ANALYTICITY OF SMOOTH CR MAPPINGS OF GENERIC SUBMANIFOLDS*

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Dedicated to Salah Baouendi for his seventieth birthday

Abstract. We consider a smooth CR mapping f from a real-analytic generic submanifold M in $\mathbb{C}^N$ into $\mathbb{C}^N$. For M of finite type and essentially finite at a point $p \in M$, and f formally finite at p, we give a necessary and sufficient condition for f to extend as a holomorphic mapping in some neighborhood of p. In a similar vein, we consider a formal holomorphic mapping H and give a necessary and sufficient condition for H to be convergent.

Key words. CR mappings, generic submanifolds, formal finite mappings

AMS subject classifications. 32H35, 32V40

1. Introduction. In this paper, we consider a smooth CR mapping f from a real-analytic generic submanifold M in $\mathbb{C}^N$ into $\mathbb{C}^N$. Assuming that M is of finite type and essentially finite at a point $p \in M$, and that f is formally finite at p (see below), we give a necessary and sufficient condition for f to extend as a holomorphic mapping (Theorem 1.1) in some neighborhood of p (or equivalently to be real-analytic near p in M). In a similar vein, we consider a formal holomorphic mapping H and give a necessary and sufficient condition for H to be convergent (Theorem 1.2).

Before stating the main results, we shall recall some definitions. Let M be a real-analytic submanifold of codimension d in $\mathbb{C}^N$. Recall that M is said to be *generic* if M is defined locally near any point $p \in M$ by defining equations $\rho_1(Z, \bar{Z}) = \ldots = \rho_d(Z, \bar{Z}) = 0$ such that $\partial\rho_1 \wedge \ldots \wedge \partial\rho_d \neq 0$ along M. A generic submanifold M is said to be of *finite type* at $p \in M$ (in the sense of Kohn [K72] and Bloom-Graham [BG77]) if the (complex) Lie algebra $\mathfrak{g}_M$ generated by all smooth $(1,0)$ and $(0,1)$ vector fields tangent to M satisfies $\mathfrak{g}_M(p_0) = \mathbb{C}T_pM$, where $\mathbb{C}T_pM$ is the complexified tangent space to M. For the definition of *essentially finite*, the reader is referred to [BER99a]; see also Section 2 for an equivalent formulation in a slightly more general setting.

A (C^∞) smooth mapping $f\colon M \to \mathbb{C}^N$ is called CR if the tangent mapping df sends the CR bundle $T^{0,1}M$ into $T^{0,1}\mathbb{C}^N$. In particular, the restriction to M of a holomorphic mapping $H\colon U \to \mathbb{C}^N$, where U is some open neighborhood of M, is CR. To define the notion of *formally finite* at a point $p \in M$, we may assume, without loss of generality, that $p = f(p) = 0$. If $f\colon (M, 0) \to (\mathbb{C}^N, 0)$ is a germ of a smooth CR mapping, then one may associate to f a formal mapping $H\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ as follows. Let x be a local coordinate on M near 0 and $x \mapsto Z(x)$ the local embedding of M into $\mathbb{C}^N$ near 0. Then, there is a unique formal mapping H such that the Taylor series of $f(x)$ at 0 equals $H(Z(x))$ (see e.g. [BER99a], Proposition 1.7.14). Recall that a formal holomorphic (or simply formal) mapping $H\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ is an N-tuple $H = (H_1, \ldots, H_N)$ with $H_j \in \mathbb{C}[\![Z_1, \ldots, Z_N]\!] = \mathbb{C}[\![Z]\!]$ such that each component H_j has no constant term. We shall use the notation $I(H_1, \ldots, H_N)$ (or

simply $I(H)$) for the ideal in $\mathbb{C}[\![Z]\!]$ generated by the components $H_1(Z), \ldots, H_N(Z)$. The formal mapping H is said to be *finite* if the ideal $I(H)$ is of finite codimension in $\mathbb{C}[\![Z]\!]$, i.e. if $\mathbb{C}[\![Z]\!]/I(H)$ is a finite dimensional vector space over $\mathbb{C}$. We shall say that the CR mapping f is *formally finite* if the associated formal mapping H is finite. The reader is referred to [BER99a] for further basic notions and properties of generic submanifolds in $\mathbb{C}^N$ and their mappings.

We may now state our first main result.

THEOREM 1.1. *Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold of dimension m that is essentially finite and of finite type at 0. Let $f \colon (M, 0) \to (\mathbb{C}^N, 0)$ be a smooth formally finite CR mapping. The following are equivalent.*

(i) *There exists an irreducible real-analytic subvariety $\tilde{X} \subset \mathbb{C}^N$ of dimension m at 0 such that $f(M) \subset \tilde{X}$ as germs at 0.*

(ii) *f is real-analytic in a neighborhood of 0.*

Our second result concerns the convergence of a formal holomorphic mapping sending a real-analytic generic submanifold into a real-analytic subvariety of the same dimension. Recall that a formal mapping $H \colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ is said to send a real-analytic generic submanifold M through 0 in $\mathbb{C}^N$ into a real-analytic subvariety $\tilde{X}$ through 0 in $\mathbb{C}^N$, denoted $H(M) \subset \tilde{X}$, if $\sigma(H(Z(x)), \overline{H(Z(x))}) \equiv 0$ as a power series in x, where x is a local coordinate on M near 0, $x \mapsto Z(x)$ the local embedding of M into $\mathbb{C}^N$ near 0, $\sigma(Z, \bar{Z}) = (\sigma_1(Z, \bar{Z}), \ldots, \sigma_k(Z, \bar{Z}))$, and $\sigma_1(Z, \bar{Z}), \ldots, \sigma_k(Z, \bar{Z})$ generate the ideal of germs at 0 of real-analytic functions vanishing on $\tilde{X}$.

THEOREM 1.2. *Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold of dimension m that is essentially finite and of finite type at 0. Let $H \colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ be a finite formal holomorphic mapping. The following are equivalent.*

(i) *There exists an irreducible real-analytic subvariety $\tilde{X} \subset \mathbb{C}^N$ of dimension m at 0 such that $H(M) \subset \tilde{X}$.*

(ii) *H is convergent in a neighborhood of 0.*

The proofs of (i) $\implies$ (ii) in Theorems 1.1 and 1.2 rest on general criteria for analyticity of smooth CR mappings and convergence of formal mappings given in [MMZ02] and [Mi02a], respectively, and a geometric result given in Lemma 3.1 below. A special case of (i) $\implies$ (ii) in Theorem 1.1, in which the additional hypothesis that the target $\tilde{X}$ is a real-analytic generic submanifold $\widetilde{M}$ of dimension m is imposed, can be proved by using known results as follows. In [Me95], it was shown that f is analytic provided that either f is CR transversal to $\widetilde{M}$ at 0, or $\widetilde{M}$ is essentially finite at 0. The desired conclusion then follows from a result in [ER05], where the CR transversality of f was proved under the hypotheses given. One of the difficulties in the general case addressed here is the fact that the target $\tilde{X}$ need not be smooth at 0 and, consequently, there is no notion of tranversality of the mapping. This is overcome by showing directly that the target must satisfy a generalized essential finiteness condition (see Lemma 3.1), and applying the result from [MMZ02] mentioned above.

