ASIAN J. MATH.
Vol. 11, No. 2, pp. 331–340, June 2007

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ON DIFFERENT NOTIONS OF HOMOGENEITY FOR CR-MANIFOLDS*

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Dedicated to M. Salah Baouendi on the occasion of his birthday

Abstract. We show that various notions of local homogeneity for CR-manifolds are equivalent. In particular, if germs at any two points of a CR-manifold are CR-equivalent, there exists a transitive local Lie group action by CR-automorphisms near every point.

Key words. CR-manifold, homogeneity, equivalence, definable, semianalytic

AMS subject classifications. 32Vxx, 32V05, 22E05

1. Introduction. The purpose of this paper is to show that various notions of local homogeneity for real-analytic CR-manifolds are in fact equivalent. The case of real-analytic hypersurfaces M in $\mathbb{C}^2$ has been considered by A. V. Loboda in [L98], where the equivalence of two different notions is shown, namely biholomorphic equivalence of germs of M at any two points and the existence of a transitive local Lie group action via biholomorphisms near every point of M. The proof is based on a refined Chern-Moser normal form [CM74] and convergence radius estimates due to V. K. Beloshapka and A. G. Vitushkin [BV81]. In this paper we extend this result to arbitrary real-analytic CR-manifolds, for which no such normal form is available in general. We also propose weaker homogeneity conditions based on the notion of "k-equivalence" introduced in [BRZ01a] and show that they still lead to an equivalent notion of local homogeneity.

These results appear to be in sharp contrast with the fact that different non-equivalent notions exist for *global* homogeneity. In fact, W. Kaup [K67] constructed an example of a domain $D \subset \mathbb{C}^2$, which is homogeneous in the sense that any two points are mapped into each other by a (global) biholomorphic automorphism of D but no (finite-dimensional) Lie group acts transitively on D via biholomorphic automorphisms.

We now briefly recall the necessary definitions to state our results. The reader is referred e.g. to the book [BER99a] for further details and related facts. An *(abstract) CR-manifold* is a real manifold M together with a formally integrable distribution $\mathcal{V}$ of its complexified tangent space $\mathbb{C}TM$ satisfying $\mathcal{V} \cap \overline{\mathcal{V}} = 0$, called the *CR-structure* (here $\overline{\mathcal{V}}$ denotes the complex conjugate subbundle). A *CR-map* between two CR-manifolds M and M' with CR-structures $\mathcal{V}$ and $\mathcal{V}'$ is any map $h\colon M \to M'$ with $h_*\mathcal{V} \subset \mathcal{V}'$, a CR-diffeomorphism is any diffeomorphism, which is CR together with its inverse, and a CR-automorphism is CR-diffeomorphisms from a manifold into itself. *All CR-manifolds and CR-maps in this paper will be assumed to be real-analytic.* This is motivated by our primary interest in homogeneous CR-manifolds and the fact that any CR-manifold admitting a transitive Lie group action by CR-automorphisms (or even a transitive local Lie group action, see below) is automatically real-analytic.

*Received July 31, 2006; accepted for publication March 2, 2007.
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It is well-known that a real-analytic CR-manifold is locally embeddable into $\mathbb{C}^N$ with suitable N such that its CR-structure is induced by the complex structure of $\mathbb{C}^N$ (which is a special case of a CR-structure with $\mathcal{V} = T^{(0,1)}\mathbb{C}^N$). This allows to pass from intrinsic to extrinsic point of view and vice versa, which we shall frequently do here.

Two germs (M, p) and (M', p') of CR-manifolds are said to be *CR-equivalent* if there exists a CR-diffeomorphism between open neighborhoods of p and M and of p' in M' sending p into p'. A weaker notion is that of a *formal CR-equivalence*, where (M, p) and (M', p') are said to be *formally CR-equivalent* if there exists an invertible formal power series map (in some and hence any local real-analytic coordinates on M and M') which sends $\mathcal{V}$ into $\mathcal{V}'$ in the formal sense. Yet more generally, (M, p) and (M', p') are said to be *k-equivalent*, where $k > 1$ is any integer, if there exists an invertible real-analytic map h between open neighborhoods of p and M and of p' in M' sending p into p' and sending $\mathcal{V}$ into $\mathcal{V}'$ "up to order k". The latter means that given a local frame $e_1(x), \ldots, e_n(x) \in \mathcal{V}_x$, $x \in M$, one can find a corresponding frame $e'_1(x'), \ldots, e'_n(x') \in \mathcal{V}_{x'}$, $x' \in M$, of $\mathcal{V}'$ such that $h_* e_j(x) = e'_j(h(x)) + O(|x|^k)$, where $x \in \mathbb{R}^{\dim M}$ is any local coordinate system vanishing at p. By a result of M. S. Baouendi, L. P. Rothschild and the author [BRZ01a, Corollary 1.2], the notions of being CR-equivalent, formally CR-equivalent and k-equivalent for all k are equivalent for germs of CR-manifolds at their points in *general position* (see Theorem 2.1 (iv)). On the other hand, a similar fact does not hold for more general real-analytic submanifolds in $\mathbb{C}^N$ in view of an example by J. Moser and S. Webster [MW83]. It is an open question whether the same conclusion holds for arbitrary CR-manifolds.

