

TOPOLOGY OF SURFACES WITH CONNECTED SHADES*

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Abstract. We prove that any closed orientable surface may be smoothly embedded in Euclidean 3-space so that when it is illuminated by parallel rays from any direction the shade cast on the surface is connected.

Key words. Shadow, shade, vision number, skew loop, tantrix, constant mean curvature, gauss map, two-piece-property, tight immersion

AMS subject classifications. Primary 53A05, 53A04; Secondary 52A15, 53C42

1. Introduction. Let M be an oriented surface smoothly immersed in Euclidean space \mathbf{R}^3 , and $\nu: M \rightarrow \mathbf{S}^2$ be the unit normal vectorfield, or the Gauss map of M . Then for any unit vector $u \in \mathbf{S}^2$ (corresponding to the direction of light) the *shade*, or *shadow*¹, cast on M is defined as

$$S_u := \{ p \in M \mid \langle \nu(p), u \rangle > 0 \},$$

where $\langle \cdot, \cdot \rangle$ is the standard innerproduct in \mathbf{R}^3 . A basic question, first considered by H. Wente in 1978 [19], is: *Does connectedness of each of the shades S_u of a closed orientable surface M imply that M is convex?* In [5] the author showed that the answer is yes provided that either M is topologically a sphere, or each of its shades is *simply* connected. Otherwise, it was proved that the answer is no by constructing *smooth* (C^∞) embedded tori with connected shades. In this paper we extend that construction to all orientable closed surfaces:

THEOREM 1.1. *Every orientable closed surface admits a smooth embedding in \mathbf{R}^3 with connected shades in all directions.*

Thus, surprisingly, topological complexity has in general no bearing on the number of components of a shade of a surface. In particular, the above theorem gives counterexamples of every genus to a conjecture of J. Choe [3, p. 210] which states that any immersion of a surface of topological genus g in \mathbf{R}^3 should have at least one shade with $g + 1$ components.

Note that M has connected shades if and only if for every great circle $C \subset \mathbf{S}^2$, $\nu^{-1}(\mathbf{S}^2 - C)$ has exactly two components. That is, the Gauss map of M satisfies a *two-piece-property* [2], or *tightness*, in the sense of Banchoff [1] or Kuiper [14]. The number of components of $\nu^{-1}(\mathbf{S}^2 - C)$ has been called the *vision number* [3] with respect to a direction perpendicular to C , and is of interest in variational problems in geometric analysis, particularly the study of the stability of surfaces of constant mean curvature (soap films) [3, 4]. The study of shades is also of substantial interest in computer vision [12, 13], where “shape from shading” problems are studied extensively.

The proof of Theorem 1.1, which is presented in Section 4, follows from a pair of preliminary results, Propositions 2.1 and 3.1, proved in the next two sections. The first proposition is concerned with the existence of closed curves without any

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¹While the author used the term “shadow” in [5], it is preferable to use “shade” as it is more suggestive, and it is consistent with the literature on computer vision.

pairs of parallel tangent lines, i.e., *skew loops*, and is an extension of a construction first discovered by B. Segre [16], see [5, Note 6.6]. It was shown in [5] that a tubular surface about a skew loop has connected shades. Here we show that one may construct a skew loop so that the corresponding tubular surface has any desired number of pairs of points which *face away* from each other, as defined in Section 3. We will then prove our second proposition which states that if a surface with connected shades has a pair of points p, q which face away from each other, then one may add a ‘handle’ to that surface and thus increase its topological genus while preserving the connectedness of each of its shades; see Figure 1.

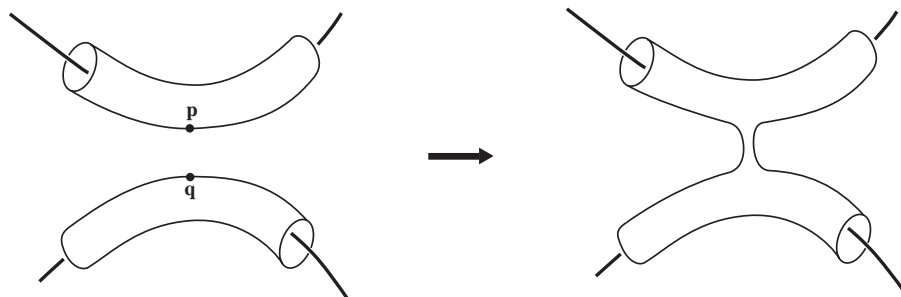


FIG. 1.

More precisely, we will delete small neighborhoods of p and q which are homeomorphic to disks and glue in their place a topological annulus. To this end we first deform neighborhoods of p and q until they coincide with pieces of spheres of the same radius, and then cut small disks from these spherical pieces. It will be shown that the resulting surface still has connected shades. Next we join the two boundary components of this surface with a surface of revolution which we call an *hour glass*. The hour glass has the crucial property that each component of each of its shades intersects its boundary. This implies that our final surface will have connected shades.

2. Skew Loops with Prescribed Points, Tangents, and Principal Normals. By a *curve* in this paper we mean a continuous mapping $\gamma: \mathbf{R} \rightarrow \mathbf{R}^3$. We say γ is *closed*, or is a *loop*, if it is periodic, and write $\gamma: \mathbf{R}/\ell \rightarrow \mathbf{R}^3$, where ℓ denotes the period. Further, γ is *simple* if $\gamma(t) \neq \gamma(s)$, whenever $0 < |t - s| < \ell$. A *regular curve* is a C^1 mapping $\gamma: \mathbf{R} \rightarrow \mathbf{R}^3$ with nonvanishing speed, i.e., $\|\gamma'\| \neq 0$. The tangential indicatrix or *tantrix* of a regular curve γ is given by $T := \gamma'/\|\gamma'\|$. If $T(t) \neq \pm T(s)$, whenever $t \not\equiv s \pmod{\ell}$, we say that γ is *skew*. When γ is C^2 and regular, its *curvature* is defined as $\kappa := \|T'\|/\|\gamma'\|$, and, if κ does not vanish, the *principal normal* of γ is given by $N := T'/\kappa$. By *smooth* in this paper we always mean C^∞ . The main aim of this section is to show:

