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# NON-UNIFORM CONTINUITY IN $H^1$ OF THE SOLUTION MAP OF THE CH EQUATION*

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**Abstract.** We show that the solution map of the Camassa-Holm equation is not uniformly continuous in the initial data in the Sobolev space of order one on the torus and the real line. The proof relies on a construction of non-smooth travelling wave solutions. We also extend to all $H^s$ an earlier result known to hold for peakons.

**Key words.** Solution map, uniform continity, Camassa-Holm equation, travelling waves, Sobolev spaces

**AMS subject classifications.** Primary: 35Q53

**1. Introduction and statement of the result.** We study the Cauchy problem for the nonlinearly dispersive Camassa-Holm equation

$$\begin{aligned} &\partial_t u + u\partial_x u + (1-\partial_x^2)^{-1}\partial_x\big(u^2 + \frac{1}{2}(\partial_x u)^2\big) = 0, \\ &u(0) = u_0, \qquad t \geq 0, \;\; x \in \mathbb{T} \text{ or } \mathbb{R}. \end{aligned} \tag{1.1}$$

This equation appeared initially in the context of hereditary symmetries studied by Fuchssteiner and Fokas [FF]. However, it was first written explicitly as a water wave equation by Camassa and Holm [CH], who also studied its "peakon" solutions (see formula (1.2)).

In order to put our work in context it will be helpful to summarize the relevant known results concerning local well-posedness of this equation. In the periodic case the Cauchy problem (1.1) is locally well-posed in the Sobolev space $H^s(\mathbb{T})$ if $s > 3/2$ (see for example [HM1], Danchin [D] or [Mi]), while if $1 \leq s \leq 3/2$ then it is locally well-posed in $H^s(\mathbb{T}) \cap \mathrm{Lip}(\mathbb{T})$ (see DeLellis, Kappeler and Topalov [DKT]) and the solution $u$ depends continuously on initial data $u_0$ in the $H^s$-norm. Furthermore, it is also known that the problem (1.1) is locally well-posed in $C^1(\mathbb{T})$ with solutions depending continuously on the data in the $C^1$-norm (see [Mi]).

Similarly, if $s > 3/2$ then the non-periodic Cauchy problem (1.1) is locally well-posed in $H^s(\mathbb{R})$ with solutions depending continuously on initial data (see Constantin and Escher [CoE], Li and Olver [LO], Rodriguez-Blanco [R], [D] or a survey in Molinet [Mo]).

On the other hand, it was recently shown in [HM3] that for $s \geq 2$ the data-to-solution map $u_0 \to u$ of (1.1) is not uniformly continuous from any bounded set in $H^s(\mathbb{T})$ into $C([0,T], H^s(\mathbb{T}))$. Therefore, in this Sobolev range continuous dependence on the data is the best one can expect. A key step in the proof of that result was a construction of a sequence of smooth travelling wave solutions of the form $u(x,t) = f(x-t)$ depending on two parameters $\varepsilon$ and $\delta$, which were related to the maximum

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and the amplitude of the solution. This construction was motivated by the fact that the CH equation has only one scaling parameter (namely if $u = u(x,t)$ is a solution then $u_c = cu(x, ct)$ is also a solution, for any constant $c$). An earlier result of [HM1] used this simple scaling and the less regular "peakon" solution

$$u(x,t) = ce^{-|x-ct|} \tag{1.2}$$

as well as its periodic analogue, to produce two sequences of solutions in $H^s$ with $s < 3/2$, whose distance in $H^s$ approached zero at the initial time while growing to infinity at any other time but which could not be confined to any ball in $H^s$. In fact, in Section 2 we will show that the assumption $s < 3/2$ can be dropped once the "peakon" solution is replaced with a suitable smooth travelling wave solution. As already mentioned, the result of [HM3] used sequences of smooth solutions. It turns out however that such sequences cannot work in $H^1$. This will also be explained in Section 2.

In this paper we will construct two appropriate sequences of non-smooth solutions and use them to provide a straightforward proof that the data-to-solution map of the CH equation is not uniformly continuous in the space $H^1$ on the torus as well as on the real line.