The case where M and $\tilde{X}$ are real-analytic (non-singular) hypersurfaces has a long history, beginning with the work of Lewy [Le77] and Pinchuk [P77]. There are also many subsequent results implying analyticity of a smooth CR mapping when the target $\tilde{X}$ is a real-analytic generic submanifold $\widetilde{M}$ of $\mathbb{C}^{N'}$ under various hypotheses on $\widetilde{M}$ and f. We mention here only, in addition to [MMZ02], the works [DW80], [Han83],

[BJT85], [BR88], [DF88], [F89], [Pu90], [BHR96], [H96], [CPS00], [D01], [MMZ03b] and refer the reader to the bibliographies of these for further references. Previous results on convergence of formal mappings were given e.g. in the papers [BER00], [Mi00], [BMR02], [Mi02a], [Mi02b], [MMZ03a].

2. The essential variety of a real-analytic subvariety in $\mathbb{C}^N$. We begin by defining the notion of essential finiteness for a real-analytic subvariety X through 0 in $\mathbb{C}^N$. Let C_0^ω denote the ring of germs of real-valued real-analytic functions at 0 and $I_\mathbb{R}(X)$ be the ideal in C_0^ω of functions vanishing on X. Let $\sigma(Z, \bar{Z}) := (\sigma_1(Z, \bar{Z}), \ldots, \sigma_d(Z, \bar{Z}))$ be (representatives of) generators of $I_\mathbb{R}(X)$. We may assume that $\sigma(Z, \zeta)$ is defined in $B \times B$, where B is a sufficiently small open ball in $\mathbb{C}^N$ centered at the origin. Define the Segre variety $\Sigma_p \subset B$ of X at p, for $p \in B$, by the holomorphic equations $\sigma(Z, \bar{p}) = 0$. Note that Σ_0 is a complex analytic variety through 0 and, by the reality of the functions $\sigma(Z, \bar{Z})$, the variety Σ_p passes through 0 for every $p \in \Sigma_0$. Let $U \subset B$ be an open neighborhood of 0 and define

$$(2.1) \qquad E_0^U := \bigcap_{p \in \Sigma_0 \cap U} \Sigma_p.$$

Observe that E_0^U is a complex analytic variety through 0, and that $E_0^{U_1} \subset E_0^{U_2}$ (as a germ at 0) if $U_2 \subset U_1$. Moreover, the germ of E_0^U at 0 depends only on the germs at 0 of the subvarieties Σ_p for $p \in \Sigma_0$ and, hence, does not depend on the ball B. We say that X is *essentially finite* at 0 if E_0^U has dimension 0 (as a germ at 0) for every open neighborhood U of 0.

We shall show (see Proposition 2.1 below) that, even if X is not essentially finite at 0, there exists a neighborhood U_0 of 0 such that $E_0^U = E_0^{U_0}$ (as germs at 0) for every $0 \in U \subset U_0$. (This is well known in the case where X is a CR manifold.) Moreover, we shall give an alternative characterization of the stabilized germ $E_0^{U_0}$ that will be used in the proof of Theorem 1.1. Let A be the subvariety in B defined by

$$(2.2) \qquad A := \{z \in B \colon \Sigma_0 \subset \Sigma_z \text{ as germs at } 0\}.$$

PROPOSITION 2.1. *Let $X \subset \mathbb{C}^N$ be a real-analytic subvariety with $0 \in X$ and let A be defined by (2.2), where B is a sufficiently small ball centered at 0. Then, there exists an open neighborhood $U_0 \subset \mathbb{C}^N$ of 0 such that $E_0^U = A$, as germs at 0, for every open U with $0 \in U \subset U_0$. In particular, $E_0^U = E_0^{U_0}$ as germs at 0.*

Proof. [Proof of Proposition 2.1] Let U be an open neighborhood of 0 contained in B. Let us temporarily, in order to distinguish between germs and subvarieties, introduce the notation D^U for the following representative in U of the germ at 0 of E_0^U

$$(2.3) \qquad D^U := \bigcap_{p \in \Sigma_0 \cap U} (\Sigma_p \cap U).$$

Observe that there exists an open neighborhood U_0 of 0, contained in B, with the property that if V is a subvariety in B and U is an open neighborhood of 0 with $U \subset U_0$, then

$$(2.4) \qquad \Sigma_0 \subset V \text{ as germs at } 0 \iff \Sigma_0 \cap U \subset V \cap U.$$

Indeed, it suffices to take U_0 so small that only the irreducible components of Σ_0 in B that contain the origin meet U_0. As a consequence, if U is an open neighborhood

of 0 with $U \subset U_0$, then the subvariety $A \cap U$ can be expressed as being those points $Z \in U$ for which $\Sigma_0 \cap U \subset \Sigma_Z \cap U$, i.e.

$$(2.5) \qquad A \cap U = \{Z \in U : \sigma(W, \bar{Z}) = 0 \ \forall W \in U \text{ such that } \sigma(W, 0) = 0\}.$$

On the other hand, the subvariety D^U, defined by (2.3), can be expressed in equations as follows

$$(2.6) \qquad D^U = \{Z \in U : \sigma(Z, \bar{W}) = 0 \ \forall W \in U \text{ such that } \sigma(W, 0) = 0\}.$$

Since $\sigma(Z, \bar{W}) = \overline{\sigma(W, \bar{Z})}$, we conclude that $A \cap U = D^U$ and, hence, that $A = E_0^U$ as germs at 0. This completes the proof of Proposition 2.1 □

We shall refer to the germ at 0 of $E_0^{U_0}$ as the *essential variety* of X at 0 and denote this germ by E_0. Thus, X is essentially finite at 0 if and only if $E_0 = \{0\}$. When X is a generic submanifold, these two notions coincide with the standard notions of essential variety and essential finiteness (see e.g. [BER99a]).

We shall need the following reformulation of essential finiteness. In what follows, we shall let $\mathcal{X} \subset \mathbb{C}^N \times \mathbb{C}^N$ denote the (local) complexification of a real-analytic subvariety X through 0 in $\mathbb{C}^N$, i.e. the complex variety through 0 in $\mathbb{C}^N \times \mathbb{C}^N$ defined as the set of points $(Z, \zeta) \in \mathbb{C}^N \times \mathbb{C}^N$, near the origin, such that

$$\sigma_1(Z, \zeta) = \ldots = \sigma_d(Z, \zeta) = 0$$

where $\sigma_1(Z, \bar{Z}), \ldots \sigma_d(Z, \bar{Z})$ are generators of the ideal $I_{\mathbb{R}}(X)$. For a complex analytic subvariety V through 0 in $\mathbb{C}^k$, we shall also denote by $I_{\mathcal{O}}(V)$ the ideal of germs at 0 of holomorphic functions vanishing on V. We shall let $I(V)$ denote the ideal generated by $I_{\mathcal{O}}(V)$ in the corresponding ring of formal power series. If I is an ideal in a ring R, then we shall write $d(I)$ for the dimension of I in R, i.e. the dimension of the ring R/I (see [Ei95]; some texts, e.g. [AM69], refer to the number $d(I)$ as the *depth* of I).

