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RATIONALLY CONNECTED VARIETIES AND LOOP SPACES*

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Abstract. We consider rationally connected complex projective manifolds M and show that their loop spaces—infinite dimensional complex manifolds—have properties similar to those of M. Furthermore, we give a finite dimensional application concerning holomorphic vector bundles over rationally connected complex projective manifolds.

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0. Introduction. Let M be a complex manifold and $r = 0, 1, \dots, \infty$. The space $C^r(S^1, M)$ of r times continuously differentiable maps $x\colon S^1 \to M$, the (free) C^r loop space of M, carries a natural complex manifold structure, locally biholomorphic to open subsets of Banach ($r < \infty$) resp. Fréchet ($r = \infty$) spaces, see [L2]. The same is true of "generalized loop spaces"—or mapping spaces— $C^r(V, M)$, where V is a compact C^r manifold, possibly with boundary; when $r = 0$, V can be just a compact Hausdorff space. A very general question is how complex analytical and geometrical properties of M and its loop spaces are related.

Our contribution to this problem mainly concerns rational connectivity. A complex projective manifold is a complex manifold, biholomorphic to a connected submanifold of some projective space $\mathbb{P}^n(\mathbb{C})$. Such a manifold M is called rationally connected if it contains rational curves (= holomorphic images of $\mathbb{P}^1(\mathbb{C})$) through any finite collection of its points. This is equivalent to requiring that for a nonempty open $U \subset M \times M$ and any $(p, q) \in U$ there should be a rational curve through p and q. For the theory of rationally connected varieties see [AK, Kl1, KMM].

Projective spaces, Grassmannians, and in general complex projective manifolds birational to projective spaces are rationally connected. In a sense rationally connected manifolds are the simplest manifolds; at the same time, general complex projective manifolds can be studied through rationally connected ones by the device of maximally rationally connected fibrations [Kl1, Theorem IV.5.4].

Here is a brief description of the results presented in this paper. For more complete formulations and for background the reader is referred to Section 1. First we prove that loop spaces $C^r(S^1, M)$ of rationally connected complex projective manifolds M contain plenty of rational curves, but in some other mapping spaces $C^r(V, M)$ rational curves are rare. Then we shall discuss holomorphic functions and, more generally, holomorphic tensor fields. Extending earlier results of Dineen–Mellon and the first author [DM,L2] we show that on mapping spaces of rationally connected complex projective manifolds M holomorphic functions are locally constant, and the same is true on submanifolds of $C^r(V, M)$ consisting of so called based maps. This infinite dimensional result has the finite dimensional corollary that holomorphic linear connections on vector bundles $E \to M$ are trivial.

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Next we consider the "trivial" component of $C^r(V, M)$ consisting of contractible maps. We show that the constancy of holomorphic functions on this component already follows once we know M is compact, connected, and all contravariant symmetric holomorphic tensor fields on M (of positive weight) vanish (i.e., the only holomorphic section the symmetric powers of T^*M admit is the zero section). This property is weaker than rational connectivity. In fact, rational connectivity implies that contravariant holomorphic tensor fields, symmetric or not, vanish, see [AK, Theorem 30]; and conjecturally the converse, "Castelnuovo's criterion", is also true for complex projective manifolds. On the other hand, Kollár pointed out in an email to us that simply connected Calabi–Yau manifolds are not rationally connected, but all contravariant symmetric holomorphic tensor fields on them are zero.—Finally we prove that for compact connected M, if on M all contravariant holomorphic tensor fields of positive weight vanish then the same holds on the trivial component of $C^r(V, M)$.

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1. Background and results.

1.1. Mapping spaces. Fix $r = 0, 1, \dots, \infty$ and a compact manifold V of class C^r, possibly with boundary (or just a compact Hausdorff space, when $r = 0$). We start by quickly describing the complex manifold structure on the mapping space $X = C^r(V, M)$ of a finite dimensional complex manifold M. For generalities on infinite dimensional complex manifolds, see [L1, Section 2]. We need

LEMMA 1.1. *There are an open neighborhood $D \subset M \times M$ of the diagonal and a C^∞ diffeomorphism F between D and a neighborhood of the zero section in TM with the following properties. Setting $D^w = \{z \in M : (z, w) \in D\}$ and $F^w = F(\cdot, w)$, for all $w \in M$ we have*
(a) F^w maps D^w biholomorphically on a convex subset of T_wM;
(b) $F^w(w) \in T_wM$ is the zero vector;
(c) $dF^w(w) : T_wD^w = T_wM \to T_wM$ is the identity.

(In (c) we have identified a tangent space to the vector space T_wM with the vector space itself.)

Proof. When M is a convex open subset of some $\mathbb{C}^n$ so that TM is identified with $\mathbb{C}^n \times M$, one can take $D = M \times M$ and $F(z, w) = (z - w, w)$. A general M being locally biholomorphic to convex open subsets of $\mathbb{C}^n$, one obtains a covering of M by open sets W and C^∞ maps $F_W : W \times W \to TW$ that satisfy (a, b, c) for $w \in W$ (with D^w replaced by W). If $\{\chi_W\}_W$ is a corresponding C^∞ partition of unity on M, one can take as $F(z, w)$ the restriction of

$$\sum_W \chi_W(w) F_W(z, w)$$

to an appropriate neighborhood of the diagonal.

