PALINDROMIC BRAIDS[*]

FLORIAN DELOUP[†], DAVID GARBER[‡], SHMUEL KAPLAN[§], AND MINA TEICHER[§]

Abstract. The braid group B_n, endowed with Artin's presentation, admits an antiautomorphism $B_n \to B_n$, such that $v \mapsto \overline{v}$ is defined by reading braids in reverse order (from right to left instead of left to right). We prove that the map $B_n \to B_n$, $v \mapsto v\overline{v}$ is injective. We also give some consequences arising due to this injectivity.

Key words. Braid, palindrome, Garside, Jacquemard

AMS subject classifications. 11E81, 11E39

1. Introduction. Let $n \geq 2$. Any free group F_{n-1} on $n-1$ generators $\sigma_1, \ldots, \sigma_{n-1}$ supports the antiautomorphism $rev : w \mapsto \overline{w}$ defined by

$$\sigma_{i_1}^{\alpha_1} \cdots \sigma_{i_r}^{\alpha_r} \mapsto \sigma_{i_r}^{\alpha_r} \cdots \sigma_{i_1}^{\alpha_1},$$

which reverses the order of the word w with respect to the prescribed set of generators. It follows that any group G presented by generators and relations admits such an antiautomorphism rev. The elements of G which are order-reversing invariant are called *palindromic*. In this paper, we consider palindromic elements of Artin's braid group B_n, equipped with Artin's presentation, which will be called *palindromic braids*. Artin's presentation of the braid group B_n consists of $n-1$ generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$(1) \qquad \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| \geq 2,$$

$$(2) \qquad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } 1 \leq i \leq n-2.$$

We distinguish between two equivalence relations on the elements of the braid group. For two braid words a, b, we write $a = b$ to denote that a and b represent the same element in the group, and $a \equiv b$ to denote that a and b are actually the same element written letter by letter (i.e., $a \equiv b$ means that a and b are equal in the free group using only the generators of the braid group, with no relations).

Palindromic braids have a particularly nice geometric interpretation. Given a geometric braid β, denote by $\widehat{\beta}$ its closure into a link inside a fixed solid torus $D^2 \times S^1$.

The solid torus admits the involution:

$$\text{inv} : D^2 \times S^1 \to D^2 \times S^1, \ \ (re^{it}, \theta) \mapsto (re^{-it}, -\theta),$$

whose set of fixed points consists of two segments ($t \equiv 0 \pmod{\pi}$ and $\theta \equiv 0 \pmod{\pi}$)), which is the intersection of the axis of the 180° rotation with the solid torus. Observe that $\widehat{rev(\beta)}$ is nothing else than $\text{inv}(\widehat{\beta})$ with the opposite orientation. In particular, if a braid $\beta \in B_n$ is palindromic, then $\widehat{\beta}$ coincides with $\text{inv}(\widehat{\beta})$ with the opposite orientation.

We prove the following rigidity result for palindromic braids:

THEOREM 1.1. *Let $\beta \in B_n$ be a palindromic braid such that*

$$(3) \qquad\qquad\qquad\qquad\qquad \beta = v\overline{v}$$

for some braid $v \in B_n$. Then the decomposition (3) *is unique. Equivalently, let $\beta = v\overline{v}$ and $\beta' = v'\overline{v'}$ be two words in Artin's generators $\sigma_1, \dots, \sigma_{n-1}$. Then, $\beta = \beta'$ in B_n if and only if $v = v'$ in B_n.*

Of course one implication is obvious. Only the "only if" part of the statement deserves a proof.

REMARK 1.2. Note that Theorem 1.1 cannot be generalized into the case of palindromic braid words of odd length. For example the two equal words $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ are of the form $w\tau\overline{w}$ and $v\sigma\overline{v}$, however $\sigma_1 = w \neq v = \sigma_2$. Moreover, not all palindromic braids of even length are of the form (3). For example, $\sigma_1\sigma_3 = \sigma_3\sigma_1$ however, $\sigma_1 \neq \overline{\sigma_3}$.