PROPOSITION 2.1. *Let $(p_i, T_i, N_i) \in \mathbf{R}^3 \times \mathbf{S}^2 \times \mathbf{S}^2$, $i = 1, \dots, n$. Suppose that $T_i \neq \pm T_j$ whenever $i \neq j$, and $\langle N_i, T_i \rangle = 0$. Then there exists a smooth skew loop $\gamma: \mathbf{R}/n \rightarrow \mathbf{R}^3$ with nonvanishing curvature such that*

$$\gamma(i) = p_i, \quad T(i) = T_i, \quad \text{and} \quad N(i) = N_i,$$

where T and N are the *tantrix* and the *principal normal* of γ respectively. Furthermore, if $p_i \neq p_j$ whenever $i \neq j$, then we may require that γ be simple.

As we mentioned earlier, the above proposition may be regarded as a generalization of an earlier work of B. Segre [16] who first proved the existence of skew loops (without prescribing specific points or frames). See [11] for some historical notes on skew loops, and see [5] for an explicit example. More recent developments on the theory of skew submanifolds may be found in [9, 8, 10, 18, 17]. To prove the above proposition we need the following lemmas:

LEMMA 2.2. *Let $(T_i, N_i) \in \mathbf{S}^2 \times \mathbf{S}^2, i = 1, \dots, n$. Suppose that $T_i \neq \pm T_j$ whenever $i \neq j$, and $\langle N_i, T_i \rangle = 0$. Then there exists a smooth simple loop $T: \mathbf{R}/n \rightarrow \mathbf{S}^2$ such that*

$$T(i) = T_i, \quad T'(i) \parallel N_i, \quad \text{and} \quad T(\mathbf{R}) \cap -T(\mathbf{R}) = \emptyset.$$

Proof. It suffices to show that there exists a simply connected open subset $U \subset \mathbf{S}^2$ such that $T_i \in U$ and $U \cap -U = \emptyset$; for one may then easily construct, by induction, a simple closed regular curve $T: \mathbf{R}/n \rightarrow U$ with $T(i) = T_i$ and $T'(i) \parallel N_i$. Any such curve would automatically satisfy $T(\mathbf{R}) \cap -T(\mathbf{R}) = \emptyset$, and thus would be the desired object.

To construct U we first note that, when $i \neq j$, every pair of points T_i, T_j determine a unique great circle C_{ij} passing through them, since $T_i \neq \pm T_j$ when $i \neq j$. Let $C := \{C_{ij}\}$ be the collection of all such circles, and pick a point $o \in \mathbf{S}^2 - C$. Then $o \neq -T_i$. So every T_i may be joined to o by a unique distance minimizing geodesic segment Γ_i . Further, since $o \notin C$, no pairs of geodesic segments Γ_i and Γ_j can be parts of the same great circle when $i \neq j$. This implies that $-\Gamma_i \cap \Gamma_j = \emptyset$ for all $i \neq j$. Further, since by assumption Γ_i is distance minimizing, i.e., its length is less than half of a great circle; we have $\Gamma_i \cap -\Gamma_i = \emptyset$. So if $\Gamma := \cup_i \Gamma_i$, then $\Gamma \cap -\Gamma = \emptyset$. In particular $\delta := \text{dist}_{\mathbf{S}^2}(\Gamma, -\Gamma) > 0$. Now let $0 < \epsilon < \delta/2$, and, for any $p \in \mathbf{S}^2$, let $U_\epsilon(p)$ denote the set of points in \mathbf{S}^2 whose (spherical) distance from p is less than ϵ . Then $U := \cup_{p \in \Gamma} U_\epsilon(p)$ has the required properties. \square

For any $X \subset \mathbf{R}^3$, the *cone* of X is defined as

$$\text{cone } X := \{ \lambda x \mid x \in X, \text{ and } \lambda \geq 0 \},$$

and the *convex hull* of X , which we denote by $\text{conv } X$, is the intersection of all convex sets containing X . By the *convex cone* of X we mean the convex hull of the cone of X .

LEMMA 2.3. *Let $T: [a, b] \rightarrow \mathbf{S}^2$ be any continuous map. Then for any point p in the interior of the convex cone of $T([a, b])$, there exists a smooth positive function $\varphi: [a, b] \rightarrow \mathbf{R}$ such that*

$$\int_a^b \varphi(t)T(t) dt = p.$$

Further, there exists $\bar{\delta} > 0$ such that for any $0 < \delta \leq \bar{\delta}$ we may set φ equal to δ on an open neighborhood of the end points of $[a, b]$.

Proof. By a theorem of Steinitz [15, p. 15], any interior point of the convex hull of $X \subset \mathbf{R}^3$ lies in the interior of the convex hull of 6 (or fewer) points of X . So, since $p \in \text{int conv cone } T([a, b]) = \text{int conv cone } T((a, b))$, there exist $t_i \in (a, b)$, and $c_i > 0, i = 1, \dots, k$, where $4 \leq k \leq 6$, such that

$$(2.1) \quad p \in \text{int conv } \{c_1 T(t_1), \dots, c_k T(t_k)\}.$$

Let $\varphi_i^\epsilon: [a, b] \rightarrow \mathbf{R}$ be a smooth nonnegative function with $\int_a^b \varphi_i^\epsilon(t) dt = 1$, and support in an ϵ -neighborhood of t_i . Note that since $t_i \in (a, b)$, we may assume that

$$(2.2) \quad 0 < \epsilon < \min_i \{|t_i - a|, |t_i - b|\},$$

which implies that φ_i^ϵ vanishes near a and b . Let

$$T_i^\epsilon := \int_a^b (\varphi_i^\epsilon(t) + \epsilon) c_i T(t) dt.$$

Then $\lim_{\epsilon \rightarrow 0} T_i^\epsilon = c_i T(t_i)$. In particular, by (2.1), we may choose ϵ small enough so that

$$(2.3) \quad p \in \text{int conv} \{T_1^\epsilon, \dots, T_k^\epsilon\}.$$

Then, by a standard result from classical convexity theory [15, Thm. 1.1.13], there are constants $\lambda_i^\epsilon > 0$, with $\sum_{i=1}^k \lambda_i^\epsilon = 1$, such that

$$p = \sum_{i=1}^k \lambda_i^\epsilon T_i^\epsilon.$$

Now set

$$\varphi^\epsilon := \sum_{i=1}^k \lambda_i^\epsilon (\varphi_i^\epsilon + \epsilon) c_i.$$