THEOREM 1.1. *In both periodic and non-periodic cases the data-to-solution map $u_o \to u$ of the Camassa-Holm equation is not uniformly continuous from any bounded set in $H^1$ into $C([0,T], H^1)$.*

An argument for the periodic case can be found in Byers [B]. The proof of Theorem 1.1 given here is more transparent and is easily extended to the case of the real line.

A few more remarks are in order. First, observe that a result of this type sets a restriction on the possible way of obtaining local well posedness results. More precisely, the fact that the data-to-solution map is not uniformly continuous on a Banach space $X$ tells us that the local wellposedness in $X$ cannot be established by a solely contraction principle argument.

In [KPV2], two parameter families of explicit solutions of the KdV, its modified version (mKdV) and the semi-linear Schrodinger equations were used to prove that the data-to-solution map is not uniformly continuous in Sobolev spaces $H^s(R)$ with indices ($s < -3/4$ for KdV, $s < 1/4$ for mKdV and $s < 0$ for 1-D cubic NLS) which are larger than the value suggested by the scaling argument ($s = -3/2$ for KdV, $s = -1/2$ for mKdV and $s = -1/2$ for the 1-D cubic NLS). In particular, this result showed that the "strong" local well posedness available for these equations (see [CW], [KPV1], and [KPV2] and references therein) are the best possible in the Sobolev scale.

In [MST], Molinet, Saut and Tzvetkov showed that for the data-to-solution map the Cauchy problem for the Benjamin–Ono equation fails to be smooth in any Sobolev space $H^s(\mathbb{R}), s \in \mathbb{R}$. As it was remarked above, this implies that the local wellposedness in those spaces cannot be obtained by a direct iteration scheme based on the Duhamel formula.

Finally, we should mention that equation (1.1) exhibits many other remarkable properties. For example, it is known to admit solutions that blow up in finite time, see McKean [Mc]. For a result on the stability of the peakon solution in $H^1$ we refer to Constantin and Strauss [CS]. Moreover, recent results on unique continuation properties of (1.1) are proved in [HMPZ]. Many other results and references can be found in the survey article [Mo].

The remainder of the paper is structured as follows. In the next section we summarize the construction of smooth travelling wave solutions and generalize a result from [HM1]. In section 3 we construct non-smooth traveling wave solutions and prove Theorem 1.1 in the periodic case. The last section contains the proof in the non-periodic case.

**2. Smooth traveling waves and dependence in $H^s$.** We begin by rewritting the CH equation in its local form

$$\partial_t u - \partial_t \partial_x^2 u + 3u\partial_x u - 2\partial_x u \partial_x^2 u - u\partial_x^3 u = 0. \tag{2.1}$$

Looking for traveling wave solutions of the form

$$u(x,t) = f(x-t)$$

we find that $f$ must satisfy the differential equation

$$(1-f){f'}^2 = -f^3 + f^2 + af + b, \quad a, b \in \mathbb{R}. \tag{2.2}$$

Substituting

$$y = 1 - f \tag{2.3}$$

and choosing the parameters $a$, $b$ appropriately we obtain

$$(y')^2 = \frac{(\delta+\varepsilon-y)(y-\delta)(2-2\delta-\varepsilon-y)}{y}. \tag{2.4}$$

In fact, equation (2.4) admits a non-constant solution of period $2\ell$, for some $\ell > 0$, which satisfies the following second order initial value problem

$$y'' = y - 1 + \frac{\delta(\delta+\varepsilon)(2-2\delta-\varepsilon)}{2y^2}, \qquad y(0) = \delta, \quad y'(0) = 0. \tag{2.5}$$

We summarize this in the following lemma, whose proof can be found in [HM3].

Lemma 2.1. *For any $0 < \varepsilon, \delta < 1/5$ there exist a positive number $\ell = \ell(\varepsilon, \delta)$ and an even $2\ell$-periodic smooth function $y = y(x)$ which solves the initial value problem (2.5) and equation (2.4). Moreover, the function $y$ satisfies the bounds (see Figure 1)*

$$\delta \le y(x) \le \delta + \varepsilon$$

*and the function $u(x,t) = f(x-t)$, where $f(x) = 1 - y(x)$, is a travelling wave solution of the CH equation. Finally, the half-period $\ell$ satisfies*

$$\ell = \int_\delta^{\delta+\varepsilon} \sqrt{\frac{y}{(\delta+\varepsilon-y)(y-\delta)(2-2\delta-\varepsilon-y)}}\, dy \simeq \sqrt{\delta+\varepsilon}.$$

