ASIAN J. MATH.
Vol. 11, No. 4, pp. 635–650, December 2007

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THE NUMBER OF RATIONAL CURVES ON K3 SURFACES*

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Abstract. Let X be a $K3$ surface with a primitive ample divisor H, and let $\beta = 2[H] \in H_2(X, \mathbf{Z})$. We calculate the Gromov-Witten type invariants n_β by virtue of Euler numbers of some moduli spaces of stable sheaves. Eventually, it verifies Yau-Zaslow formula in the non primitive class β.

Key words. Rational curve, K3 surface, stable sheaf, Euler number

AMS subject classifications. 14N35, 14D20

Introduction. Let X be a $K3$ surface with an ample divisor H, and let $C \in |H|$ be a reduced curve. By adjunction formula, the arithmetic genus of C is $g = \frac{1}{2}H^2 + 1$. Under the assumption that the homology class $[H] \in H_2(X, \mathbf{Z})$ is primitive, Yau and Zaslow [18] showed that the number of rational curves in the linear system $|H|$ is equal to the coefficient of q^g in the series

$$\begin{aligned}\frac{q}{\Delta(q)} = \prod_{k>0} \frac{1}{(1-q^k)^{24}} &= \sum_{d \geq 0} G_d q^d \\ &= 1 + 24q + 324q^2 + 3200q^3 + 25650q^4 + 176256q^5 + \cdots\end{aligned}$$

Here a multiplicity $e(\bar{J}C)$ is assigned to each rational curve C in the counting([1]).

In [5], Fantechi, Göttsche and van Straten gave an interpretation of the multiplicity $e(\bar{J}C)$. Let $M_{0,0}(X, [H])$ be the moduli space of genus zero stable maps $f : \mathbf{P}^1 \to X$ with $f_*([\mathbf{P}^1]) = [H] \in H_2(X, \mathbf{Z})$. $M_{0,0}(X, [H])$ is a zero dimensional scheme which is in general nonreduced. Let $\iota : C \hookrightarrow X$ be a rational curve in the class $[H]$, and $n : \mathbf{P}^1 \to C$ its normalization. Then $f = \iota \circ n : \mathbf{P}^1 \to X$ is a closed point of $M_{0,0}(X, [H])$ and $e(\bar{J}C)$ is equal to the multiplicity of $M_{0,0}(X, [H])$ at f.

There is another formulation and generalization of Yau and Zaslow's formula by virtue of Gromov-Witten invariants. For $K3$ surfaces, the usual genus 0 Gromov-Witten invariants vanish. To remedy this, one can use the notion of twistor family developed by Bryan and Leung in [2] provided that β is a primitive class. In general, there is an algebraic geometric approach proposed by Jun Li [11] using virtual moduli cycles. Roughly speaking, he defines Gromov-Witten type invariants $N_g(\beta)$ on $K3$ surfaces by modifying the usual tangent-obstruction complex. When β is primitive, these invariants coincide with those defined by twistor family. Geometrically, $N_g(\beta)$ can be thought as Gromov-Witten invariants of a one dimensional family of $K3$ surfaces, which actually count curves in the original surface. For the rigorous definitions, see [2],[11].

Bryan and Leung [2] proved a formula for $N_g(\beta)$ when β is primitive. Let $n_\beta = N_0(\beta)$. Then $n_\beta = G_d$ with $d = \frac{1}{2}\beta^2 + 1$. It recovers the formula of Yau and Zaslow. For a non primitive class β, the numbers $N_g(\beta)$ are still unknown. However, there is a conjectural formula for $N_0(\beta)$([11]). Using the notation n_β, it says

$$n_\beta = \sum_k \frac{1}{k^3} G_{\frac{1}{2}(\frac{\beta}{k})^2+1}$$

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where the sum runs over all integers $k > 0$ such that $\frac{\beta}{k}$ is an integral homology class(see also [6]). The case $\beta = 2[H]$ with $[H]$ primitive and $H^2 = 2$ was proved by Gathmann in [6].

In this paper, we will prove the following result.

THEOREM 0.1. *Let X be a K3 surface with an ample divisor H. Assume $[H] \in H_2(X, \mathbf{Z})$ is primitive. Let $\beta = 2[H]$ and $g = \frac{1}{2}H^2 + 1$. Then*

$$n_\beta = G_{4g-3} + \frac{1}{8}G_g.$$

Now we sketch the proof of this theorem. It can be divided into two steps. First, we deform the pair (X, H) to general position and then reduce the calculation of n_β to N_β, which is the number of reduced and irreducible rational curves in β. The second step is the calculation of N_β. In Gathmann's approach [6], the assumption $H^2 = 2$ is essential in this step. In this paper, we will generalize the approach of Yau and Zaslow [18] according to the suggestion in [11].

Next, we describe these two steps in details.

We begin with the first step. Let (X, H) be a pair of a $K3$ surface X and a primitively polarization H on X. It is well known that two pairs (X, H) and (X', H') with $H^2 = H'^2$ are deformation equivalent. One can choose a general primitively polarized $K3$ surface (X, H), such that $\operatorname{Pic} X = \mathbf{Z} \cdot [H]$ and every rational curve in the linear system $|H|$ is nodal [3]. Moreover, using a generalization of the method in [3], one can also assume that any two rational curves in the system $|H|$ intersect transversely [4]. Now we fix such a pair (X, H) once and for all. Since n_β is a deformation invariant, we only need to calculate n_β for such a surface.

By the enumerative interpretation of n_β, and follow up a similar argument as in [6], all stable maps $f : C \to X$ with $f_*([C]) = \beta$ can be decomposed into the following three types:

1) The domain C is $\mathbf{P}^1$, and the image $f(C) \subset X$ is a reduced and irreducible curve in the linear system $|2H|$. We denote the number of such maps by N_β. The multiplicity of such f is the Euler number of the compactified Jacobian of the image $f(C)$, as shown in [1].

2) The domain is a union of two $\mathbf{P}^1$ that intersect at one point P. In this case, the image is a union of two rational nodal curves that intersect at H^2 points. The image of P has to be one of the intersections, hence there are H^2 such maps. Since the number of rational curves in the system $|H|$ is G_g, the total number of such maps is $\frac{1}{2}G_g(G_g - 1)H^2$.

3) $f : C \to X$ is a double cover onto the image $f(C)$. There are two different cases:

(a) Double covers that factor through the normalization of $f(C)$, this space has dimension 2.

(b) Double covers that do not factor through the normalization. In this case, the domain must be a union of two $\mathbf{P}^1$, which intersect at one point P. The image of P is a node on the image curve $f(C)$, and there is only one map for each choice of node. Note that the number of nodes on $f(C)$ is equal to the arithmetic genus g, so there are totally gG_g such maps.

By Lemma 4.1 in [6], the contribution of type (3a) is $\frac{1}{8}G_g$. Therefore,

$$n_\beta = N_\beta + \frac{1}{2}G_g(G_g - 1)H^2 + gG_g + \frac{1}{8}G_g. \tag{1}$$

Since the first step of the proof is already known, in this paper, we will focus on the second step, namely, the calculation of the number N_β of reduced and irreducible rational curves in the linear system $|2H|$. To this end, we will work with the moduli space of sheaves on a $K3$ surface.

Let (X, H) be the pair we fixed previously. Let $\mathfrak{M}$ be the moduli scheme of stable sheaves $\mathcal{F}$ on X such that $\dim \mathcal{F} = 1$, $c_1(\mathcal{F}) = 2H$ and $\chi(\mathcal{F}) = 1$. The Hilbert polynomial of $\mathcal{F}$ with respect to the polarization H is $2H^2 \cdot n + 1$. Since there is no strictly semistable sheaf in $\mathfrak{M}$, by [13], $\mathfrak{M}$ is a smooth projective variety, and its Euler number $e(\mathfrak{M})$ is G_{2H^2+1}([19]). In section 1, we will construct a morphism $\Phi : \mathfrak{M} \to |2H|$ that sends $\mathcal{F} \in \mathfrak{M}$ to its support in $|2H|$. For $D \in |2H|$, we denote by $\mathfrak{M}_D$ the fiber of Φ over D with the reduced subscheme structure. When D is reduced and irreducible, $\mathfrak{M}_D$ is the compactified Jacobian $\bar{J}D$ of D. In section 2, we will show that $e(\mathfrak{M}_D) = 0$ if D has an irreducible component whose geometric genus is positive.

Therefore only divisors with rational components contribute to the Euler number $e(\mathfrak{M})$. Since H is primitive, we have three types of these divisors in the linear system $|2H|$.

1) $D = C$, C is a rational curve in homology class $\beta(= 2[H])$. In this case, $\mathfrak{M}_D \cong \bar{J}D$. The number of such divisors D, counted with multiplicity $e(\bar{J}D)$, is equal to N_β.

2) $D = C_1 + C_2$, where C_1 and C_2 are different rational nodal curves. In this case, both C_i are contained in the linear system $|H|$. There are totally $\frac{1}{2}G_g(G_g - 1)$ divisors of this type. We will show that $e(\mathfrak{M}_D) = H^2$ in section 3.