Then $\varphi^\epsilon > 0$, because $c_i, \lambda_i^\epsilon, \epsilon > 0$. Further

$$(2.4) \quad \int_a^b \varphi^\epsilon(t) T(t) dt = p.$$

Also note that, whenever (2.2) is satisfied, φ^ϵ is equal to

$$\bar{\delta}(\epsilon) := \epsilon \sum_{i=1}^k \lambda_i^\epsilon c_i$$

near a and b , because φ_i^ϵ vanishes near a and b .

Let ϵ be small enough so that (2.2) and (2.3) are satisfied, and set

$$\bar{T}_i^\epsilon := \int_a^b \varphi_i^\epsilon(t) c_i T(t) dt.$$

Then $\lim_{\epsilon \rightarrow 0} \bar{T}_i^\epsilon = c_i T(t_i)$, and just as we had argued above, we may choose ϵ small enough so that, in addition to (2.2) and (2.3), the following condition is also satisfied:

$$(2.5) \quad p \in \text{int conv} \{\bar{T}_1^\epsilon, \dots, \bar{T}_k^\epsilon\}.$$

Then, again by [15, Thm. 1.1.13], there are constants $\bar{\lambda}_i^\epsilon > 0$, with $\sum_{i=1}^k \bar{\lambda}_i^\epsilon = 1$, such that

$$p = \sum_{i=1}^k \bar{\lambda}_i^\epsilon \bar{T}_i^\epsilon.$$

Now if we set

$$\overline{\varphi}^\epsilon := \sum_{i=1}^k \overline{\lambda}_i^\epsilon \varphi_i^\epsilon c_i,$$

then,

$$(2.6) \quad \int_a^b \overline{\varphi}^\epsilon(t) T(t) dt = p.$$

Finally, for fixed ϵ satisfying (2.2), (2.3), (2.5) and any $0 < \delta \leq \overline{\delta}(\epsilon)$, set

$$\varphi := \frac{\delta}{\overline{\delta}(\epsilon)} \varphi^\epsilon + \left(1 - \frac{\delta}{\overline{\delta}(\epsilon)}\right) \overline{\varphi}^\epsilon.$$

Then $\int_a^b \varphi(t) T(t) dt = p$ by (2.4) and (2.6). Also recall that $\varphi_\epsilon > 0$; thus $\varphi > 0$. Finally note that $\varphi = \delta$ near a and b , because in that region $\overline{\varphi}_i^\epsilon$ and consequently $\overline{\varphi}^\epsilon$ vanish by (2.2) and $\varphi^\epsilon = \overline{\delta}(\epsilon)$. \square

Finally we need to recall the following basic fact, which will be used to prove the last assertion in the statement of Proposition 2.1.

LEMMA 2.4 ([11]). *C^2 skew loops with nonvanishing curvature form an open subset in the space of all C^2 closed curves $\gamma: \mathbf{R}/\ell \rightarrow \mathbf{R}^3$, with respect to the C^2 topology.* \square

Now we are ready to prove the main result of this section:

Proof. [Proof of Proposition 2.1] Let $v_j \in \mathbf{S}^2$, $j = 1, \dots, 4$, be four points which contain the origin of \mathbf{R}^3 in the interior of their convex hull,

$$o \in \text{int conv}\{v_1, v_2, v_3, v_4\}.$$

In particular, $v_i \neq \pm v_j$, whenever $i \neq j$. Further suppose that $v_j \neq \pm T_i$. Then there exist open neighborhoods U_j of v_j in \mathbf{S}^2 such that $U_i \cap \pm U_j = \emptyset$ whenever $i \neq j$, $\pm T_i \notin U_j$ for all i and j , and if $y_j \in U_j$ then the interior of the convex hull of $\{y_1, \dots, y_4\}$ contains the origin. Now let $v_{ij} \in U_j$, $i = 1, \dots, n$, be a sequence of distinct points, and $\overline{T}_k \in \mathbf{S}^2$, $k = 1, \dots, 5n$ be the sequence whose elements are

$$T_1, v_{11}, v_{12}, v_{13}, v_{14}, T_2, v_{21}, v_{22}, v_{23}, v_{24}, \dots, T_n, v_{n1}, v_{n2}, v_{n3}, v_{n4}.$$

Then $\overline{T}_i \neq \pm \overline{T}_j$, whenever $i \neq j$. Let $\overline{N}_k \in T_{\overline{T}_k} \mathbf{S}^2$, $k = 1, \dots, 5n$, be any sequence with $\overline{N}_{5i-4} := N_i$, $i = 1, \dots, n$. Applying Lemma 2.2 to \overline{T}_k and \overline{N}_k , we obtain, after a reparameterization, a smooth simple closed curve $T: \mathbf{R}/n \rightarrow \mathbf{S}^2$ such that, for $i = 1, \dots, n$,

$$T(i) = T_i, \quad T'(i) \parallel N_i \quad \text{and} \quad \{v_{i1}, v_{i2}, v_{i3}, v_{i4}\} \subset T([i, i + 1]).$$