LEMMA 2.2. *Let X be a real-analytic variety through 0 in $\mathbb{C}^N$ and let Σ_0 be its Segre variety at 0. The following are equivalent:*

(a) *X is not essentially finite at 0.*

(b) *There is a positive dimensional complex variety Γ through 0 in $\mathbb{C}^N$ such that $\Gamma \times \Sigma_0^* \subset \mathcal{X}$ as germs at 0, where $\mathcal{X} \subset \mathbb{C}^N \times \mathbb{C}^N$ denotes the complexification of X and * denotes the complex conjugate of a set (i.e. $S^* := \{Z \in \mathbb{C}^N : \bar{Z} \in S\}$).*

(c) *There is an ideal $J \subset \mathbb{C}[[Z]]$ with positive dimension such that $I(\mathcal{X}) \subset I(J) + I(\mathbb{C}^N \times \Sigma_0^*)$.*

(d) *There is a non-trivial formal holomorphic mapping $\mu \colon (\mathbb{C}, 0) \to (\mathbb{C}^N, 0)$ and a neighborhood $U \subset \mathbb{C}^N$ such that $\mu(\mathbb{C}) \subset E_0^U$ (where E_0^U is defined by (2.1)), i.e. if $\sigma(Z, \bar{Z}) = (\sigma_1(Z, \bar{Z}), \ldots, \sigma_d(Z, \bar{Z}))$ are generators of the ideal $I_{\mathbb{R}}(X)$, then, for every fixed $p \in \Sigma_0 \cap U$,*

$$(2.7) \qquad s \mapsto \sigma(\mu(s), \bar{p}) \text{ is identically zero as a power series.}$$

Proof. Recall that we use the coordinates (Z, ζ) in $\mathbb{C}^N \times \mathbb{C}^N$. Observe that for $p \in \mathbb{C}^N$, we have

$$(2.8) \qquad \mathcal{X} \cap \{\zeta = \bar{p}\} = \Sigma_p \times \{\bar{p}\}, \quad \mathcal{X} \cap \{Z = p\} = \{p\} \times \Sigma_p^*.$$

Hence, $\Gamma \times \Sigma_0^* \subset \mathcal{X}$ means that for every $p \in \Sigma_0$,

$$\Gamma \times \{\bar{p}\} \subset \mathcal{X} \cap \{\zeta = \bar{p}\} = \Sigma_p \times \{\bar{p}\},$$

i.e. $\Gamma \subset \Sigma_p$. The equivalence of (a) and (b) is a simple consequence of this observation.

The implication (b) $\implies$ (c) is easy. Simply observe that $\Gamma \times \Sigma_0^* = (\Gamma \times \mathbb{C}^N) \cap (\mathbb{C}^N \times \Sigma_0^*)$. The conclusion of (c) now follows from (b) by taking $J = I(\Gamma)$.

To prove the implication (c) $\implies$ (d), we let $\mu \colon (\mathbb{C}, 0) \to (\mathbb{C}^N, 0)$ be a germ at 0 of a non-trivial holomorphic mapping such that $f(\mu(t)) \equiv 0$ for all $f \in J$ (such exist, by [BER00], Lemma 3.32, since $d(J) \geq 1$). It follows from the hypothesis in (c) that there are formal power series $d_{ij}(Z, \zeta)$ such that

$$(2.9) \qquad \sigma_i(\mu(t), \zeta) = \sum_{j=1}^{k} d_{ij}(\mu(t), \zeta) h_j(\zeta),$$

where $h_1(\zeta), \ldots, h_k(\zeta)$ generate the ideal $I(\Sigma_0^*) \subset \mathbb{C}[[\zeta]]$. Let us Taylor expand $\sigma_i(\mu(t), \zeta)$ in t,

$$(2.10) \qquad \sigma_i(\mu(t), \zeta) = \sum_{l=0}^{\infty} a_{il}(\zeta) t^l,$$

and note that the coefficients $a_{il}(\zeta)$ are all holomorphic functions of ζ in some open neighborhood V of 0. If we Taylor expand both sides of (2.9) in t and compare coefficients, we conclude that the coefficents a_{il} all belong to the ideal $I(\Sigma_0^*) \subset \mathbb{C}[[\zeta]]$. Consequently, they vanish on $\Sigma_0^* \cap U$ for some open neighborhood $U \subset V$ of 0. This proves (d).

To show that (d) implies (a), let $\mu \colon (\mathbb{C}, 0) \to (\mathbb{C}^N, 0)$ be a non-trivial formal mapping such that $\mu(\mathbb{C}) \subset E_0^U$ for some U. Then $I(E_0^U)$ does not have finite codimension (see [BER00], Lemma 3.32) and, hence, the dimension of E_0^U is positive, i.e. (a) holds. $\square$

3. Main lemma. Recall that a formal mapping $H \colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ is said to send (a germ at 0 of) a real-analytic subvariety $X \subset \mathbb{C}^N$ into another $\tilde{X} \subset \mathbb{C}^N$, denoted $H(X) \subset \tilde{X}$, if

$$(3.1) \qquad \tilde{\sigma}\big(H(Z), \overline{H(Z)}\big) = a(Z, \bar{Z}) \sigma(Z, \bar{Z})$$

holds as formal power series in Z and $\bar{Z}$, where $\sigma = (\sigma_1, \ldots, \sigma_d)^t$ and $\tilde{\sigma} = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_{\tilde{d}})^t$ generate $I_{\mathbb{R}}(X)$ and $I_{\mathbb{R}}(\tilde{X})$, respectively, and $a(Z, \bar{Z})$ is a $\tilde{d} \times d$ matrix of formal power series.

Before stating our main lemma, we review some notation and results concerning homomorphisms induced by mappings. If $h \colon (\mathbb{C}_z^k, 0) \to (\mathbb{C}_{\tilde{z}}^k, 0)$ is a formal mapping, then h induces a homomorphism $\varphi_h \colon \mathbb{C}[[\tilde{z}]] \to \mathbb{C}[[z]]$ defined by $\varphi_h(\tilde{f})(z) := \tilde{f}(h(z))$. If J is an ideal in $\mathbb{C}[[z]]$, then $\varphi_h^{-1}(J)$ is an ideal in $\mathbb{C}[[\tilde{z}]]$, and $\varphi_h^{-1}(J)$ is prime if J is prime. If $\tilde{J}$ is an ideal in $\mathbb{C}[[\tilde{z}]]$, we denote by $I(\varphi_h(\tilde{J}))$ the ideal in $\mathbb{C}[[z]]$ generated by $\varphi_h(\tilde{J})$. If $\mathrm{Jac}\, h \not\equiv 0$, where $\mathrm{Jac}\, h$ denotes the Jacobian determinant of h, then φ_h is injective. Indeed, in that case any $f \in \mathbb{C}[[\tilde{Z}, \tilde{\zeta}]]$ for which $f \circ h \equiv 0$ must be identically zero (see e.g. [BER99a], Proposition 5.3.5).

We may now state the geometric result needed to prove Theorem 1.1.

