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006

A NOTE ON COMPLEX MONGE-AMPÈRE EQUATION IN STEIN MANIFOLDS*

ALIREZA BAHRAINI†

Abstract. We study in this note the Dirichlet problem for complex Monge-Ampère equation in compact Stein manifolds with boundary. As far as we know among the global results for Monge-Ampère equations, compact manifolds with boundary have been less discussed.

Key words. Monge-Ampère equation, Stein manifold, Continuity method, Pluri-subharmonic functions

AMS subject classifications. 58J32, 35J25

Introduction. We begin this note with a very brief review on some of the aspects of Monge-Ampère equation which have been motivating for the present work. Complex Monge-Ampère equation and its applications have been the subject of extensive studies by several mathematicians since more than 3 decades ago. Thirty years ago S.T.Yau solved complex Monge-Ampère equation on a compact Kähler manifold to prove a conjecture of Calabi:

THEOREM 1. *([15]) Let X be a compact connected Kähler manifold of complex dimension n, equipped with a Kähler form ω. If μ is a smooth volume form satisfying $\mu(X) = \int_X \omega^n$, then there exists a unique (upto a constant) $\phi \in C^\infty(X)$ such that:*

$$(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = \mu. \tag{1}$$

Since then different variants of the equation for compact or non-compact manifolds have been studied (see [4] [9][13] for example). The solutions of this equation provide us with examples of hyper-Kähler manifolds. In complex dimension n=2 the moduli of Hodge structures on K3 surfaces can be characterized locally as well as globally using hyper-Kähler metrics [7] [14]. As a result special Lagrangian submanifolds of $K3$ surfaces and their properties are much better known and studied. In the category of super-manifolds, an important class of super-symmetric geometries are constructed by the aid of Ricci-flat metrics. This leads to a re-interpretation of special-Lagrangian sub-manifolds in complex dimension 3 and through the works of physicists as bosonic part of super-symmetric objects ([2]).

The problem of studying special Lagrangian representatives for duals of cohomology classes in certain Stein surfaces [2][10] led us to the study of complex Monge-Ampère equations in compact Stein manifolds with boundary and to prove the following theorem which seems to be missing in the current literature :

THEOREM 2. *Let X be a compact Stein manifold with boundary, ω a (1,1)-Kähler form on X , f a real smooth function in X and ϕ a real smooth function defined only in ∂X. Then there is a unique smooth function u on X such that $\omega + \sqrt{-1}\partial\bar{\partial}u > 0$ and*

*Received August 9, 2006; accepted for publication July 17, 2007.
†Department of Mathematical Sciences, Sharif University of Technology, P.O. Box 11365-9415, Tehran, Iran (bahraini@sharif.edu).

$$(2) \qquad (\omega + \sqrt{-1}\partial\bar{\partial}u)^n = e^f \omega^n$$
$$u|_{\partial X} = \phi.$$

Proof of Theorem 2. In order to prove this theorem we follow the method of Caffarelli, Kohn, Nirenberg and Spruck as in [3] and we find an estimate of the norm $C^{2+\alpha}$ of u for $0 < \alpha < 1$:

$$|u|_{\alpha+2} \leq K.$$

It turns out that for estimates of order 0 ,1 and $2+\alpha$, some global difficulties arise and the method of Caffarelli et al. needs some modifications.

Estimate of order zero. To show that $|u|_0 < C$ we need the following lemma:

LEMMA 1. *Let u and v be two functions in X fulfilling:*

$$\frac{(\omega + \sqrt{-1}\partial\bar{\partial}u)^n}{\omega^n} \geq \frac{(\omega + \sqrt{-1}\partial\bar{\partial}v)^n}{\omega^n}$$

and

$$u \leq v \ \text{ on } \partial X$$

then $u \leq v$ in $\bar{X}$.

