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COMPLEX PRODUCT MANIFOLDS CANNOT BE NEGATIVELY CURVED*

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Abstract. We show that if $M = X \times Y$ is the product of two complex manifolds (of positive dimensions), then M does not admit any complete Kähler metric with bisectional curvature bounded between two negative constants. More generally, a locally-trivial holomorphic fibre-bundle does not admit such a metric.

Key words. Kähler manifolds, product manifolds, bisectional curvature, negative curvature

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1. Introduction. The classical theorem of Preissmann states that for any compact Riemannian manifold N with negative sectional curvature, any non-trivial abelian subgroup of the fundamental group $\pi_1(N)$ is cyclic. In particular, N cannot be (topologically) a product manifold, since otherwise $\pi_1(N)$ will contain $\mathbf{Z}^2$ as a subgroup.

For Kähler manifolds, the more "natural" notion of curvature is that of *bisectional* curvature B. Note that the condition $B \leq 0$ is weaker than nonpositive sectional curvature and, in particular, does not imply that the manifold is a $K(\pi, 1)$. In fact, it is not known if this curvature condition has any topological implications. Nevertheless, the negativity of B does impose restrictions on the complex structure of the underlying manifold: For a compact Kähler manifold M with negative B, the cotangent bundle is ample and thus M cannot be biholomorphic to a product of two (positive dimensional) complex manifolds. In fact, one can classify all Kähler metrics of nonpositive bisectional curvature on complex product manifolds [Z2].

It is a general belief that the complex product structure would prevent the existence of a metric with negative curvature, even in the non-compact case. That is, any complex product manifold cannot admit a complete Kähler metric with bisectional curvature bounded between two negative constants. The main result of this note is to confirm just that. In fact, the Kählerness assumption on the metric is not important, and can be relaxed to Hermitian with bounded torsion. Here "torsion" refers to the torsion of the Chern connection associated to a Hermitian metric.

THEOREM 1. *Let $M = X \times Y$ be the product of two complex manifolds of positive dimensions. Then M does not admit any complete Hermitian metric with bounded torsion and bisectional curvature bounded between two negative constants.*

The first result along these lines was obtained by P. Yang [Yn]. Yang proved that the polydisc does not admit such Kähler metrics. In [Z1], the second author obtained the above result under certain assumptions on the factors. For instance, if X and Y are both bounded domains of holomorphy in Stein manifolds, then the result holds. More recently, the first named author proved in [S] that any complex product manifold M cannot admit a complete Kähler metric with *sectional* curvature bounded between

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two negative constants. In this case, the negativity of sectional curvature allows one to use the δ-hyperbolicity criterion of Gromov [G], and no restriction is needed on the factor manifolds.

In light of Theorem 1, it would be interesting to understand which holomorphic fibrations (see **§3** for definitions) admit negatively curved metrics. In this connection, we prove:

PROPOSITION 2. *Let M be the total space of a locally trivial holomorphic fibre-bundle with positive-dimensional fibres. Then M does not admit any complete Hermitian metric with bounded torsion and bisectional curvature bounded between two negative constants.*

2. Proofs. The main idea of our proof is a combination of the ideas involved in our earlier results [S] and [Z1]. In the latter, a construction originated by Paul Yang in [Yn] was the starting point. The crucial observation in this paper is that Yau-Schwarz lemma can be used in two different ways to compare the induced metrics on different slices as done in [S]. This allows us to drop the extra assumptions made in [Z1].

The major tools in the proof are the following two classical results of Yau, the generalized Schwarz lemma [Y1] and the generalized maximum principle [Y2]. Since there are various version of these results, for completeness' sake and for the convenience of the reader, let us give two precise statements along with their references below. The second one is directly from Yau's paper [Y2], while the first one is the generalization of Yau's Schwarz lemma to the Hermitian case, due to Zhihua Chen and Hongcang Yang in [CY] in 1981.

THEOREM 3 ([Y1], [CY]). Suppose (M,g) is a complete Hermitian manifold with bounded torsion, and with second Ricci curvature $\geq -K_1$. Let (N,h) be a Hermitian manifold with non-positive bisectional curvature and with holomorphic sectional curvature $\leq -K_2 < 0$. Then for any holomorphic map $f : M \longrightarrow N$, one has $f^*(h) \leq \frac{K_1}{K_2} g$.

THEOREM 4 ([Y2]). Let (M,g) be a complete Riemannian manifold with Ricci curvature bounded from below, and φ a C^2 function on M bounded from above. Then for any $\varepsilon > 0$, there exists $x \in M$ such that: $\varphi(x) > sup\ \varphi(M) - \varepsilon$, $|\nabla\varphi(x)| < \varepsilon$, $\Delta\varphi(x) < \varepsilon$.