LEMMA 3.1. *Let $X, \tilde{X} \subset \mathbb{C}^N$ be irreducible real-analytic subvarieties of dimension m through 0 and Σ_0, $\tilde{\Sigma}_0$ their Segre varieties at 0. Let $H: (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ be a finite formal mapping such that $H(X) \subset \tilde{X}$. Then the following hold.*

(i) *$\varphi_H^{-1}(I(\Sigma_0)) = I(\tilde{\Sigma}_0)$, where φ_H denotes the homomorphism $\varphi_H: \mathbb{C}[[\tilde{Z}]] \to \mathbb{C}[[Z]]$ induced by H. In particular, $\dim \tilde{\Sigma}_0 = \dim \Sigma_0$, and if Σ_0 is irreducible, then so is $\tilde{\Sigma}_0$.*

(ii) *If Σ_0 is irreducible at 0, then X is essentially finite at 0 if and only if $\tilde{X}$ is essentially finite at 0.*

Proof. [Proof of Lemma 3.1] We begin by proving statement (i). We denote by $\mathcal{H}: (\mathbb{C}^N \times \mathbb{C}^N, 0) \to (\mathbb{C}^N \times \mathbb{C}^N, 0)$ the complexified formal mapping $\mathcal{H}(Z, \zeta) = (H(Z), \bar{H}(\zeta))$ and by $\Phi = \varphi_{\mathcal{H}}: \mathbb{C}[[\tilde{Z}, \tilde{\zeta}]] \to \mathbb{C}[[Z, \zeta]]$ the induced homomorphism, i.e. $\Phi(\tilde{f})(Z, \zeta) := \tilde{f}(\mathcal{H}(Z, \zeta))$. The fact that H sends X into $\tilde{X}$ can be rephrased as saying that $I(\Phi(I(\tilde{\mathcal{X}}))) \subset I(\mathcal{X})$ or equivalently $I(\tilde{\mathcal{X}}) \subset \Phi^{-1}(I(\mathcal{X}))$, where $\tilde{\mathcal{X}}$ denotes the complexification of $\tilde{X}$. Since $\operatorname{Jac} \mathcal{H} \not\equiv 0$, Φ is injective. We also claim that $\mathbb{C}[[Z, \zeta]]$ is integral over $\Phi(\mathbb{C}[[\tilde{Z}, \tilde{\zeta}]])$. To see this, note, as is well known, that $\mathbb{C}[[Z, \zeta]]$ is finitely generated over $\Phi(\mathbb{C}[[\tilde{Z}, \tilde{\zeta}]])$ (any $f_1, \ldots, f_p \in \mathbb{C}[[Z, \zeta]]$ such that their images form a basis for the finite dimensional vector space $\mathbb{C}[[Z, \zeta]]/I(\mathcal{H}_1, \ldots, \mathcal{H}_{2N})$ generate $\mathbb{C}[[Z, \zeta]]$ over $\Phi(\mathbb{C}[[\tilde{Z}, \tilde{\zeta}]])$). The fact that $\mathbb{C}[[Z, \zeta]]$ is integral over $\Phi(\mathbb{C}[[\tilde{Z}, \tilde{\zeta}]])$ is now a general fact about finitely generated modules (see e.g. [AM69], Proposition 5.1). It follows that $d(\Phi^{-1}(I(\mathcal{X}))) = d(I(\mathcal{X}))$ (see e.g. [Ei95], Proposition 9.2). Moreover, $\Phi^{-1}(I(\mathcal{X}))$ is a prime ideal that contains the prime ideal $I(\tilde{\mathcal{X}})$. Since the dimensions of $\mathcal{X}$ and $\tilde{\mathcal{X}}$ are the same, we conclude that $d(\Phi^{-1}(I(\mathcal{X}))) = d(I(\tilde{\mathcal{X}}))$ and, hence,

$$\Phi^{-1}(I(\mathcal{X})) = I(\tilde{\mathcal{X}}). \tag{3.2}$$

Observe that $\mathcal{X} \cap \{\zeta = 0\} = \Sigma_0 \times \{0\}$. By using the specific form of the mapping $\mathcal{H}$, we see that

$$\varphi^{-1}(I(\Sigma_0)) = I(\tilde{\Sigma}_0) \iff \Phi^{-1}(I(\Sigma_0 \times \{0\})) = I(\Sigma_0 \times \{0\}), \tag{3.3}$$

where $\varphi = \varphi_H$. By the Nullstellensatz,

$$I(\Sigma_0 \times \{0\}) = \sqrt{I(\mathcal{X}) + I(\zeta)}. \tag{3.4}$$

Thus, the first identity in statement (i) is equivalent to

$$\sqrt{I(\tilde{\mathcal{X}}) + I(\tilde{\zeta})} = \Phi^{-1}(\sqrt{I(\mathcal{X}) + I(\zeta)}). \tag{3.5}$$

We remark that the inclusion $I(\tilde{\mathcal{X}}) \subset \Phi^{-1}(I(\mathcal{X}))$ implies $I(\tilde{\mathcal{X}}) + I(\tilde{\zeta}) \subset \Phi^{-1}(I(\mathcal{X}) + I(\zeta))$ and hence the inclusion

$$\sqrt{I(\tilde{\mathcal{X}}) + I(\tilde{\zeta})} \subset \Phi^{-1}(\sqrt{I(\mathcal{X}) + I(\zeta)}). \tag{3.6}$$

To prove the opposite inclusion, it suffices to show that $\Phi^{-1}(I(\mathcal{X}) + I(\zeta)) \subset \sqrt{I(\tilde{\mathcal{X}}) + I(\tilde{\zeta})}$. Thus, we suppose that $\Phi(\tilde{f}) \in I(\mathcal{X}) + I(\zeta)$ and must prove $\tilde{f}^k \in I(\tilde{\mathcal{X}}) + I(\tilde{\zeta})$ for some k. Since $\bar{H}$ is a formal finite mapping, there is an integer m such that $I(\zeta)^m \subset I(\Phi(I(\tilde{\zeta})))$. We conclude that, for any $k \geq m$,

$$\Phi(\tilde{f})^k = \Phi(\tilde{f}^k) \in I(\mathcal{X}) + I(\zeta)^k \subset I(\mathcal{X}) + I(\Phi(I(\tilde{\zeta}))). \tag{3.7}$$

Hence, we have

$$(3.8) \qquad \Phi(\tilde{f}^k) = g_k(Z, \zeta) + \sum_{j=1}^{N} a_{jk}(Z, \zeta) \bar{H}_j(\zeta),$$

where $g_k \in I(\mathcal{X})$ and $a_{jk} \in \mathbb{C}[[Z, \zeta]]$. Let (α_j, β_j), for $j = 0, \ldots, p$, be multi-indices such that the images of $Z^{\alpha_j} \zeta^{\beta_j}$ generate the finite dimensional vector space $\mathbb{C}[[Z, \zeta]]/I(\mathcal{H})$. Observe that $(0, 0)$ is necessarily one of these; we order the multi-indices in such a way that $(\alpha_0, \beta_0) = (0, 0)$. We may then, as is easy to verify, write each $a_j(Z, \zeta)$ in the following way

$$(3.9) \qquad a_{jk}(Z, \zeta) = \sum_{l=0}^{p} b_{jlk}(\mathcal{H}(Z, \zeta)) Z^{\alpha_l} \zeta^{\beta_l},$$

where $b_{00k}(0) = 0$ (since $\Phi(\tilde{f}^k)(0) = 0$). By substituting (3.9) in (3.8), it follows that

$$(3.10) \qquad \Phi(\tilde{f}^k) = g_k(Z, \zeta) + \sum_{l=0}^{p} \sum_{j=1}^{N} b_{jlk}(\mathcal{H}(Z, \zeta)) \bar{H}_j(\zeta) Z^{\alpha_l} \zeta^{\beta_l}.$$

Let S and $\tilde{S}$ denote rings $\mathbb{C}[[Z, \zeta]]/I(\mathcal{X})$ and $\mathbb{C}[[\tilde{Z}, \tilde{\zeta}]]/I(\tilde{\mathcal{X}})$, respectively. We shall denote by f^* and A^* the images of any element $f \in \mathbb{C}[[Z, \zeta]]$ and subset $A \subset \mathbb{C}[[Z, \zeta]]$ in S, and similarly for images in $\tilde{S}$. Since $\Phi(I(\tilde{\mathcal{X}}) \subset I(\mathcal{X})$, the homomorphism Φ induces a homomorphism $\Phi^*\colon \tilde{S} \to S$. The facts that Φ is injective and $\Phi^{-1}(I(\mathcal{X})) = I(\tilde{\mathcal{X}})$ imply that Φ^* is injective. Let $\tilde{R}$ denote the ring with identity generated by $I(\tilde{\zeta})^* \subset \tilde{S}$, i.e.