Proof. In local coordinates we can write

$$det(g_{i\bar{j}} + u_{i\bar{j}}) - det(g_{i\bar{j}} + v_{i\bar{j}}) = \int_0^1 \frac{d}{dt} det(t(g_{i\bar{j}} + u_{i\bar{j}}) + (1-t)(g_{i\bar{j}} + v_{i\bar{j}})dt$$
$$= \sum (\int_0^1 B^{i\bar{j}}(t)dt)(u-v)_{i\bar{j}} \geq 0.$$

$B^{i\bar{j}}(t)$ are the co-factors of the matrix $(tu_{i\bar{j}} + (1-t)v_{i\bar{j}} + g_{i\bar{j}})$ which constitute themeselves a positive definite matrix. So according to the maximum principle $(v-u)$ attains its maximum on ∂X and the desired result follows. ◻

Now let ϕ_0 be a pluri-subharmonic function in X such that:

$$\frac{(\omega + \sqrt{-1}\partial\bar{\partial}\phi_0)^n}{\omega^n} \geq e^f = \frac{(\omega + \sqrt{-1}\partial\bar{\partial}u)^n}{\omega^n}$$
$$\phi_0|_{\partial X} = \phi.$$

To show the existence of ϕ_0 we take two strictly pluri-subharmonic functions ϕ_1 and ϕ_2 such that $\phi_1|_{\partial X} = 0$ and $\phi_2|_{\partial X} = \phi$ and we set $\phi_0 = \lambda\phi_1 + \phi_2$. It's clear that for $\lambda \in \mathbf{R}$ sufficiently large ϕ_0 satisfies the above inequality. Now according to lemma 1, $u \geq \phi_0$ and we obtain a lower bound for u. On the other hand if we calculate the trace of $(\omega + \sqrt{-1}\partial\bar{\partial}u)$ with respect to ω we find $n + \Delta_\omega u \geq 0$. So according to the maximum principle we obtain an upper bound for u. ◻

First order estimates. We would like to show the existence of a bound for norm 1:

$$|u|_1 < C. \tag{3}$$

Let F be the application defined by

$$F(u) = \log \frac{(\omega + \sqrt{-1}\partial\bar{\partial}u)^n}{\omega^n} - f.$$

We take a unit tangent vector ξ in a given point of X. Let D be a vector field extending ξ which can be described in an open dense holomorphic coordinate subset U of X as a vector field with constant coefficients. Such a field exists: it suffices to imbed X properly in some $\mathbb{C}^N$ and then to project over a generic vector sub-space of dimension n . Let h be a potential of the metric ω in some local coordinates near a point $x \in U$. We have:

$$F(u) = \log det((h+u)_{i\bar{j}}) - \log det(h_{i\bar{j}}) - f. \tag{4}$$

Set $k = h+u$ and suppose that $F(u) = 0$. It can be easily seen that $F^{i\bar{j}} = \frac{\partial F}{\partial u_{i\bar{j}}} = (k^{i\bar{j}})$ the inverse of the matrix $(k_{i\bar{j}})$ and the linearisation of the operator F at u is written as follows:

$$(\tilde{L})v, \quad \tilde{L} = k^{i\bar{j}}\partial_{i\bar{j}}.$$

We have $DF = 0$ and since D is described almost everywhere in local coordinates by a vector field with constant coefficients, a.e. we get:

$$\tilde{L}(Dk) = DH$$

where $H = f + \log det(h_{i\bar{j}})$. On the other hand we know that $det(k^{i\bar{j}}) = e^H$, so

$$\frac{1}{n}\sum k^{i\bar{i}} \geq e^{H/n} \tag{5}$$

and for some constant B_x :

$$|DH| \leq B_x e^{H/n} \text{near } x.$$

In order to establish the inequality:

$$\max_{\bar{X}} |Du| \leq \max_{\partial X} |Du| + C \tag{6}$$

let p be a strictly pluri-subharmonic function in X and consider locally defined functions $w_0^{\pm} = \pm Dk + e^{\lambda p}$ near x. If η denotes the least eigenvalue of $(p_{i\bar{j}})$ we obtain

$$\begin{aligned}\tilde{L}w_0^{\pm} &= \pm\tilde{L}Dk + \tilde{L}e^{\lambda p} = \pm DH + k^{i\bar{j}}(e^{\lambda p})_{i\bar{j}} \\ &= \pm DH + k^{i\bar{j}}(\lambda p_{i\bar{j}} + \lambda^2 p_i p_{\bar{j}})e^{\lambda p} \geq -Be^{H/n} + (\lambda\eta\sum k^{i\bar{i}})e^{\lambda p}.\end{aligned}$$

Hence:

$$\tilde{L}(\pm Dh) = \pm k^{i\bar{j}} Dh_{i\bar{j}} \geq -\gamma \sum k^{i\bar{i}}$$

where γ is a constant which depends only on the metric ω. Furthermore for globally defined functions $w_1^{\pm} = \pm Du + e^{\lambda p}$ in X , we have $w_1^{\pm} = w_0^{\pm} \mp Dh$, thus for all x in U, in a neighborhood of x one gets:

$$\tilde{L} w_1^{\pm} \geq -B_x e^{H/n} + \sum k^{i\bar{i}} (\lambda \eta e^{\lambda p} - \gamma).$$

The compactness of X allows us to find a finite number of points x in U and some neighborhoods of these points covering U on which the metric has local potentials. By using the inequality (5) it follows that for λ sufficiently large

$$\tilde{L} w_1 \geq 0$$

and the inequality (6) follows with the aid of the maximum principle.