3) $D = 2C_0$, where C_0 is a rational nodal curve and contained in $|H|$. The number of such divisors is G_g. In the last two sections we will prove $e(\mathfrak{M}_D) = g$, which is equal to the number of nodes of C_0.

Since $e(\mathfrak{M}) = \sum e(\mathfrak{M}_D)$, where the sum runs over all divisors D with rational components, we get

$$
\begin{aligned}
N_\beta &= e(\mathfrak{M}) - \frac{1}{2}G_g(G_g - 1)H^2 - gG_g \\
&= G_{4g-3} - \frac{1}{2}G_g(G_g - 1)H^2 - gG_g.
\end{aligned} \tag{2}
$$

Together with (1), we prove

$$
n_\beta = G_{4g-3} + \frac{1}{8}G_g.
$$

Recently, J. Li and the author [12] proved the conjectured formula for non primitive class $\beta = n[H]$ with $n < 6$, under the assumption that the transversality of rational curves still holds.

I am most grateful to Jun Li, from whom I learned moduli spaces of sheaves and Gromov-Witten invariants. During the preparation of this paper, his constant encouragement and discussions are invaluable. After finishing the manuscript, the author is informed that J.Lee and N.C.Leung [9] proved the same result using degeneration method and also counted genus 1 curves in $K3$ surfaces [10].

1. Decomposition of the moduli scheme $\mathfrak{M}$. We start with some definitions and notations([15],[8]).

Let X be a complex projective scheme with an ample line bundle $\mathcal{O}(1)$. For a coherent sheaf $\mathcal{E}$ of $\mathcal{O}_X$-module, the Hilbert polynomial $p(\mathcal{E}, n)$ of $\mathcal{E}$ is defined as

$$
p(\mathcal{E}, n) = \dim H^0(X, \mathcal{E}(n)), \ n \gg 0.
$$

The dimension of the support of $\mathcal{E}$ is equal to the degree of $p(\mathcal{E}, n)$. A coherent sheaf $\mathcal{E}$ is pure of dimension d if for any nonzero coherent subsheaf $\mathcal{F} \subset \mathcal{E}$, $\dim \mathcal{F} = d$.

The Hilbert polynomial $p(\mathcal{E}, n)$ can be written as

$$p(\mathcal{E}, n) = \frac{a_0}{d!} n^d + \frac{a_1}{(d-1)!} n^{d-1} + \cdots$$

with integral coefficients $a_i = a_i(\mathcal{E})$. We define the slope of $\mathcal{E}$ to be

$$\mu(\mathcal{E}) = a_0(\mathcal{O}_X) \frac{a_1(\mathcal{E})}{a_0(\mathcal{E})} - a_1(\mathcal{O}_X).$$

Definition 1.1. *A coherent sheaf $\mathcal{E}$ is stable (resp. semistable) if it is pure, and if for any nonzero proper subsheaf $\mathcal{F} \subset \mathcal{E}$, there exists an N, such that for $n > N$,*

$$\frac{p(\mathcal{F}, n)}{a_0(\mathcal{F})} < \frac{p(\mathcal{E}, n)}{a_0(\mathcal{E})} \quad (resp. \leq).$$

Definition 1.2. *A coherent sheaf $\mathcal{E}$ is μ-stable (resp. μ-semistable) if it is pure, and if for any nonzero proper subsheaf $\mathcal{F} \subset \mathcal{E}$,*

$$\mu(\mathcal{F}) < \mu(\mathcal{E}) \quad (resp. \leq).$$

Theorem 1.3. *[15]Let X be a complex projective scheme with an ample line bundle $\mathcal{O}(1)$. There is a projective coarse moduli scheme whose closed points represent the S-equivalence classes of semistable sheaves with Hilbert polynomial $P(n)$.*

Let X be a K3 surface with an ample line bundle H. By Riemann-Roch theorem, the Hilbert polynomial of a torsion free sheaf $\mathcal{E}$ is

$$p(\mathcal{E}, n) = \frac{r}{2} H^2 n^2 + (c_1 \cdot H) n + r\chi(\mathcal{O}_X) + \frac{1}{2}(c_1^2 - 2c_2),$$

where r is the rank of $\mathcal{E}$ and $c_i = c_i(\mathcal{E})$. Let $\mathcal{F}$ be a pure sheaf of dimension 1 on X. By a locally free resolution, one can verify that the Hilbert polynomial of $\mathcal{F}$ is

$$p(\mathcal{F}, n) = (c_1(\mathcal{F}) \cdot H) n + \frac{1}{2}(c_1^2(\mathcal{F}) - 2c_2(\mathcal{F})).$$

It is clear that for such sheaves the notion of stability and μ-stability coincide.

From now on, we fix a pair (X, H) of a $K3$ surface X and a polarization H of X, such that

1) $\operatorname{Pic} X = \mathbf{Z} \cdot [H]$;
2) every rational curve in $|H|$ is nodal; and
3) any two distinct rational curves in $|H|$ intersect transversely.

We let $\beta = 2[H] \in H_2(X, \mathbf{Z})$. Our immediate goal is to calculate N_β, the number of reduced and irreducible rational curves in $|2H|$ counted with multiplicity. To this end, we consider the moduli scheme $\mathfrak{M}$ of stable sheaves $\mathcal{F}$ of $\mathcal{O}_X$-modules that satisfy $\dim \mathcal{F} = 1$, $c_1(\mathcal{F}) = \beta$ and $\chi(\mathcal{F}) = 1$.

Theorem 1.4. *[19] $\mathfrak{M}$ is a smooth projective variety. The Euler number $e(\mathfrak{M})$ is G_{2H^2+1}.*

Next we define the morphism $\Phi : \mathfrak{M} \to |2H|$ mentioned earlier.

Let $\mathcal{F}$ be a sheaf in $\mathfrak{M}$. Since $\mathcal{F}$ is pure of dimension 1, it admits a length 1 locally free resolution

$$0 \longrightarrow \mathcal{E}_1 \xrightarrow{f} \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0,$$

with $r(\mathcal{E}_1) = r(\mathcal{E}_0)$. The homomorphism $f : \mathcal{E}_1 \to \mathcal{E}_0$ induces a homomorphism $\wedge^r f : \wedge^r \mathcal{E}_1 \to \wedge^r \mathcal{E}_0$, and a nonzero global section $s \in H^0(\wedge^r \mathcal{E}_1)^{-1} \otimes (\wedge^r \mathcal{E}_0))$, that defines an effective divisor $D = s^{-1}(0)$ on X. Since $(\wedge^r \mathcal{E}_1)^{-1} \otimes (\wedge^r \mathcal{E}_0) = c_1(\mathcal{F}) = 2H$, D is contained in the linear system $|2H|$. The assignment $\mathcal{F} \to D$ defines a morphism $\Phi : \mathfrak{M} \to |2H|$.

We now give a specific decomposition of the projective space $|2H|$ according to the topological type of $D \in |2H|$.

We let $\mathcal{W}_1$ be the set of divisors D which is reduced and irreducible. The arithmetic genus of D is $p_a(D) = 2H^2 + 1$, which is an invariant for all $D \in \mathcal{W}_1$. We further stratify $\mathcal{W}_1$ according to the geometric genus of curves, $\mathcal{W}_1 = \sqcup_k \mathcal{W}_1^k$, where $\mathcal{W}_1^k$ consists of those D that have geometric genus k. Clearly $\sqcup_{k \le a} \mathcal{W}_1^k$ is closed in $\mathcal{W}_1$. Let $\mathcal{W}_2$ be the stratum of divisors $D = C_1 + C_2$ with $C_1 \ne C_2$ and $C_i \in |H|$. Without loss of generality, we can assume $p_g(C_1) \le p_g(C_2)$. For $a \le b$, we let $\mathcal{W}_2^{a,b} \subset \mathcal{W}_2$ be the subset of divisors D with $p_g(C_1) = a$ and $p_g(C_2) = b$. Then $\mathcal{W}_2 = \sqcup \mathcal{W}_2^{a,b}$. Let $\mathcal{W}_3$ be the subset of divisors $D = 2C_0$ with $C_0 \in |H|$. Similarly, $\mathcal{W}_3 = \sqcup_k \mathcal{W}_3^k$, where $\mathcal{W}_3^k$ consists of $D = 2C_0$ with $p_g(C_0) = k$.

Put together,

$$|2H| = (\sqcup_k \mathcal{W}_1^k) \bigsqcup (\sqcup_{a \le b} \mathcal{W}_2^{a,b}) \bigsqcup (\sqcup_k \mathcal{W}_3^k).$$

This induces a decomposition on $\mathfrak{M}$,

$$\mathfrak{M} = (\sqcup_k \Phi^{-1}(\mathcal{W}_1^k)) \bigsqcup (\sqcup \Phi^{-1}(\mathcal{W}_2^{a,b})) \bigsqcup (\sqcup_k \Phi^{-1}(\mathcal{W}_3^k)).$$

Now we state a general fact on the Euler number of varieties.