$$\tilde{R} := \{\tilde{g}^* + c^* \colon \tilde{g} \in I(\tilde{\zeta}), \ c \in \mathbb{C}\}.$$

Denote by R the ring $\Phi^*(\tilde{R}) \subset S$ and let $\mathcal{N}$ denote the R-module generated by $(Z_j^\alpha \zeta^{\beta_j})^*$ for $j = 0, \ldots, p$. Clearly, $\mathcal{N}$ is not annihilated by any elements of S. Let s denote $\Phi(\tilde{f}^m)^* \in S$ and observe that

$$(3.11) \qquad s = \Phi(\tilde{f}^m)^* = \sum_{l=0}^{p} \sum_{j=1}^{N} \left(b_{jlm}(\mathcal{H}(Z, \zeta)) \bar{H}_j(\zeta) Z^{\alpha_l} \zeta^{\beta_l} \right)^*.$$

We claim that $s\mathcal{N} \subset \mathcal{N}$. Indeed, if $f^* \in \mathcal{N}$, then

$$f^* = \sum_{i=0}^{p} \left(d_i(\mathcal{H}(Z, \zeta)) Z^{\alpha_i} \zeta^{\beta_i} \right)^*,$$

where $d_l = \tilde{g} + c$ for some $\tilde{g} \in I(\tilde{\zeta})$ and $c \in \mathbb{C}$, and hence

$$(3.12) \qquad s f^* = \sum_{l=0}^{p} \sum_{i=0}^{p} \sum_{j=1}^{N} \left(b_{jlm}(\mathcal{H}(Z, \zeta)) d_i(\mathcal{H}(Z, \zeta)) \bar{H}_j(\zeta) Z^{\alpha_l + \alpha_i} \zeta^{\beta_l + \beta_i} \right)^*.$$

Any monomial $Z^\alpha \zeta^\beta$ can be written

$$(3.13) \qquad Z^\alpha \zeta^\beta = \sum_{q=0}^{p} e_{\alpha\beta q}(\mathcal{H}(Z, \zeta)) Z^{\alpha_q} \zeta^{\beta_q}.$$

By substituting (3.13) in (3.12), we conclude that $sf^* \in \mathcal{N}$, as claimed. It follows from [Ei95], Corollary 4.6, that s is integral over R, i.e.

$$
(3.14) \qquad s^r + \sum_{k=0}^{r-1} (h_k(\mathcal{H}(Z, \zeta)))^* s^k = 0,
$$

where $h_k = \tilde{g}_k + c_k$ for $\tilde{g}_k \in I(\tilde{\zeta})$ and $c_k \in \mathbb{C}$. Since Φ^* is injective (and $s = \Phi(\tilde{f}^m)^*$), we conclude that

$$
(3.15) \qquad (\tilde{f}^{rm}(\tilde{Z}, \tilde{\zeta}))^* + \sum_{k=0}^{r-1} (h_k(\tilde{Z}, \tilde{\zeta}))^* (\tilde{f}^{km}(\tilde{Z}, \tilde{\zeta}))^* = 0.
$$

Since $h_k = \tilde{g}_k + c_k$, we can rewrite this as

$$
(3.16) \quad (\tilde{f}^{rm}(\tilde{Z}, \tilde{\zeta}))^* + \sum_{k=0}^{r-1} (g_k(\tilde{Z}, \tilde{\zeta}))^* (\tilde{f}^{km}(\tilde{Z}, \tilde{\zeta}))^* + \sum_{k=1}^{r-1} c_k^* (\tilde{f}^{km}(\tilde{Z}, \tilde{\zeta}))^* = -c_0^*.
$$

First, observe that the left hand side of (3.16) is in the maximal ideal of $\tilde{S}$ and, hence, $c_0^* = 0$. Let us write $c_r^* = 1$, and let k_0 be the smallest integer in $\{1, \ldots, r\}$ such that $c_{k_0}^* \neq 0$. We may then rewrite (3.16) as

$$
(3.17) \quad (\tilde{f}^{mk_0}(\tilde{Z}, \tilde{\zeta}))^* \left(\sum_{k=k_0}^{r} c_k^* (\tilde{f}^{m(k-k_0)}(\tilde{Z}, \tilde{\zeta}))^* \right) = -\sum_{k=0}^{r-1} (g_k(\tilde{Z}, \tilde{\zeta}))^* (\tilde{f}^{km}(\tilde{Z}, \tilde{\zeta}))^*.
$$

The right hand side of (3.17) is in $I(\tilde{\zeta})^*$ and, hence, so is then the left hand side. Observe that

$$
\tilde{f}^{m(r-k_0)}(\tilde{Z}, \tilde{\zeta}))^* + \sum_{k=k_0}^{r-1} c_k^* (\tilde{f}^{m(k-k_0)}(\tilde{Z}, \tilde{\zeta}))^*
$$

is a unit, since $c_{k_0}^* \neq 0$. We conclude that $(\tilde{f}^{mk_0})^* \in I(\tilde{\zeta})^*$ and, hence, $\tilde{f}^{mk_0} \in I(\tilde{\mathcal{X}}) + I(\tilde{\zeta})$, as desired. This completes the proof of the identity $\varphi_H^{-1}(I(\Sigma_0)) = I(\tilde{\Sigma}_0)$. We observe that if $I(\Sigma_0)$ is prime, then so is $I(\tilde{\Sigma}_0)$. The fact that $d(I(\tilde{\Sigma}_0)) = d(I(\Sigma_0))$ (or equivalently $\dim \Sigma_0 = \dim \tilde{\Sigma}_0$) follows from the assumption that H is a finite mapping (cf. the use of [Ei95], Proposition 9.2 above). This completes the proof of (i) in Lemma 3.1.

4. Proof of (ii) **of Lemma 3.1.** We now proceed with the proof of statement (ii) in Lemma 3.1. The fact that Σ_0 is assumed to be irreducible at 0 implies that $I(\Sigma_0)$ is a prime ideal. By part (i) of Lemma 3.1, it follows that $I(\tilde{\Sigma}_0)(= \varphi_H^{-1}(I(\Sigma_0)))$ is a prime ideal and $d(\varphi_H^{-1}(I(\Sigma_0))) = d(I(\Sigma_0))$. Equivalently, $\tilde{\Sigma}_0$ is an irreducible complex analytic variety of the same dimension as Σ_0.

We begin by proving that if X is essentially finite at 0, then $\tilde{X}$ is also essentially finite at 0. We shall show the logical negation of this statement. Thus, we shall assume that $\tilde{X}$ is not essentially finite at 0. By Lemma 2.2, there exists a positive dimensional complex analytic variety $\tilde{\Gamma} \subset \mathbb{C}^N$ through 0 such that $\tilde{\Gamma} \times \tilde{\Sigma}_0^* \subset \tilde{\mathcal{X}}$. We may assume that $\tilde{\Gamma}$ is irreducible.