Given D and F, define the complex structure on $X = C^r(V, M)$ as follows. A coordinate neighborhood of $y \in X$ consists of those $x \in X$ for which $(x(t), y(t)) \in D$ for all $t \in V$. This neighborhood is mapped to an open subset of $C^r(y^*TM)$, the space of C^r sections of the induced bundle $y^*TM \to V$, by the map

$$\varphi_y : x \mapsto \xi, \qquad \xi(t) = F(x(t), y(t)).$$

It is straightforward that the local charts φ_y are holomorphically related and so define a complex manifold structure on $C^r(V,M)$; also, this structure is independent of the choice of D and F. Both facts can be verified as in [L2, p. 38].

The above construction is slightly simpler then the one in [L2, Section 2] (but it defines the same complex structure). Its drawback is that it does not generalize to infinite dimensional manifolds M, that may not admit C^∞ partitions of unity. By contrast, the construction in [L2] does generalize, since it uses partitions of unity on V only.

Lemma 1.2. *Let N be a (finite dimensional) complex manifold, and with a map $h: N \to C^r(V,M)$ associate the map*

$$g: N \times V \to M, \qquad g(s,t) = h(s)(t).$$

Then h is holomorphic if and only if g is C^r and $g(\cdot,t)$ is holomorphic for each $t \in V$.

Proof. We sketch a proof for $r > 0$. When M and N are open subsets of $\mathbb{C}^m$, resp. $\mathbb{C}^n$, and V is the closure of a domain with C^r boundary in $\mathbb{R}^p$, the manifold $C^r(V,M)$ is an open subset of the Fréchet space $C^r(V,\mathbb{C}^m)$. In this case the "if" direction follows from Morera's theorem. Indeed, if $\Delta \subset N$ is a holomorphically embedded closed disc and ζ a holomorphic coordinate along Δ, then

$$\int_{\partial\Delta} h(\zeta)\, d\zeta = \int_{\partial\Delta} g(\zeta,\cdot)\, d\zeta = 0;$$

since h is obviously continuous, it must be holomorphic. In the "only if" direction we only verify that g is C^r. Let f denote a partial derivative of g along V, of some order $\le r$. This is a continuous function, holomorphic along $\{t\} \times N$, $t \in V$, for it is the locally uniform limit of certain difference quotients that do have these properties. It follows that partial derivatives of f along N, of any order, are also continuous, since they can be represented by an integral according to Cauchy. Hence g is indeed C^r.

The general case can be reduced to the one just discussed by shrinking M, N, and V to coordinate neighborhoods.

A closed $A \subset V$ and $x_0 \in X$ determine a subspace of "based" maps. Denoting the r–jet of x by $j^r x$, the subspace in question is

$$Z = C^r_{A,x_0}(V,M) = \{x \in X : j^r x | A = j^r x_0 | A\}, \tag{1.1}$$

a complex submanifold of X. As explained in [L2, Sections 2,3], for $x \in X$ the tangent space $T_x X$ is naturally isomorphic to $C^r(x^*TM)$; if $x \in Z$, under this isomorphism $T_x Z \subset T_x X$ corresponds to

$$C^r_A(x^*TM) = \{\xi \in C^r(x^*TM) : j^r \xi | A = 0\}. \tag{1.2}$$

Up to this point TX, TZ are real vector bundles. However, as in finite dimensions, the local charts endow the real tangent bundles of X and Z with the structure of a locally trivial holomorphic vector bundle, and we shall always regard TX and TZ as such.

1.2. Rational connectivity. While our principal interest is in complex manifolds, we will have to deal with projective (or quasiprojective) varieties defined over fields other than $\mathbb{C}$ as well. Then we shall use the language of algebraic geometry, in particular the topology implied will be Zariski's. If M is a variety defined over a field

k, we write $M(k)$ for its points over k. When k is algebraically closed we shall ignore the difference between M and $M(k)$, so, for instance, a smooth projective variety M over $\mathbb{C}$ will be thought of as a complex projective manifold determined by $M(\mathbb{C})$.

Definition 1.3. *Let M be a smooth projective variety defined over a field k of characteristic 0. When k is algebraically closed, M is rationally connected if there is a morphism $f\colon \mathbb{P}^1 \to M$ defined over k (i.e., a rational curve) such that the induced bundle f^*TM is ample: $f^*TM \approx \bigoplus \mathcal{O}_{\mathbb{P}^1}(d_j)$, with all $d_j > 0$. In general, M is rationally connected if it is such when considered over some (and then over an arbitrary) algebraically closed field $K \supset k$.*

Over a field of positive characteristic the above property defines the so called separably rationally connected varieties. When $k = \mathbb{C}$, rational connectivity is equivalent to requiring that there be a nonempty open $U \subset M(\mathbb{C}) \times M(\mathbb{C})$ such that for $(p, q) \in U$ there is a rational curve through p and q; and also to requiring that through any finite collection of points in $M(\mathbb{C})$ there be a rational curve. For all this, see [AK, Definition–Theorem 29]. (In [AK, Definition-Theorem 29], unlike in Definition 1.3 above, it is assumed that k is uncountable, but it is for the other five definitions—this particular definition works for all k. By the way, all our fields in this paper contain the complex numbers, so they are all uncountable.)