Another type of notion of local homogeneity is based on local Lie group actions. As is customary we always assume a Lie group to have at most countably many connected components. Recall that a (real-analytic) *local action* of a Lie group G with unit e on a manifold M is a neighborhood O of $\{e\} \times M$ and a real-analytic map $\varphi\colon O \to M$, $(g, x) \mapsto g \cdot x$, satisfying $e \cdot x = x$ and $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ whenever both sides are defined (see [P57] for further details on local group actions). A local Lie group action is said to be *transitive* if for every $p, q \in M$, there exists a finite sequence $g_1, \ldots, g_s \in G$ such that all expressions $R_j := g_j \cdot (g_{j-1} \cdot \ldots (g_1 \cdot p) \ldots)$ are defined for $1 \le j \le s$ and $R_s = q$. It is easy to see that if M is connected, a local Lie group action φ is transitive if and only if the differential φ_* sends $T_e G \subset T_{(e,p)}(G \times M)$ onto $T_p M$ for every $p \in M$.

Another, a priori weaker notion is based on the following generalization of a transitive local Lie group action that we state in a local form for germs:

DEFINITION 1.1. We say that a germ of CR-manifold (M, p) admits a *transitive family of local CR-automorphisms* if there exists a germ of a CR-map $\varphi\colon (M, p) \times (\mathbb{R}^{\dim M}, 0) \to (M, p)$ with $\varphi_*(T_p M) = \varphi_*(T_0 \mathbb{R}^{\dim M}) = T_p M$, where φ_* is taken at $(p, 0)$.

Finally we make use of (real-analytic) *infinitesimal CR-automorphisms* of M, which are (real-analytic) real vector fields on M whose local flows are 1-parameter families of CR-automorphisms (see e.g. [BER99a, §12.4] for more details).

We can now state our global result:

THEOREM 1.2. *Let M be a connected (real-analytic) CR-manifold. Then the following are equivalent:*

(i) for every $p, q \in M$, the germs (M, p) and (M, q) are k-equivalent for every k;
(ii) for every $p, q \in M$, the germs (M, p) and (M, q) are formally CR-equivalent;

(iii) *for every* $p, q \in M$*, the germs* (M, p) *and* (M, q) *are CR-equivalent;*
(iv) *for every* $p \in M$*, the germ* (M, p) *admits a transitive family of local CR-automorphisms;*
(v) *for every* $p \in M$*, the germs of all infinitesimal CR-automorphisms of* (M, p) *span the tangent space* T_pM*;*
(vi) *for every* $p \in M$*, there exists a finite-dimensional Lie algebra of germs of infinitesimal CR-automorphisms of* (M, p) *that spans the tangent space* T_pM*;*
(vii) *there exists a Lie group* G *and, for every* $p \in M$*, a transitive local action of* G *by CR-automorphisms on an open neighborhood of* p *in* M*.*

In our second main result we refine Theorem 1.2 stating all local homogeneity conditions for a germ of a CR-manifold (M, p), where the homogeneity means that some representative of the germ is locally homogeneous. It turns out that the weakest condition (i) in Theorem 1.2 can be here further weakened by requiring that only the germs of M at sufficiently many points are equivalent rather than all germs. A more precise definition is as follows.

Definition 1.3. Let M be a real-analytic CR-manifold and $p \in M$ be an arbitrary point. The *weak equivalence orbit* of p in M is the set of all $q \in M$ such that the germs (M, q) and (M, p) are k-equivalent for all k. We say that the germ (M, p) *satisfies condition* $(*)$ if for any open neighborhood U of p in M, the weak equivalence orbit of p in U is not contained in a real-analytic submanifold of U of smaller dimension.

We can now state our local result.

Theorem 1.4. *Let* (M, p) *be a germ of a real-analytic CR-manifold, where we write* M *for any representative. Then the following are equivalent:*
(i) (M, p) *satisfies condition* $(*)$*;*
(ii) *the weak equivalence orbit of* p *in* M *contains an open neighborhood of* p *in* M*;*
(iii) *for every* $q \in M$ *sufficiently close to* p*, the germs* (M, q) *and* (M, p) *are formally CR-equivalent;*
(iv) *for every* $q \in M$ *sufficiently close to* p*, the germs* (M, q) *and* (M, p) *are CR-equivalent;*
(v) (M, p) *admits a transitive family of local CR-automorphisms;*
(vi) *the germs of all infinitesimal CR-automorphisms of* (M, p) *span the tangent space* T_pM*;*
(vii) *there exists a finite-dimensional Lie algebra of germs of infinitesimal CR-automorphisms of* (M, p) *that spans the tangent space* T_pM*;*
(viii) *there exists a Lie group* G *and a transitive local action of* G *by CR-automorphisms on an open neighborhood of* p *in* M*.*

The proofs of Theorems 1.2 and 1.4 are given in §5. In §2 we state basic structure results for general CR-manifolds and their maps that play crucial role in the proofs. In §3 we recall a few definitions and facts about sets definable in terms of certain rings of functions, also needed for the proofs. In §4 we prove a proposition that represents the main technical core of the proofs of Theorems 1.2 and 1.4.