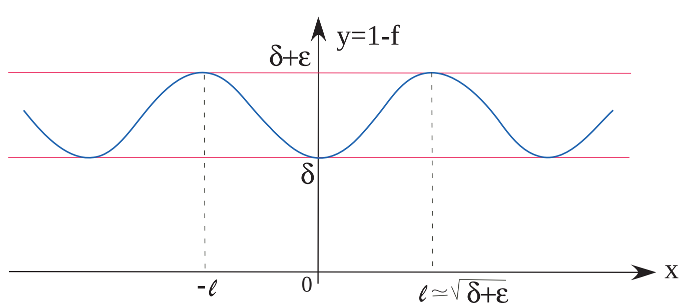


Fig. 1

For a given positive integer $n$ choosing $\delta$ and $\varepsilon$ such that

$$\frac{\pi}{n} = \ell \simeq \sqrt{\delta + \varepsilon}$$

we obtain a $2\pi$ periodic solution with frequency equal to $n$.

It is also worth pointing out that these traveling wave solutions are in fact globally analytic functions in both variables $x$ and $t$, since the solution $y$ to the initial value problem (2.5) is an analytic function on the torus. Using a method of Baouendi and Goulaouic [BG] it was shown in [HM2] that solutions to the CH equation with analytic initial data are globally analytic in $x$ and locally analytic in $t$.

Next, with the help of one of the smooth traveling wave solutions described in Lemma 2.1 we can now generalize Theorem 1 in [HM1] so that it holds for arbitrary Sobolev index $s$.

THEOREM 2.2. *There exist two sequences of smooth traveling wave solutions such that at time $t = 0$ their distance in $H^s$-norm goes to zero, while for any $t > 0$ the distance goes to infinity.*

*Proof.* For fixed $\delta$ and $\varepsilon$ let $f$ be the solution constructed in Lemma 2.1. Then $u_c(x,t) = cf(x-ct)$ is a smooth traveling wave solution of the CH equation (also, see Lenells [L] for a similar construction of such solutions). A simple computation shows

$$\widehat{u}_c(\xi,t) = \frac{c}{\sqrt{2\pi}} \int_0^{2\pi} e^{-ix\xi} f(x-ct)dx = ce^{-ict\xi}\widehat{f}(\xi).$$

Therefore, for any positive constants $c_1$ and $c_2$ we have

$$\|u_{c_2}(\cdot,t) - u_{c_1}(\cdot,t)\|^2_{H^s} =$$

$$\|u_{c_1}(\cdot,0) - u_{c_2}(\cdot,0)\|^2_{H^s} + 2c_1c_2 \sum_{\xi\epsilon Z} \left(1+\xi^2\right)^s \left|\widehat{f}(\xi)\right|^2 \left(1 - \cos\left(c_2 - c_1\right)t\xi\right)$$

and if we choose $c_1 = c_1(n)$ and $c_2 = c_2(n)$ such that

$$c_2 - c_1 = \frac{1}{n}$$

then at $t = 0$ we have

$$\|u_{c_1}(\cdot,0) - u_{c_2}(\cdot,0)\|_{H^s} = \frac{1}{n}\|f\|_{H^s} \longrightarrow 0, \quad \text{as } n \to \infty.$$

Next, define

$$g(n) = \sum_{\xi\epsilon Z} \left(1+\xi^2\right)^s \left|\widehat{f}(\xi)\right|^2 \left(1 - \cos\frac{t\xi}{n}\right)$$

and note that $g(n) \longrightarrow 0$ as $n \to \infty$ by the dominated convergence theorem. Choosing for each $n$, such that $g(n) \neq 0$, the constant

$$c_1(n) = \frac{1}{g(n)}$$

and substituting into the expression for the $H^s$ norm of the difference we get

$$\left\|u_{c_2}(\cdot,t)-u_{c_1}(\cdot,t)\right\|_{H^s}^2 \geq 2\frac{1}{g(n)}\left(\frac{1}{n}+\frac{1}{g(n)}\right)g(n) \longrightarrow \infty, \quad \text{as} \quad n\to\infty,$$

for any positive $t$. It is clear that with the choices of the constants made above the $H^s$ norms of $u_{c_1}$ and $u_{c_2}$ must grow without bound. □