After this work was finished, F. Deloup communicated to us an alternative proof for Theorem 1.1, which is presented in [2], and is derived from the properties of the Dehornoy ordering of braids. The construction of the latter is a long process that requires rather sophisticated methods. In this paper we give an elementary proof, based on Garside normal form and its variant as developed by Jacquemard.

The paper is organized as follows. In Section 2, we give some preliminaries about the braid group and introduce the Jacquemard algorithm. In Section 3, we prove Theorem 1.1.

2. Preliminaries and the Jacquemard Algorithm. This section is devoted to the building blocks we use in order to prove Theorem 1.1. Mainly, this section is intended to fix notations and recall some of the algorithms we use in this paper (for more information, see [1]).

The monoid B_n^+ of *positive braids* consists of braids which admit a word representative which does not contain σ_i^{-1}, $1 \leq i \leq n-1$.

Among positive braids, we can consider those whose number of crossings between any two strands is less or equal to 1: they form the subset $S_n^+ \subset B_n^+$ of *positive permutation braids*.

There is a canonical epimorphism $B_n \to S_n$. The image of a braid γ is the permutation *associated to* γ. In particular, it is known that S_n^+ is in canonical bijection with the symmetric group S_n, which justifies the name of positive permutation braids (the inverse map $S_n \to S_n^+$ can be defined as follows: take an element in S_n, write it as a minimal length word in the standard generators $(i, i+1)$ of S_n, then replace each instance of $(i, i+1)$ by the generator σ_i of B_n).

Note that in a permutation braid, each pair of strands crosses at most once. There is only one positive braid $\Delta \in B_n^+$ in which any pair of strings crosses exactly once. It corresponds geometrically to a generalized half-twist which consists of all the strands $1, \ldots, n$, and is called the *Garside element*. Δ is given by the formula:

$$(4) \qquad \Delta = (\sigma_1 \sigma_2 \cdots \sigma_{n-1}) \, (\sigma_1 \sigma_2 \cdots \sigma_{n-2}) \cdots (\sigma_1 \sigma_2) \, \sigma_1.$$

A basic result asserts that the center of B_n is generated by Δ^2. Abelianization of B_n yields a canonical homomorphism $B_n \to \mathbb{Z}$ which, when restricted to B_n^+, coincides with word length with respect to Artin generators. We denote by $|\beta|$ the length of $\beta \in B_n^+$; we have $|\sigma_i| = 1$ where $1 \leq i \leq n-1$ and the trivial braid e is the only positive braid such that $|e| = 0$.

We recall the algorithm given by Jacquemard [5], which manipulates a positive braid word $w \in B_n^+$ in order to write it using a given leading letter σ_i. The output of the algorithm is an equivalent positive braid word $\sigma_i w' = w$ or an indication that no w' exists such that equality holds.

The basic nature of the algorithm is greedy. It starts by asking whether $w \equiv \sigma_i w'$ and stops if it does. If not, it looks for σ_i inside w. In case σ_i is not one of the letters of w, the algorithm returns *false* which indicates the non-existence of $w' \in B_n^+$ such that $\sigma_i w' = w$.

When the algorithm found the leftmost σ_i it works in two steps:
1. Switch σ_i with its left neighbor σ_j as long as $|i - j| \geq 2$. If σ_i becomes the first letter of the word, we are done. However, in case that $|i - j| = 1$, the word is of the form $w = w_0 \sigma_j \sigma_i w_1$, and so in order to move σ_i to the left one must use the triple relation (2) between $\sigma_j \sigma_i$ and the leftmost letter of w_1. If this is the case, the algorithm goes to Step (2).
2. The algorithm calls itself recursively with the word w_1 and the letter σ_j. Upon success of the recursive call, the word looks like $w = w_0 \sigma_j \sigma_i \sigma_j w_1'$ and therefore, we activate Relation (2) on $\sigma_j \sigma_i \sigma_j$ resulting with $w = w_0 \sigma_i \sigma_j \sigma_i w_1'$, and return to Step (1). However, if the recursive call fails to extract σ_j to the left of w_1 the algorithm returns *false*.