1.3. Rational connectivity of loop spaces.

Theorem 1.4. *Let M be a rationally connected complex projective manifold and V a one real dimensional manifold (always compact!). Then the space $C^r(V, M)$ is rationally connected in the sense that for any $n \in \mathbb{N}$ there is a dense open $O \subset C^r(V, M^n)$ such that through any n–tuple of maps $(x_1, \dots, x_n) \in O$ there is a rational curve, i.e., a holomorphic image of $\mathbb{P}^1(\mathbb{C})$, in $C^r(V, M)$.*

The theorem would not hold for higher dimensional V. First, the space $C^r(V, M)$ may be disconnected, which precludes rational connectivity. But even within components rational curves will be scarce, typically. Let us call the component of $C^r(V, M)$ containing constant maps the trivial component.

Theorem 1.5. *If V is an oriented closed connected surface, and $h\colon \mathbb{P}^1(\mathbb{C}) \to C^r(V, \mathbb{P}^1(\mathbb{C}))$ holomorphic, then h is constant or else maps into the trivial component of $C^r(V, \mathbb{P}^1(\mathbb{C}))$.*

We do not know whether in the trivial component generic n–tuples can be connected with rational curves; but at least a nonempty open set of n–tuples can be. We shall not prove this, but it follows along the lines of Section 2 (with Lemma 2.2 slightly modified), even for arbitrary V and the mapping space $C^r(V, M)$ of a rationally connected M instead of $\mathbb{P}^1(\mathbb{C})$.

1.4. Holomorphic functions. A simple consequence of Theorem 1.4 is that on loop spaces of rationally connected complex projective manifolds holomorphic functions are constant. It turns out that this generalizes to spaces Z of based maps, see (1.1), even though a typical Z will not contain compact subvarieties, let alone rational curves according to [L2, Theorem 3.4]. For the rest, we consider general mapping spaces as introduced in 1.1.

Theorem 1.6. *If M is a rationally connected complex projective manifold, $A \subset V$ is closed, and $x_0 \in C^r(V, M)$, then holomorphic functions $Z = C^r_{A,x_0}(V, M) \to \mathbb{C}$ are locally constant.*

The case when M is a projective space was known earlier, see [DM, Theorems 7, 11] for $r = 0$ and [L2, Theorem 4.2] in general. The theorem has the following

COROLLARY 1.7. *If a holomorphic vector bundle (possibly with fibers Banach spaces) $E \to M$ over a rationally connected complex projective manifold admits a holomorphic (linear) connection, then both E and the connection are trivial.*

In fact, in Section 4 we shall prove a rather more general result. However, something far more general may also be true that has nothing to do with rational connectivity. We conjecture that Corollary 1.7 is true for all simply connected compact Kähler manifolds M.—As Kollár noted, when E has finite rank and M is not only rationally connected but Fano, Corollary 1.7 immediately follows from [AW, Proposition 1.2]. Holomorphic connections have been studied first by Atiyah [A]. Among other things he proved a Lefschetz type theorem on generic hyperplanes, and completely classified holomorphic connections over Riemann surfaces in terms of the fundamental group.

1.5. Holomorphic tensor fields. All tensor fields will be contravariant, without explicit mentioning. Over a finite dimensional manifold N these tensor fields are sections of tensor powers of T^*N. To avoid, in the infinite dimensional case, dealing with the ambiguous notion of tensor product of Banach or Fréchet spaces/bundles, we simply define a holomorphic tensor field on a complex manifold N as a holomorphic function

$$g\colon T^jN = TN \oplus \ldots \oplus TN \to \mathbb{C},$$

multilinear on each fiber. The integer $j = 0, 1, \ldots$ is the weight of the tensor field; a tensor field of weight 0 is just a holomorphic function on N. If g is symmetric on the fibers we speak of a symmetric tensor field. Examples of symmetric holomorphic tensor fields are the zero fields (for all weights) and the constant fields of weight 0. We call these fields trivial, and we shall be interested in manifolds M on which all holomorphic tensor fields are trivial (this implies M is connected). As said earlier, rationally connected complex projective manifolds are of this kind.

In the next theorem M can be a complex manifold locally biholomorphic to open sets in Banach spaces.

THEOREM 1.8. *Let M be a complex manifold and $Y \subset C^r(V, M)$ a connected neighborhood of the space of constant maps.*

(a) If on M all symmetric holomorphic tensor fields are trivial, then holomorphic functions on Y are constant.

(b) If on M all holomorphic tensor fields are trivial, then the same holds on Y.

As already said, for complex projective manifolds having only trivial holomorphic tensor fields is conjecturally equivalent to rational connectivity. According to Theorems 1.4, 1.8, both properties are inherited by loop spaces, a fact we consider a mild additional evidence in favor of the conjecture.

2. Rational connectivity of loop spaces. In this section we shall prove Theorem 1.4. The key is the following.

LEMMA 2.1. *Let k be a field of characteristic zero and M a smooth, rationally connected projective variety over k. Given distinct points $p_1, \ldots, p_n \in \mathbb{P}^1$ defined over k, there are a smooth variety W and a morphism $f\colon \mathbb{P}^1 \times W \to M$ over k such that the map*

$$\varphi = (f(p_\nu, \cdot))_{\nu=1}^n \colon W \to M^n \tag{2.1}$$

is a surjective submersion on a dense open $U \subset M^n$ and its fibers are irreducible.

Proof. This is more or less a reformulation of [KSz, Theorem 16]. We give here a short, standard argument. For more details, we refer to the Internet: [LSz].