Let Z be a complex variety. Let $Z = \sqcup Z_i$ be a decomposition into locally closed subset Z_i. Then the Euler number $e(Z) = \sum e(Z_i)$.

Apply this to the decomposition of $\mathfrak{M}$, we have

$$\begin{aligned} e(\mathfrak{M}) &= e(\Phi^{-1}(\mathcal{W}_1)) + e(\Phi^{-1}(\mathcal{W}_2)) + e(\Phi^{-1}(\mathcal{W}_3)) \\ &= \sum_k e(\Phi^{-1}(\mathcal{W}_1^k)) + \sum_{a \le b} e(\Phi^{-1}(\mathcal{W}_2^{a,b})) + \sum_k e(\Phi^{-1}(\mathcal{W}_3^k)). \end{aligned}$$

Proposition 1.5. *[1] Let $h : Y \to Z$ be a surjective morphism between complex algebraic varieties. Suppose that $e(h^{-1}(z)) = 0$ for every closed point $z \in Z$. Then $e(Y) = 0$.*

The following proposition will be proved in the next section.

Proposition 1.6. *Suppose D is a divisor that has one irreducible component whose geometric genus is positive, then $e(\mathfrak{M}_D) = 0$.*

Combine these results, we have

$$e(\mathfrak{M}) = e(\Phi^{-1}(\mathcal{W}_1^0)) + e(\Phi^{-1}(\mathcal{W}_2^{0,0})) + e(\Phi^{-1}(\mathcal{W}_3^0)).$$

Because $e(\Phi^{-1}(\mathcal{W}_1^0))$ is equal to N_β, and $e(\mathfrak{M}) = G_{2H^2+1}$, To calculate N_β, it suffices to find the Euler numbers $e(\Phi^{-1}(\mathcal{W}_2^{0,0}))$ and $e(\Phi^{-1}(\mathcal{W}_3^0))$. The number $e(\Phi^{-1}(\mathcal{W}_2^{0,0}))$ is essentially known, which is equal to $\frac{1}{2}G_g(G_g-1)H^2$ as will be shown in section 3. The main body of the remainder of the paper is to show that $e(\Phi^{-1}(\mathcal{W}_3^0)) = gG_g$. Therefore,

$$N_\beta = G_{2H^2+1} - \frac{1}{2}G_g(G_g-1)H^2 - gG_g.$$

Apply equality (1) in the introduction, we obtain the formula in the main theorem.

2. Proof of Proposition 1.6. We state a basic fact about the Euler number of a variety. Let X be a quasi-projective variety. If there exists a finite group action on X which is free of fixed point, then $e(X)$ is divisible by the order of this group. Therefore, if for any positive integer N, there is a finite group G_N whose order is greater than N, and a free G_N action on X, then $e(X)$ is zero.

If D is a reduced and irreducible curve, then $\mathfrak{M}_D \cong \bar{J}D$. Since the geometric genus of D is positive, $e(\bar{J}D) = 0$(see[1]). Now if $D = C_1 + C_2$ with $C_i \in |H|$ and by assumption the geometric genus $p_g(C_2) > 0$. From the restriction homomorphism $\alpha : \operatorname{Pic} D \to \operatorname{Pic} C_2$, we can choose a subgroup $G \subset \operatorname{Pic} D$, such that for $\mathcal{L} \in G$, $\mathcal{L}|_{C_1} \cong \mathcal{O}_{C_1}$ and $\alpha(\mathcal{L}) = \mathcal{L}|_{C_2}$ is trivial if and only if $\mathcal{L}$ is trivial. Next we show that the G-action on $\mathfrak{M}_D$ defined by tensorization is free, i.e., for any sheaf $\mathcal{F} \in \mathfrak{M}_D$ and $\mathcal{L} \in G$, $\mathcal{F} \otimes \mathcal{L} \cong \mathcal{F}$ if and only if $\mathcal{L}$ is trivial. To this end, suppose $\mathcal{F} \otimes \mathcal{L} \cong \mathcal{F}$ for some $\mathcal{F}$ and $\mathcal{L}$. Let $\mathcal{F}_2$ be the torsion free part of the restriction $\mathcal{F}|_{C_2}$. We obtain $\mathcal{F}_2 \otimes \alpha(\mathcal{L}) \cong \mathcal{F}_2$ and therefore $\alpha(\mathcal{L})$ is trivial by the same argument as in case 1. Finally, it implies $\mathcal{L}$ is trivial by the choice of the subgroup G. Finally, $D = 2C_0$ is a divisor whose associated subscheme is a nonreduced curve C, and a closed point in $\mathfrak{M}_D$ is a sheaf of $\mathcal{O}_C$-modules. To prove this case, we first recall some facts on nonreduced curves.

Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. An infinitesimal extension([7], Exer II 8.7) of X by $\mathcal{F}$ is a scheme X', with an ideal sheaf $\mathcal{I}$, such that $\mathcal{I}^2 = 0$ and $(X', \mathcal{O}_{X'}/\mathcal{I}) \cong (X, \mathcal{O}_X)$ and such that $\mathcal{I}$ with the induced structure of $\mathcal{O}_X$-module is isomorphic to the given sheaf $\mathcal{F}$. Let S be a smooth projective surface, and $C_0 \subset S$ be a reduced and irreducible curve. There is an associated closed subscheme $C \subset S$ to the divisor $2C_0$. In fact, C is an infinitesimal extension of C_0 by $\mathcal{I} = \mathcal{O}_S(-C_0)|_{C_0}$.

Next we discuss the Picard group of C([7], Exer III 4.6). From the exact sequence of sheaves of abelian groups

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_C^* \longrightarrow \mathcal{O}_{C_0}^* \longrightarrow 0,$$

there is an induced exact sequence

$$0 \longrightarrow H^1(C, \mathcal{I}) \longrightarrow \operatorname{Pic} C \longrightarrow \operatorname{Pic} C_0 \longrightarrow 0.$$

Notice that $H^1(C, \mathcal{I})$ is a vector space and hence an injective $\mathbf{Z}$-module, it implies that $\operatorname{Pic} C \cong \operatorname{Pic} C_0 \oplus H^1(C, \mathcal{I})$ as groups. For $\mathcal{L} \in \operatorname{Pic} C$, we let $\mathcal{L}_0 \in \operatorname{Pic} C_0$ be the restriction of $\mathcal{L}$ to C_0.

Now we continue the proof. Let $\pi : \tilde{C}_0 \to C_0$ be the normalization of C_0. Then $\operatorname{Pic}^0 C_0 \cong \operatorname{Pic}^0 \tilde{C}_0 \oplus A$, where A is an affine commutative group. Since the genus of $\tilde{C}_0$ is positive, $\operatorname{Pic}^0 \tilde{C}_0$ is nontrivial. For any odd prime p, we can choose an order p subgroup $G \subset \operatorname{Pic}^0 C$, such that for $\mathcal{L} \in G$, $\mathcal{L}$ is trivial if and only if $\tilde{\mathcal{L}} = \pi^*\mathcal{L}_0$ is trivial. There is a G-action on $\mathfrak{M}_D$ defined by tensorization. Next we show that this group action is free and therefore $e(\mathfrak{M}_D) = 0$.

Suppose $\mathcal{L} \otimes \mathcal{E} \cong \mathcal{E}$ for some sheaf $\mathcal{E} \in \mathfrak{M}_D$ and $\mathcal{L} \in G$. Restrict to C_0 and let $\mathcal{E}_0$ be the torsion free part of $\mathcal{E} \otimes \mathcal{O}_{C_0}$, we get $\mathcal{L}_0 \otimes \mathcal{E}_0 \cong \mathcal{E}_0$.

1) If $\mathcal{I}\mathcal{E} \neq 0$, $\mathcal{E}_0$ is a rank 1 torsion free sheaf on C_0. Using the same argument as in case 1, we obtain $\mathcal{L}_0 \cong \mathcal{O}_{C_0}$, and hence $\mathcal{L} \cong \mathcal{O}_C$, i.e., the group action is free.

2) If $\mathcal{I}\mathcal{E} = 0$, $\mathcal{E}_0$ is a rank 2 torsion free sheaf on C_0. Let $\tilde{\mathcal{E}}_0$ be the torsion free part of $\pi^*\mathcal{E}_0$. Then we have $\tilde{\mathcal{L}}_0 \otimes \tilde{\mathcal{E}}_0 \cong \tilde{\mathcal{E}}_0$. Take top wedge on both sides, we get $\tilde{\mathcal{L}}_0^{\otimes 2} \otimes \wedge^2\tilde{\mathcal{E}}_0 \cong \wedge^2\tilde{\mathcal{E}}_0$. Since $\wedge^2\tilde{\mathcal{E}}_0$ is invertible, $\tilde{\mathcal{L}}_0^{\otimes 2} \cong \mathcal{O}_{\tilde{C}_0}$. Note that $\mathcal{L} \in G$ and G has odd prime order p, it implies that $\mathcal{L} \cong \mathcal{O}_C$. Hence the group action is free.