We conclude by mentioning that (locally) homogeneous CR-manifolds can be described in purely algebraic terms, e.g. in terms of the so-called “CR-algebras” considered in [MN05], see also [F06] for a local description.

2. Structure results for CR-manifolds and jet parametrization of CR-diffeomorphisms. We recall here some basic definition and structure results from [BRZ01a] for real-analytic CR-manifolds. We first note that any real-analytic CR-manifold can be locally embedded as a real-analytic generic submanifold into $\mathbb{C}^N$ for suitable N (see e.g. [BER99a, Chapter II]). (Recall that a real submanifold $M \subset \mathbb{C}^N$ is *generic* if $T_pM + iT_pM = T_p\mathbb{C}^N$ for every $p \in M$.) We thus give the extrinsic definitions for embedded generic submanifolds of $\mathbb{C}^N$ following [BRZ01a] that will suffice for our purposes (see e.g. [BER99a, Chapter XI] for an intrinsic approach). Let $\rho(Z,\overline{Z}) = (\rho^1(Z,\overline{Z}),\ldots,\rho^d(Z,\overline{Z}))$ be a vector-valued local defining function of M near a point p, i.e. with the rank of $\frac{\partial\rho}{\partial Z}$ being equal to the codimension of M. Recall that a $(0,1)$ vector field on M is any vector field of the form $L = \sum a_j(Z,\overline{Z})\frac{\partial}{\partial \overline{Z}_j}$ with $(L\rho)(Z,\overline{Z}) \equiv 0$ on M. In our case when M is real-analytic, it will be sufficient to consider only real-analytic vector fields.

Following [BRZ01a, §2.3], consider the vector subspace

$$E(p) := \operatorname{span}_{\mathbb{C}} \left\{ (L_1 \ldots L_s \rho^j_Z)(p,\overline{p}) : 1 \le j \le d; 0 \le s < \infty \right\} \subset \mathbb{C}^N, \tag{2.1}$$

where $L_1,\ldots,L_s$ run through all collections of $(0,1)$ vector fields and $\rho^j_Z(Z,\overline{Z}) \in \mathbb{C}^N$ denotes the complex gradient of ρ^j with respect to Z. The number $r_2(p) := N - \dim_{\mathbb{C}} E(p)$ is said to be the *degeneracy* of M at p and M is said to be of *minimum degeneracy* at a point $p_0 \in M$ if p_0 is a local minimum of the integer function $p \mapsto r_2(p)$. Recall that (M,p) is *finitely nondegenerate* if and only if $r_2(p) = 0$ (i.e. $E(p) = \mathbb{C}^N$) and is *l-nondegenerate* if l is the smallest integer such that $\mathbb{C}^N$ is spanned by the vectors $(L_1 \ldots L_s \rho^j_Z)(p,\overline{p})$ with $s \le l$.

Similarly consider the vector subspace $\mathfrak{g}_M(p)$ of the complexified tangent space $\mathbb{C}T_pM$ generated by the values at p of all $(0,1)$ vector fields, their conjugates and all finite order commutators involving $(0,1)$ vector fields and their conjugates. The corresponding number $r_3(p) := \dim_{\mathbb{R}} M - dim_{\mathbb{C}}\mathfrak{g}_M(p)$ is said to be the *orbit codimension* of M at p and M is said to be of *minimum orbit codimension* at a point $p_0 \in M$ if p_0 is a local minimum of the function $p \mapsto r_3(p)$. Recall that (M,p) is of *finite type* (in the sense of Kohn and and Bloom-Graham) if and only if $r_3(p) = 0$ (i.e. $\mathfrak{g}_M(p) = \mathbb{C}T_pM$).

The following theorem summarizes some of the results by M. S. Baouendi, L. P. Rothschild and the author [BRZ01a] that will be crucial for the proofs of both Theorems 1.2 and 1.4.

THEOREM 2.1. *Let M be a connected real-analytic CR-manifold and $V \subset M$ be the subset of all points $p \in M$ such that M is either not of minimum degeneracy or not of minimum orbit codimension at p. Then V is a (closed) proper real-analytic subvariety of M and there exist nonnegative integers N_1, N_2, N_3 and, for every $p \in M \setminus V$, a generic real-analytic submanifold $M' \subset \mathbb{C}^{N_1} \times \mathbb{R}^{N_2} \subset \mathbb{C}^{N_1+N_2}$ passing through 0 such that the following hold:*

- *(i) (M,p) is CR-equivalent to $(M' \times \mathbb{C}^{N_3}, 0)$;*
- *(ii) $(M',0)$ is finitely nondegenerate;*
- *(iii) for every $u \in \mathbb{R}^{N_2}$ near 0, one has $(0,u) \in M'$ and $M' \cap (\mathbb{C}^{N_1} \times \{u\})$ is a CR-manifold of finite type at $(0,u)$;*
- *(iv) if $(\widetilde{M},q)$ is another germ of a real-analytic CR-manifold such that (M,p) and $(\widetilde{M},q)$ are k-equivalent for any k, then they are also CR-equivalent.*