At the price of replacing M by $\mathbb{P}^3 \times M$ it can be assumed that $\dim(M) \geq 3$, and we may also assume $n \geq 3$, and $p_1 = \infty \in \mathbb{P}^1$. The n-tuples of points on M are parameterized by M^n, hence the generic n-tuple is defined over the function field $K(M^n)$. We shall apply [KSz, Theorem 16] to M over the field $K(M^n)$, and get a family of smooth rational curves (defined over $K(M^n)$) going through the generic n-tuple, parameterized by a smooth, geometrically irreducible variety. This amounts to having a dense open subset $U \subset M^n$, a family $\omega : W \to U$ of smooth, irreducible varieties, a subfamily $R \xrightarrow{\pi} W$ of $W \times M \to W$, whose fibers R_w are smooth rational curves in $\{w\} \times M$, and n sections $q_i : W \to R$ such that the composite morphism $W \overset{(q_1, \dots q_n)}{\longrightarrow} R^n \overset{\pi \times \cdots \times \pi}{\longrightarrow} M^n$ is just ω. We shrink U and W so that for $w \in W$ the points $q_i(w) \in R_w$ are disjoint, and choose coordinates on R_w such that $q_1(w) = \infty$, $q_2(w) = 0$ and $q_3(w) = 1$. Then there is a unique polynomial morphism $\mathbb{P}^1 \to R_w$ of degree $n-2$ which sends each p_i to $q_i(w)$. These morphisms glue together to a global morphism $\mathbb{P}^1 \times W \to R$, and the composition $f : \mathbb{P}^1 \times W \to R \to M$ will satisfy the condition of the lemma. Indeed, φ is just our ω, hence it has all required properties.

Next we need a result from differential geometry. Let V be a one dimensional compact manifold.

Lemma 2.2. *Let $\varphi\colon W \to U$ be a surjective C^∞ submersion between finite dimensional C^∞ differential manifolds, whose fibers are connected. For any r and $y \in C^r(V, U)$ there is such an $\eta \in C^r(V, W)$ that $\varphi \circ \eta = y$.*

Proof. First observe that any compact subset C of a fiber $\varphi^{-1}(u)$ has an open neighborhood $W_0 \subset W$ such that $\varphi|W_0$ is a trivial fiber bundle with connected fibers. To verify this we can assume $U = \mathbb{R}^m$ and $u = 0$. A partition of unity argument gives a connection on W, i.e. a subbundle $H \subset TW$ complementary to the tangent spaces of the fibers of φ. Fix a relatively compact, connected open neighborhood $G \subset \varphi^{-1}(u)$ of C. Connect an arbitrary $v \in \mathbb{R}^m$ with $0 \in \mathbb{R}^m$ by a curve γ consisting of m segments, the μ'th segment parallel to the μ'th coordinate axis. If v is in a sufficiently small neighborhood $U_0 \subset \mathbb{R}^m$ of u and $c \in G$ then γ can be uniquely lifted to a piecewise smooth curve Γ, tangent to H and starting at c. Let $\psi(c, v)$ denote the endpoint of Γ. Then ψ is a fiberwise diffeomorphism of $G \times U_0$ on an open neighborhood W_0 of C, as claimed.

It follows that there are closed arcs $A_1, \dots, A_n$ covering V and C^r maps $\eta_\nu\colon A_\nu \to W$ such that $\varphi \circ \eta_\nu = y$. We show that there is a C^r map $\overline{\eta}\colon A_1 \cup A_2 \to W$ such that $\varphi \circ \overline{\eta} = y$. Indeed, $A_1 \cap A_2$ is empty or consists of one or two components. In the first case $\overline{\eta} = \eta_\nu$ on A_ν, $\nu = 1, 2$, will do. Otherwise choose points b_i from each component of $A_1 \cap A_2$; thus $A_1 \cup A_2 \backslash \{b_i\}_i$ is the disjoint union of two arcs $\tilde{A}_\nu \subset A_\nu$, $\nu = 1, 2$. Using the neighborhoods of $C = C_i = \{\eta_1(b_i),\ \eta_2(b_i)\}$ from our initial observation, it is straightforward to construct the required $\overline{\eta}$; it will agree with η_ν on $\tilde{A}_\nu$, away from a small neighborhood of b_i.

Now one can continue in the same spirit, fusing more and more arcs, eventually to obtain the η of the lemma.

Proof of Theorem 1.4. Fix distinct $p_1, \dots, p_n \in \mathbb{P}^1(\mathbb{C})$ and apply Lemma 2.1, with $k = \mathbb{C}$. We obtain a holomorphic map $f\colon \mathbb{P}^1(\mathbb{C}) \times W \to M$ of complex manifolds so that

$$\varphi = (f(p_\nu, \cdot))_{\nu=1}^n : W \to M^n$$

is a surjective submersion on a Zariski dense open $U \subset M^n$, with irreducible, hence connected fibers (see [M, 4.16 Corollary]). Since the complement of U is of real codimension 2 in M^n, $O = C^r(V, U)$ is dense in $C^r(V, M^n)$. Given $x_1, \ldots, x_n \in C^r(V, M)$ such that $y = (x_1, \ldots, x_n) \in O$, there is a holomorphic map $h\colon \mathbb{P}^1(\mathbb{C}) \to C^r(V, M)$ with $h(p_\nu) = x_\nu,\ \nu = 1, \ldots, n$. Indeed, using Lemma 2.2 one finds $\eta\colon V \to W$ such that $\varphi \circ \eta = y$. Setting

$$F(p,t) = f(p, \eta(t)), \quad p \in \mathbb{P}^1(\mathbb{C}),\ t \in V,$$

the map h given by $h(p) = F(p, \cdot)$ will do.