The last condition implies that the origin lies in the interior of the convex hull of $T([i, i + 1])$. Thus $\text{conv cone } T([i, i + 1]) = \mathbf{R}^3$, and, in particular,

$$(p_{i+1} - p_i) \in \text{int conv cone } T([i, i + 1]).$$

So it follows from Lemma 2.3 that for each i there exists a smooth positive function $\varphi_i: [i, i+1] \rightarrow \mathbf{R}$ such that

$$\int_i^{i+1} \varphi_i(t)T(t) dt = p_{i+1} - p_i,$$

where we set $p_{n+1} := p_1$. Further, on an open neighborhood of the end points of $[i, i+1]$ we may set φ_i equal to any positive constant smaller than or equal to some positive constant $\bar{\delta}_i$. In particular we may assume that φ_i is equal to $\bar{\delta} := \min_i \delta_i$ near i and $i+1$. Then, $\varphi: \mathbf{R}/n \rightarrow \mathbf{R}$ defined by

$$\varphi|_{[i, i+1]} := \varphi_i$$

is a smooth function. Now define $\gamma: \mathbf{R}/n \rightarrow \mathbf{R}$ by

$$\gamma(t) := p_1 + \int_1^t \varphi(s)T(s) ds.$$

Then $\gamma(i) = p_i$, and $\gamma'(i) \parallel T_i$, $i = 1, \dots, n$. Further $\gamma'/\|\gamma'\| = T$. So T is the tantrix of γ . Thus, since by Lemma 2.2 T is regular, i.e., $\|T'\| \neq 0$, it follows that the curvature κ of γ does not vanish. Furthermore, by Lemma 2.2, T is one-to-one on \mathbf{R}/n and $T(\mathbf{R}) \cap -T(\mathbf{R}) = \emptyset$. Thus T is skew. Finally recall that $N := T'/\|T'\|$. So, since $T'(i) \parallel N_i$, it follows that $N(i) = N_i$.

To prove the last assertion of the proposition, it only remains to note that, using the Wierstrauss approximation theorem followed by gluings, we may construct a smooth closed curve $\tilde{\gamma}: \mathbf{R}/n \rightarrow \mathbf{R}^3$ which coincides with γ in a neighborhood of each integer and is \mathcal{C}^2 -close to γ . In particular, if $p_i \neq p_j$, for all $i \neq j$, then we may assume that $\tilde{\gamma}$ is simple (since embeddings are dense in the space of immersions of an n -manifold into a $2n+1$ -manifold). Further, Lemma 2.4 ensures that $\tilde{\gamma}$ is skew. \square

3. Adding Handles to Surfaces with Connected Shades. Let $M \subset \mathbf{R}^3$ be a \mathcal{C}^2 immersed surface and $p, q \in M$. We say that p and q *face away* from each other provided that (i) the line segment pq is orthogonal to M at p and q , (ii) the Gaussian curvature of M is positive at p and q , and (iii) if H is the mean curvature vector of M , then $H(p) \parallel p - q$ and $H(q) \parallel q - p$; see Figure 2. The main result of this section is:

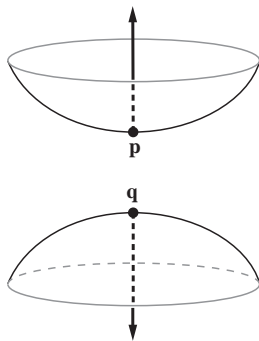


FIG. 2.

PROPOSITION 3.1. *Let $M \subset \mathbf{R}^3$ be a smooth closed embedded surface with connected shades. Suppose M has a pair of points p, q which face away from each other,*

and M is disjoint from the interior of the segment pq . Then there exists a smooth closed embedded surface $\widetilde{M} \subset \mathbf{R}^3$ with connected shades, and

$$g(\widetilde{M}) = g(M) + 1.$$

Further, for any open neighborhood U of $\{p, q\}$ in M and $\epsilon > 0$, we may require that $\widetilde{M} \supset (M - U)$, and $\widetilde{M} - M$ lie within an ϵ neighborhood of the line segment pq .

The proof of the above result rests on the following lemmas, the first of which is concerned with a simple topological fact:

LEMMA 3.2. *Let M be a closed surface which is obtained by gluing a pair of surfaces M_1 and M_2 along their boundary. Suppose there exists an open set $U \subset M$ such that $U \cap M_1$ is connected and each component of $U \cap M_2$ intersects ∂M_2 . Then U is connected.*

Proof. If $U \cap M_2 = \emptyset$ we are done. Otherwise, let V be a nonempty component of $U \cap M_2$. Then, by assumption, V intersects ∂M_2 . Let W be the component of U which contains V . Then $W \cap \partial M_2 \neq \emptyset$. So, since W is open in M , it must contain a point of M_1 . In particular, W intersects $U \cap M_1$. Therefore, since $U \cap M_1$ is connected, W must contain $U \cap M_1$. Thus every component of $U \cap M_2$ lies in the (unique) component of U which contains $U \cap M_1$. So U is connected. \square

The next four lemmas are concerned with local deformations of a surface and their effects on the connectedness of shades of that surface. We say that a C^2 hypersurface M^n immersed in \mathbf{R}^{n+1} is *strictly convex*, if (i) M has positive Gauss curvature, (ii) through every point p of M there passes a plane Π such that $\Pi \cap M = \{p\}$, and (iii) M lies on one side of Π . Note that, in our terminology, a strictly convex surface may be disconnected.

LEMMA 3.3 ([7]). *Every smooth compact strictly convex hypersurface $M \subset \mathbf{R}^{n+1}$ may be extended to a smooth closed embedded hypersurface O of positive curvature. Further, we may require that O lie within an arbitrary small distance of the convex hull of M . \square*

The above lemma yields:

LEMMA 3.4. *Let $M \subset \mathbf{R}^3$ be a smooth embedded surface, and $p \in M$ be an interior point with positive Gauss curvature. Then for any open neighborhood U of p in M which has positive Gauss curvature there exists a smooth embedded surface M' homeomorphic to M , and arbitrarily C^1 -close to M , such that $M' \supset M - U$, $M' - (M - U)$ has positive Gauss curvature, $p \in M'$, $T_p M' = T_p M$, and an open neighborhood of p in M' lies on a sphere of radius r , for any given $r > 0$.*

Proof. First note that, without loss of generality, we may replace U by any open neighborhood of p contained in U . In particular, we may assume that \overline{U} is homeomorphic to a disk, is strictly convex, and is a graph over $T_p M$. Further, for definiteness we may assume, after a rigid motion, that $T_p M$ coincides with the xy plane and \overline{U} lies above xy -plane.