3. Calculation of $e(\Phi^{-1}(\mathcal{W}_2^{0,0}))$**.** Recall that $\mathcal{W}_2^{0,0}$ is a finite set of divisors $D = C_1 + C_2$ with $C_1, C_2 \in |H|$ being rational nodal curves and intersect transversally, $\Phi^{-1}(\mathcal{W}_2^{0,0}) = \sqcup \mathfrak{M}_{D_i}$ with $D_i \in \mathcal{W}_2^{0,0}$. We will calculate $e(\mathfrak{M}_D)$ for $D \in \mathcal{W}_2^{0,0}$ and then the Euler number $e(\Phi^{-1}(\mathcal{W}_2^{0,0}))$ follows.

A closed point in $\mathfrak{M}_D$ is a stable sheaf $\mathcal{E}$ of $\mathcal{O}_D$-modules, such that the restrictions $\mathcal{E}|_{C_i}$ are rank 1 sheaves of $\mathcal{O}_{C_i}$-modules respectively. Let $x_1, x_2, \cdots, x_s$ be a list of intersections of C_1 and C_2. Then $s = H^2 > 0$. Since $\mathcal{E}$ is stable, there is at least one point x_k, so that the stalk $\mathcal{E}_{x_k}$ is isomorphic to $\mathcal{O}_{x_k}$. For otherwise, $\mathcal{E}$ is the direct image of some sheaf on the disjoint union of C_1 and C_2, which violates the stability of $\mathcal{E}$.

We let $S_{ij} \subset \mathfrak{M}_D$ be the subset of stable sheaves $\mathcal{E}$ such that $\mathcal{E}_{x_i} \cong \mathcal{O}_{x_i}$ and $\mathcal{E}_{x_j} \cong \mathcal{O}_{x_j}$ for two intersection points x_i and x_j. We can find a subgroup $G \subset \operatorname{Pic} D$ coming from the gluing of $\mathcal{O}_{C_1}$ and $\mathcal{O}_{C_2}$ at x_i and x_j. $G \cong \mathbf{C}^*$. Now follow a similar argument as in the previous section, the G-action on S_{ij} is free. Therefore, the contribution to the Euler number $e(\mathfrak{M}_D)$ come from stable sheaves $\mathcal{E}$ whose stalks are not $\mathcal{O}$ at all nodes but one intersection point. Since both C_i are rational curves, there is only one such stable sheaf corresponds to an intersection point. We have

Proposition 3.1. *Let D be a divisor in the set $\mathcal{W}_2^{0,0}$. Then $e(\mathfrak{M}_D) = H^2$.*

Since the number of rational curves in $|H|$ is G_g, $\mathcal{W}_2^{0,0}$ is a finite set with cardinality $\frac{1}{2}G_g(G_g - 1)$.

Corollary 3.2. $e(\Phi^{-1}(\mathcal{W}_2^{0,0})) = \frac{1}{2}G_g(G_g - 1)H^2$.

4. Calculation of $e(\Phi^{-1}(\mathcal{W}_3^{0}))$**, Part I.** In the remainder of this paper, we will calculate the Euler number $e(\Phi^{-1}(\mathcal{W}_3^0))$. Remember that $\mathcal{W}_3^0$ is a finite set of divisors $D = 2C_0$ with $C_0 \in |H|$ being rational nodal curves, there is a decomposition $\Phi^{-1}(\mathcal{W}_3^0) = \sqcup \mathfrak{M}_{D_i}$. It suffices to calculate $e(\mathfrak{M}_D)$ for $D \in \mathcal{W}_3^0$.

Recall that for every effective divisor, there is an associated subscheme. Let C be the nonreduced curve associated to $D = 2C_0$. Then every closed point in $\mathfrak{M}_D$ corresponds to a stable sheaf $\mathcal{E}$ of $\mathcal{O}_C$-modules, such that the Hilbert polynomial $P_{\mathcal{E}}(n)$ is $2H^2 + 1$.

There are two kinds of these sheaves. A sheaf $\mathcal{E}$ in the first type satisfies $\mathcal{I}\mathcal{E} = 0$, where $\mathcal{I} \subset \mathcal{O}_C$ is the nilpotent ideal sheaf. That is to say, $\mathcal{E}$ is a rank 2 sheaf on C_0, the reduced part of C. Let $\mathfrak{M}_D^1 \subset \mathfrak{M}_D$ be the subset of sheaves of this type. The second type consists of sheaves $\mathcal{E}$ satisfy $\mathcal{I}\mathcal{E} \neq 0$. It is direct to verify that for sheaves of this type, $\mathcal{E}_\eta \cong \mathcal{O}_\eta$ with η the generic point of C. Let $\mathfrak{M}_D^2$ be the subset of sheaves of the second type. Then $e(\mathfrak{M}_D) = e(\mathfrak{M}_D^1) + e(\mathfrak{M}_D^2)$.

In this section, we calculate $e(\mathfrak{M}_D^1)$. The discussion of $\mathfrak{M}_D^2$ is left to the next section. The result has been obtained by T. Teodorescu in his PhD thesis [16] which deals with a more general problem. It is also proved in [17] independently. Now we use a slightly different approach.

We recall some standard facts about sheaves on a nodal curve. Since we will not talk about nonreduced curves in the next part of this section, we use C, instead of C_0, to denote a nodal curve. We always work on the complex topology.

Let C be a projective curve with n ordinary nodes $x_1, x_2, \cdots, x_n$ as singularities, and let $\pi : \tilde{C} \to C$ be the normalization of C. A torsion free sheaf $\mathcal{E}$ is locally free away from the nodes. It has the following nice local structure at each node $x_i \in C$([14])

$$\mathcal{E}_{x_i} \cong \mathcal{O}_{x_i}^{\oplus a_i} \oplus m_{x_i}^{\oplus(r-a_i)},$$

where $m_{x_i} \subset \mathcal{O}_{x_i}$ is the maximal ideal, and r is the rank of $\mathcal{E}$. Let $\hat{\pi} : \hat{C} \to C$ be a partial normalization of C at one node x. Then there exists a torsion free sheaf $\mathcal{F}$ on $\hat{C}$ such that $\mathcal{E} \cong \hat{\pi}_*\mathcal{F}$ if and only if $\mathcal{E}_x \cong m_x^{\oplus r}$.

Let $r \geq 1$ be an integer and choose n such that $(r, n) = 1$. There is a smooth projective variety $\mathfrak{M}_C(r, n)$ whose closed points correspond to isomorphism classes of stable $\mathcal{O}_C$-modules $\mathcal{E}$, such that $r(\mathcal{E}) = r$ and $\chi(\mathcal{E}) = n$.

Next we introduce the notion of admissible quotients. It will be used to determine whether two torsion free sheaves $\mathcal{E}_1$, $\mathcal{E}_2$ are isomorphic. Let $\mathcal{E}$ be a torsion free sheaf and $(\pi^*\mathcal{E})^\sharp$ be the torsion free part of $\pi^*\mathcal{E}$. There is a canonical exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \pi_*(\pi^*\mathcal{E})^\sharp \longrightarrow \mathcal{T} \longrightarrow 0,$$

where $\mathcal{T} \cong \oplus \mathbf{C}_{x_i}^{\oplus a_i}$ is a skyscraper sheaf supported at the nodes.

Definition 4.1. *Let $\mathcal{V}$ be a rank r locally free sheaf on $\tilde{C}$, and $\mathcal{Q} = \oplus \mathbf{C}_{x_i}^{\oplus a_i}$ be a skyscraper sheaf supported at the set of nodes on C. Let $\rho : \pi_*\mathcal{V} \to \mathcal{Q}$ be a surjective morphism and $\mathcal{E}$ be the kernel of ρ. ρ is said to be an admissible quotient if there is a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E} & \longrightarrow & \pi_*\mathcal{V} & \xrightarrow{\rho} & \mathcal{Q} & \longrightarrow & 0 \\
 & & \big\downarrow{=} & & \big\downarrow{\cong} & & \big\downarrow{\cong} & & \\
0 & \longrightarrow & \mathcal{E} & \longrightarrow & \pi_*(\pi^*\mathcal{E})^\sharp & \longrightarrow & \mathcal{T} & \longrightarrow & 0
\end{array}$$

where the second row is the canonical exact sequence.