As already mentioned in the introduction, the constructions in [HM3] lead to sequences of smooth solutions which yield a non-uniform continuity of the data-to-solution map $u_0 \to u$ on bounded subsets of $H^s$ whenever $s \geq 2$. Next, we show that such sequences cannot work in $H^1$. For this it suffices to show the following

PROPOSITION 2.3. *For any smooth travelling wave solution $u(x,t) = f(x-t)$, where $f$ is as in Lemma 2.1, the norm $\|f'\|_{L^2} \to 0$ as the parameters $\delta$ and $\varepsilon$ approach zero.*

*Proof.* Given any $\delta$ and $\varepsilon$ in $(0,1/5)$ let $y = 1-f$ be the corresponding solution to the second order differential equation in (2.5). Integrating by parts and using (2.5) we obtain

$$\begin{aligned}
\|y'\|_{L^2(0,2\pi)}^2 &= \int_0^{2\pi} y'(x)y'(x)dx = -\int_0^{2\pi} y(x)y''(x)dx \\
&= -\int_0^{2\pi} y(x)\Big(y(x)-1+\frac{\delta(\delta+\varepsilon)(2-2\delta-\varepsilon)}{2y(x)^2}\Big)dx \\
&= \int_0^{2\pi} y(x)dx - \int_0^{2\pi} y^2(x)dx - \frac{1}{2}\delta(\delta+\varepsilon)(2-2\delta-\varepsilon)\int_0^{2\pi}\frac{1}{y(x)}dx.
\end{aligned}$$

Since $\delta \leq y \leq \delta+\varepsilon$ from the calculation above we get

$$\|y'\|_{L^2(0,2\pi)}^2 \leq 2\pi(\delta+\varepsilon),$$

showing that it is impossible to obtain a positive lower bound on $\|y'\|_{L^2}$ which is independent of $\delta$ and $\varepsilon$ when these two parameters go to zero. □

Now, choosing the sequence as in [HM3], that is

$$u_n(x,t) \doteq f_n(x-t) \quad \text{and} \quad v_n(x,t) \doteq c_n f_n(x-c_n t),$$

where $c_n = 1+1/n$, and computing the $H^1$ norm of their difference we have

$$\|v_n(t_0)-u_n(t_0)\|_{H^1(\mathbb{T})}^2 \lesssim (\delta+\varepsilon)^2 + \frac{1}{n^2} \longrightarrow 0$$

as $\delta$, $\varepsilon$ go to zero and (consequently) $n$ goes to infinity.

**3. Non-smooth waves and Non-uniform dependence in $H^1(\mathbb{T})$.** In this section we prove Theorem 1.1 in the periodic case. Using the same substitution $y = 1-f$ of the dependent variable in equation (2.2) but choosing the parameters $a$ and $b$ differently (to ensure an infinite slope of the graph at one of the two endpoints of the half-period) we obtain the differential equation

$$y'^2 = \frac{(\varepsilon-y)(\beta+y)(2+\beta-\varepsilon-y)}{y}, \tag{3.1}$$

where $0<\varepsilon<1/5$ and $\beta$ is any fixed constant in the interval $(0,1/5]$.

The next result provides the non-smooth analogue of Lemma 2.1.

LEMMA 3.1. *For any $0 < \varepsilon < 1/5$ there is a $2\ell$-periodic, even and continuous function $0 \le y(x) \le \varepsilon$ solving (3.1) such that $y \in C^\infty(\mathbb{R} \setminus 2\ell\,\mathbb{Z})$ and $y'(\ell) = 0$, $y'(0^\pm) = \pm\infty$ (see Figure 2). Moreover*

$$\|y'\|^2_{L^2(-\ell,\ell)} \simeq \varepsilon,$$

*where*

$$\ell = \int_0^\varepsilon \sqrt{\frac{y}{(\varepsilon - y)(\beta + y)(2 + \beta - \varepsilon - y)}}\, dy \simeq \varepsilon.$$