Now let $A \subset \overline{U}$ be an open neighborhood of ∂U , such that $p \notin \overline{A}$, and let S be a sphere of radius of r which is tangent to M at p and lies above the xy -plane; see Figure 3. Note that, since \overline{U} is strictly convex, every tangent plane of \overline{A} is disjoint from p . Further, $T_p S = T_p U$ is disjoint from \overline{A} . So, since \overline{A} is compact, it follows that there exists an open neighborhood V of p in S such that $\overline{V} \cup \overline{A}$ is strictly convex. Consequently, by Lemma 3.3, there exists a smooth closed surface of positive Gauss

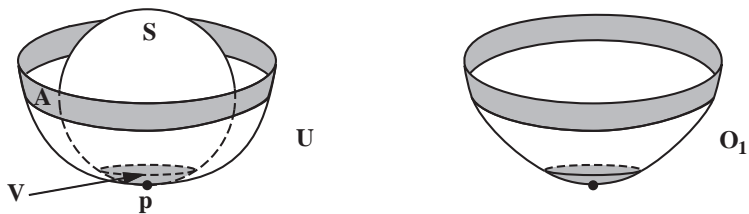


FIG. 3.

curvature $O \supset \overline{A} \cup \overline{V}$. By Jordan’s curve theorem, $O - \partial U$ consists of precisely two components. Let O_1 be the component of $O - \partial U$ which contains p , and M' be the closed surface obtained from gluing $M - U$ to O_1 along their common boundary.

Note that $A - \partial U \subset O_1$. Thus M' is smooth. Further, by Schoenflies theorem, O_1 is homeomorphic to a disk. Thus, since U is also homeomorphic to a disk, and $M' - O_1 = M - U$, it follows that M' is homeomorphic to M . Next note that, $V \subset O_1 \subset M'$. So $T_p M = T_p M'$. Finally note that, since $T_p M = T_p M'$ and O_1 is convex, we may choose U small enough so that M' is as close to M as desired in the sense of C^1 -topology. Thus, since M is compact and embedded, we can make sure that M' is embedded as well. \square

The proof of the next lemma employs the following simple but useful observation, which will be invoked a number of times in the remainder of this paper. For every $u \in \mathbf{S}^2$ let

$$H_u := \{ p \in \mathbf{S}^2 \mid \langle p, u \rangle > 0 \}$$

be the (open) hemisphere of \mathbf{S}^2 centered at u . Then if M is any oriented surface immersed in \mathbf{R}^3 and $\nu: M \rightarrow \mathbf{S}^2$ is its Gauss map, we have

$$(3.1) \quad S_u = \nu^{-1}(H_u)$$

for all $u \in \mathbf{S}^2$. We say that a set $X \subset \mathbf{S}^2$ is *geodesically convex* if it lies in an open hemisphere and every pairs of points of X may be joined by a geodesic segment which lies in X .

LEMMA 3.5. *Let $M \subset \mathbf{R}^3$ be a smooth oriented closed surface with connected shades, and $U \subset M$ be an open set such that $\nu|_{\overline{U}}$ is a diffeomorphism into a geodesically convex set. Then $M - U$ has connected shades.*

Proof. Let \tilde{S}_u be a shade of $\tilde{M} := M - U$. Then $\tilde{S}_u := S_u - U$. In particular, \tilde{S}_u is connected if $U \cap S_u = \emptyset$. So we may assume that $U \cap S_u \neq \emptyset$; further, we may assume $S_u \not\subset U$, for otherwise \tilde{S}_u would be empty. These two assumptions yield that $S_u \cap \partial U \neq \emptyset$. Since S_u is connected, this implies that every component of \tilde{S}_u must intersect ∂U . So, to show that \tilde{S}_u is connected, it suffices to check that $\tilde{S}_u \cap \partial U$ is connected. But $\tilde{S}_u \cap \partial U = S_u \cap \partial U$, and $S_u \cap \partial U = \nu^{-1}(H_u \cap \nu(\partial U))$. Thus, since by assumption $\nu^{-1}: \nu(\overline{U}) \rightarrow U$ is continuous, we just need to check that $H_u \cap \nu(\partial U)$ is connected. This follows from the assumption that $\nu(\overline{U})$ is geodesically convex. To see this assume that $\nu(\overline{U})$ lies in $H_{(0,0,1)}$, and project radially from the center of \mathbf{S}^2 to obtain a homeomorphism $\pi: H_{(0,0,1)} \rightarrow T_{(0,0,1)}\mathbf{S}^2 \simeq \mathbf{R}^2$. Then $\pi(\nu(\overline{U}))$ is a convex planar set, and $\pi(H_{(0,0,1)} \cap H_u)$ is an open half-space. Thus, by basic convexity theory, $\pi(\nu(\partial U) \cap H_u) = \pi(\nu(\partial U)) \cap \pi(H_{(0,0,1)} \cap H_u)$ is connected, which yields that $\nu(\partial U) \cap H_u$ is connected. \square

LEMMA 3.6. *Let $M \subset \mathbf{R}^3$ be a smooth oriented closed surface with connected shades, and $U \subset M$ be an open set such that \overline{U} has positive Gauss curvature and $\nu|_{\overline{U}}$ is a diffeomorphism into a geodesically convex set. If M' is any closed smooth surface such that $M' \supset M - U$ and $M' - (M - U)$ has positive Gauss curvature, then M' has connected shades.*