LEMMA 4.1. *Let $\tilde{\Gamma}$ be as above and let $\tilde{\mathfrak{P}}$ be the (prime) ideal in $\mathbb{C}[\![\tilde{Z}, \tilde{\zeta}]\!]$ generated by the ideal of $\Gamma \times \tilde{\Sigma}_0^*$ (in the ring $\mathbb{C}\{\tilde{Z}, \tilde{\zeta}\}$ of convergent power series). Let $\Phi := \varphi_{\mathcal{H}}$ be the homomorphism defined in the beginning of the proof of Lemma 3.1. If*

$$(4.1) \qquad\qquad I(\Phi(\tilde{\mathfrak{P}})) = \bigcap_{j=1}^{m} P_j$$

is a primary decomposition of $I(\Phi(\tilde{\mathfrak{P}}))$, then there exists j_0 such that

$$(4.2) \qquad\qquad I(\mathcal{X}) \subset \sqrt{P_{j_0}} \ \text{ and } \ I(P_{j_0}) = d(\tilde{\mathfrak{P}}).$$

Proof. Suppose, in order to reach a contradiction, that this is not the case. Then, each of the ideals $I(\mathcal{X}) + P_j$ must satisfy

$$d(I(\mathcal{X}) + P_j) \leq d(P_j) - 1 \leq d(\tilde{\mathfrak{P}}) - 1.$$

It follows that

$$(4.3) \qquad\qquad d\left(\bigcap_{i=1}^{m} (I(\mathcal{X}) + P_j) \right) \leq d(\tilde{\mathfrak{p}}) - 1.$$

Consider the induced homomorphism

$$(4.4) \qquad\qquad \Phi^*\colon \mathbb{C}[\![\tilde{Z}, \tilde{\zeta}]\!]/I(\tilde{\mathcal{X}}) \to \mathbb{C}[\![Z, \zeta]\!]/I(\mathcal{X}).$$

If we, as above, use J^* to denote the image of an ideal $J \subset \mathbb{C}[\![\tilde{Z}, \tilde{\zeta}]\!]$ in $\mathbb{C}[\![\tilde{Z}, \tilde{\zeta}]\!]/I(\tilde{\mathcal{X}})$ and similarly for images in $\mathbb{C}[\![Z, \zeta]\!]/I(\mathcal{X})$, then we have

$$(4.5) \qquad\qquad I(\Phi^*(\tilde{\mathfrak{P}}^*)) = \left(\bigcap_{j=1}^{m} P_j \right)^*.$$

Observe that

$$\left(\bigcap_{j=1}^{m} P_j^* \right)^m \subset \left(\bigcap_{j=1}^{m} P_j \right)^* \subset \bigcap_{j=1}^{m} P_j^*.$$

Since $P_j^* = (P_j + I(\mathcal{X}))^*$ and, hence, $d(P_j^*) \leq d(\tilde{\mathfrak{P}}^*) - 1$, we conclude that

$$(4.6) \qquad\qquad d(I(\Phi^*(\tilde{\mathfrak{P}}^*))) \leq d(\tilde{\mathfrak{P}}^*) - 1.$$

We shall show that this is a contradiction. Recall that the homomorphism Φ^* is injective (see the proof of (i) above). Clearly, $\mathbb{C}[\![Z, \zeta]\!]/I(\mathcal{X})$ is integral over $\Phi^*(\mathbb{C}[\![\tilde{Z}, \tilde{\zeta}]\!]/I(\tilde{\mathcal{X}}))$, since $\mathbb{C}[\![Z, \zeta]\!]$ is integral over $\Phi(\mathbb{C}[\![\tilde{Z}, \tilde{\zeta}]\!])$. Now, the fact that (4.6) cannot hold is an immediate consequence of Lemma 4.2 below. $\square$

The proof of Lemma 4.1 follows, as mentioned above, from the following commutative algebra lemma. For the reader's convenience, we include a proof; we have been unable to find an exact reference for this statement.

LEMMA 4.2. *Let A and B be (commutative) rings, and $\psi\colon A \to B$ an injective homomorphism such that B is integral over $\psi(A)$. Then, for any ideal $J \subset A$,*

$$d(I(\psi(J))) = d(J).$$

Proof. [Proof of Lemma 4.2] Since ψ is injective, we may identify A with the subring $\psi(A) \subset B$. Thus, J is identified with its image $\psi(J)$ and $I(J)$ is the ideal in B generated by J. Let

$$(4.7) \qquad I(J) \subset \mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \ldots \subsetneq \mathfrak{q}_r,$$

be a chain of prime ideals. Then $\mathfrak{p}_i = \mathfrak{q}_i \bigcap A \ (= \psi^{-1}(\mathfrak{q}_i))$ are prime ideals and since $J \subset I(J) \bigcap A$, we obtain

$$(4.8) \qquad J \subset \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r.$$

By Corollary 4.18 in [Ei95], we also have strict inclusions $\mathfrak{p}_i \subsetneq \mathfrak{p}_{i+1}$ for $i = 0, \ldots, r-1$. It follows that $d(J) \geq d(I(J))$.

To prove the opposite inequality, let

$$(4.9) \qquad J \subset \mathfrak{p}'_0 \subsetneq \mathfrak{p}'_1 \subsetneq \ldots \subsetneq \mathfrak{p}'_s,$$

be a chain of prime ideals. By the Going Up Lemma (see [Ei95], Proposition 4.15), there is a prime ideal $\mathfrak{q}'_0$ in B such that $\mathfrak{p}'_0 = \mathfrak{q}'_0 \bigcap A$. Hence, $I(J) \subset \mathfrak{q}'_0$. By inductively applying the Going Up Lemma, we find prime ideals $\mathfrak{q}'_1, \ldots, \mathfrak{q}'_s$ such that $\mathfrak{p}'_i = \mathfrak{q}_i \bigcap A$ and

$$(4.10) \qquad I(J) \subset \mathfrak{q}'_0 \subset \mathfrak{q}'_1 \subset \ldots \subset \mathfrak{q}'_s.$$

Clearly, we have strict inclusions $\mathfrak{q}'_i \subsetneq \mathfrak{q}'_{i+1}$ for $i = 0, \ldots, s-1$ (since $\mathfrak{p}'_i = \mathfrak{q}_i \bigcap A$ and the $\mathfrak{p}_i$ are distinct). This proves the opposite inequality $d(I(J)) \geq d(j)$, which completes the proof of the lemma. $\square$

To complete the proof of statement (ii) in Lemma 3.1, we shall need the following lemma.

LEMMA 4.3. *Let* $\varphi := \varphi_H \colon \mathbb{C}[\![\tilde{Z}]\!] \to \mathbb{C}[\![Z]\!]$ *denote the homomorphism induced by the formal mapping* $H \colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ *and* $\psi := \varphi_{\bar{H}} \colon \mathbb{C}[\![\tilde{\zeta}]\!] \to \mathbb{C}[\![\zeta]\!]$ *the homomorphism induced by* $\bar{H} \colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$. *Every minimal prime* $\sqrt{P} = \sqrt{P_j}$ *in the factorization* (4.1) *is of the form*

$$(4.11) \qquad \sqrt{P} = I(\mathfrak{p}) + I(\mathfrak{q}),$$

where $\mathfrak{p}$ *is a minimal prime of* $I(\varphi(I(\tilde{\Gamma}))) \subset \mathbb{C}[\![Z]\!]$ *and* $\mathfrak{q}$ *is a minimal prime of* $I(\psi(I(\tilde{\Sigma}_0^*))) \subset \mathbb{C}[\![\zeta]\!]$.