3. The Proof of Theorem 1.5. The proof we give here is simpler than our original proof, that used more complex geometry. It depends on the following topological observation. If $g : S^2 \times V \to S^2$ is continuous, then either $g(a, \cdot)$ or $g(\cdot, b)$ is homotopic to constant, for all $a \in S^2$, resp. $b \in V$. This was pointed out to us by Kollár and Z. Szabó in the case $V = S^2$, but an even simpler argument than theirs takes care of the general statement. Indeed, pull back a generator of $H^2(S^2)$ by g to a class $w \in H^2(S^2 \times V)$; then $w^2 = 0$ in the cohomology ring. By Künneth's formula $H^2(S^2 \times V)$ is the direct sum of the pullbacks of $H^2(S^2)$ and $H^2(V)$ under the projection maps. If $w = (u, v)$ in this direct decomposition, then $w^2 = 0$ implies $u = 0$ or $v = 0$. This means that g restricted to $\{a\} \times V$ or to $S^2 \times \{b\}$ induces the zero map on H^2, whence it is homotopic to constant by Hopf's theorem, see [S, Chapter 8, Section 1].

From this Theorem 1.5 follows by considering the map $g : \mathbb{P}^1(\mathbb{C}) \times V \to \mathbb{P}^1(\mathbb{C})$ given by $g(s,t) = h(s)(t)$. If h does not map into the trivial component, then, as seen above, $g(\cdot, b)$ is homotopic to constant. This map being holomorphic, itself must be constant, for all $b \in V$; in other words, h is constant.

4. Holomorphic Functions on the Manifold of Based Loops. In this section we shall consider a rationally connected complex projective manifold M, the space $C^r_{A,x_0}(V, M) = Z$ of based maps, $A \subset V,\ x_0 \in C^r(V, M)$, and we shall show that complex valued holomorphic functions on Z are locally constant, Theorem 1.6. We shall also derive Corollary 1.7, in a more general form.

Lemma 4.1. *Given $p \in M$ and $v \in T_pM$, there are a neighborhood U of p and a holomorphic map $\varphi\colon \mathbb{P}^1(\mathbb{C}) \times U \to M$ such that*

$$\varphi(\infty, \cdot) = id_U \quad \textit{and} \quad \varphi_* T_{(\infty,p)}(\mathbb{P}^1(\mathbb{C}) \times \{p\}) \ni v.$$

Proof. In [LSz] the reader will find a proof using only basic deformation theory. Upon repeated prodding by our referees, here we give a shorter argument. Let $q \in M$ be a very general point. By [Kl2, Theorem 4.1.2.4] there is a rational curve $g : \mathbb{P}^1 \to M$ through p and q whose tangent direction at p is v. This curve is free by [AK, Corollary 11]. By II.3.5.4.2 of [Kl1] the deformations of this curve form a family $f : \mathbb{P}^1 \times W \to M$, with the evaluation map f a submersion. Hence on some (metric) neighborhood $U \subset M$ of p it has a holomorphic section $(\sigma_1, \sigma_2) : U \to \mathbb{P}^1 \times W$. We can choose the section so that $f(\cdot, \sigma_2(p)) = g$. Then the family $\varphi : \mathbb{P}^1 \times U \to M$ given by $\varphi(t, u) = f(t, \sigma_2(u))$ has the property that each member $\varphi(\cdot, u)$ goes through the point u, and the tangent direction of $\varphi(\cdot, p) = g$ at p is just v. After reparameterizing these curves we can achieve that $\phi(\infty, u) = u$ for all $u \in U$. This proves the lemma.

Proof of Theorem 1.6. Let $f\colon Z \to \mathbb{C}$ be holomorphic; we have to prove $df(\xi) = 0$ for all $x \in Z$ and $\xi \in T_x Z$. Fix x. Given $\tau \in V$ and nonzero $v \in T_{x(\tau)}M$, construct U and φ as in Lemma 4.1, and with a sufficiently small neighborhood $B \subset V$ of τ define a C^r map

$$\Phi\colon \mathbb{P}^1(\mathbb{C}) \times B \ni (s, t) \mapsto \varphi(s, x(t)) \in M,$$

holomorphic in s. Note that

$$\Phi(\infty, t) = x(t) \quad , \quad t \in B.$$

We take B compact and (when $r \geq 1$) a C^r manifold with boundary. We also arrange that $\Phi^t = \Phi(\cdot, t)$ is an immersion near ∞, when $t \in B$.