In fact, for $u \in \mathbb{R}^{N_2}$ near 0, the slice $(M' \cap (\mathbb{C}^{N_1} \times \{u\})$ represents the so-called *local CR-orbit* of M' at $(0,u)$. Recall that the local *CR orbit* of a point $q \in M'$ is the germ at q of a (real-analytic) submanifold of M' through q of smallest possible dimension to which all the $(0,1)$ vector fields on M are tangent. (The existence and uniqueness of a local CR-orbit is a consequence of a theorem of Nagano [N66], see also [BER99a, §3.1].) Note that in general M' cannot be locally written as a product of its CR-orbit and $\mathbb{R}^{N_2}$ since different CR-orbits may not be CR-equivalent (see [BRZ01b, §2] for an example).

Remark 2.2. The integers N_1, N_2, N_3 are uniquely determined by M, where N_2 is the minimum degeneracy and N_3 the minimum orbit codimension of M, see [BRZ01a].

We shall also need the following result from [BRZ01a] (see also [BRZ01b]) describing the behavior of CR-equivalences with respect to the decomposition provided by Theorem 2.1.

Theorem 2.3. *Let $M, M' \subset \mathbb{C}^{N_1} \times \mathbb{R}^{N_2}$ be generic real-analytic submanifolds of the same dimension passing through* 0*, both satisfying* (ii) *and* (iii) *of Theorem 2.1, i.e. such that both $(M,0)$ and $(M',0)$ are finitely nondegenerate and for every $u \in \mathbb{R}^{N_2}$ near* 0*, one has $(0,u) \in M \cap M'$ and $M \cap (\mathbb{C}^{N_1} \times \{u\})$ and $M' \cap (\mathbb{C}^{N_1} \times \{u\})$ are both of finite type at $(0,u)$. Let $H\colon (\mathbb{C}^{N_1} \times \mathbb{C}^{N_2} \times \mathbb{C}^{N_3}, 0) \to (\mathbb{C}^{N_1} \times \mathbb{C}^{N_2} \times \mathbb{C}^{N_3}, 0)$ be a germ of a biholomorphic map fixing* 0 *and sending $M \times \mathbb{C}^{N_3} \times \{0\}$ into $M' \times \mathbb{C}^{N_3} \times \{0\}$. Then H is of the form*

$$H(Z_1, Z_2, Z_3) = (H_1(Z_1, Z_2), H_2(Z_2), H_3(Z_1, Z_2, Z_3)), \quad (Z_1, Z_2, Z_3) \in \mathbb{C}^{N_1} \times \mathbb{C}^{N_2} \times \mathbb{C}^{N_3}, \tag{2.2}$$

where H_2 is a local biholomorphic map of $\mathbb{C}^{N_2}$ preserving $\mathbb{R}^{N_2}$ and for $u \in \mathbb{R}^{N_2}$ near 0, $H_1(\cdot, u)$ a local biholomorphic map of $\mathbb{C}^{N_1}$ sending $(M \cap (\mathbb{C}^{N_1} \times \{u\}), (0,u))$ into $(M' \cap (\mathbb{C}^{N_1} \times \{H_2(u)\}), (0, H_2(u))$ (both regarded as generic submanifolds of $\mathbb{C}^{N_1}$).

Our next main ingredient is a parametrization result from [BER99b] for local biholomorphisms between generic manifolds with parameters. (Further parametrization results of this kind can be found in [E01, ELZ03, KZ05, LM06, LMZ06].) Here we consider a *real-analytic family of generic submanifolds of $\mathbb{C}^N$*, which is a collection $\{M_x\}$ of generic submanifolds of $\mathbb{C}^N$ with parameter x from another real-analytic manifold X such that for every $x_0 \in X$ and $p \in M_{x_0}$, all manifolds M_x near p with $x \in X$ near x_0 can be defined by a family of defining functions $\rho(Z, \bar{Z}, x)$, which is real-analytic in all its arguments. We also write $J^k_{0,0}(\mathbb{C}^N, \mathbb{C}^N)$ for the space of all k-jets of holomorphic maps from $\mathbb{C}^N$ into itself with both source and target being 0.