The algorithm terminates since after each iteration, either the length of the word is smaller, or the length remains the same but the position of σ_i, as counted from the left, is reduced.

Note that for a permutation braid $\gamma \in S_n^+$, the condition that σ_i can be extracted on the left in B_n^+ is equivalent to that the length of $\sigma_i \gamma$ is one less than that of γ, considered as permutations in S_n. It is also equivalent to the geometric condition that the i-th strand and the $(i+1)$-th strand (counted from the left) crosses in γ.

To finish this section, we recall another result on the decomposition of braids, due to Garside [4] and later refined by Thurston [6], and by Elrifai and Morton [3].

DEFINITION 2.1. We say that a product $\alpha_1 \cdots \alpha_r$ satisfies *Thurston's condition* if each α_i is a nontrivial positive permutation braid, and for any $1 \leq i \leq r - 1$ we have that any j such that $\alpha_{i+1} = \sigma_j \gamma_{i+1}$ also satisfies $\alpha_i = \gamma_i \sigma_j$ where $\gamma_i, \gamma_{i+1} \in S_n^+$.

PROPOSITION 2.2 (Left-canonical form of a braid). *Given any braid $\beta \in B_n$, there exists a unique decomposition*

$$(5) \qquad \beta = \Delta^k \alpha_1 \cdots \alpha_r,$$

where $\alpha_i \in S_n^+$, $\alpha_1 \neq \Delta$ and the product $\alpha_1 \cdots \alpha_r$ satisfies Thurston's condition. The α_i are called canonical factors.

3. Proof of Theorem 1.1. Our first step consists of looking at the behavior of Δ and permutation braids under the antiautomorphism *rev*.

LEMMA 3.1. *The following properties hold:*
(1) $\overline{\Delta} = \Delta$
(2) The set of permutation braids is invariant under $\mathrm{rev} : w \mapsto \overline{w}$.
(3) $\overline{v^{-1}} = \overline{v}^{-1}$ *for all* $v \in B_n$.

We now identify the basic problem. Let $\beta = \Delta^k \alpha_1 \cdots \alpha_r$ be the left-canonical form for a braid $\beta \in B_n$. We cannot assume that the decomposition $v = \alpha_1 \cdots \alpha_r$ remains in left-canonical form when viewed in $v\overline{v}$. Indeed, after multiplying on the right by the reversed braid, the product $\alpha_1 \cdots \alpha_r$ (viewed in $v\overline{v}$) may cease to satisfy Thurston's condition. A simple example is provided by

$$\alpha_1 = \sigma_1 \sigma_3, \quad \alpha_2 = \sigma_3.$$

Both α_1 and α_2 are positive permutation braids and the product $\gamma = \alpha_1 \alpha_2$ satisfies Thurston's condition. However, when we write $\beta = \alpha_1 \alpha_2 \overline{\alpha_2 \alpha_1}$ in left-canonical form, we find:

$$\beta = \underbrace{\sigma_3 \sigma_1}_{\alpha_1} \underbrace{\sigma_3 \sigma_1}_{\alpha_2'} \underbrace{\sigma_3}_{\alpha_2''} \underbrace{\sigma_3}_{\alpha_1''},$$

so that the second canonical factor α_2' does *not* coincide with α_2.

We start by proving Theorem 1.1 for positive braids.

3.1. Proof of Theorem 1.1 for positive braid words. We start by proving the following lemma:

LEMMA 3.2. *Let v, w' be positive braid words and let $\sigma = \sigma_i$ for some i. Suppose $v\overline{v} = \sigma w' \overline{w'} \sigma$ and let $v\overline{v} \equiv w_0 = w_1 = \cdots = w_k \equiv \sigma v'$ be a sequence of positive braid words such that each w_{i+1} is the outcome of the activation of one relation out of the relations in the semigroup B_n^+ on w_i according to Jacquemard's algorithm. Then, all relations are performed only within the first half of the word w_i which implies they all involve only letters from v.*

Proof. Notice that since $v\overline{v} = \sigma w' \overline{w'} \sigma$, the success of the Jacquemard's algorithm is guaranteed. Hence we know that σ is one of the letters of v. Now, we need to prove that in each step of Jacquemard's algorithm that uses a relation, it occurs in the first half of the word $v\overline{v}$.