Proof. We first observe that if $\mathfrak{p} \subset \mathbb{C}[\![Z]\!]$ and $\mathfrak{q} \subset \mathbb{C}[\![\zeta]\!]$ are prime ideals, then $I(\mathfrak{p}) + I(\mathfrak{q}) \subset \mathbb{C}[\![Z, \zeta]\!]$ is also prime. For, $\mathbb{C}[\![Z, \zeta]\!] = \mathbb{C}[\![Z]\!] \otimes \mathbb{C}[\![\zeta]\!]$ (where the tensor product is over $\mathbb{C}$) and, by Theorem III.14.35 of [ZS58],

$$(4.12) \qquad \mathbb{C}[\![Z, \zeta]\!]/(I(\mathfrak{p}) + I(\mathfrak{q})) = (\mathbb{C}[\![Z]\!]/\mathfrak{p}) \otimes (\mathbb{C}[\![\zeta]\!]/\mathfrak{q}).$$

Since $\mathfrak{p}$ and $\mathfrak{q}$ are prime, $\mathbb{C}[\![Z]\!]/\mathfrak{p}$ and $\mathbb{C}[\![\zeta]\!]/\mathfrak{q}$ are integral domains. It follows that $\mathbb{C}[\![Z]\!]/\mathfrak{p}) \otimes (\mathbb{C}[\![\zeta]\!]/\mathfrak{q})$ is an integral domain, since $(\mathbb{C}[\![Z]\!]/\mathfrak{p}) \otimes (\mathbb{C}[\![\zeta]\!]/\mathfrak{q}) \subset K \otimes K'$, where K and K' denote the quotient fields of $\mathbb{C}[\![Z]\!]/\mathfrak{p}$ and $\mathbb{C}[\![\zeta]\!]/\mathfrak{q}$ respectively, and $K \otimes K'$ is an integral domain by Corollary III.15.1 of [ZS58]. This proves that $I(\mathfrak{p}) + I(\mathfrak{q})$ is prime. By considering maximal chains of prime ideals containing $I(\mathfrak{p}) + I(\mathfrak{q})$ constructed in an obvious way from maximal chains of prime ideals containing $\mathfrak{p}$ and $\mathfrak{q}$, we also deduce that $d(I(\mathfrak{p}) + I(\mathfrak{q})) = d(\mathfrak{p}) + d(\mathfrak{q})$. Now, let $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ and

$\mathfrak{q}_1, \ldots, \mathfrak{q}_l$ be the minimal primes of $I(\varphi(I(\tilde{\Gamma})))$ and $I(\psi(I(\tilde{\Sigma}_0^*)))$, respectively, i.e. $d(\mathfrak{p}_i) = d(I(\varphi(I(\tilde{\Gamma}))))$, $d(\mathfrak{q}_j) = d(I(\psi(I(\tilde{\Sigma}_0^*))))$ and

$$\sqrt{I(\varphi(I(\tilde{\Gamma})))} = \bigcap_{i=1}^{k} \mathfrak{p}_i, \quad \sqrt{I(\psi(I(\tilde{\Sigma}_0^*)))} = \bigcap_{i=j}^{l} \mathfrak{q}_j.$$

To prove the decomposition (4.11), we shall show that

(4.13) $$\sqrt{I(\Phi(\tilde{\mathfrak{P}}))} = \bigcap_{i=1}^{k} \bigcap_{j=1}^{l} (I(\mathfrak{p}_i) + I(\mathfrak{q}_j)).$$

The uniqueness of the minimal primes of an ideal then implies that (4.11) must hold. The inclusion

$$\sqrt{I(\Phi(\tilde{\mathfrak{P}}))} \subset \bigcap_{i=1}^{k} \bigcap_{j=1}^{l} (I(\mathfrak{p}_i) + I(\mathfrak{q}_j))$$

is easy to prove and the details of this are left to the reader. Now, let $h \in \cap_{i,j}(I(\mathfrak{p}_i) + I(\mathfrak{q}_j))$. Fix a $j \in \{1, \ldots, l\}$ and let $f_i \in I(\mathfrak{p}_i)$, $g_i \in I(\mathfrak{q}_j)$ such that $h = f_i + g_i$ for $i = 1, \ldots, k$. It follows that

$$h^k = \prod_{i=1}^{k} (f_i + g_i).$$

Since $f_{i_1} \ldots f_{i_{k-r}} g_{i_{k-r+1}} \ldots g_{i_k}$ belongs to $I(\mathfrak{q}_j)$ whenever $r \geq 1$ and $f_1 \ldots f_k$ belongs to $\cap_i I(\mathfrak{p}_i)$, we conclude that

(4.14) $$h^k = f_j' + g_j',$$

where $f_j' \in \cap_i I(\mathfrak{p}_i)$ and $g_j' \in I(\mathfrak{q}_j)$ for every $j = 1, \ldots, l$. A similar argument shows that $h^{kl} = f + g$ with $f \in \cap_i I(\mathfrak{p}_i)$ and $g \in \cap_j I(\mathfrak{q}_j)$. We conclude that

$$h^{kl} \in \cap_i I(\mathfrak{p}_i) + \cap_j I(\mathfrak{q}_j)$$

or, equivalently,

(4.15) $$h \in \sqrt{\sqrt{I(\varphi(I(\tilde{\Gamma})))} + \sqrt{I(\psi(I(\tilde{\Sigma}_0^*)))}} = \sqrt{I(\Phi(\tilde{\mathfrak{P}}))}.$$

This proves (4.13) and, hence, also the lemma. $\square$

We now return to the proof of statement (ii) in Lemma 3.1. Let $P := P_{j_0}$ be a primary ideal in (4.1) as given by Lemma 4.1. Let $\mathfrak{p}$ and $\mathfrak{q}$ be as in Lemma 4.3 such that (4.11) is satisfied. In particular, $d(\mathfrak{p}) \geq 1$. By evaluating at $Z = 0$, we deduce that $I(\Sigma_0^*) \subset \mathfrak{q}$. Since

$$d(\mathfrak{q}) = d(I(\tilde{\Sigma}_0^*)) = d(I(\Sigma_0^*)),$$

where the latter identity follows from statement (i) in Lemma 3.1, and both $\mathfrak{q}$ and $I(\Sigma_0^*)$ are primes, we conclude that, in fact, $\mathfrak{q} = I(\Sigma_0^*)$. Hence, $I(\mathcal{X}) \subset I(\mathfrak{p}) + I(\mathbb{C}^N \times \Sigma_0^*)$. The fact that $\mathcal{X}$ is not essentially finite at 0 now follows from Lemma 2.2, part

(c) with $J = \mathfrak{p}$. This completes the proof of the implication "X is essentially finite at 0" $\implies$ "$\tilde{X}$ is essentially finite at 0".