Theorem 2.4. *Let M_x, $x \in X$, and $M'_{x'}$, $x' \in X'$, be real-analytic families of generic submanifolds through* 0 *in $\mathbb{C}^N$ of codimension d. Assume that, for some fixed points $x_0 \in X$ and $x'_0 \in X_0$,*

(i) M_{x_0} is of finite type at 0*;*

(ii) M'_{x_0} is l-nondegenerate at 0 *for some l.*

Set $k := l(d+1)$. Then for every invertible jet $\Lambda_0 \in J^k_{0,0}(\mathbb{C}^N, \mathbb{C}^N)$, there exist open neighborhoods Ω' of 0 *in $\mathbb{C}^N$, Ω'' of Λ_0 in $J^k_{0,0}(\mathbb{C}^N, \mathbb{C}^N)$, U of x_0 in X and U' of x'_0 in X', and a real-analytic map $\Psi\colon \Omega' \times \Omega'' \times U \times U' \to \mathbb{C}^N$ such that the identity*

$$H(Z) = \Psi(Z, j^k_0 H, x, x') \tag{2.3}$$

holds for any $x \in U$, $x' \in U'$, any local biholomorphism H of $\mathbb{C}^N$ fixing 0 *and sending M_x into $M'_{x'}$ and any $Z \in \Omega'$ sufficiently close to* 0*.*

Finally, in the setting of Theorem 2.4, it will be important to describe the sets of those jets that actually arise as jets of local biholomorphisms between $(M_x, 0)$ and $(M'_{x'}, 0)$. We shall make use of the following result, similar to [BER99b, Theorem 5.2.9] whose proof can be obtained by repeating the corresponding arguments in [BER99b]:

THEOREM 2.5. *Under the assumptions of Theorem 2.4, there exist open neighborhoods U of x_0 in X and U' of x'_0 in X', and finite sets of polynomials $a_j(\Lambda, \overline{\Lambda}, x, x')$ and $b_s(\Lambda, x, x')$ in $(\Lambda, \overline{\Lambda}) \in J^k_{0,0}(\mathbb{C}^N, \mathbb{C}^N) \times \overline{J^k_{0,0}(\mathbb{C}^N, \mathbb{C}^N)}$ with real-analytic coefficients in $(x, x') \in U \times U'$ such that the set of all $(\Lambda, x, x') \in J^k_{0,0}(\mathbb{C}^N, \mathbb{C}^N) \times U \times U'$, for which there exists a local biholomorphism H of $\mathbb{C}^N$ fixing 0 and sending M_x into $M'_{x'}$ with $j^k_0 H = \Lambda$, is given by*

$$\{a_j(\Lambda, \overline{\Lambda}, x, x') = 0 \text{ for all } j\} \setminus \{b_s(\Lambda, x, x') = 0 \text{ for all } k\}. \tag{2.4}$$

3. Definable and semianalytic sets. Here we collect some basic definitions and properties of sets definable over rings, in particular, of semianalytic sets. The readers is referred e.g. to [BM88] and the extensive literature cited there for proofs and further related facts.

Let $\mathcal{R}$ be a ring of real-valued functions on a set E. A subset $A \subset E$ is said to be *definable* over $\mathcal{R}$ if A can be written as $\cup_{j=1}^s \cap_{k=1}^r A_{jk}$, where each A_{jk} is either $\{f_{jk} = 0\}$ or $\{f_{jk} > 0\}$ with $f_{jk} \in \mathcal{R}$. In particular, a subset A in a real-analytic manifold M is called *semianalytic* if every point $p \in M$ has an open neighborhood U such that $A \cap U$ is definable over the ring of all real-analytic functions on U. It is elementary to see that any real-analytic subset is always semianalytic and that finite unions, intersections and complements of semianalytic sets are again semianalytic.

The following is a fundamental structure theorem for semianalytic sets:

THEOREM 3.1. *Every semianalytic set $A \subset M$ admits a stratification into a locally finite disjoint union of real-analytic submanifolds A_j of M, each being a semianalytic subset of M, and satisfying the "frontier condition": if $A_j \cap \overline{A_k} \neq \emptyset$, then $A_j \subset \overline{A_k}$ and $\dim A_j < \dim A_k$.*

As a consequence, the Hausdorff dimension $\dim A$ equals to the maximum stratum dimension. We shall use the following Lojaciewicz's version of the Tarski-Seidenberg theorem (see e.g. [BM88, §2]):

THEOREM 3.2. *Let $\mathcal{R}$ be a ring of functions on a set E and $\mathcal{R}[x_1, \ldots, x_k]$ be the corresponding polynomial ring on $E \times \mathbb{R}^k$. Denote by $\pi\colon E \times \mathbb{R}^k \to E$ the canonical projection. Then, if $A \subset E \times \mathbb{R}^k$ is definable over $\mathcal{R}[x_1, \ldots, x_k]$, its projection $\pi(A) \subset E$ is definable over $\mathcal{R}$.*

In particular, if M is a real-analytic manifold and $A \subset M \times \mathbb{R}^k$ is definable over the ring of polynomials in $(x_1, \ldots, x_k) \in \mathbb{R}^k$ with real-analytic coefficients in M, then its projection $\pi(A) \subset M$ is semianalytic.

Note that it is essential in Theorem 3.2 that A is definable over the ring of polynomials with real-analytic coefficients in M rather than the ring of all real-analytic functions on $M \times \mathbb{R}^k$, for which the corresponding conclusion would fail (see e.g. [BM88, §2] for an example).

4. Weak equivalence orbits and their properties. Here we consider weak equivalence orbits and condition $(*)$ as defined in Definition 1.3 and obtain its implications that will be crucial for the proofs of Theorems 1.2 and 1.4. As before M denotes a connected real-analytic CR-manifold and $p \in M$ its arbitrary point.