For Step (1) of the algorithm, this is obvious: all relations involve σ and *left* neighbors of σ; since σ is in v, all relations occur inside v. Moreover, relations can be activated mirror-like on $\overline{v}$ as well; Hence, we maintain the palindromic structure of the word. This implies that when we need to move to Step (2) of the algorithm, we have

$$v\overline{v} = v_1 \tau \sigma v_2 \overline{v} = v_1 \tau \sigma v_2 \overline{v_2} \sigma \tau \overline{v_1},$$

where τ and σ do not commute, and σ is not in v_1.

If this is the case, the algorithm calls itself recursively using $v_2 \overline{v} = v_2 \overline{v_2} \sigma \tau \overline{v_1}$ and τ, trying to extract τ to the left, first by looking for the leftmost τ letter in $v_2 \overline{v} = v_2 \overline{v_2} \sigma \tau \overline{v_1}$.

Assume by contradiction that τ is not a letter of v_2 (hence is not a letter of $\overline{v_2}$). Then the leftmost τ letter in $v_2\overline{v} = v_2\overline{v_2}\sigma\tau\overline{v_1}$ appears to the right of σ and to the left of $\overline{v_1}$. In order to extract this τ to the left of v_2, we need to activate another recursive call of the algorithm on $\overline{v_1}$ with the letter σ (since in our case σ and τ do not commute and we have to use Step (2) of the algorithm). But, σ is not in $\overline{v_1}$ (since it is not in v_1). Therefore, σ cannot be extracted to the left of $\overline{v_1}$. This implies that Jacquemard's algorithm failed, and this is a contradiction.

If τ is found within v_2 and during the process of extracting it to the left we do not encounter the need to use letters from the right half of $v\overline{v}$ we are finished. Therefore, assume by contradiction that at some point in the process we encounter a relation involving a letter of $\overline{v}$. Again, since until this step all relations were activated only in the left half of the word their mirror image can be activated on the right half of the word, so the palindromic structure of the word is preserved.

Suppose that we performed k recursive steps of the algorithm. Then, our word looks like:

$$v\overline{v} = v_1\tau_1\sigma v_2\tau_2\tau_1 v_3\tau_3\tau_2\cdots v_{k-1}\tau_{k-1}\tau_{k-2}v_k\tau_k\tau_{k-1}v_{k+1} \cdot$$

$$\overline{v_{k+1}}\tau_{k-1}\tau_k\overline{v_k}\tau_{k-2}\tau_{k-1}\overline{v_{k-1}}\cdots\tau_2\tau_3\overline{v_3}\tau_1\tau_2\overline{v_2}\sigma\tau_1\overline{v_1},$$

where τ_i and τ_{i+1} are two non-commuting letters, σ does not commute with τ_1 and is not a letter of v_1. Moreover, τ_i is not a letter of v_{i+1} for any $1 \leq i \leq k$.

Now, in this recursion step, we have called the algorithm with the letter τ_k and the word $v_{k+1}\overline{v_{k+1}}\tau_{k-1}\tau_k\overline{v_k}\tau_{k-2}\tau_{k-1}\overline{v_{k-1}}\cdots\tau_2\tau_3\overline{v_3}\tau_1\tau_2\overline{v_2}\sigma\tau_1\overline{v_1}$. However, since τ_k is not a letter of v_{k+1} and of $\overline{v_{k+1}}$, the leftmost τ_k in this recursion call is to the right of τ_{k-1} and to the left of $\overline{v_k}$. Since τ_{k-1} does not commute with τ_k, another recursion call is needed with the letter τ_{k-1} and the word $\overline{v_k}\tau_{k-2}\tau_{k-1}\overline{v_{k-1}}\cdots\tau_2\tau_3\overline{v_3}\tau_1\tau_2\overline{v_2}\sigma\tau_1\overline{v_1}$. Again, τ_{k-1} is not a letter of $\overline{v_k}$ hence the leftmost τ_{k-1} in this recursion call is to the right of τ_{k-2} and to the left of $\overline{v_{k-1}}$. Similarly τ_{k-2} does not commute with τ_{k-1}, so we continue $k-3$ recursion calls until we reach a recursion call with the letter τ_1 and the word $\overline{v_2}\sigma\tau_1\overline{v_1}$. Since τ_1 is not a letter in $\overline{v_2}$, the leftmost τ_1 in this call is to the right of σ and to the left of $\overline{v_1}$. This implies that another recursion call is needed in order to extract the letter σ from the word $\overline{v_1}$. However, this contradicts the hypothesis on v_1.