To complete the demonstration of (3) we should find upper bounds for $|Du|$ on ∂X. Let ϕ_0 be a pluri-subharmonic function in X s.t.

$$\frac{(\omega + \sqrt{-1}\partial\bar{\partial}\phi_0)^n}{\omega^n} \geq e^f = \frac{(\omega + \sqrt{-1}\partial\bar{\partial}u)^n}{\omega^n}$$
$$\phi_0|_{\partial X} = \phi$$

and let ψ denote a solution of the equation $\Delta_\omega \psi = -n$, $\psi|_{\partial X} = \phi$ then :

$$\phi_0 \leq u \leq \psi$$

and we obtain

$$|\nabla u(z)| \leq \max\{|\nabla \phi_0(z)|, |\nabla \psi(z)|\}, z \in \partial X.$$

Estimates of second order derivatives. Following [3] it is sufficient to show:

LEMMA 2. *There exists a constant $C > 0$ such that:*

$$\max_{\bar{X}} |\nabla^2 u| \leq \max_{\partial X} |\nabla^2 u| + C.$$

Proof: Let D be a vector field which can be described almost everywhere by constant coefficients in appropriate coordinate systems . Using the concavity of F as a function of $k_{i\bar{j}}$ locally we get:

$$k^{i\bar{j}} (D^2 k)_{i\bar{j}} \geq D^2 H$$

$$\tilde{L} D^2 k \geq -C H^{-1/n}.$$

Consequently as we did in the estimations for the first order derivatives, for λ sufficiently large we can prove the following inequality:

$$\tilde{L}(D^2 k + e^{\lambda p}) \geq 0$$

and then use the maximum principle to complete the proof of the lemma. □

On the other hand we have

Lemma 3. *([2]) There exists a constant C such that*

$$\max_{\partial X} |\nabla^2 u| \leq C.$$

This gives the required bound on second order derivatives.

Estimates of order (2+α). The argument contains the following steps:

Lemma 4. *([3],[4]) For all $X' \subset \bar{X}' \subset \bar{X} \setminus \partial X$ there exists $K(X')$ such that:*

$$|u|_{2+\alpha} \leq K(X').$$

Lemma 5. *([2]) There exists a constant K such that for all $x, y \in \partial X$:*

$$|u_{ij}(x) - u_{ij}(y)| \leq \frac{K}{1 + |\log|x-y||} \text{ for } x, y \in \partial X. \tag{7}$$

Using this lemma we prove that:

Lemma 6. *There exists a constant K such that for all $x \in \partial X, y \in \bar{X}$ we have:*

$$|u_{ij}(x) - u_{ij}(y)| \leq \frac{K}{1 + |\log|x-y||}. \tag{8}$$

Proof. The idea is again the maximum principle, this time by constructing two functions: one denoted by w, and defined in (11), which contains second order derivatives, and the other v, defined in (12) and fulfilling $\tilde{L}h < \tilde{L}v$. By (7) they may be so chosen that on ∂X we have $w \leq v$. In this way we obtain a control for w leading to the inequality (8).

Let $\{U_\alpha\}$ be a covering of X such that each U_α is biholomorphic with an open set in $\mathbb{C}^n$, and let $\{f_\alpha\}$ be an associated partition of unity. According to I.Motzkin and W.Wasow [11] in each U_α there exist vector fields of length 1 with constant coefficients ξ_α^1,...,ξ_α^N and constants $c_1, ..., c_N$ such that the linear approximation $\tilde{L}_\alpha$ of F can be written as:

$$\tilde{L}_\alpha = \sum_1^N b_i^\alpha(x) \partial^2_{\xi_\alpha^i} \text{ with } c_i \leq b_i^\alpha \leq c_i^{-1}. \tag{9}$$