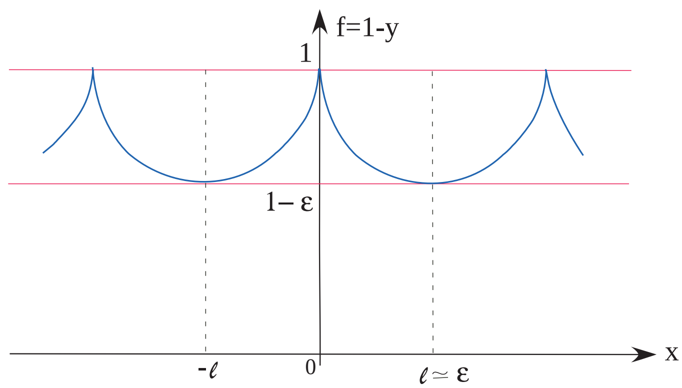


FIG. 2

*Proof.* To construct $y = y(x)$, first pick any $0 < y_0 < \varepsilon$ and apply the fundamental ODE theorem to equation(3.1) with the initial condition $y(x_0) = y_0$. Next, translating and reflecting with respect to the $y$-axis we obtain the solution $y$ over its basic period $(-\ell, \ell)$. Extending periodically to the whole line gives the solution with the desired properties.

Next, from equation (3.1) we have

$$\ell = \int_0^\varepsilon \sqrt{\frac{y}{(\varepsilon - y)(\beta + y)(2 + \beta - \varepsilon - y)}}\, dy$$

and therefore

$$\ell \le \sqrt{\frac{\varepsilon}{\beta}} \int_0^\varepsilon \frac{dy}{\sqrt{\varepsilon - y}} \simeq 2\frac{\varepsilon}{\sqrt{\beta}}. \tag{3.2}$$

On the other hand, we similarly get

$$\ell \ge \frac{1}{\sqrt{3}\sqrt{\beta + \varepsilon}} \int_0^\varepsilon \frac{\sqrt{y}\, dy}{\sqrt{\varepsilon - y}} \ge \frac{\varepsilon}{\sqrt{3}\sqrt{\beta + \varepsilon}}. \tag{3.3}$$

These two inequalities imply that $\ell \simeq \varepsilon$, since $\beta > 0$ is fixed (for simplicity, in what follows, we will assume $\beta = 1/5$).

To estimate the $L^2$-norm of the derivative we have

$$\begin{aligned}
\|y'\|^2_{L^2(-\ell,\ell)} &= 2\|y'\|^2_{L^2(0,\ell)} = 2\int_0^\ell y'(x)\,dy(x) \\
&= 2\int_0^\varepsilon \sqrt{\frac{(\varepsilon-y)(\beta+y)(2+\beta-\varepsilon-y)}{y}}\,dy \\
&\le 2\sqrt{2}\,\sqrt{\beta+\varepsilon}\int_0^\varepsilon \sqrt{\frac{\varepsilon-y}{y}}\,dy \le 8\varepsilon\sqrt{\beta+\varepsilon}.
\end{aligned} \tag{3.4}$$

Estimating from below we obtain

$$\|y'\|^2_{L^2(-\ell,\ell)} \ge 2\sqrt{\beta}\int_0^\varepsilon \sqrt{\frac{\varepsilon-y}{y}}\,dy \ge \varepsilon\sqrt{\beta}. \tag{3.5}$$

This completes the proof. ☐

**Construction of non-smooth solutions $u_n$ and $v_n$.** Let $f_n = 1 - y_n$ be the $2\ell = \frac{2\pi}{n}$ periodic non-smooth solution constructed in Lemma 3.1, where the frequency $n$ satisfies

$$n \simeq 1/\varepsilon.$$

Define the following two sequences of solutions

$$u_n(x,t) = f_n(x-t) \quad \text{and} \quad v_n(x,t) = c_n f_n(x - c_n t),$$

and for any fixed $t_0 > 0$ choose

$$c_n = 1 + \frac{\pi}{nt_0}.$$

Now we are ready to show boundedness of these solutions. From Lemma 3.1 we have $\|y_n'\|^2_{L^2(-\frac{\pi}{n},\frac{\pi}{n})} \simeq 1/n$. It follows that

$$\begin{aligned}
\|v_n(t)\|^2_{H^1(-\pi,\pi)} &= \|c_n f_n\|^2_{H^1(-\pi,\pi)} \lesssim 1 + c_n^2\|y_n\|^2_{H^1(-\pi,\pi)} \\
&\lesssim 1 + \|y_n'\|^2_{L^2(-\pi,\pi)} \lesssim 1 + n\|y_n'\|^2_{L^2(-\frac{\pi}{n},\frac{\pi}{n})} \\
&\lesssim 1 + n\,\frac{1}{n} = 2.
\end{aligned}$$