This concludes the proof of all cases, hence all relations are activated inside the left half of the word $v\overline{v}$ as claimed. $\square$

Now we are ready to prove the theorem for positive braid words.

THEOREM 3.3. *Let $\beta, \beta' \in B_n^+$ be two palindromic positive braid words of even length such that $\beta = v\overline{v}$ and $\beta' = w\overline{w}$ for some braids $v, w \in B_n^+$. Then, $\beta = \beta'$ in B_n if and only if $v = w$ in B_n.*

Proof. The 'if' part is obvious. We prove that $v\overline{v} = w\overline{w}$ implies $v = w$ by induction on the length $l = |w|$ of w. Assume that $w \equiv \sigma w'$ where $w' \in B_n^+$, i.e., σ is the first letter in w. Then, $w\overline{w} = \sigma w'\overline{w'}\sigma$. This means that $v\overline{v}$ can be written such that its first letter is σ, that is, $v\overline{v} = \sigma v'$ for some $v' \in B_n^+$.

By the embedding theorem of Garside [4] it follows that there is a sequence of words $v\overline{v} \equiv w_0 = w_1 = \cdots = w_k \equiv \sigma v'$, such that each w_{i+1} is obtained from w_i by activating one relation out of the relations in the semigroup B_n^+.

One possible sequence is the one which uses the relations suggested by the algorithm of Jacquemard given in [5], where we pick v' to be the braid word produced by

Jacquemard's algorithm (not necessarily $w'\overline{w'}\sigma$). Now, Lemma 3.2 shows that every relation used in the sequence is fully contained in v (the left half of the word), and does not affect $\overline{v}$. Therefore, it is possible to activate all relations described in the sequence in a mirror-like image on $\overline{v}$ and get $w\overline{w} = \sigma w'\overline{w'}\sigma = v\overline{v} = \sigma v' = \sigma v''\overline{v''}\sigma$, where $v = \sigma v''$.

However, if this is the case $w'\overline{w'} = v''\overline{v''}$ where $|w'\overline{w'}| = |v''\overline{v''}| = l - 2$. Therefore, by the induction hypothesis we have $w' = v''$ which implies that $w = \sigma w' = \sigma v'' = v$. $\square$

Next, we use the above, and give the proof of Theorem 1.1.

3.2. Proof of Theorem 1.1 for the general case. Suppose $v\overline{v} = w\overline{w}$ for two braid words v and w. Write v and w in left-canonical form as $v = \Delta^k\alpha_1\cdots\alpha_r$ and $w = \Delta^j\beta_1\cdots\beta_p$. Without loss of generality, $j \leq k$ (otherwise interchange v and w). Multiply the equality $v\overline{v} = w\overline{w}$ on both the left and right by Δ^{-j} to obtain:

$$\Delta^{k-j}\alpha_1\cdots\alpha_r\overline{\alpha_r}\cdots\overline{\alpha_1}\Delta^{k-j} = \beta_1\cdots\beta_p\overline{\beta_p}\cdots\overline{\beta_1}.$$

These are two equal positive braid words and have the form $v'\overline{v'} = w'\overline{w'}$, where $v' = \Delta^{k-j}\alpha_1\cdots\alpha_r$ and $w' = \beta_1\cdots\beta_p$ respectively. Therefore, Theorem 3.3 applies and we conclude that $v = w$. $\square$