First suppose that $\xi \in T_x Z \approx C^r_A(x^* TM)$ is supported in the (relative) interior of B, and

$$\xi(t) \in \Phi^t_* T_\infty \mathbb{P}^1(\mathbb{C}), \quad \text{for all } t \in B. \tag{4.1}$$

To show that $df(\xi) = 0$, consider the map $\nu\colon C^r_{(A\cap B)\cup\partial B,\infty}(B, \mathbb{P}^1(\mathbb{C})) \to Z$,

$$\nu(y)(t) = \begin{cases} \Phi(y(t), t), & \text{if } t \in B \\ x(t), & \text{if } t \in V \backslash B. \end{cases}$$

By [L2, Propositions 2.3, 3.1] ν is holomorphic, and so is

$$f \circ \nu\colon C^r_{(A\cap B)\cup\partial B,\infty}(B, \mathbb{P}^1(\mathbb{C})) \to \mathbb{C}.$$

Therefore $f \circ \nu$ is locally constant by [L2, Theorem 4.2]; for the case $r = 0$, see the earlier [DM]. Now ξ is in the range of ν_*; indeed, $\xi = \nu_* \eta$, if $\eta(t) \in T_\infty \mathbb{P}^1(\mathbb{C})$ is defined by $\eta(t) = 0$ when $t \in V \backslash B$ and $\Phi^t_* \eta(t) = \xi(t)$ when $t \in B$, cf. (4.1). It follows that

$$df(\xi) = d(f \circ \nu)(\eta) = 0.$$

Next choose a basis $v = v_1, \ldots, v_m$ of $T_{x(\tau)}M$ and construct corresponding maps

$$\Phi = \Phi_1, \Phi_2, \ldots, \Phi_m\colon \mathbb{P}^1(\mathbb{C}) \times B \to M.$$

If B is sufficiently small then

$$T_{x(t)}M = \bigoplus_j \Phi^t_{j*} T_\infty \mathbb{P}^1(\mathbb{C}) \quad , \quad t \in B.$$

For each j, if $\xi_j \in T_x Z \approx C^r_A(x^* TM)$ has support in $\operatorname{int} B$ and satisfies (4.1), with j appended, then $df(\xi_j) = 0$. Since any $\xi \in C^r_A(x^* TM)$ supported in $\operatorname{int} B$ is the sum of such ξ_j's, we conclude each $\tau \in V$ has a neighborhood B so that $df(\xi) = 0$ when $\operatorname{supp} \xi \subset \operatorname{int} B$. But then a partition of unity gives $df(\xi) = 0$ for all $\xi \in T_x Z$, as needed.

We shall apply Theorem 1.6 to study holomorphic connections in the following setting. Let $\pi\colon E \to N$ be a holomorphic map of complex manifolds locally biholomorphic to open subsets of Banach spaces. Assume π is a submersion, i.e. $\pi_*(e)\colon T_eE \to T_{\pi(e)}N$ is surjective for all $e \in E$. A holomorphic connection on E (or on π) is a holomorphic subbundle $D \subset TE$ such that D_e is complementary to Ker $\pi_*(e)$, $e \in E$. The connection is complete if curves in N can be lifted to horizontal curves in E, i.e., for any $x \in C^1([0,1],N)$ and $e \in \pi^{-1}(x(0))$ there is a $y \in C^1([0,1],E)$ such that $y(0) = e$, $\pi \circ y = x$, and $y'(t) \in D_{y(t)}$ for all $0 \le t \le 1$. The lift is unique by the uniqueness theorem for ODE's. For example, linear connections on Banach bundles and G–invariant connections on principal G bundles—G a Banach–Lie group—are complete.

The simplest example of a connection is on a trivial bundle $\pi\colon E = F \times N \to N$, with $D_{(f,n)} = T_{(f,n)}(\{f\} \times N)$. Connections isomorphic to such a connection are called trivial. Corollary 1.7 follows from

Theorem 4.2. *Let M be a rationally connected complex projective manifold, E a complex manifold locally biholomorphic to open subsets of Banach spaces, and $\pi\colon E \to M$ a holomorphic submersion such that on each fiber holomorphic functions separate points. If π admits a complete holomorphic connection D then the connection is trivial.*

Proof. The mapping space $C^1([0,1],E)$ has a natural structure of a complex manifold—the construction in [L2, Section 2] carries over to Banach manifolds. Horizontal lift defines a map Λ of the manifold

$$\{(e,x) \in E \times C^1([0,1],M)\colon \pi(e) = x(0)\}$$

into $C^1([0,1],E)$. This map is holomorphic. To see this, note that for (e,x) in a small neighborhood of a fixed (e_0,x_0), and for small $\tau \in (0,1]$, finding $y = \Lambda(e,x)$ over the interval $[0,\tau]$ amounts to solving an ODE. Doing this by the standard iterative scheme of Picard–Lindelöf (see [Hm, p. 8]) shows the local lift $y|[0,\tau] \in C^1([0,\tau],E)$ depends holomorphically on (e,x). Since the full lift y is obtained by concatenating local lifts, Λ is indeed holomorphic. It is also equivariant with respect to reparametrizations: if $\sigma\colon [0,1] \to [0,1]$ is a C^1 map, $\sigma(0) = 0$, then

$$\Lambda(e,x) \circ \sigma = \Lambda(e, x \circ \sigma), \qquad x \in C^1([0,1],M). \tag{4.2}$$

With fixed $p \in M$ and variable $q \in M$ consider

$$Y = \{x \in C^1([0,1],M)\colon x(0) = p\},\ Y_q = \{x \in Y\colon x(1) = q\},\ \text{and}$$

$$Z_q = \{x \in Y_q\colon x'(0) \in T_pM \text{ and } x'(1) \in T_qM \text{ are both zero}\},$$

connected manifolds since M is simply connected by [C]. Therefore Theorem 1.6 implies that $\mathbb{C}$–valued holomorphic functions on Z_q are constant. In particular, for any $e \in \pi^{-1}(p)$ and holomorphic function $h\colon \pi^{-1}(q) \to \mathbb{C}$, $h(\Lambda(e,x)(1))$ is independent of $x \in Z_q$. Since holomorphic functions separate points of $\pi^{-1}(q)$, $\Lambda(e,x)(1)$ itself is independent of $x \in Z_q$. It follows from (4.2) that $\Lambda(e,x)(1)$ is even independent of $x \in Y_q$ (take e.g. $\sigma(t) = 3t^2 - 2t^3$, then $x \circ \sigma \in Z_q$), and so there is a holomorphic map $\Psi\colon \pi^{-1}(p) \times M \to E$ such that

$$\Lambda(e,x)(1) = \Psi(e, x(1)). \tag{4.3}$$

One checks that Ψ is biholomorphic and maps $\pi^{-1}(p) \times \{q\}$ to $\pi^{-1}(q)$, $q \in M$.