Let $p_i, q_i \in \tilde{C}$ be the inverse images of the node x_i. Then $(\pi_*\mathcal{V})_{x_i} = \mathcal{V}_{p_i} \oplus \mathcal{V}_{q_i}$. The homomorphism ρ is given by $\rho_{x_i} : \mathcal{V}_{p_i} \oplus \mathcal{V}_{q_i} \to \mathcal{Q}_{x_i}$. Let $\iota_i^1 : \mathcal{V}_{p_i} \to \mathcal{V}_{p_i} \oplus \mathcal{V}_{q_i}$ and $\iota_i^2 : \mathcal{V}_{q_i} \to \mathcal{V}_{p_i} \oplus \mathcal{V}_{q_i}$ be the natural injections. By the definition of admissible quotients, $\rho_{x_i}^k = \rho_{x_i} \circ \iota_i^k$ are both surjective. Conversely, we have

Proposition 4.2. *Let $\rho : \pi_*\mathcal{V} \to \mathcal{Q}$ be a quotient such that $\rho_{x_i}^k$ defined above are surjective for all $i = 1, 2 \cdots, n$ and $k = 1, 2$. Then ρ is an admissible quotient.*

Proof. Let $\mathcal{E}$ be the kernel of ρ. Apply the functor π^* to the exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \pi_*\mathcal{V} \xrightarrow{\rho} \mathcal{Q} \longrightarrow 0,$$

we get

$$\pi^*\mathcal{E} \xrightarrow{\psi_1} \pi^*(\pi_*\mathcal{V}) \xrightarrow{\psi_2} \pi^*\mathcal{Q} \longrightarrow 0.$$

Since $\rho_{x_i}^k$ are surjective, the restriction of ψ_2 on the torsion part $\mathcal{T}' \subset \pi^*(\pi_*\mathcal{V})$ is surjective, i.e. $\psi_2(\mathcal{T}') = \pi^*\mathcal{Q}$. It implies that the homomorphism $\pi^*\mathcal{E} \to (\pi^*(\pi_*\mathcal{V}))^\sharp$induced by ψ_1 is surjective. Because the kernel of ψ_1 is a skyscraper sheaf, ψ_1 induces an isomorphism $(\pi^*\mathcal{E})^\sharp \to (\pi^*(\pi_*\mathcal{V}))^\sharp$. Since every step is functorial, the result follows from the canonical isomorphism $(\pi^*(\pi_*\mathcal{V}))^\sharp \cong \mathcal{V}$. □

Proposition 4.3. *Let $\rho_1, \rho_2 : \pi_*\mathcal{V} \to \mathcal{Q}$ be two admissible quotients and let $\mathcal{E}_1 = \ker\rho_1, \mathcal{E}_2 = \ker\rho_2$. Every isomorphism $u : \mathcal{E}_1 \cong \mathcal{E}_2$ can be extended to an isomorphism $\psi : \pi_*\mathcal{V} \cong \pi_*\mathcal{V}$, i.e. we have a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \pi_*\mathcal{V} & \xrightarrow{\rho_1} & \mathcal{Q} & \longrightarrow & 0 \\
 & & \Big\downarrow u & & \Big\downarrow \psi & & \Big\downarrow \cong & & \\
0 & \longrightarrow & \mathcal{E}_2 & \longrightarrow & \pi_*\mathcal{V} & \xrightarrow{\rho_2} & \mathcal{Q} & \longrightarrow & 0
\end{array}$$

The next proposition deals with the automorphism group of $\pi_*\mathcal{V}$.

Proposition 4.4. *Let $\mathcal{V}$ be a locally free sheaf on $\tilde{C}$. Every automorphism of $\pi_*\mathcal{V}$ as an $\mathcal{O}_C$-module can be induced from an automorphism of $\mathcal{V}$ as an $\mathcal{O}_{\tilde{C}}$-module. Hence there is a canonical isomorphism $Aut_{\mathcal{O}_C}(\pi_*\mathcal{V}) \cong Aut_{\mathcal{O}_{\tilde{C}}}(\mathcal{V})$.*

Proof. Let $u : \pi_*\mathcal{V} \to \pi_*\mathcal{V}$ be an automorphism of $\pi_*\mathcal{V}$ as an $\mathcal{O}_C$-module. It induces canonically an automorphism $\bar{u} : \pi^*\pi_*\mathcal{V} \to \pi^*\pi_*\mathcal{V}$ as an $\mathcal{O}_{\tilde{C}}$-module. Let $\mathcal{T}' \subset \pi^*\pi_*\mathcal{V}$ be the torsion part. Then $\bar{u}(\mathcal{T}') = \mathcal{T}'$, and it induces an automorphism $u^\sharp : (\pi^*\pi_*\mathcal{V})^\sharp \to (\pi^*\pi_*\mathcal{V})^\sharp$. Since $\mathcal{V}$ is locally free, there is a canonical isomorphism $(\pi^*\pi_*\mathcal{V})^\sharp \cong \mathcal{V}$. We obtain an automorphism $\tilde{u} : \mathcal{V} \to \mathcal{V}$ as an $\mathcal{O}_{\tilde{C}}$-module. Since every step is functorial, it establishes an isomorphism $Aut_{\mathcal{O}_C}(\pi_*\mathcal{V}) \cong Aut_{\mathcal{O}_{\tilde{C}}}(\mathcal{V})$. □

Next we assume C is a rational nodal curve with n nodes. We describe a method to calculate $e(\mathfrak{M}_C(r, n))$.

Let $\mathcal{E}$ be a stable sheaf in $\mathfrak{M}_C(r, n)$, and let $\mathcal{V} = (\pi^*\mathcal{E})^\sharp$ be the torsion free part of $\pi^*\mathcal{E}$. Then $\mathcal{V}$ be a locally free sheaf of rank r on $\tilde{C}$. Since $\tilde{C} \cong \mathbf{P}^1$, by Grothendieck's Lemma, $\mathcal{V} \cong \mathcal{O}(l_1) \oplus \mathcal{O}(l_2) \oplus \cdots \oplus \mathcal{O}(l_r)$ for some integers $l_1 \le l_2 \le \cdots \le l_r$. There is a decomposition of $\mathfrak{M}_C(r, n)$,

$$\mathfrak{M}_C(r, n) = \bigsqcup \mathfrak{M}^{l_1,\cdots,l_r}_{a_1,a_2,\cdots,a_n},$$

such that $[\mathcal{E}] \in \mathfrak{M}^{l_1,\cdots,l_r}_{a_1,a_2,\cdots,a_n}$ if and only if

$$(\pi^*\mathcal{E})^\sharp \cong \mathcal{O}(l_1) \oplus \mathcal{O}(l_2) \oplus \cdots \oplus \mathcal{O}(l_r)$$

and

$$\mathcal{E}_{x_i} \cong \mathcal{O}_{x_i}^{\oplus a_i} \oplus m_{x_i}^{\oplus(r-a_i)}.$$

Let $\mathcal{E}_1$, $\mathcal{E}_2$ be the kernels of two admissible quotients $\rho_1, \rho_2 : \pi_*\mathcal{V} \to \mathcal{Q}$ respectively. The automorphism group of $\mathcal{Q}$ is a direct sum of automorphism groups of $\mathcal{Q}_{x_i}$. Let $G_i = Aut(\mathcal{Q}_i)$. Then $G_i \cong GL(a_i, \mathbf{C})$. There is an $Aut(\mathcal{V}) \times \prod G_i$ action on $\mathrm{Hom}(\pi_*\mathcal{V}, \mathcal{Q})$, $\rho \longrightarrow g \circ \rho \circ u$, where $\rho \in \mathrm{Hom}(\pi_*\mathcal{V}, \mathcal{Q})$, $u \in Aut(\mathcal{V})$ and $g \in \prod G_i$. Proposition 4.3 says that $\mathcal{E}_1 \cong \mathcal{E}_2$ if and only if ρ_1 and ρ_2 lie in the same orbit of $\mathrm{Hom}(\pi_*\mathcal{V}, \mathcal{Q})$ under this group action.

Next we work out a matrix form of these results under suitable bases.

Let V_i and W_i be the fibres of $\mathcal{V}$ at p_i and q_i respectively. Then $(\pi_*\mathcal{V}) \otimes \mathbf{C}_{x_i} \cong V_i \oplus W_i$. Since $\mathcal{Q}_{x_i} = \mathbf{C}^{\oplus a_i}$, every homomorphism $\rho : \pi_*\mathcal{V} \to \mathcal{Q}$ gives an element in the vector space

$$U = \oplus_{i=1}^n (\mathrm{Hom}(V_i, \mathbf{C}^{\oplus a_i}) \oplus \mathrm{Hom}(W_i, \mathbf{C}^{\oplus a_i})).$$

Fix an isomorphism $\mathcal{V} \cong \mathcal{O}(l_1) \oplus \mathcal{O}(l_2) \oplus \cdots \oplus \mathcal{O}(l_r)$ once and for all. For any summand $\mathcal{O}(l_i)$, there is an isomorphism of stalks $\mathcal{O}(l_i)_x \cong \mathcal{O}_x$ by the locally freeness of $\mathcal{O}(l_i)$. Those isomorphisms at p_i and q_i give rise to bases $e_i^k \in V_i$ and $f_i^k \in W_i$. Fix all these choices once and for all. Now an element $\rho \in U$ corresponds to a set of $a_i \times r$ matrixes

$$\{A_i, B_i\}_{i=1,2,\cdots,n}.$$

Let $\rho_i' \in Hom(V_i, \mathbf{C}^{\oplus a_i}), \rho_i'' \in Hom(W_i, \mathbf{C}^{\oplus a_i})$ and $v_i = (v_i^1, v_i^2, \cdots, v_i^r)^t \in V_i$, $w_i = (w_i^1, w_i^2, \cdots, w_i^r)^t \in W_i$. Then

$$\rho_i'(v_i) = A_i \begin{pmatrix} v_i^1 \\ v_i^2 \\ \cdots \\ v_i^r \end{pmatrix}, \rho_i''(w_i) = B_i \begin{pmatrix} w_i^1 \\ w_i^2 \\ \cdots \\ w_i^r \end{pmatrix}.$$

Corollary 4.5. *A quotient $\{A_i, B_i\}$ is admissible if and only if the ranks of A_i and B_i are both equal to a_i for all i. In particular, for an admissible quotient $\{A_i, B_i\}$, one has $a_i \leq r$.*

Proof. Follows from proposition 4.2. □

Now we consider the $Aut(\mathcal{V}) \times \prod G_i$ action on $\mathrm{Hom}(\pi_*\mathcal{V}, \mathcal{Q})$.