Proof of Theorem 1. Suppose $M = X \times Y$ is the product of two complex manifolds X and Y of complex dimensions n and m, respectively. Assume that M admits a complete Hermitian metric g with bounded torsion and with its bisectional curvature B bounded between two negative constants, say

$$-c_1 \ \leq \ B \ \leq \ -c_2 \ < \ 0.$$

We want to derive a contradiction from this. Fix a point $q \in Y$, and let $(y_{n+1}, \ldots, y_{n+m})$ be a local holomorphic coordinates in a neighborhood $q \in U \subseteq Y$ such that q is the origin. Let $D = \{t \in \mathbf{C} : |t| < 1\}$ be the unit disc in $\mathbf{C}$, and $\iota : D \to Y$ be the holomorphic embedding which sends t to $(t, 0, \ldots, 0)$ in U.

Next, for any $x \in X$, denote by $\iota_x : Y \to M$ the inclusion which sends $y \in Y$ to $(x,y) \in M$, and denote by $\phi_x : D \to M$ the composition of ι with ι_x.

Take a cutoff function $\rho \in C_0^\infty(D)$ in D such that ρ is smooth, non-trivial, with compact support, and $0 \leq \rho \leq 1$. Now define a function f on M by assigning

$$f(x,y) = f(x) = \int_D \rho \ \phi_x^* \omega_g$$

where ω_g is the Kähler form of the metric g. This is a smooth, positive function on M and is constant in the Y directions. Denote by g^0 the Poincaré metric on the unit disk D, with constant curvature -1. Then by applying Yau's Schwarz lemma to the holomorphic map $\phi_x : D \to M$, we get

$$\phi_x^* \omega_g \ \leq \ \frac{1}{c_2} \omega_{g^0}.$$

From this we conclude that our function f is bounded from above.

Now we want to compute the Laplacian of the function f under the metric g. Fix an arbitrary point $p = (x_0, y_0)$ in M. Since f depends on x alone, we can assume that $y_0 = q$. Let $(x_1, \ldots, x_n)$ be local holomorphic coordinates in a neighborhood of x_0 in X, such that $x_0 = (0, \ldots, 0)$ and let $(y_{n+1}, \ldots y_{n+m})$ be the local holomorphic coordinates on Y which was chosen earlier in a neighborhood of q. Then $(x_1, \ldots, x_n, y_{n+1}, \ldots, y_{n+m})$ becomes local holomorphic coordinates near p in M, with p being the origin.

Write $\partial_i = \frac{\partial}{\partial x_i}$ if $1 \leq i \leq n$, and $\partial_i = \frac{\partial}{\partial y_i}$ if $n+1 \leq i \leq n+m$. Denote by $g_{i\bar{j}} = g(\partial_i, \overline{\partial_j})$.

By a constant linear change of the coordinates x if necessary, we may assume that at the center p, we have

$$g_{i\bar{j}}(0) \ = \ \delta_{ij}$$

for any $1 \leq i, j \leq n$.

First let us compute the value $f_{i\bar{i}} = \frac{\partial^2 f}{\partial x_i \partial \overline{x_i}}$ at p for any fixed i between 1 and n. For the sake of convenience in writing, we will write v for ∂_{n+1}. We have

$$\phi_x^* \omega_g \ = \ g_{v\bar{v}} \ dt d\bar{t}.$$

From the curvature formula

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \overline{z}_l} + \sum_{\alpha,\beta} g^{\overline{\alpha}\beta} \frac{\partial g_{i\overline{\alpha}}}{\partial z_k} \frac{\partial g_{\bar{j}\beta}}{\partial \overline{z}_l}$$

we get

$$g_{v\bar{v},i\bar{i}} \ \geq \ -R_{v\bar{v}i\bar{i}} \ = \ -B(v, \partial_i) g_{v\bar{v}} g_{i\bar{i}} \ \geq \ c_2 g_{v\bar{v}} g_{i\bar{i}}.$$

Next, if h is the Hermitian metric to X obtained by restricting g on $X \times \{y_0\}$, then the bisectional curvature of h is again bounded from above by the negative constant $-c_2$. So if we consider the holomorphic map $\pi_1 : (M, g) \to (X, h)$, where π_1 is the projection to the first factor, then by Theorem 3 we know that $\pi_1^* h \leq c_3 g$, where $c_3 = \frac{(n+m)c_1}{c_2}$. In particular,

$$g_{i\bar{i}}(x, y_0) \ \leq \ c_3 \ g_{i\bar{i}}(x, y)$$

for any point (x, y) near p and any $1 \leq i \leq n$.