To conclude, note that with $\tau \in [0,1]$ and $\sigma(t) = \tau t$ (4.2), (4.3) imply

$$\Lambda(e,x)(\tau) = \Psi(e, x(\tau)),$$

i.e. Ψ maps curves (e,x) to horizontal curves in E. It follows that the induced connection $\Psi_*^{-1}D$ on the bundle $\pi^{-1}(p) \times M \to M$ is trivial, hence so is D.

5. Holomorphic Tensor Fields. To prove Theorem 1.8 we first discuss the notion of order of vanishing. Let Y be a complex manifold, locally biholomorphic to open sets in Banach or even Fréchet spaces, $y \in Y$, and $f : Y \to \mathbb{C}$ holomorphic. We say that f vanishes at y to order n if for arbitrary $0 \le k < n$ and vector fields $v_1, \ldots, v_k$ on Y, holomorphic near y

$$(v_1 v_2 \ldots v_k f)(y) = 0.$$

If Y is connected and f vanishes at y to all orders then $f \equiv 0$.

Suppose f vanishes at y to order n. To see if it vanishes to order $n+1$, one is led to consider holomorphic vector fields $v_1, \ldots, v_n$ in a neighborhood of y and

$$(v_1 v_2 \ldots v_n f)(y). \tag{5.1}$$

Observe first that (5.1) is independent of the order in which the vector fields are applied (since e.g.

$$v_2 v_1 v_3 \ldots v_n f = v_1 v_2 \ldots v_n f - [v_1, v_2] v_3 \ldots v_n f = v_1 v_2 \ldots v_n f$$

at y); next that (5.1) vanishes if some v_i vanishes at y (since this is clearly so if $v_1(y) = 0$). It follows that (5.1) depends only on the values that the v_i take at y, and so (5.1) induces a symmetric n–linear map

$$d^n f(y) : T_y^n Y = T_y Y \oplus \ldots \oplus T_y Y \to \mathbb{C}.$$

Proof of Theorem 1.8(a). Constant maps $V \to M$ form a submanifold of Y, biholomorphic to M; we shall simply denote this manifold by $M \subset Y$. If $f : Y \to \mathbb{C}$ is holomorphic then by assumption $f|M$ is constant. At the price of subtracting this constant from f we can assume f vanishes at each point of M to first order. We shall prove by induction it vanishes at each $p \in M$ to arbitrary order.

Suppose f is already known to vanish to order $n \ge 1$ at each $p \in M$, so that the differentials $d^n f(p)$ are defined on $T_p^n Y$. We want to show $d^n f(p) = 0$, i.e.,

$$d^n f(p)(\eta_1, \ldots, \eta_n) = 0, \qquad \eta_i \in T_p Y,\ p \in M. \tag{5.2}$$

Note that by Subsection 1.1 $T_p Y$ is naturally isomorphic to $C^r(V, T_p M)$. With fixed $\varphi_1, \ldots, \varphi_n \in C^r(V, \mathbb{C})$ define a homomorphism $\Phi_n : T^n M \to T^n Y|M$ of holomorphic vector bundles

$$\Phi_n : T^n M \ni (\xi_1, \ldots, \xi_n) \mapsto (\varphi_1 \xi_1, \ldots, \varphi_n \xi_n) \in T^n Y|M; \tag{5.3}$$

the pullback of $d^n f$ by Φ_n is a symmetric holomorphic tensor field on M, of weight $n \ge 1$, hence vanishes. Therefore (5.2) holds when each η_i is of form $\varphi_i \xi_i$, and also when each η_i is a linear combination of such tangent vectors. When $\dim T_p M < \infty$, linear combinations

$$\sum_{j=1}^{k} \varphi^{(j)} \xi^{(j)}, \qquad k \in \mathbb{N},\ \varphi^{(j)} \in C^r(V, \mathbb{C}),\ \xi^{(j)} \in T_p M,$$

constitute all of $C^r(V, T_p M)$, and in general a dense subspace; whence indeed $d^n f(p) = 0$, $p \in M$. This means f vanishes to order $n+1$ along M, hence to all orders, and therefore $f = 0$ on Y.

For the rest of Theorem 1.8 we first extend the notions of vanishing order and higher differentials to tensor fields. Let now f be a holomorphic tensor field of weight j on the manifold Y. We say that f vanishes at $y \in Y$ to order $n \geq 0$ if for all $0 \leq k < n$ and holomorphic vector fields $v_1, \dots, v_k, w_1, \dots, w_j$, defined near y

$$v_1 \dots v_k f(w_1, \dots, w_j) = 0 \qquad \text{at } y.$$

Note that vanishing to order 0 is automatic. Suppose f does vanish to order n. As before,

$$(v_1 \dots v_n f(w_1, \dots, w_j))(y) \tag{5.4}$$

is symmetric in the v_l, and for fixed w_i, depends only on the values $v_l(y)$.