LEMMA 4.1. *Let (M,p) satisfy condition $(*)$. Then the weak equivalence orbit of p in M is not contained in any semianalytic subset $A \subset M$ with* $\dim A < \dim M$.

Proof. Without loss of generality, M is connected. Assume, by contradiction, that the weak equivalence orbit O of p in M is contained in a semianalytic subset $A \subset M$ of lower dimension. Fix a stratification of A into a locally finite disjoint union of real-analytic submanifolds A_j, that exists due to Theorem 3.1. Let m, $0 \le m < \dim M$, be the minimum integer such that O is contained in the union $\widetilde{A}$ of all strata of dimension not greater than m. Then there exists a point $q \in O$ which is contained in a stratum A_j of dimension precisely m. Now the "frontier condition" in Theorem 3.1 implies that q is not contained in the closure of any stratum A_k with $\dim A_k \le \dim A_j = m$. Hence, by the choice of m, there exists an open neighborhood Ω of q in M such that $O \cap \Omega \subset A_j$. Finally, by the definition of the weak equivalence orbit, the germs (M,q) and (M,p) are k-equivalent for any k. Hence they are also CR-equivalent in view of Theorem 2.1 (iv). Let $\varphi\colon U \to V$ be any CR-equivalence between open neighborhoods U and V of q and p respectively. Without loss of generality, $U \subset \Omega$. Then φ sends $O \cap U$ onto $O \cap V$ and therefore $O \cap V$ is contained in the low dimensional submanifold $\varphi(A_j \cap U)$ of V, which is a contradiction with condition $(*)$. The proof is complete. ☐

PROPOSITION 4.2. *Let (M,p) satisfy condition $(*)$. Then there exist integers N_1, N_2, N_3 and a generic real-analytic submanifold $\widetilde{M} \subset \mathbb{C}^{N_1}$ passing through* 0 *such that the following hold:*

(i) (M,p) is CR-equivalent to $(\widetilde{M} \times \mathbb{C}^{N_2} \times \mathbb{R}^{N_3}, 0)$;

(ii) $(\widetilde{M}, 0)$ is finitely nondegenerate and of finite type;

(iii) $(\widetilde{M}, 0)$ admits a transitive family of local CR-automorphisms.

Proof. Let $V \subset M$ be the proper real-analytic subvariety considered in Theorem 2.1. Since V is also a semianalytic subset of M of a smaller dimension, Lemma 4.1 implies that the weak equivalence orbit O of p in M is not contained in V. Let $q \in O \setminus V$ be any point. Then (M,q) is CR-equivalent to a germ $(M' \times \mathbb{C}^{N_3}, 0)$ as in Theorem 2.1. But since $q \in O$, the germs (M,q) and (M,p) are k-equivalent for any k and therefore also CR-equivalent by Theorem 2.1 (iv). Hence also (M,p) is CR-equivalent to $(M' \times \mathbb{C}^{N_3}, 0)$. Without loss of generality, we may assume $(M,p) = (M' \times \mathbb{C}^{N_3}, 0)$. Since (M,p) is assumed to satisfy condition $(*)$, $(M' \times \mathbb{C}^{N_3}, 0)$ also does and hence also $(M', 0)$ satisfies condition $(*)$ in view of Theorem 2.3.

We next consider for every $(q,u) \in M' \subset \mathbb{C}^{N_1} \times \mathbb{R}^{N_2}$, the submanifold

$$\widetilde{M}_{(q,u)} := \{Z_1 - q \in \mathbb{C}^{N_1} : (Z_1, u) \in M'\}, \tag{4.1}$$

passing through 0. It follows from our construction that $\widetilde{M}_{(q,u)}$, $(q,u) \in M'$, is a real-analytic family of generic submanifolds through 0 in $\mathbb{C}^{N_1}$ and that $\widetilde{M}_{(0,0)}$ is finitely nondegenerate and of finite type. Hence we can apply Theorem 2.5. As its consequence, we conclude that there exist an open neighborhood U' of $(0,0)$ in M' such that the set A of all $(\Lambda, x') \in J^k_{0,0}(\mathbb{C}^{N_1}, \mathbb{C}^{N_1}) \times U'$, for which there exists a local biholomorphism of $\mathbb{C}^{N_1}$ sending $(\widetilde{M}_{(0,0)}, 0)$ into $(\widetilde{M}_{x'}, 0)$ with $j^k_0 H = \Lambda$, is definable (in the sense of §3) over the ring of polynomials in $(\Lambda, \overline{\Lambda})$ with real-analytic coefficients in $x' \in U'$. Here as in Theorem 2.5 we set $k := l(d+1)$, where d is the codimension of $\widetilde{M}_{(0,0)}$ in $\mathbb{C}^{N_1}$ and l is such that $\widetilde{M}_{(0,0)}$ is l-nondegenerate at 0.