To finish the proof of (ii), we must show the converse implication, namely "$\tilde{X}$ is essentially finite at 0" $\implies$ "X is essentially finite at 0". Again, we shall prove the logical negation of this statement. Thus, suppose that X is not essentially finite at 0 and let Γ be an irreducible complex analytic variety through 0 in $\mathbb{C}^N$ such that $\Gamma \times \Sigma_0^* \subset I(\mathcal{X})$. Let $\mathfrak{p} := I(\Gamma)$, $\mathfrak{q} := I(\Sigma_0^*)$, and observe, as above, that $\mathfrak{P} := I(\mathfrak{p}) + I(\mathfrak{q})$ is a prime ideal such that $I(\mathcal{X}) \subset \mathfrak{P}$. Let $\tilde{\mathfrak{P}}$ be the prime ideal $\Phi^{-1}(\mathfrak{P})$ and observe that $I(\tilde{\mathcal{X}}) \subset \tilde{\mathfrak{P}}$. Recall the homomorphisms $\varphi \colon \mathbb{C}[[\tilde{Z}]] \to \mathbb{C}[[Z]]$ and $\psi \colon \mathbb{C}[[\tilde{\zeta}]] \to \mathbb{C}[[\zeta]]$ induced by $H(z)$ and $\bar{H}(\zeta)$. We claim that

$$(4.16) \qquad \tilde{\mathfrak{P}} = I(\phi^{-1}(\mathfrak{p})) + I(\psi^{-1}(\mathfrak{q})).$$

Indeed, we have $\Phi^{-1}(I(\mathfrak{p})) + \Phi^{-1}(I(\mathfrak{q})) \subset \Phi^{-1}(\mathfrak{P})$, $\Phi^{-1}(I(\mathfrak{p})) = I(\phi^{-1}(\mathfrak{p}))$, and $\Phi^{-1}(I(\mathfrak{q})) = I(\psi^{-1}(\mathfrak{q}))$, which together imply the inclusion

$$(4.17) \qquad I(\phi^{-1}(\mathfrak{p})) + I(\psi^{-1}(\mathfrak{q})) \subset \tilde{\mathfrak{P}}.$$

Both sides of (4.17) are prime ideals and the following chain of identities of dimensions follow from the results above

$$d(\tilde{\mathfrak{P}}) = d(\mathfrak{P}) = d(\mathfrak{p}) + d(\mathfrak{q}) = d(\phi^{-1}(\mathfrak{p})) + d(\psi^{-1}(\mathfrak{q})) = d(I(\phi^{-1}(\mathfrak{p})) + I(\psi^{-1}(\mathfrak{q}))).$$

This implies the desired identity (4.16). By (i) of the lemma, we have $\psi^{-1}(\mathfrak{q})) = I(\tilde{\Sigma}_0^*)$. Since $d(\phi^{-1}(\mathfrak{p})) \geq 1$, an argument analogous to that used to complete the proof of the implication "X is essentially finite at 0" $\implies$ "$\tilde{X}$ is essentially finite at 0" above now shows that $\tilde{X}$ is not essentially finite at 0. This completes the proof of Lemma 3.1. $\square$

5. Proofs of Theorems 1.1 and 1.2.

Proof. [Proof of Theorem 1.1] We first prove the implication (i) $\implies$ (ii) in Theorem 1.1. Let $H \colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ be the formal holomorphic mapping associated to f sending M into $\tilde{X}$, and $\phi_H \colon \mathbb{C}[[\tilde{Z}]] \to \mathbb{C}[[Z]]$ the homomorphism induced by H. We shall use the following sufficient criterion from [MMZ02] (see Theorem 1.1 and Remark 1.2 in [MMZ02]) for f to be real-analytic at 0 (or, equivalently, extend holomorphically near 0): f is real-analytic near 0 if the germ at 0 of the variety

$$(5.1) \qquad C := \{\tilde{Z} \colon \phi_H(I(\tilde{\Sigma}_{\tilde{Z}})) \subset I(\Sigma_0)\}$$

reduces to the single point $\{0\}$. We should perhaps point out that if 0 does not belong to $\tilde{\Sigma}_{\tilde{Z}}$, then $I(\tilde{\Sigma}_{\tilde{Z}})$ is the whole ring $\mathbb{C}[[\tilde{Z}]]$ (and, hence, $\tilde{Z}$ does not belong to C). Observe that C can also be expressed as

$$(5.2) \qquad C := \{\tilde{Z} \colon I(\tilde{\Sigma}_{\tilde{Z}})) \subset \phi_H^{-1}(I(\Sigma_0))\}$$

By using Lemma 3.1 (i), we observe that $I(\tilde{\Sigma}_{\tilde{Z}})) \subset \phi_H^{-1}(I(\tilde{\Sigma}_0))$ implies that $I(\tilde{\Sigma}_{\tilde{Z}})) \subset I(\tilde{\Sigma}_0)$, i.e. $\tilde{\Sigma}_0 \subset \tilde{\Sigma}_{\tilde{Z}}$. It follows that $C \subset \tilde{A}$ as germs at 0, where $\tilde{A}$ is given by (2.2) using $\tilde{X}$ instead of X. Since M is assumed to be essentially finite at 0, it follows that $\tilde{X}$ is also essentially finite at 0 by Lemma 3.1 (ii). By Proposition 2.1, the germ of $\tilde{A}$ at 0, and hence that of C, reduces to the point $\{0\}$. The real-analyticity of f near 0 now follows from the above mentioned result from [MMZ02].

To prove (ii) $\implies$ (i), we let $H\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ be the finite holomorphic mapping extending f near 0. We further let $\mathcal{M} \subset \mathbb{C}^N \times \mathbb{C}^N$ denote the local complexification of M near 0 and $\mathcal{H}\colon (\mathbb{C}^N \times \mathbb{C}^N, 0) \to (\mathbb{C}^N \times \mathbb{C}^N, 0)$ the complexification of the mapping H. Since $\mathcal{H}$ is a finite holomorphic mapping and $\mathcal{M}$ a complex submanifold, we conclude, by Remmert's proper mapping theorem, that the image $\tilde{\mathcal{X}} := \mathcal{H}(\mathcal{M})$ is an irreducible complex analytic subvariety of the same (complex) dimension as $\mathcal{M}$. Let $\tilde{X}$ denote the real-analytic subvariety of $\mathbb{C}^N$ obtained by intersecting $\tilde{\mathcal{X}}$ with the anti-diagonal $\{(\tilde{Z}, \tilde{\zeta})\colon \tilde{\zeta} = \bar{\tilde{Z}}\}$. It is easy to check that H maps M into $\tilde{X}$. Moreover, since $\det(\partial H/\partial Z)$ does not vanish identically on M, the dimension of $\tilde{X}$ is at least that of M. On the other hand, since $\tilde{X}$ is the intersection between the totally real manifold $\{(\tilde{Z}, \tilde{\zeta})\colon \tilde{\zeta} = \bar{\tilde{Z}}\}$ and $\tilde{\mathcal{X}}$, its dimension is at most equal to the complex dimension of $\tilde{\mathcal{X}}$. Since

$$(5.3) \qquad \dim_{\mathbb{C}} \tilde{\mathcal{X}} = \dim_{\mathbb{C}} \mathcal{M} = \dim_{\mathbb{R}} M,$$

we conclude that $\dim_{\mathbb{R}} \tilde{X} = \dim_{\mathbb{R}} M$. The fact that $\tilde{X}$ is irreducible now follows from the fact that $\tilde{\mathcal{X}}$ is its complexification and $\tilde{\mathcal{X}}$ is irreducible. This completes the proof of Theorem 1.1. ▢

Proof. [Proof of Theorem 1.2] The proof of (ii) $\implies$ (i) in Theorem 1.2 is exactly the same as the proof of (ii) $\implies$ (i) in Theorem 1.1. The proof of (i) $\implies$ (ii) in Theorem 1.2 is follows the same reasoning as that of the proof of (i) $\implies$ (ii) in Theorem 1.1, except that we apply a result from [Mi02a] instead of the result from [MMZ02] used in the proof above. Indeed, Theorem 9.1 in [Mi02a] combined with the remark preceding it shows that the formal mapping H in Theorem 1.2 is convergent if the variety C given by (5.1) reduces to the single point $\{0\}$. This completes the proof of Theorem 1.2. ▢

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