Proof. Let $\nu' : M' \rightarrow \mathbf{S}^2$ be the Gauss map of M' , and $U' := M' - (M - U)$. Note that $\partial U' = \partial U$, which is homeomorphic to a circle, since $\nu|_{\overline{U}}$ is a diffeomorphism into a geodesically convex set. So \overline{U}' is a topological manifold with boundary. Next note that, since U' has positive Gauss curvature by assumption, and $\partial U' = \partial U \subset \overline{U}$, it follows that \overline{U}' has positive Gauss curvature. So, by the inverse function theorem, ν' is locally one-to-one on \overline{U}' . Further ν' is one-to-one on $\partial U'$, since $\nu'|_{\partial U'} = \nu|_{\partial U}$. So it follows that ν' is one-to-one on \overline{U}' ; see [6] where it is proved that any locally one-to-one map from a compact n -manifold, $n \geq 2$, into a sphere of the same dimension is one-to-one everywhere, if it is one-to-one on the boundary of the manifold. Consequently, since \overline{U}' is compact and has nonvanishing Gauss curvature, $\nu' : \overline{U}' \rightarrow \nu'(\overline{U}')$ is a diffeomorphism. Now, since $\nu'|_{\partial U'} = \nu|_{\partial U}$, it follows that $\nu'(U') = \nu(U)$. In particular, $\nu'(U')$ is geodesically convex.

Let $S'_u := (\nu')^{-1}(H_u)$ be a shade of M' . Note that if $S'_u \subset U'$, then $H_u \subset \nu(U')$, which yields that $\nu(U') = H_u$; because, by definition, a geodesically convex set lies in an open hemisphere. But if $\nu(U') = H_u$, then $S'_u = U'$ which is connected, since U' is diffeomorphic to a geodesically convex set. Now suppose that $S'_u \not\subset U'$. Since $\nu'(U')$ and H_u are geodesically convex, $H_u \cap \nu'(U')$ is geodesically convex. In particular $H_u \cap \nu'(U')$ is connected, which yields that $S'_u \cap U'$ is connected. Further, if $S'_u \cap U' \neq \emptyset$, then S'_u intersects $\partial U'$, because $S'_u \not\subset U'$. So to prove that S'_u is connected, it is now enough to show that $S'_u \cap (M - U')$ is connected. To see this note that, since $M - U' = M - U$, we have $S'_u \cap (M - U') = S_u \cap (M - U)$, where $S_u = \nu^{-1}(H_u)$. So it remains to check that $S_u \cap (M - U)$ is connected, which is indeed the case, since, by Lemma 3.5, $M - U$ has connected shades. \square

By an *hour glass* $\mathcal{H} \subset \mathbf{R}^3$ we mean a smooth embedded surface of revolution, homeomorphic to an annulus, whose profile curve, when viewed as the graph of a positive function $f : [a, b] \rightarrow \mathbf{R}$, satisfies the following properties:

1. $f'(a) < 0$ and $f'(b) > 0$,
2. $f'' < 0$ on $[a, x_1] \cup (x_2, b]$ and $f'' > 0$ on (x_1, x_2) , for some $x_1, x_2 \in (a, b)$;

see Figure 4.



FIG. 4.

In other words, $\mathcal{H} \subset \mathbf{R}^3$ is an hour glass if after a rigid motion it may be parametrized by

$$[a, b] \times \mathbf{R} \ni (z, \theta) \longmapsto (f(z) \sin(\theta), f(z) \cos(\theta), z) \in \mathbf{R}^3,$$

where $f : [a, b] \rightarrow \mathbf{R}$ is a smooth positive function which satisfies the properties enumerated above. The main property of an hour glass, as far as the illuminations are concerned, is stated in the next result:

LEMMA 3.7. *If $\mathcal{H} \subset \mathbf{R}^3$ is an hour glass, then every component of each shade of \mathcal{H} intersects $\partial\mathcal{H}$.*

Proof. By definition there are precisely two inflection points in the profile curve of \mathcal{H} . These points generate a pair of meridians which divide \mathcal{H} into three regions. Supposing, after a rigid motion, that \mathcal{H} is positioned vertically, i.e., its axis of rotation is parallel to the z -axis, we let R_1, R_2 and R_3 be the top, middle, and bottom regions respectively; see Figure 5

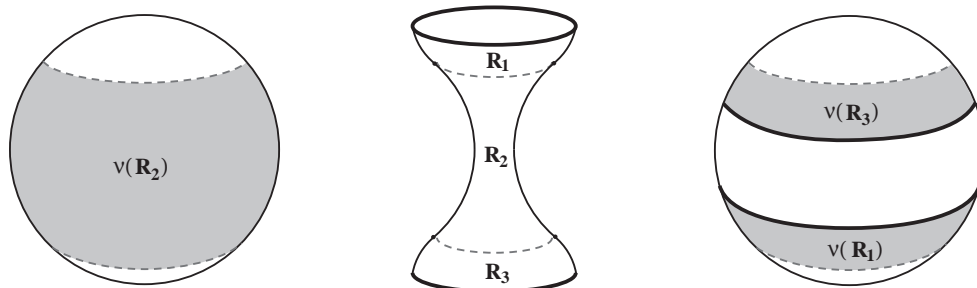


FIG. 5.

Let $\nu: \mathcal{H} \rightarrow \mathbf{S}^2$ be the outward unit normal of \mathcal{H} , and note that ν embeds each R_i into an annular region of \mathbf{S}^2 bounded by a pair of horizontal circles or meridians. In particular, the intersection of $\nu(R_i)$ with any hemisphere H_u of \mathbf{S}^2 is connected. So, by (3.1), the intersection of any shade S_u of \mathcal{H} with R_i must be connected.