Evaluated at a closed point $x \in \tilde{C}$, every automorphism $u \in Aut(\mathcal{V})$ gives rise to an automorphism in $Aut(V_x)$, where V_x is the fibre of $\mathcal{V}$ at x. Therefore, every $u \in Aut(\mathcal{V})$ gives rise to an element

$$\prod_i u(p_i) \times \prod_i u(q_i) \in \prod_i Aut(V_i) \times \prod_i Aut(W_i).$$

Let $G' \subset \prod_i Aut(V_i) \times \prod_i Aut(W_i)$ be the subgroup of elements derived in this way. Since $G_i \cong GL(a_i, \mathbf{C})$, there is an $G' \times \prod GL(a_i, \mathbf{C})$ action on the vector space U of all quotients $\{A_i, B_i\}_{i=1,2,\cdots,n}$, which is given by

$$\{A_i, B_i\} \longrightarrow \{g_i A_i u(p_i), g_i B_i u(q_i)\},$$

where $\prod u(p_i) \times \prod u(q_i) \in G'$ and $g_i \in GL(a_i, \mathbf{C})$. Two quotients $\{A_i, B_i\}$ and $\{A_i', B_i'\}$ are equivalent if they lie in one and the same orbit under this group action.

An application of this formulation is to determine whether $\mathcal{E} \otimes \mathcal{L} \cong \mathcal{E}$ for $\mathcal{L} \in \mathrm{Pic}^0 C$.

Let $\rho : \pi_*\mathcal{V} \to \mathcal{Q}$ be an admissible quotient with corresponding matrixes $\{A_i, B_i\}$ and let $\mathcal{E} = \ker \rho$. Let $\mathcal{L} \in \mathrm{Pic}^0 C$ be given by the matrixes $\{1, t_i\}$, where $t_i \in \mathbf{C}^*$. The exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \pi_*\mathcal{V} \xrightarrow{\rho} \mathcal{Q} \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow \mathcal{E} \otimes \mathcal{L} \longrightarrow \pi_*\mathcal{V} \otimes \mathcal{L} \xrightarrow{\rho \otimes 1} \mathcal{Q} \otimes \mathcal{L} \longrightarrow 0.$$

Note that $\pi_*\mathcal{V}\otimes\mathcal{L}\cong\pi_*\mathcal{V}$ and the quotient $\rho\otimes 1:\pi_*\mathcal{V}\otimes\mathcal{L}\to\mathcal{Q}\otimes\mathcal{L}$ is also admissible. Fix an isomorphism $\mathcal{Q}\otimes\mathcal{L}\cong\mathcal{Q}$ and choose corresponding bases, $\rho\otimes 1$ is given by the matrixes $\{A_i, t_iB_i\}$.

We are now ready to calculate the Euler number $e(\mathfrak{M}_C(r,n))$. For the purpose of this paper, we only consider the case $r=2$ and $n=1$.

Proposition 4.6. *Let $\mathfrak{M}^{l_1,l_2}_{a_1,a_2,\cdots,a_n}$ be a stratum in $\mathfrak{M}_C(2,1)$ such that $\sum a_i\geq 2$. Then $e(\mathfrak{M}^{l_1,l_2}_{a_1,a_2,\cdots,a_n})=0$.*

Proof. For simplicity, we consider only the stratum $\mathfrak{M}^{l_1,l_2}_{1,1,0,\cdots,0}$ as illustration. Let $[\mathcal{E}]\in\mathfrak{M}^{l_1,l_2}_{1,1,0,\cdots,0}$ be the kernel of an admissible quotient $\{A_i,B_i\}$. We can choose a suitable base such that $A_1=A_2=(1,0), B_1=B_2=(0,1)$. For an odd prime p, let $\mathcal{L}$ be given by $t_1=1, t_2=\zeta$, where ζ is a p-th primitive root of unity. Then $\mathcal{L}^{\otimes p}=\mathcal{O}_C$ and $\mathcal{E}\otimes\mathcal{L}$ corresponds to the quotient $\{A_i,t_iB_i\}$. It is direct to verify that $\{A_i,t_iB_i\}$ and $\{A_i,B_i\}$ are not equivalent, hence $\mathcal{E}\otimes\mathcal{L}$ and $\mathcal{E}$ are not isomorphic. So we get a free $\mathbf{Z}/(p)$ action on $\mathfrak{M}^{l_1,l_2}_{1,1,0,\cdots,0}$. Because p can be chose arbitrarily large, $e(\mathfrak{M}^{l_1,l_2}_{1,1,0,\cdots,0})=0$. ◻

Since $\mathfrak{M}^{l_1,l_2}_{0,0,\cdots,0}$ is empty, by this proposition, the contribution to the Euler number $e(\mathfrak{M}_C(2,1))$ comes from strata $\mathfrak{M}^{l_1,l_2}_{a_1,a_2,\cdots,a_n}$ with $\sum a_i=1$. Because $\chi(\mathcal{E})=1$ and $\mathcal{E}$ fits into an exact sequence

$$0\longrightarrow\mathcal{E}\longrightarrow\pi_*(\mathcal{O}(l_1)\oplus\mathcal{O}(l_2))\longrightarrow\mathbf{C}_{x_i}\longrightarrow 0,$$

the stability of $\mathcal{E}$ forces $l_1=l_2=0$. Every $\mathfrak{M}^{0,0}_{0,\cdots,1,\cdots,0}$ is a set of a single point. Therefore,

Proposition 4.7. *The Euler number $e(\mathfrak{M}_C(2,1))$ is equal to n, which is the number of nodes on C.*

Let $D=2C_0$ be a divisor in the set $\mathcal{W}^0_3$. Then $\mathfrak{M}^1_D$ is isomorphic to $\mathfrak{M}_{C_0}(2,1)$. The number of nodes on C_0 is equal to the arithmetic genus $g=\frac{1}{2}H^2+1$ of C_0. Therefore,

Proposition 4.8. $e(\mathfrak{M}^1_D)=g$.

5. Calculation of $e(\Phi^{-1}(\mathcal{W}^0_3))$, Part II. This is the second part of the calculation of $e(\Phi^{-1}(\mathcal{W}^0_3))$. As we mentioned in the previous section, $\mathfrak{M}_D$ is a disjoint union of $\mathfrak{M}^1_D$ and $\mathfrak{M}^2_D$ for $D\in\mathcal{W}^0_3$. We have calculated $e(\mathfrak{M}^1_D)$. In this section, we will show that $e(\mathfrak{M}^2_D)=0$.

Let $C_0\subset S$ be a nodal curve, and let C be the associated nonreduced curve to the divisor $2C_0$. Let p be a node on C_0, and $\pi_0:\hat{C}_0\to C_0$ be the partial normalization of C_0 at p. Now we construct a curve $\hat{C}$, which is an infinitesimal extension of $\hat{C}_0$, and a finite morphism $\pi:\hat{C}\to C$ called a partial normalization of C.

We pick a small neighborhood U around p on the surface S, such that C is defined by $x^2y^2=0$ in U. Let $\mathbf{C}\{x,y\}$ be the ring of holomorphic functions on U. Then $\mathcal{O}_C(U\cap C)=\mathbf{C}\{x,y\}/(x^2y^2)$. The injective homomorphism

$$\psi:\mathbf{C}\{x,y\}/(x^2y^2)\to\mathbf{C}\{x,u\}/(u^2)\oplus\mathbf{C}\{v,y\}/(v^2)$$

is a local isomorphism except at p. Remove the point p on C and glue the pieces defined by the ringed space $\mathbf{C}\{x,u\}/(u^2)\oplus\mathbf{C}\{v,y\}/(v^2)$ along ψ, we get a curve $\hat{C}$, and a finite map $\pi:\hat{C}\to C$. There is a canonical exact sequence

$$0\longrightarrow\mathcal{O}_C\longrightarrow\pi_*\mathcal{O}_{\hat{C}}\longrightarrow\mathcal{A}\longrightarrow 0,$$

where $\mathcal{A} \cong \mathbb{C}[x,y]/(x^2,y^2)$ is a skyscraper sheaf supported at p. Moreover, there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & \pi_*\mathcal{O}_{\hat{C}} & \longrightarrow & \mathcal{A} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_{C_0} & \longrightarrow & \pi_*\mathcal{O}_{\hat{C}_0} & \longrightarrow & \mathbf{C}_p & \longrightarrow & 0. \end{array}$$

Let $\mathcal{I}$ and $\hat{\mathcal{I}}$ be the nilpotent ideal sheaves of $\mathcal{O}_C$ and $\mathcal{O}_{\hat{C}}$ respectively. Then $\chi(\hat{\mathcal{I}}) = \chi(\mathcal{I}) + 3$. It implies that $\deg \hat{\mathcal{I}} = \deg \mathcal{I} + 2$. Let C be a rational nodal curve on a $K3$ surface and let $\tilde{C} \to C$ be the normalization of C. Then $\deg \mathcal{I} = 2-2g = H^2$. Because the number of nodes on C is equal to g, $\deg \tilde{\mathcal{I}} = \deg \mathcal{I} + 2g = 2$.