We now consider the natural projection $\pi\colon J^k_{0,0}(\mathbb{C}^{N_1}, \mathbb{C}^{N_1}) \times U' \to U'$. Then $\pi(A)$ is a semianalytic subset of U' by Theorem 3.2. We claim that $\pi(A)$ contains the weak equivalence orbit of 0 in M'. Indeed, let $x' \in U'$ be in the orbit. Note that, by our construction, M' is both of minimum degeneracy (in fact finitely nondegenerate) and of the minimum orbit codimension d at 0. Then, in view of Theorem 2.1 (iv), there exists a CR-equivalence H' between germs $(M', 0)$ and (M', x'), which extends to a biholomorphic map of $\mathbb{C}^{N_1} \times \mathbb{C}^{N_2}$ sending $(M', 0)$ into (M', x') (see e.g. [BER99a, Corollary 1.7.13] for the latter fact). By Theorem 2.3, H is of the form (2.2) (with $N_3 = 0$). Furthermore, it follows from the property of the component H_1 in the decomposition (2.2) and our construction (4.1) that $\widetilde{H}(Z_1) := H(Z_1) - q$ is a local biholomorphism of $\mathbb{C}^{N_1}$ sending $(\widetilde{M}_{(0,0)}, 0)$ into $(\widetilde{M}_{x'}, 0)$, where $x' = (q, u)$. But then $(j^k_0 \widetilde{H}, x') \in A$ and hence $x' \in \pi(A)$, proving our claim.

We can now make use of Lemma 4.1 and conclude that the semianalytic subset $\pi(A) \subset U'$ must have the top dimension $\dim U'$. Equivalently, $\pi(A)$ has a nonempty interior in U'. Furthermore, the set $A \in J^k_{0,0}(\mathbb{C}^{N_1}, \mathbb{C}^{N_1}) \times U'$ is also semianalytic and hence admits itself a stratification in the sense of Theorem 3.1. It follows that, in order for $\pi(A)$ to have a nonempty interior in U', there must exist a stratum A_j of A such that $\pi|_{A_j}\colon A_j \to U'$ is a submersion at some point of A_j. By the implicit function theorem, there exists an open set $\Omega \subset U'$ and a real-analytic map $\nu\colon \Omega \to J^k_{0,0}(\mathbb{C}^{N_1}, \mathbb{C}^{N_1})$ with $(\nu(x'), x') \in A_j \subset A$ for $x' \in \Omega$.

We next apply Theorem 2.4 giving a parametrization Ψ of local biholomorphisms H sending $(\widetilde{M}_x, 0)$ into $(\widetilde{M}_{x'}, 0)$, where we set $x_0 := 0$, pick arbitrary $x'_0 \in \Omega$ and set $\Lambda_0 := \nu(x'_0)$. Since for $x' \in \Omega$, we have $(\nu(x'), x') \in A$, there exists a local biholomorphism $H_{x'}$ of $\mathbb{C}^{N_1}$ sending $(\widetilde{M}_{(0,0)}, 0)$ into $(\widetilde{M}_{x'}, 0)$, which must therefore be given by the formula

$$H_{x'}(Z_1) = \Psi(Z_1, \nu(x'), 0, x').$$

Then, in view of (4.1), for $x' = (q, u) \in M' \subset \mathbb{C}^{N_1} \times \mathbb{R}^{N_2}$ close to x'_0, the map

$$Z_1 \mapsto \Psi(Z_1, \nu(x'), 0, x') + q \tag{4.2}$$

defines a local biholomorphism of $\mathbb{C}^{N_1}$ sending $(\widetilde{M}_{(0,0)}, 0)$ into $(M' \cap (\mathbb{C}^{N_1} \times \{u\}), x')$, the latter being regarded as a submanifold of $\mathbb{C}^{N_1}$.

Let $S \subset M'$ be any real-analytic submanifold through x'_0 satisfying

$$T_{x'_0} M' = T_{x'_0}\big(M' \cap (\mathbb{C}^{N_1} \times \{u\})\big) \oplus T_{x'_0} S.$$

Note that near x'_0, S is automatically totally real and its projection to $\mathbb{R}^{N_2}$ defines a local diffeomorphism at x'_0 between S and $\mathbb{R}^{N_2}$. Then the map

$$(Z_1, x') \in \mathbb{C}^{N_1} \times S \mapsto \big(\Psi(Z_1, \nu(x'), 0, x') + q, u\big) \in \mathbb{C}^{N_1} \times \mathbb{R}^{N_2},$$

where we keep the notation $x' = (q, u)$ as before, defines a local CR-equivalence between $(\widetilde{M}_{(0,0)}, 0) \times (S, x'_0)$ and (M', x'_0). Since S is totally real of dimension N_2, we conclude that (M', x'_0) is CR-equivalent to $(\widetilde{M}_{(0,0)} \times \mathbb{R}^{N_2}, (0,0))$. Furthermore, by our construction, $(\nu(x'_0), x'_0) \in A$ implying that $(M', 0)$ is CR-equivalent to (M', x'_0) and therefore to $(\widetilde{M}_{(0,0)}, 0) \times (\mathbb{R}^{N_2}, 0)$. This shows (i) and (ii) with $\widetilde{M} := \widetilde{M}_{(0,0)}$.