Now note that $\nu(R_2)$ contains the equator of \mathbf{S}^2 in its interior. Thus every great circle of \mathbf{S}^2 must intersect the interior of $\nu(R_2)$ (because no two great circles of \mathbf{S}^2 are disjoint). Since ∂H_u is a great circle, it follows that H_u intersects $\nu(R_2)$. But $\nu(R_2) \not\subset H_u$, because $\nu(R_2)$ contains a great circle and H_u is an open hemisphere. So H_u must intersect $\partial\nu(R_2) = \nu(\partial R_2)$. This yields that S_u intersects ∂R_2 . But, as we argued above, $S_u \cap R_2$ is connected. Thus no component S_u^1 of S_u may be trapped in the interior of ∂R_2 . Since $\partial R_2 \subset (\partial R_1 \cup \partial R_3)$, it follows that S_u^1 must intersect R_1 or R_3 .

Suppose that S_u^1 intersects R_1 . Then H_u intersects $\nu(R_1)$. If $\nu(R_1) \subset H_u$, then it follows that S_u^1 intersects $\partial\mathcal{H}$, because a boundary component of $\nu(R_1)$ is the image of a boundary component of $\partial\mathcal{H}$ under ν . So if $\nu(R_1) \subset H_u$ we are done. If $\nu(R_1) \not\subset H_u$, then, since H_u intersects $\nu(R_1)$, it follows that ∂H_u must intersect $\nu(R_1)$. But $\nu(R_1)$ is contained in the bottom hemisphere of \mathbf{S}^2 , and ∂H_u is a great circle. Thus ∂H_u must intersect the upper boundary of $\nu(R_1)$ (otherwise, by the Jordan curve theorem, ∂H_u would have to be contained entirely in the interior of the bottom hemisphere of \mathbf{S}^2 , which would be a contradiction). But the upper boundary of $\nu(R_1)$ is the image of the top boundary component of \mathcal{H} under ν . So, since $S_u \cap R_1$ is connected, we conclude that S_u^1 must intersect $\partial\mathcal{H}$.

Similarly, one can show that if S_u^1 intersects R_3 , then S_u^1 must intersect $\partial\mathcal{H}$, which will complete the proof. \square

LEMMA 3.8. *Let $S_1, S_2 \subset \mathbf{R}^3$ be a pair of spheres of radius r , and $p_1 \in S_1, p_2 \in S_2$ be a pair of points which face away from each other. Then, for every $\epsilon > 0$, there exists an hour glass \mathcal{H} such that an open neighborhood of $\partial\mathcal{H}$ lies on $S_1 \cup S_2$ and \mathcal{H} lies within an ϵ neighborhood of the line segment $p_1 p_2$.*

Proof. After a rigid motion we may assume that $p_1 = (-a, 0, 0)$ and $p_2 = (a, 0, 0)$. Since, by assumption, $p_1 p_2$ is orthogonal to S_1 and S_2 , it follows then that S_1 and S_2 are centered at $(-a - r, 0, 0)$ and $(a + r, 0, 0)$ respectively. Thus the intersection of S_1 and S_2 with the xy -plane consists of a pair of circles of radius of r centered at $(-a - r, 0)$ and $(a + r, 0)$ respectively. It suffices to show that, for every $0 < \delta < r$, there exists a smooth positive function $f : [-a - \delta, a + \delta] \rightarrow \mathbf{R}$ with precisely two inflection points such that the graph of f coincides with the given circles near the end points of $[-a - \delta, a + \delta]$, see Figure 6. Revolving the graph of f around the x -axis

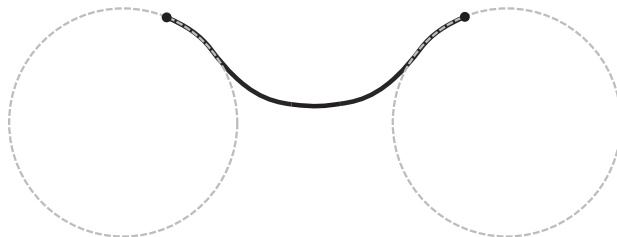


FIG. 6.

then yields an hour glass \mathcal{H} such that an open neighborhood of $\partial\mathcal{H}$ lies on $S_1 \cup S_2$, as desired. Further, choosing δ sufficiently small, we can make sure that the graph of f is within a distance ϵ of $[-a, a]$ which in turn yields that \mathcal{H} is within an ϵ neighborhood of pq . Since the existence of f is pretty obvious we omit further details with regard to its formal construction. \square

Now we are ready to prove the main result of this section:

Proof of Proposition 3.1. Since M has positive Gauss curvature at p, q it follows that, given any open neighborhood U of $\{p, q\}$ in M , we may let $V_p, V_q \subset U$ be open neighborhoods of p and q respectively such that $V_p \cap V_q = \emptyset$, $\overline{V_p}$ and $\overline{V_q}$ are strictly convex, and $\nu|_{\overline{V_p}}, \nu|_{\overline{V_q}}$ are diffeomorphisms into geodesically convex subsets of \mathbf{S}^2 , where ν is the Gauss map of M . By Lemma 3.4, we may deform M inside V_p and V_q , without changing the sign of curvature there or perturbing the tangent planes at p, q , so as to obtain a smooth embedded surface M' , homeomorphic to M , which is spherical in neighborhoods of p, q , and contains $M - U$. Since (i) M' meets the end points of the segment pq transversally, (ii) M' can be made arbitrarily close to M , and by assumption (iii) M does not intersect the interior of the segment pq , it follows that we can make sure M' does not intersect the interior of pq . Thus p, q face away from each other as points of M . Finally note that, by Lemma 3.6, M' has connected shades.

Now let W_p, W_q be small open neighborhoods of p, q in M' which are convex caps cut off from M' by planes parallel to $T_p M'$ and $T_q M'$ respectively. We may assume that W_p and W_q are so small that their closures lie inside the spherical open neighborhoods of M' at p and q . Then, by Lemma 3.5, $M'' := M' - (W_p \cup W_q)$ has connected shades. By Lemma 3.8 we may construct an hour glass \mathcal{H} , bounded by $\partial M'' = \partial W_p \cup \partial W_q$, such that a neighborhood of $\partial\mathcal{H}$ coincides with a pair of collars of $\overline{W_p}$ and $\overline{W_q}$. Then, gluing M'' to \mathcal{H} along their common boundary yields a smooth closed surface \widetilde{M} , which, by Lemma 3.2, has connected shades.