Proposition 5.1. *Let C be a nonreduced curve with nilpotent ideal sheaf $\mathcal{I}$. Suppose $\mathcal{I}$ is invertible as a sheaf of $\mathcal{O}_{C_0}$-modules and $\deg \mathcal{I} > 0$. Let $\mathcal{E}$ be a pure sheaf of $\mathcal{O}_C$-modules such that $\mathcal{E}_\eta \cong \mathcal{O}_\eta$ at the generic point η of C. Then $\mathcal{E}$ is not stable.*

Proof. Let $\mathcal{E}$ be such a sheaf and let $\mathcal{E}_0^\sharp$ be the torsion free part of $\mathcal{E}_0 = \mathcal{E} \otimes \mathcal{O}_{C_0}$, considered as a sheaf of $\mathcal{O}_{C_0}$-modules. There is a canonical homomorphism $\mathcal{E} \to \mathcal{E}_0^\sharp$. Every quotient $\mathcal{E} \to \mathcal{F}$ with $\mathcal{F}$ a torsion free $\mathcal{O}_{C_0}$-module is equivalent to $\mathcal{E} \to \mathcal{E}_0^\sharp$. Therefore, for the stability of $\mathcal{E}$, it is enough to check the quotient $\mathcal{E} \to \mathcal{E}_0^\sharp$.

We start with the exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_{C_0} \longrightarrow 0.$$

Tensoring with $\mathcal{E}$, we obtain

$$\mathcal{E}_0 \otimes \mathcal{I} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_0 \longrightarrow 0.$$

Let $\mathcal{T}'$ be the torsion part of $\mathcal{E}_0 \otimes \mathcal{I}$, and let $(\mathcal{E}_0 \otimes \mathcal{I})^\sharp = (\mathcal{E}_0 \otimes \mathcal{I})/\mathcal{T}'$. There is an exact sequence

$$0 \longrightarrow (\mathcal{E}_0 \otimes \mathcal{I})^\sharp \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_0 \longrightarrow 0.$$

On the other hand, we have

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_0^\sharp \longrightarrow 0$$

Because $\mathcal{I}$ is an invertible sheaf of $\mathcal{O}_{C_0}$-modules, the torsion part of $\mathcal{E}_0$ is isomorphic to $\mathcal{T}'$. $\chi(\mathcal{K}) = \chi(\mathcal{E}) - \chi(\mathcal{E}_0^\sharp) = \chi(\mathcal{E}_0 \otimes \mathcal{I})$. Because $\chi(\mathcal{E}_0 \otimes \mathcal{I}) = \chi(\mathcal{E}_0) + \deg \mathcal{I} > \chi(\mathcal{E}_0) > \chi(\mathcal{E}_0^\sharp)$, $\mathcal{E}$ is not stable. ☐

Let $\pi : \hat{C} \to C$ be the partial normalization of C at p. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_C$-modules which is pure of dimension 1. Then there is a canonical homomorphism $\mathcal{F} \to \pi_*(\pi^*\mathcal{F})$. Let $T_0 \subset \pi^*\mathcal{F}$ be the maximal subsheaf of dimension 0, we get a sheaf $(\pi^*\mathcal{F})^\sharp = \pi^*\mathcal{F}/T_0$ which is pure of dimension 1, and there is a canonical injective homomorphism $\mathcal{F} \to \pi_*(\pi^*\mathcal{F})^\sharp$. The cokernel $\mathcal{T}$ is a skyscraper sheaf supported at p. Note that if $\mathcal{F}$ satisfies $\mathcal{I}\mathcal{F} \neq 0$, so does $(\pi^*\mathcal{F})^\sharp$ as a sheaf of $\mathcal{O}_{\hat{C}}$-modules.

The notion of admissible quotients can be defined in the same way as in section 4, and propositions 4.2-4.4 are also true in this case.

Let $\rho : \pi_*\mathcal{E} \to \mathcal{Q}$ be an admissible quotient, and let p be a node on C with $\pi^{-1}(p) = \{q_1, q_2\}$. Then $(\pi_*\mathcal{E})_p = \mathcal{E}_{q_1} \oplus \mathcal{E}_{q_2}$. We let $\iota_i : \mathcal{E}_{q_i} \to (\pi_*\mathcal{E})_p$ and p_i :

$(\pi_*\mathcal{E})_p \to \mathcal{E}_{q_i}$ be the natural injections and projections respectively. Define ρ^i as compositions

$$\rho^i : \mathcal{E}_{q_i} \xrightarrow{\iota_i} (\pi_*\mathcal{E})_p \xrightarrow{\rho} \mathcal{Q}_p.$$

Because ρ is admissible, ρ^i are both surjective homomorphisms. Clearly $\rho = \rho^1 p_1 + \rho^2 p_2$. For $t \in \mathbf{C}^*$, we define $\rho_t = \rho^1 p_1 + t\rho^2 p_2$. It gives rise to a surjective homomorphism $\rho_t : \pi_*\mathcal{E} \to \mathcal{Q}$ which is also admissible.

Let $\pi : \hat{C} \to C$ be the partial normalization of C at p. Apply the above construction to the canonical exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \pi_*\mathcal{O}_{\hat{C}} \xrightarrow{\rho} \mathcal{A} \longrightarrow 0,$$

and let $\mathcal{L}_t = \ker \rho_t$. Then $\mathcal{L}_t$ is invertible for every $t \in \mathbf{C}^*$. The set of these invertible sheaves form a subgroup $G_p \subset \operatorname{Pic}^0 C$. Clearly $G_p \cong \mathbf{C}^*$.

For any admissible quotient $\rho : \pi_*\mathcal{E} \to \mathcal{Q}$, let $\mathcal{K}_t$ be the kernel of ρ_t.

Lemma 5.2. *There is a canonical isomorphism between $\mathcal{L}_t \otimes \mathcal{K}_s$ and $\mathcal{K}_{st}$.*

Now we give a decomposition on $\mathfrak{M}^2_D$ for $D \in \mathcal{W}^0_3$. Let $\pi : \tilde{C} \to C$ be the normalization of C. Let $\mathfrak{M}_{\tilde{\mathcal{F}},\mathcal{T}}$ be the subset consists of stable sheaves $\mathcal{F}$ such that $(\pi^*\mathcal{F})^\sharp \cong \tilde{\mathcal{F}}$ and $\pi_*\tilde{\mathcal{F}}/\mathcal{F} \cong \mathcal{T}$. We get a decomposition $\mathfrak{M}^2_D = \sqcup \mathfrak{M}_{\tilde{\mathcal{F}},\mathcal{T}}$. In fact, for every nonempty stratum $\mathfrak{M}_{\tilde{\mathcal{F}},\mathcal{T}}$, $\mathcal{T}$ is nonzero. Because if $\mathfrak{M}_{\tilde{\mathcal{F}},0}$ is nonempty, a sheaf $\mathcal{F}$ in $\mathfrak{M}_{\tilde{\mathcal{F}},0}$ is the direct image of a sheaf on $\tilde{C}$, i.e. $\mathcal{F} \cong \tilde{\pi}_*\mathcal{E}$ for a sheaf $\mathcal{E}$ of $\mathcal{O}_{\tilde{C}}$-modules. By proposition 5.1, $\mathcal{E}$ is not stable, which violates the stability of $\mathcal{F}$.

Let $\mathfrak{M}_{\tilde{\mathcal{F}},\mathcal{T}}$ be a stratum, and let $p \in C$ be a node such that $\mathcal{T}_p \neq 0$. There is a subgroup $G_p \subset \operatorname{Pic}^0 C$ defined as above, and a G-action on $\mathfrak{M}_{\tilde{\mathcal{F}},\mathcal{T}}$ defined by tensorization. Next we will show that this group action is free on $\mathfrak{M}_{\tilde{\mathcal{F}},\mathcal{T}}$. The following lemma is useful in the proof.

Let $\mathcal{E}$ be a pure sheaf of $\mathcal{O}_C$-modules such that $\mathcal{E}_\eta \cong \mathcal{O}_\eta$ at the generic point η of C. Let $\mathcal{E}''$ be the torsion free part of $\mathcal{E}_0 = \mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{O}_{C_0}$, and let $\mathcal{E}'$ be the kernel of the restriction homomorphism $f : \mathcal{E} \to \mathcal{E}''$. Since f is not an isomorphism, $\mathcal{E}'$ and $\mathcal{E}''$ are both rank 1 torsion free sheaves of $\mathcal{O}_{C_0}$-modules whose automorphism groups are $\mathbf{C}^*$.