Further one can suppose that the operators $\partial_{\xi_\alpha^i}$ contain all the operators $\partial/\partial x_i$ as well as $1/\sqrt{2}((\partial/\partial x_i) \pm (\partial/\partial x_j))$ for $i \neq j$ in the coordinates of U_α. Then we calculate $\tilde{L}_\alpha(\partial_{\xi_\alpha})^2 k^\alpha$, for a unit vector $\xi_\alpha = (\xi_{\alpha 1}, ..., \xi_{\alpha n})$, by applying $\partial^2_{\xi_\alpha}$ on both sides of the equation (4). In this way, with the previous notations can write:

$$\tilde{L}(\partial_{\xi_\alpha})^2 k^\alpha + F^{ij,pl} \partial_{\xi_\alpha} k_{ij}^\alpha \partial_{\xi_\alpha} k_{pl}^\alpha + \partial_{\xi_\alpha} H = 0.$$

Hence by concavity of F and using the inequality $|u|_2 < C$ we can find some constants c_{jp} such that

$$\tilde{L}(\partial_{\xi_\alpha})^2 k^\alpha \geq -C - \sum c_{jp}\partial_{\xi_\alpha} k^\alpha_{jp}.$$

Let ϵ be a positive number such that :

$$\epsilon|\nabla^2 k^\alpha| \leq \frac{1}{4}$$

where $|\nabla^2 k^\alpha|$ represents the norm of the hessian matrix (k^α_{ij}). We have

$$\tilde{L}(\partial^2_{\xi_\alpha} k^\alpha + \epsilon(\partial^2_{\xi_\alpha} k^\alpha)^2) \geq -C - C\sum_{ijp}|k^\alpha_{ijp}| + 2\epsilon c_0 \sum_i |\partial^2_{\xi_\alpha} k^\alpha_i|^2 \tag{10}$$

thus if we define

$$h = \sum_\alpha \sum_1^N f_\alpha(\partial^2_{\xi^i_\alpha} k^\alpha + \epsilon(\partial^2_{\xi^i_\alpha} k^\alpha)^2)$$

we get the inequality:

$$\tilde{L}h \geq -C - C\sum_\alpha \sum_{ijp}|k^\alpha_{ijp}| + 2\epsilon c_0 \sum_\alpha \sum_{i=1}^n \sum_{j=1}^N |\partial^2_{\xi^j_\alpha} k^\alpha_i|^2$$

which can be deduced from the inequality (9) and from a second order estimation on k.

Now since:

$$\sum_i \sum_j |\partial^2_{\xi^j_\alpha} k^\alpha_i|^2 \geq c_1 \sum |k^\alpha_{ijp}|^2 \text{ for } c_1 \text{ positive}$$

one finds,

$$\tilde{L}h \geq c_0\epsilon \sum |k^\alpha_{ijp}|^2 - \frac{C}{\epsilon}.$$

Thus if we define w^i by

$$w^i = \sum_\alpha f_\alpha \partial^2_{\xi^i_\alpha} k^\alpha + \epsilon h \tag{11}$$

for $i = 1, ..., N$ we obtain

$$\tilde{L}w^i \geq -\frac{C}{\epsilon^2}.$$

Now set $w = w^i$. Let $y_0 \in \partial X$ be fixed and suppose that X is imbedded in $\mathbb{C}^r$ such that y_0 coincides with 0. Let $\delta = |y|^{1/3}$ for fixed y and g be a smooth function in $\bar{X}$ vanishing on ∂X and satisfying $\tilde{L}g \leq -1$ (so $g > 0$ in X). Define:

$$v(x) = w(0) + \frac{M}{|\log|\delta||} + M\frac{|x|^2}{\delta^2} + \frac{Ag}{\delta^2}, \tag{12}$$

where $|.|$ is the norm of $\mathbb{C}^r$. We have,

$$\tilde{L}v \le \frac{CM}{\delta^2} - \frac{Ag}{\delta^2}$$

and so for $A = CM + C/\epsilon^2$,

$$\tilde{L}v \le -\frac{C}{\epsilon^2} < \tilde{L}w$$

according to (7) for $x \in \partial X$, $|x| < \delta$,

$$|w(x) - w(0)| \le \frac{C}{|\log|\delta||}$$

hence for M sufficiently large we get $w \le v$ on ∂X, and the maximum principle yields:

$$w \le v \text{ in } X.$$

In particular,

$$w(y) - w(0) \le \frac{3M}{|\log|y||} + M|y|^{4/3} + CA|y|^{1/3} \le \frac{C}{|\log|y||}.$$

Here we use the fact that g is a C^∞ function vanishing on the boundary to estimate the last term.