Furthermore, with the choices made above we now easily compute

$$\|v_n(0) - u_n(0)\|^2_{H^1(\mathbb{T})} = (c_n - 1)^2\|f_n\|^2_{H^1} \simeq \frac{1}{(nt_0)^2} \to 0,$$

as $n \to \infty$.

On the other hand, for any positive time $t_0$, we have

$$\begin{aligned}
\|v_n(t_0) - u_n(t_0)\|^2_{H^1(\mathbb{T})} &= \Big\|\Big(1+\frac{\pi}{nt_0}\Big)f_n\Big(\cdot - t_0 - \frac{\pi}{n}\Big) - f_n(\cdot - t_0)\Big\|^2_{H^1(\mathbb{T})} \\
&= \Big\|\Big(1+\frac{\pi}{nt_0}\Big)f_n\Big(\cdot - \frac{\pi}{n}\Big) - f_n(\cdot)\Big\|^2_{H^1(\mathbb{T})} \\
&\gtrsim \Big\|\Big(1+\frac{\pi}{nt_0}\Big)f_n'\Big(\cdot - \frac{\pi}{n}\Big) - f_n'(\cdot)\Big\|^2_{L^2(\mathbb{T})} \\
&\gtrsim \|f_n'(\cdot)\|^2_{L^2(\mathbb{T})} \simeq 1.
\end{aligned}$$

The last estimate from below in these inequalities was possible because the function $f$ has been constructed to satisfy the crucial property

$$f_n'(x) \cdot f_n'(x - \frac{\pi}{n}) \leq 0,$$

as shown in Figure 3.

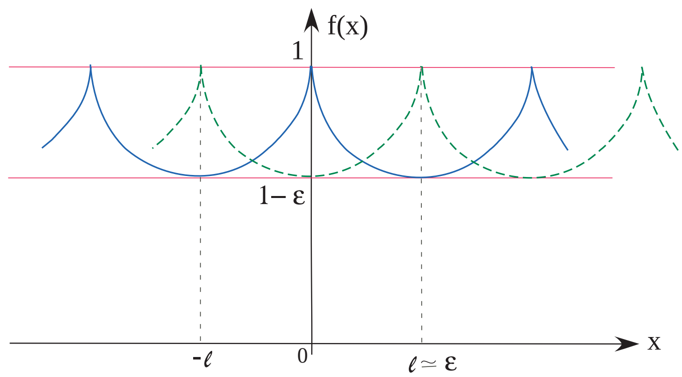


FIG. 3

This completes the proof ot Theorem 1.1 in the periodic case.

**4. Non-uniform dependence in $H^1(\mathbb{R})$.** Finally, we turn to the non-periodic case. The main idea is to exploit the periodic construction of the previous section. Starting with the periodic solution we modify it outside the interval $[-2\pi, 2\pi]$ to look like a "peakon" solution. More precisely, for each $\varepsilon = \varepsilon_n$ we extend the periodic function $f_n$ given in Lemma 3.1 as follows

$$g_n(x) = \begin{cases} f_n(x), & \text{if } |x| \leq 2\pi \\ e^{2\pi - |x|}, & \text{if } |x| > 2\pi. \end{cases}$$

For $n = 3$, the graph of $g_n$ is shown in Figure 4 below.