Lemma 5.3. *Let $\psi : \mathcal{E} \to \mathcal{E}$ be an automorphism and let $c : \mathcal{E}' \to \mathcal{E}'$ and $d : \mathcal{E}'' \to \mathcal{E}''$ be the induced automorphisms. Then they fit into the commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}'' & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle c} & & \Big\downarrow{\scriptstyle \psi} & & \Big\downarrow{\scriptstyle d} & & \\
0 & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}'' & \longrightarrow & 0
\end{array}$$

and $c = d$.

Proof. Consider $\psi' = \psi - c \cdot id : \mathcal{E} \to \mathcal{E}$. Clearly $\psi'(\mathcal{E}') = 0$. It induces a homomorphism $u : \mathcal{E}'' \to \mathcal{E}$. Composed with $\mathcal{E} \to \mathcal{E}''$, we get $h : \mathcal{E}'' \to \mathcal{E}''$. Since $\mathcal{E}''$ is torsion free and has rank 1 as an $\mathcal{O}_{C_0}$-module, h is a multiplication by $(d - c)$. If $h \neq 0$, then after scaling u by $\frac{1}{h}$, u splits the exact sequence and hence $\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{E}''$, which contradicts to $\mathcal{E}_\eta \cong \mathcal{O}_\eta$. Therefore $h = 0$, i.e. $c = d$. □

PROPOSITION 5.4. *Let $\mathfrak{M}_{\tilde{\mathcal{F}},\mathcal{T}}$ be a stratum such that $\mathcal{T}_p \neq 0$ for a node $p \in C$. Then the associated G_p-action on $\mathfrak{M}_{\tilde{\mathcal{F}},\mathcal{T}}$ is free. Therefore, by the decomposition of $\mathfrak{M}^2_D$, $e(\mathfrak{M}^2_D) = 0$.*

Proof. Let $\pi : \hat{C} \to C$ be the partial normalization of C at p. Let $\mathcal{F}$ be a stable sheaf in $\mathfrak{M}_{\tilde{\mathcal{F}},\mathcal{T}}$. Then $\mathcal{F}$ fits into the exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \pi_*(\pi^*\mathcal{F})^\sharp \longrightarrow \mathcal{T} \longrightarrow 0.$$

Let $\mathcal{E} = (\pi^*\mathcal{F})^\sharp$. Then $\rho : \pi_*\mathcal{E} \longrightarrow \mathcal{T}$ is clearly an admissible quotient. We let $\mathcal{K}_t$ be the kernel of ρ_t. Then $\mathcal{F} = \mathcal{K}_1$ and $\mathcal{F} \otimes \mathcal{L}_t = \mathcal{K}_t$. Suppose $\mathcal{F} \otimes \mathcal{L}_t \cong \mathcal{F}$ for some $\mathcal{L}_t \in G_p$, there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}_1 & \longrightarrow & \pi_*\mathcal{E} & \xrightarrow{\rho_1} & \mathcal{T} & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle\cong} & & \big\downarrow{\scriptstyle\psi} & & \big\downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & \mathcal{K}_t & \longrightarrow & \pi_*\mathcal{E} & \xrightarrow{\rho_t} & \mathcal{T} & \longrightarrow & 0.
\end{array}$$

It induces the following diagram on the stalks at the node p,

$$\begin{array}{ccccc}
\mathcal{E}_{q_1} \oplus \mathcal{E}_{q_2} & \xrightarrow{\rho_1} & \mathcal{T}_p & \longrightarrow & 0 \\
\big\downarrow{\scriptstyle\psi} & & \big\downarrow{\scriptstyle h} & & \\
\mathcal{E}_{q_1} \oplus \mathcal{E}_{q_2} & \xrightarrow{\rho_t} & \mathcal{T}_p & \longrightarrow & 0
\end{array}$$

where $\{q_1, q_2\} = \pi^{-1}(p)$.

Recall that there is a canonical exact sequence for $\mathcal{E}$,

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0,$$

where $\mathcal{E}'$ and $\mathcal{E}''$ are nonzero torsion free sheaves of $\mathcal{O}_{C_0}$-modules. Let $\mathcal{T}^i \subset \mathcal{T}_p$ be the images of $\mathcal{E}'_{q_i}$ under the surjective homomorphisms $\rho^i : \mathcal{E}_{q_i} \to \mathcal{T}_p$. Then we have the following three cases.

Case 1. $\mathcal{T}^1 \neq 0$ and $\mathcal{T}^2 \neq 0$.

Since $\psi : \pi_*\mathcal{E} \to \pi_*\mathcal{E}$ is induced from an automorphism of $\mathcal{E}$, $\psi(\mathcal{E}_{q_i}) = \mathcal{E}_{q_i}$. Consider the restriction of the diagram to $\mathcal{E}'_{q_i}$ respectively, by Lemma 5.3, there are commutative diagrams

$$\begin{array}{ccc}
\mathcal{E}'_{q_1} & \xrightarrow{\rho^1} & \mathcal{T}^1 \\
\big\downarrow{\scriptstyle c} & & \big\downarrow{\scriptstyle h_1} \\
\mathcal{E}'_{q_1} & \xrightarrow{\rho^1} & \mathcal{T}^1,
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{E}'_{q_2} & \xrightarrow{\rho^2} & \mathcal{T}^2 \\
\big\downarrow{\scriptstyle c} & & \big\downarrow{\scriptstyle h_2} \\
\mathcal{E}'_{q_2} & \xrightarrow{t\rho^2} & \mathcal{T}^2.
\end{array}$$

If $\mathcal{T}^1 \cap \mathcal{T}^2 \neq \{0\}$, let $0 \neq x \in \mathcal{T}^1 \cap \mathcal{T}^2$. Then from the left diagram, $h(x) = h_1(x) = cx$, and from the right diagram, $h(x) = h_2(x) = ctx$. It implies that $t = 1$ and therefore the group action is free. Next we assume $\mathcal{T}^1 \cap \mathcal{T}^2 = \{0\}$. Let $x \in \mathcal{T}^1$ be a nonzero element. Then the image $\bar{x}$ of x in $\mathcal{T}_p/\mathcal{T}^2$ is nonzero. From the commutative diagram

$$\begin{array}{ccc}
\mathcal{E}''_{q_2} & \xrightarrow{\rho^2} & \mathcal{T}_p/\mathcal{T}^2 \\
\big\downarrow{\scriptstyle c} & & \big\downarrow{\scriptstyle h_3} \\
\mathcal{E}''_{q_2} & \xrightarrow{t\rho^2} & \mathcal{T}_p/\mathcal{T}^2
\end{array}$$

we have $h_3(\bar{x}) = ct\bar{x}$. Because $h(x) = h_1(x) = cx$, $h_3(\bar{x}) = c\bar{x}$. It implies that $t = 1$.

Case 2. $\mathcal{T}^1 \neq 0$ and $\mathcal{T}^2 = 0$(Or equivalently $\mathcal{T}^1 = 0$ and $\mathcal{T}^2 \neq 0$).

Since $\mathcal{T}^2 = 0$ and $\mathcal{E}_{q_2} \xrightarrow{\rho^2} \mathcal{T}_p$ is surjective, there is a surjective morphism $\rho^2 : \mathcal{E}''_{q_2} \to \mathcal{T}_p$ and a commutative diagram

$$\begin{array}{ccc} \mathcal{E}''_{q_2} & \xrightarrow{\rho^2} & \mathcal{T}_p \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle h} \\ \mathcal{E}''_{q_2} & \xrightarrow{t\rho^2} & \mathcal{T}_p. \end{array}$$

Let $0 \neq x \in \mathcal{T}^1$. We already have $h(x) = h_1(x) = cx$. However, from the above diagram, $h(x) = ctx$. Hence $t = 1$.

Case 3. $\mathcal{T}^1 = 0$ and $\mathcal{T}^2 = 0$.

Since $\mathcal{T}^1 = 0$ and $\mathcal{T}^2 = 0$, we get commutative diagrams

$$\begin{array}{ccc} \mathcal{E}''_{q_1} & \xrightarrow{\rho^1} & \mathcal{T}_p \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle h_1} \\ \mathcal{E}''_{q_1} & \xrightarrow{\rho^1} & \mathcal{T}_p, \end{array} \qquad \begin{array}{ccc} \mathcal{E}''_{q_2} & \xrightarrow{\rho^2} & \mathcal{T}_p \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle h_2} \\ \mathcal{E}''_{q_2} & \xrightarrow{t\rho^2} & \mathcal{T}_p. \end{array}$$

Apply the same argument as in case 1, we get $t = 1$. ◻

Because the cardinality of the finite set $\mathcal{W}_3^0$ is G_g, combine proposition 4.8 and 5.4, we conclude

Proposition 5.5. $e(\Phi^{-1}(\mathcal{W}_3^0)) = gG_g$.

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