Next note that \mathcal{H} lies within the convex hull of $W_p \cup W_q$. Thus, choosing W_p, W_q sufficiently small, we can make sure that \mathcal{H} is arbitrarily close to pq , which is disjoint from M'' . So we can make sure that \widetilde{M} is embedded. Finally, since

$\widetilde{M} - M \subset (V_p \cup V_q \cup \mathcal{H})$, choosing V_p and V_q sufficiently small, ensures that $\widetilde{M} - M$ is as close to pq as desired. \square

4. Proof of Theorem 1.1. We need only one more lemma. Let $\gamma: \mathbf{R}/\ell \rightarrow \mathbf{R}^3$ be a smooth simple loop, and $\text{dist}_\gamma: \mathbf{R}^3 \rightarrow \mathbf{R}$, given by

$$\text{dist}_\gamma(p) := \inf \{ \|p - \gamma(t)\| \mid t \in \mathbf{R}/\ell \},$$

be the *distance function* of γ . Then it follows from the tubular neighborhood theorem that for sufficiently small r

$$\text{Tube}_r(\gamma) := \text{dist}_\gamma^{-1}(r),$$

is a smooth embedded surface.

LEMMA 4.1. *Let $\gamma: \mathbf{R}/\ell \rightarrow \mathbf{R}^3$ be a smooth simple loop with nonvanishing curvature, r be sufficiently small so that $M := \text{Tube}_r(\gamma)$ is a smooth embedded surface, and K be the Gauss curvature of M . Then for every $t \in \mathbf{R}/\ell$,*

$$K(\gamma(t) - rN(t)) > 0$$

where N is the principal normal of γ .

Since it is a routine exercise to establish the last lemma, we omit its proof. Now we are ready to prove Theorem 1.1. First recall that the case of $g(M) = 0$ is trivial, since the standard sphere \mathbf{S}^2 has connected shades, and the case of $g(M) = 1$ was proved in [5], by showing that a tube around a skew loop has connected shades. For the case of $g(M) = n \geq 2$, let

$$p_i := (0, 0, i), \quad \text{and} \quad N_i := (0, 0, (-1)^i),$$

for $i = 1, \dots, 2n - 2$. Then, by Proposition 2.1, there exists a smooth simple skew loop $\gamma: \mathbf{R}/(2n - 2) \rightarrow \mathbf{R}^3$ such that $\gamma(i) = p_i$ and $N(i) = N_i$ (to apply Proposition 2.1, we only need to note that one may easily find $T_i \in \mathbf{S}^2$ such that $\langle T_i, N_i \rangle = 0$, and $T_i \neq \pm T_j$, whenever $i \neq j$; for instance, let $T_i := (\cos(\pi i/4n), \sin(\pi i/4n), 0)$). By Lemma 2.4, after a perturbation of γ , which keeps γ fixed on an open neighborhood of each integer, we may assume that the z -axis does not intersect the image of γ at any points other than p_i .

Since γ is a smooth simple skew loop, $M := \text{Tube}_r(\gamma)$ is a smoothly embedded surface with connected shades, for r sufficiently small [5, Prop. 6.3]. Now, for $i = 1, \dots, n - 1$, set

$$x_i := p_{2i} - rN_{2i}, \quad y_i := p_{2i-1} - rN_{2i-1};$$

see Figure 7. Then $x_i, y_i \in M$. We claim that x_i, y_i ‘face away’ from each other, as defined in the beginning of Section 3. To see this first note that if K denotes the Gauss curvature of M , then $K(x_i), K(y_i) > 0$ by Lemma 4.1. Further, the segments $x_i y_i$ are pairwise disjoint from each other, and their interiors are disjoint from M , if we assume that r is sufficiently small. Furthermore, if ν is the inward unit normal of M , then $\nu(x_i) = N_{2i}$ and $\nu(y_i) = N_{2i-1}$. So the segment $x_i y_i$ is orthogonal to M at its end points. Finally note that, since by definition N_{2i} is parallel to the positive direction of the z -axis, p_{2i} is a local minimum for the height of γ (with respect to the xy -plane). This yields that x_i is a local minimum for the height of M . Thus if

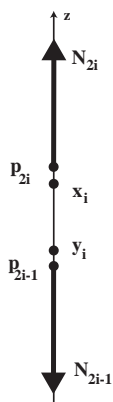


FIG. 7.

H is the mean curvature vector of M , then $H(x_i)$ points up, i.e., $H(x_i) \parallel x_i - y_i$. Similarly, since N_{2i-1} is parallel to the negative direction of the z -axis, it follows that $H(y_i) \parallel y_i - x_i$. Thus we conclude that M has $n - 1$ pairs of points $\{x_i, y_i\}$ which face away from each other.

Let U_i be an open neighborhood of $\{x_i, y_i\}$ in M chosen sufficiently small so that $U_i \cap U_j = \emptyset$, when $i \neq j$, and let $\epsilon > 0$ be smaller than the smallest distance between the segments $x_i y_i$. By Proposition 3.1 there exists a smooth embedded surface M_1 with connected shades, and $g(M_1) = g(M) + 1 = 2$, such that $M - U_1 \subset M_1$ and $M_1 - M$ lies within a distance ϵ of $x_1 y_1$. In particular, M_1 has $n - 2$ pairs of points $\{x_i, y_i\}$, $i = 2, \dots, n - 1$, which face away from each other, and M_1 is disjoint from the interiors of $x_i y_i$. So we may apply Proposition 3.1 again. Repeating this procedure, we eventually obtain a smooth embedded surface M_{n-1} with connected shades and $g(M_{n-1}) = n$.

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