As a consequence, we obtain the following corollaries:

COROLLARY 3.4. *Let $\beta = x\overline{x} \in B_n^+$ be a positive palindromic braid of even length, and let the left-canonical form of x be $\alpha_1\cdots\alpha_r$ such that $\alpha_1 \neq \Delta$. Then, the left-canonical form of $x\overline{x}$ is $\beta_1\cdots\beta_p$ where $\beta_1 \neq \Delta$.*

Proof. Otherwise because the process of Jacquemard's algorithm may be used to transform the word $\alpha_1\cdots\alpha_r\overline{\alpha_r}\cdots\overline{\alpha_1}$ into its left-canonical form, and since it extracts letters to the left only from the first half of the word, we might have

$$\beta = \alpha_1\cdots\alpha_r\overline{\alpha_r}\cdots\overline{\alpha_1} = \Delta\gamma_1\cdots\gamma_q\overline{\gamma_q}\cdots\overline{\gamma_1}\Delta,$$

which means, by Theorem 3.3, that $\alpha_1\cdots\alpha_r = \Delta\gamma_1\cdots\gamma_q$. By the uniqueness of the left-canonical normal form, we deduce that $\alpha_1 = \Delta$, which is a contradiction.

We still have to rule out the possibility that x is a proper initial segment of Δ, such that $x\overline{x} = \Delta z$, where $z \in B_n^+$. Since the signed crossing number of two strands is invariant under the braid relations, and these crossing numbers in Δ are 1 for each pair of strands, and 0 or 1 in x (where at least one 0 exists, since x is a proper initial segment of Δ). Hence, in $z = \Delta^{-1}x\overline{x}$ there is at least one -1, which contradicts the positivity of z. $\square$

We may generalize Corollary 3.4:

COROLLARY 3.5. *Let $v \in B_n^+$ be a positive braid, and let $n(v)$ denote the number of leading permutation braids which are Δ when v is written in its left-canonical form. Then, for $\beta = v\overline{v}$, we have $n(v\overline{v}) = 2n(v)$.*

Proof. Since v is written in left-canonical form as $\Delta^{n(v)}\alpha_1\cdots\alpha_r$ where $\alpha_1 \neq \Delta$, we have that $\beta = \Delta^{n(v)}\alpha_1\cdots\alpha_r\overline{\alpha_r}\cdots\overline{\alpha_1}\Delta^{n(v)}$. Note that since Δ almost commutes with any permutation braid, we may write

$$\beta = \Delta^{2n(v)}\alpha_1'\cdots\alpha_r'\overline{\alpha_r'}\cdots\overline{\alpha_1'},$$

where $\alpha_i' = \alpha_i$ if $n(v)$ is even and α_i' is obtained from α_i by replacing each σ_j by σ_{n-j} in case that $n(v)$ is odd. In any of the cases, the product $\alpha_1'\cdots\alpha_r'$ keeps its left-canonical form. Hence, by Corollary 3.4, we get that $n(v\overline{v}) = 2n(v)$. $\square$

REFERENCES

[1] J. S. BIRMAN, *Braids, Links and Mapping Class Groups*, Ann. Math. Studies 86, Princeton University Press, 1975.

[2] F. DELOUP, *Involutive braids*, preprint (math.GT/0410275).

[3] E. A. ELRIFAI AND H.R. MORTON, *Algorithms for positive braids*, Oxford Quart. J. Math. (2), 45 (1994), pp. 479–497.

[4] F. A. GARSIDE, *The Braid group and other groups*, Oxford Quart. J. of Math., 20 (1969), pp. 235–254.

[5] A. JACQUEMARD, *About the effective classification of conjugacy classes of braids*, J. Pure. Appl. Alg., 63 (1990), pp. 161–169.

[6] W. P. THURSTON, *Braid Groups*, Chapter 9 in: "Word Processing in Groups", pp. 181–209, D.B.A. Epstein (Ed.), Jones and Bartlett, Boston, 1992.