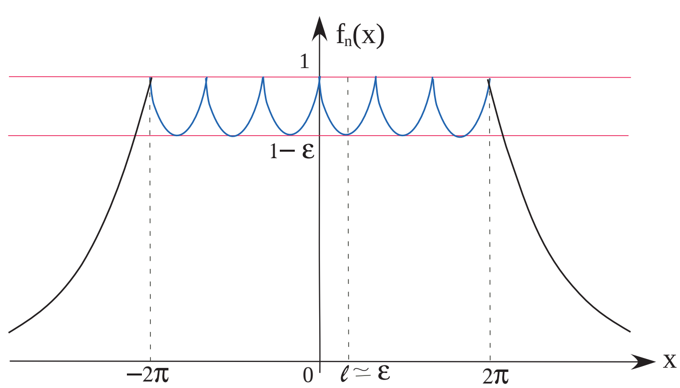


FIG. 4

One can check that $y = 1 - g_n$ satisfies the equation

$$(y')^2 = \frac{(\varepsilon - y)(\beta + y)(2 + \beta - \varepsilon - y)}{y}, \tag{4.1}$$

in the interval $(-2\pi, 2\pi)$ and the equation

$$(y')^2 = (1-y)^2 \tag{4.2}$$

in the intervals $|x| > 2\pi$, where as before $0 < \varepsilon < 1/5$ and $\beta = 1/5$. Note that each $g_n$ is a continuous function on $\mathbb{R}$, which is smooth except at finitely many points $\{x_k\}$ and is a classical solution in $\mathbb{R} \smallsetminus \{x_k\}$.

The next lemma follows from the definition of $g_n$ and Lemma 3.1.

LEMMA 4.1. *Let $0 < \varepsilon < 1/5$. For any sufficiently large positive integer $n = n(\varepsilon)$ there is a continuous weak solution $g_n$ to the differential equations (4.1) and (4.2) with $0 \leq g_n(x) \leq 1$ and such that*

$$\|g_n'\|^2_{L^2(\mathbb{R})} \simeq 1,$$

*and $g_n \in C^\infty(\mathbb{R} \smallsetminus \{x_k\})$.*

Given $g_n$ as in Lemma 4.1 we again define the following two sequences of weak solutions to the CH equation

$$u_n(x,t) = g_n(x-t) \quad \text{and} \quad v_n(x,t) = c_n g_n(x - c_n t)$$

and, for any fixed $t_0 > 0$, we choose $c_n = 1 + \dfrac{\pi}{nt_0}$.

To see that that the solution defined above stay bounded we estimate as follows

$$\begin{aligned}
\|v_n(t)\|^2_{H^1(\mathbb{R})} &= \|c_n g_n\|^2_{H^1(\mathbb{R})} \\
&= c_n^2 \|f_n\|^2_{H^1(-2\pi,2\pi)} + 2c_n^2 e^{4\pi} \int_{|x|>2\pi} e^{-2|x|} dx \\
&\lesssim c_n^2 \simeq 1.
\end{aligned}$$

It remains to estimate the distance between the two sequences. At $t = 0$, we have

$$\|v_n(0) - u_n(0)\|_{H^1(\mathbb{R})} = (c_n - 1)^2 \|g_n\|^2_{H^1} \simeq \frac{1}{(nt_0)^2} \to 0.$$

On the other hand at any time $t_0 > 0$ we have

$$\begin{aligned}
\|v_n(t_0) - u_n(t_0)\|^2_{H^1(\mathbb{R})} &= \Big\| \big(1 + \frac{\pi}{nt_0}\big) g_n\big(\,\cdot - t_0 - \frac{\pi}{n}\big) - g_n(\cdot - t_0) \Big\|^2_{H^1(\mathbb{R})} \\
&= \Big\| \big(1 + \frac{\pi}{nt_0}\big) g_n\big(\,\cdot - \frac{\pi}{n}\big) - g_n(\cdot) \Big\|^2_{H^1(\mathbb{R})} \\
&\gtrsim \Big\| \big(1 + \frac{\pi}{nt_0}\big) g_n\big(\,\cdot - \frac{\pi}{n}\big) - g_n(\cdot) \Big\|^2_{H^1(-\pi,\pi)} \\
&\gtrsim \Big\| \big(1 + \frac{\pi}{nt_0}\big) f_n'\big(\,\cdot - \frac{\pi}{n}\big) - f_n'(\cdot) \Big\|^2_{L^2(-\pi,\pi)} \\
&\gtrsim \|f_n'(\cdot)\|^2_{L^2(-\pi,\pi)} \simeq 1,
\end{aligned}$$

since $f_n'(x) \cdot f_n'(x - \frac{\pi}{n}) \leq 0$. The proof of Theorem 1.1 is complete.

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