^\alpha h - \sum_{|\alpha|=4} f_\alpha A_k^{1/2}(\partial^\alpha h) + \dots$$

where $\dots$ stand for terms in $C^0((-2,2), H^{\ell-k-3})$. Moreover, the assumptions above imply that, for any $-2 < s < 2$, each of the entries of the fundamental matrix solutions Φ^s and Ψ^s of (6.1), respectively, (6.3) is a continuous function of t with values in $H^{\ell-1}$. At this point it is crucial that we assume that $k \geq 1$. It guarantees that $A_k^{-1} \circ R_{\varphi(s)}^{-1} \circ \Psi^s(A_k v_0)$ is in $C^0((-2,2), H^\ell)$. Thus, for any $\alpha \in \mathbb{Z}_{\geq 0}^2$ with $|\alpha| = 4$, it follows that $g_\alpha := \partial^\alpha (A_k^{-1} \circ R_{\varphi(s)}^{-1} \circ \Psi^s(A_k v_0)) \in C^0((-2,2), H^{\ell-4})$. Hence, by Lemma B.3 applied to $f_\alpha \in H^{\ell-3}$ and $g_\alpha \in H^{\ell-4}$ for any $\alpha \in \mathbb{Z}_{\geq 0}^2$ with $|\alpha| = 4$,

$$[A_k^{1/2}, \mathcal{P}_\lambda^\dagger \mathcal{P}_\lambda] \circ A_k^{-1} \circ R_{\varphi(s)}^{-1} \circ \Psi^s(A_k v_0) \in C^0((-2,2), H^{\ell-k-3}).$$

As a consequence,

$$\Phi^t \int_0^t (\Phi^s)^{-1} \circ R_{\varphi(s)} \circ A_k^{-1/2} \circ [A_k^{1/2}, \mathcal{P}_\lambda^\dagger(s) \mathcal{P}_\lambda(s)] \circ A_k^{-1} \circ R_{\varphi(s)}^{-1} \circ \Psi^s ds (A_k v_0)$$

is in $C^0((-2,2), H^{\ell-3})$.

Appendix C. Euler-Lagrange equation. For the convenience of the reader we review in this appendix the derivation of the Euler-Lagrange equation of the action functional $\mathcal{E}_k^T$ induced by the right-invariant, weak Riemannian metric $\nu^{(k)}$ on the group $\mathcal{D}_+$ for any $k \geq 0$. Recall that the H^k-Sobolev type inner product is given by

$$\langle u, v\rangle_k := \int_{\mathbb{T}^2} \langle u, A_k v\rangle dx, \quad \forall u, v \in T_{\mathrm{id}}\mathcal{D}_+$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^2$, and

$$A_k := \mathrm{Id} + \sum_{j=1}^{k} (-\Delta)^j.$$

It induces the right-invariant metric $\nu^{(k)}$ on $\mathcal{D}_+$ which, at any φ in $\mathcal{D}_+$, is defined by

$$\nu_\varphi^{(k)}(\xi, \eta) := \langle (d_{\mathrm{id}} R_\varphi)^{-1}\xi, (d_{\mathrm{id}} R_\varphi)^{-1}\eta\rangle_k, \quad \forall \xi, \eta \in T_\varphi \mathcal{D}_+$$

where $R_\varphi : \mathcal{D}_+ \to \mathcal{D}_+, \psi \mapsto \psi \circ \varphi$ denotes the right translation by φ. Note that $d_{\mathrm{id}} R_\varphi : T_{\mathrm{id}}\mathcal{D}_+ \to T_\varphi \mathcal{D}_+$ is given by $(d_{\mathrm{id}} R_\varphi)u = u \circ \varphi$. In particular, for any curve $\gamma : [0, T] \to \mathcal{D}_+$, $t \mapsto \gamma(t)$, emanating from the identity,

$$(d_{\mathrm{id}} R_{\gamma(t)})^{-1}\dot\gamma(t) = \dot\gamma(t) \circ \gamma^{-1}(t).$$

The action functional $\mathcal{E}_k^T$ induced by $\nu^{(k)}$ on the space of C_F^2-curves is given by (1.1b) i.e.,

$$\mathcal{E}_k^T(\gamma) = \frac{1}{2}\int_0^T \int_{\mathbb{T}^2} \left\langle \dot\gamma(t) \circ \gamma^{-1}(t),\, A_k \left(\dot\gamma(t) \circ \gamma^{-1}(t)\right) \right\rangle dt dx. \tag{C.1}$$

To derive the Euler-Lagrange equation of (C.1), let $\gamma : (-\varepsilon, \varepsilon) \times [0, T] \to \mathcal{D}_+$, $(s, t) \mapsto \gamma(s, t)$ be a C_F^2-smooth variation of the C_F^2-smooth curve $\varphi : [0, T] \to \mathcal{D}_+$, $t \mapsto \gamma(0, t) := \varphi(t)$ with fixed end points

$$\gamma(s, 0) \equiv \mathrm{id}; \;\; \gamma(s, T) \equiv \varphi(T), \quad \text{for every} \quad -\varepsilon < s < \varepsilon. \tag{C.2}$$

For clarity of exposition, we will omit the explicit time and/or parameter dependence, e.g. we will write γ and φ instead of $\gamma(s, t)$ and $\varphi(t)$ respectively, and so on.