To show (iii), consider the family of local diffeomorphisms $\Phi_{x'}(Z_1) := \Psi(Z_1, \nu(x'), 0, x') + q$. We write $x'_0 = (q_0, u_0)$ and let $x' = (q, u_0) \in M' \cap (\mathbb{C}^{N_1} \times \{u_0\})$. Then, after a local identification of $(M' \cap (\mathbb{C}^{N_1} \times \{u_0\}, x'_0))$ with $(\mathbb{R}^m, 0)$, where m is the corresponding dimension, the map

$$(Z_1, q) \mapsto \Phi_{(q,u_0)} \circ \Phi^{-1}_{(q_0,u_0)}(Z_1)$$

defines a transitive family of local CR-automorphisms for $(M' \cap (\mathbb{C}^{N_1} \times \{u_0\}, x'_0))$. Finally, since $(M' \cap (\mathbb{C}^{N_1} \times \{u_0\}, x'_0))$ is CR-equivalent to $(\widetilde{M}, 0)$, the latter also admits a transitive family of local CR-automorphisms as desired. The proof is complete. □

5. Proofs of Theorems 1.2 and 1.4. We begin with Theorem 1.4. The implications (viii) ⇒ (vii) ⇒ (vi) ⇒ (v) ⇒ (iv) ⇒ (iii) ⇒ (ii) ⇒ (i) are obvious. The implication (i) ⇒ (v) is a consequence of Proposition 4.2.

To show (v) ⇒ (vi) set $m := \dim M$ and consider any transitive family $\varphi\colon (M, p) \times (\mathbb{R}^m, 0) \to (M, p)$ of CR-automorphisms as in Definition 1.1. Since (M, p) is real-analytic, we may assume it is embedded as a generic submanifold of $\mathbb{C}^N$. Then the germ of a CR-map φ extends to a germ of a holomorphic map $\Phi\colon (\mathbb{C}^N, p) \times (\mathbb{C}^m, 0) \to (\mathbb{C}^N, p)$. Differentiating Φ in the second component in the direction of the standard unit vectors in $\mathbb{R}^m$, we obtain m holomorphic vector fields $\mathbb{C}^N$ whose real parts are tangent to M. Hence their restrictions to M are infinitesimal CR-automorphisms. Furthermore, by the assumption $\varphi_*(T_0\mathbb{R}^m) = T_pM$ in Definition 1.1, the values of these vector fields at p span T_pM. This proves (vi).

To show (vi) ⇒ (vii), we note that, by Proposition 4.2, (M, p) is CR-equivalent to $(\widetilde{M} \times \mathbb{C}^{N_2} \times \mathbb{R}^{N_3}, 0)$ with suitable N_2 and N_3 such that $(\widetilde{M}, 0)$ satisfies (v) and hence also (vi) by the argument just before. Furthermore, $(\widetilde{M}, 0)$ is both finitely nondegenerate and of finite type in view of Proposition 4.2 (ii). Then a result by M.S. Baouendi, P. Ebenfelt and L.P. Rothschild [BER98] implies that the Lie algebra of all germs of infinitesimal CR-automorphisms of $(\widetilde{M}, 0)$ is finite-dimensional. Since $(\widetilde{M}, 0)$ satisfies (vi), this Lie algebra must span T_pM. Adding constant vector fields in the directions of $\mathbb{C}^{N_2}$ and $\mathbb{R}^{N_3}$ to this algebra, we easily conclude that also $(\widetilde{M} \times \mathbb{C}^{N_2} \times \mathbb{R}^{N_3}, 0)$ satisfies (vii). Since the latter germ is CR-equivalent to (M, p), we also have (vii) for (M, p).

Finally, given a finite-dimensional Lie algebra $\mathfrak{g}$ as in (vii), let G be the corresponding connected and simply connected Lie group. Then it is a well-known fact (Lie's Second Fundamental Theorem) that $\mathfrak{g}$ induces a local action of G in a neighborhood of p in M such that the transformations by elements of G correspond to local flows of the vector fields from $\mathfrak{g}$. Since $\mathfrak{g}$ consists of infinitesimal CR-automorphisms, we conclude that the action obtained is by CR-automorphisms as desired. The fact that the action of G is transitive easily follows from the assumption in (vii) that $\mathfrak{g}$ spans T_pM. This proves (viii), completing the proof of Theorem 1.4.

To prove Theorem 1.2, we first note that it follows directly from Theorem 1.4 that conditions (iv) – (vii) in Theorem 1.2 are equivalent. Furthermore, by the equivalence of (v) and (iv) in Theorem 1.4, it follows that (iv) in Theorem 1.2 implies that every p has a neighborhood $U(p)$ in M such that (M, q) is CR-equivalent to (M, p) for every $q \in U(p)$. Since M is connected, it is easy to see that its germs at any two points are CR-equivalent, proving (iii). Hence we have the implication (iv) ⇒ (iii) and the implications (iii) ⇒ (ii) ⇒ (i) are obvious. Finally, applying again Theorem 1.4, we see that (i) in Theorem 1.2 implies (iv) there. Hence all conditions in Theorem 1.2 are equivalent and the proof is complete.

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