Proposition 5.1. *If some w_i vanishes at y then (5.4) vanishes.*

Proof. First observe that if F is a Fréchet space, h an F valued holomorphic function defined in a neighborhood of 0 in some $\mathbb{C}^q$, and $h(0) = 0$, then there are holomorphic functions $h_1, \dots, h_q$ such that

$$h(z_1, \dots, z_q) = \sum_{s=1}^{q} z_s h_s(z_1, \dots, z_q)$$

in a neighborhood of 0. Indeed,

$$h(z) = \int_0^1 \frac{d}{d\lambda} h(\lambda z) d\lambda = \sum_s z_s \int_0^1 \frac{\partial h}{\partial z_s}(\lambda z) d\lambda.$$

Now suppose, for concreteness, that $w_1(y) = 0$. Since as far as the v_l are concerned, (5.4) depends only on $v_l(y)$, we can assume that all v_l are tangent to a finite, say q, dimensional submanifold $Q \subset Y$ passing through y. After a local trivialization of TY the above observation gives holomorphic functions (local coordinates) $\zeta_1, \dots, \zeta_q$ on Q and holomorphic sections $h_1, \dots, h_q$ of $TY|Q$ near y, such that

$$\zeta_1(y) = \ldots = \zeta_q(y) = 0 \qquad \text{and} \qquad w_1|Q = \sum_s \zeta_s h_s.$$

Then Leibniz's rule implies

$$v_1 \dots v_n f(w_1, \dots, w_j) = \sum_s v_1 \dots v_n \{\zeta_s f(h_s, w_2, \dots, w_n)\} = 0$$

at y.

It follows that (5.4) depends only on (f and) the values that $v_1, \dots, w_j$ take at y; therefore (5.4) induces a multilinear map

$$d^n f(y) \colon T_y^{n+j} Y \to \mathbb{C}.$$

Proof of Theorem 1.8(b). Assume now f is a holomorphic tensor field of weight $j \geq 1$ on Y. Suppose we know f vanishes at all $p \in M$ to order $n \geq 0$. As before, a choice of $\varphi_1, \dots, \varphi_{n+j} \in C^r(V, \mathbb{C})$ defines a homomorphism $\Phi_{n+j} \colon T^{n+j} M \to T^{n+j} Y|M$, cf. (5.3). The pullback of $d^n f$ by Φ_{n+j} is a holomorphic tensor field on M, hence 0; from which it follows, as earlier, that $d^n f(p) = 0$, $p \in M$. Thus f vanishes to order $n + 1$ along M, so to all orders. This implies $f = 0$ on Y as claimed.

REFERENCES

[AW] M. Andreatta, J. Wiśniewski, *On manifolds whose tangent bundle contains an ample subbundle*, Invent. Math., 146 (2001), pp. 209–217.

[AK] C. Araujo, J. Kollár, *Rational curves on varieties*, Higher Dimensional Varieties and Rational Points, Bolyai Soc. Math. Studies 12, eds. K. Böröczky Jr., J. Kollár, T. Szamuely. Springer (2003), pp. 13–68.

[A] M. F. Atiyah, *Complex analytic connections in fibre bundles*, Trans. Amer. Math. Soc., 85 (1957), pp. 181–207.

[C] F. Campana, *On twistor spaces of class C*, J. Diff. Geom., 33 (1991), pp. 541–549

[DM] S. Dineen, P. Mellon, *Holomorphic functions on symmetric Banach manifolds of compact type are constant*, Math. Z., 229 (1998), pp. 753–765.

[Hm] P. Hartman, *Ordinary Differential Equations*, 2nd edition, Birkhäuser, 1982, Boston.

[Hs] R. Hartshorne, *Algebraic Geometry*, Springer, 1997, New York.

[Kd] K. Kodaira, *A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds*, Ann. of Math. (2), 75 (1962), pp. 146–162.

[Kl1] J. Kollár, *Rational Curves on Algebraic Varieties*, Springer, 1996, Berlin.

[Kl2] J. Kollár, *Low degree polynomial equations: arithmetic, geometry and topology*, European Congress of Mathematics, Vol. I (Budapest, 1996), pp. 255–288, Progr. Math., 168, Birkhäuser, Basel, 1998.

[KMM] J. Kollár, Y. Miyaoka, S. Mori, *Rationally connected varieties*, J. Alg. Geom., 1 (1992), pp. 429–448.

[KSz] J. Kollár, E. Szabó, *Rationally connected varieties over finite fields*, Duke Math. J., 120 (2003), pp. 251–167.

[L1] L. Lempert, *The Dolbeault complex in infinite dimensions I*, J. Amer. Math. Soc., 11 (1998), pp. 485–520.

[L2] L. Lempert, *Holomorphic functions on (generalised) loop spaces*, Math. Proc. Royal Irish Acad., 104A (2004), pp. 35–46

[LSz] L. Lempert, E. Szabó, *Rationally Connected Varieties and Loop Spaces*, manuscript: `http://www.arxiv.org/abs/math.AG/0507449`.

[M] D. Mumford, *Algebraic Geometry I. Complex Projective Varieties*, Springer, 1976, Berlin.

[S] E. Spanier, *Algebraic Topology*, McGraw–Hill, 1966, New York.