As A_k is symmetric, it follows from (C.1) that

$$\begin{aligned} \delta\mathcal{E}_k^T(\varphi) :=& \frac{d}{ds}\Big|_{s=0} \mathcal{E}_k^T(\gamma) \\ =& \int_0^T \int_{\mathbb{T}^2} \left\langle \delta\left(\dot\varphi \circ \varphi^{-1}\right), A_k\left(\dot\varphi \circ \varphi^{-1}\right) \right\rangle dt dx. \end{aligned} \tag{C.3}$$

By the chain rule we have

$$\delta(\dot\varphi \circ \varphi^{-1}) = (\delta\varphi)^{\cdot} \circ \varphi^{-1} + [(d\dot\varphi) \circ \varphi^{-1}]\delta\varphi^{-1}.$$

The variation of the inverse diffeomorphism $\delta\varphi^{-1}$ may be expressed in terms of $\delta\varphi$ by an application of the chain rule to the identity $\varphi \circ \varphi^{-1} = \mathrm{Id}$,

$$\delta\varphi^{-1} = -[(d\varphi)^{-1}\delta\varphi] \circ \varphi^{-1}$$

where $(d\varphi)^{-1}$ is the section $x \mapsto (d\varphi)^{-1}(x) = (d_x\varphi)^{-1}$. Hence

$$\delta(\dot\varphi \circ \varphi^{-1}) = \left[(\delta\varphi)^{\cdot} - (d\dot\varphi)(d\varphi)^{-1}\delta\varphi\right] \circ \varphi^{-1}.$$

Similar arguments show that

$$d(\dot\varphi \circ \varphi^{-1}) = -\left[(d\dot\varphi)(d\varphi)^{-1}\right] \circ \varphi^{-1}.$$

Hence, with

$$u := \dot\varphi \circ \varphi^{-1}$$

one gets

$$\delta u = \delta(\dot\varphi \circ \varphi^{-1}) = \{(\delta\varphi)^{\cdot} - [(du) \circ \varphi]\delta\varphi\} \circ \varphi^{-1}.$$

Substituting the latter identity into formula (C.3), yields

$$\begin{aligned} \delta\mathcal{E}_k^T(\varphi) &= \int_0^T \int_{\mathbb{T}^2} \left\langle \left\{(\delta\varphi)^{\cdot} - [(du) \circ \varphi]\delta\varphi\right\} \circ \varphi^{-1}, A_k u\right\rangle dtdx \\ &= \int_0^T \int_{\mathbb{T}^2} \left\langle (\delta\varphi)^{\cdot} - [(du) \circ \varphi]\delta\varphi, (A_k u) \circ \varphi\right\rangle \det(d\varphi) dtdy \end{aligned}$$

where the latter identity results from the change of variables $x = \varphi(t, y)$. Clearly, $\langle [(du) \circ \varphi]\delta\varphi, (A_k u) \circ \varphi\rangle = \langle \delta\varphi, [(du)^\dagger (A_k u)] \circ \varphi\rangle$. Hence, upon integrating by parts with respect to t keeping in mind that the end points of the curves γ are held fixed (see (C.2)), we get

$$\delta\mathcal{E}_k^T(\varphi) = -\int_0^T\int_{\mathbb{T}^2} \langle \delta\varphi, \{\det(d\varphi)[(A_k u) \circ \varphi]\}^{\cdot} + \det(d\varphi)\left(\left[(du)^\dagger A_k u\right] \circ \varphi\right)\rangle dtdx.$$

It follows that the critical points φ of the action functional satisfy the Euler-Lagrange equation

$$\{\det(d\varphi)[(A_k u) \circ \varphi]\}^{\cdot} + \det(d\varphi)\left(\left[(du)^\dagger A_k u\right] \circ \varphi\right) = 0. \tag{C.4}$$

This equation can be simplified as follows. Using the well-known identity relating determinant and trace of any one parameter family of regular square matrices depending smoothly on the parameter,

$$[\mathrm{Log}(\det(d\varphi))]^{\cdot} = \mathrm{Tr}\left[(d\dot\varphi)(d\varphi)^{-1}\right] = \mathrm{Tr}\left[(du) \circ \varphi\right] = (\mathrm{div}\, u) \circ \varphi,$$

one gets

$$\{\det(d\varphi)[(A_k u) \circ \varphi]\}^{\cdot} = \det(d\varphi)\left([A_k \dot u + (dA_k u)u + \mathrm{div}\, u \cdot A_k u] \circ \varphi\right).$$

Thus (C.4) reads

$$\det(d\varphi)\left(\left[A_k \dot u + (dA_k u)u + \left(\mathrm{div}\, u \cdot \mathrm{Id} + (du)^\dagger\right) A_k u\right] \circ \varphi\right) = 0.$$

Finally, since the factor $\det(d\varphi)$ is everywhere positive, we can drop it from the latter identity. Moreover if we factor out the right translation R_φ, we end up with the following equivalent more compact formulation of (C.4):

$$A_k\dot{u} + (dA_k u)u + \left(\operatorname{div} u \cdot \mathrm{Id} + (du)^\dagger\right) A_k u = 0. \tag{C.5}$$

We remark that (C.5) is a transport equation for $A_k u$. Indeed, upon observing that $(dA_k u)u = (u \cdot \nabla)A_k u$, where $u \cdot \nabla$ is the vector field $u_1\partial_{x_1} + u_2\partial_{x_2}$ acting componentwise, it reads

$$(A_k u)^{\cdot} + \left[(u \cdot \nabla) + \operatorname{div} u \cdot \mathrm{Id} + (du)^\dagger\right] A_k u = 0.$$

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