Thus if we choose the partition of unity in such a way that in a neighborhood of $0 \in \partial U_\alpha \cap \partial X$, $f_\alpha = 1$ then for $j = 1, ..., N$ we get :

$$\partial^2_{\xi^j_\alpha} k(y) - \partial^2_{\xi^j_\alpha} k(0) + \epsilon \sum_{j=1}^{N} (\partial^2_{\xi^j_\alpha} k(y) - \partial^2_{\xi^j_\alpha} k(0))(1 + \epsilon\partial^2_{\xi^j_\alpha} k(y) + \epsilon\partial^2_{\xi^j_\alpha} k(0)) \le \frac{C}{|\log|y||} \tag{13}$$

After multiplying the relation (13) by $1 + \epsilon\partial^2_{\xi^i_\alpha} k(y) + \epsilon\partial^2_{\xi^i_\alpha} k(0)$ and summing over i we find:

$$h(y) - h(0) \le \frac{C}{|\log|y||}$$

but according to our choice of the partition of unity we know that in a neighborhood of 0 in U_α

$$h = \sum_{j=1}^{N} (\partial^2_{\xi^j_\alpha} k^\alpha + \epsilon(\partial^2_{\xi^j_\alpha} k^\alpha)^2).$$

In order to obtain an inequality in the opposite direction we use the concavity of F as a function of $D^2 k^\alpha$:

$$\begin{aligned} F(x, D^2k^\alpha(x)) + F^{ij}(x, D^2k^\alpha(x))(k^\alpha_{ij}(y) - k^\alpha_{ij}(x)) &\ge F(x, D^2k^\alpha(y)) \\ &\ge F(y, D^2k^\alpha(y)) - C|x - y|. \end{aligned}$$

By using the representation (9) we can rewrite the last inequality as:

$$\sum_{i=1}^{N} b_i^\alpha(x)(\partial_{\xi_\alpha^i} k^\alpha(y) - \partial_{\xi_\alpha^i} k^\alpha(x)) \geq -C_\alpha|x-y|.$$

Setting $x = 0$, for $p \leq N$ and by using (13) we find :

$$\begin{aligned} b_p^\alpha(0)(\partial_{\xi_\alpha^p} k^\alpha(y) - \partial_{\xi_\alpha^p} k^\alpha(0)) &\geq -\sum_{i\neq p} b_i^\alpha(0)(\partial_{\xi_\alpha^i} k^\alpha(y) - \partial_{\xi_\alpha^i} k^\alpha(0)) - C|y| \\ &= -\sum_{i\neq p} b_i^\alpha(0)(w^i(y) - w^i(0) \\ &\quad +\epsilon(h(y)-h(0))\sum_{i\neq p} b_i^\alpha(0) - C|y| \\ &\geq \epsilon \sum_{i\neq k} b_i^\alpha(0).(h(y)-h(0)) - \frac{C}{|\log|y||} \end{aligned}$$

from which we get

$$h(y) - h(0) \geq -\frac{C}{|\log|y||}.$$

Thus we have shown:

$$|h(y) - h(0)| \leq \frac{C}{|\log|y||}$$

and therefore:

$$|\partial_{\xi_\alpha^p} k^\alpha(y) - \partial_{\xi_\alpha^p} k^\alpha(0)| \leq \frac{C}{|\log|y||}.$$

Since the $\partial_{\xi_\alpha^i}$ contain all ∂_{x_i} and $1/\sqrt{2}(\partial_{x_i} \pm \partial_{x_j})$ for $i \neq j$ the desired inequality (8) is implied.

The proof of theorem 1 can now be completed by:

LEMMA 7. *([2]) If the inequality (8) holds then for a positive number $\alpha < 1$ we have:*

$$|u_{ij}(x) - u_{ij}(y)| \leq K|x-y|^\alpha.$$

As a result we obtain:

CORROLARY 1. *Let X be a Stein manifold with boundary s.t. $K_X \cong O_X$ then in each class of metric $[\omega] \in A^{1,1}(X) \cap H^2(X,\mathbb{R})$ and for every function $\phi \in C^\infty(\partial X)$ there exists a unique Ricci-flat metric ω' in the same class as $[\omega]$ such that $\omega' = \omega + \sqrt{-1}\partial\bar{\partial}u$ with $u|_{\partial X} = \phi$*

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