Combining the above observations, we then have

$$\begin{aligned} f_{i\bar{i}}(x,y) &= \int_D \rho \ g_{v\bar{v},i\bar{i}} \ dtd\bar{t} \\ &\geq \int_D \rho c_2 g_{v\bar{v}} g_{i\bar{i}} \ dtd\bar{t} \\ &\geq \int_D \rho \frac{c_2}{c_3} g_{v\bar{v}} g_{i\bar{i}}(x, y_0) dtd\bar{t} \\ &= \frac{c_2}{c_3} g_{i\bar{i}}(x, y_0) f(x). \end{aligned}$$

In particular, at the point p, we have

$$f_{i\bar{i}} \geq \alpha f$$

for each $1 \leq i \leq n$ where we wrote $\alpha = \frac{c_2}{c_3}$. This leads to $\Delta f \geq n\alpha f$ at p if the metric g is Kähler, since in this case the Laplacian is just the trace of $\partial\overline{\partial} f$ with respect to ω_g. When g is only Hermitian, then Δf differs from the trace of $\partial\overline{\partial} f$ by a term that involves the torsion of g. Under our assumption, g has bounded torsion, so there exists a positive constant β, again independent of the choice of p, such that

$$\Delta f + \beta |\nabla f| \geq \alpha f$$

at the point p. Since p is arbitrary, the above inequality holds everywhere on M.

On the other hand, since the smooth positive function f is bounded from above, we may apply Yau's maximum principle Theorem 4 to the function $\varphi = \log f$, which is again bounded from the above. The theorem says that, for any prescribed $\epsilon > 0$, there exists a point in M at which

$$|\nabla f| < \epsilon f, \quad \Delta f < 2\epsilon f.$$

So the inequality we obtained above leads to $(2 + \beta)\epsilon \geq \alpha$, which is impossible when ϵ is sufficiently small. This contradiction establishes the non-existence of a complete Hermitian metric with bounded torsion on $M = X \times Y$ with bisectional curvature bounded between two negative constants. $\square$

Proof of Proposition 2. Let $f : M \to B$ be a locally-trivial holomorphic fibre bundle with fibre F. By definition, this means the following: B and F are complex manifolds, f is a surjective holomorphic map with maximal rank and there exists an (locally finite) open covering $\{U_i\}$ of N, such that there is a fibre-preserving biholomorphism h_i of $f^{-1}(U_i)$ with $U_i \times F$. As usual, whenever $U_i \cap U_j \neq \phi$, we have a map $\phi_{ij} : U_i \cap U_j \to Aut(F)$, where $Aut(F)$ is the (real) Lie group of holomorphic automorphisms of F. This map is "holomorphic" in the sense that $\phi_{ij}(x, , .) : U_{ij} \to F$ is holomorphic for each $x \in F$. Applying a result of H. Fujimoto [F], it follows that there is a *complex* Lie subgroup $G \subset Aut(F)$, $h \in Aut(F)$ and a holomorphic map $\psi : U_{ij} \to G$ such that $\phi_{ij} = h\psi$.

Now assume that M admits a metric as in the theorem. We first claim that each ϕ_{ij} is constant. If not, by the discussion above, there would be a positive-dimensional complex Lie subgroup of $Aut(F)$. This would imply that there is a nonconstant holomorphic map from $\mathbb{C}$ to F. But this contradicts (by Yau's Schwarz Lemma) the fact that the metric induced on F has holomorphic sectional curvature bounded above by a negative constant.

Since the ϕ_{ij} are constant, the universal cover $\tilde{M}$ is biholomorphic to $\tilde{F} \times \tilde{B}$ and we can invoke Theorem 1. $\square$

3. Remarks. (i) It would be interesting to find necessary and sufficient conditions for a holomorphic fibration to admit a Kähler metric with pinched negative bisectional curvature. By a holomorphic fibration we mean a complex manifold M which admits a surjective holomorphic map f onto a complex manifold N, such that the derivative of f has maximal rank everywhere.

Note that the unit ball in $\mathbb{C}^n$, with the projection map onto a lower-dimensional ball, is an example of a holomorphic fibration which does support such a metric. On the other hand, it is unknown if the Kodaira fibrations (these are certain compact complex surfaces which are holomorphic fibrations over compact Riemann surfaces) admit such metrics.

In some special cases one can rule out such metrics. For instance, if the fibres are all compact, connected and biholomorphic then M is a locally-trivial holomorphic fibre bundle, according to the Fischer-Grauert theorem [FG]. Hence, Proposition 2 applies.

(ii) In another direction, one can ask if Theorem 1 holds under weaker curvature restrictions. For instance, it is not clear if the lower curvature bound is necessary (the upper bound is necessary since $\mathbb{C}^n$ admits metrics with strictly negative bisectional curvature, cf. [S]). The following question, which was raised by N. Mok, is still open: Does the bidisc admit a complete Kähler metric with bisectional curvature ≤ -1 ?

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