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THE TWISTOR CONSTRUCTION AND PENROSE TRANSFORM IN SPLIT SIGNATURE*

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To Salah Baouendi on the occasion of his 70th birthday

Abstract. The twistor construction in Euclidean 4-space may be based on the algebra of quaternions. A counterpart to this construction is established in split signature by using the split quaternions.

Key words. Twistor, Penrose transform, Split signature

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1. Introduction. Let $Z = \mathbb{R}^4 \times \mathbb{CP}_1$. The twistor construction in the Euclidean setting provides a complex structure on Z. The projection $Z \to \mathbb{CP}_1$ onto the second factor is holomorphic and, if L denotes the pull-back of the line bundle Ω^1 of holomorphic 1-forms on $\mathbb{CP}_1$ to Z, then the Penrose transform provides a canonical isomorphism

$$H^1(Z, \mathcal{O}(L)) \xrightarrow{\cong} \{\mathbb{C}\text{-valued harmonic functions on } \mathbb{R}^4\}. \tag{1.1}$$

In split signature, i.e. if $\mathbb{R}^4$ is endowed with a quadratic form of type $(+,+,-,-)$, then an entirely parallel construction gives a cohomological description of the solutions of the ultrahyperbolic wave equation in four variables. The main difference, however, is that Z is no longer a complex manifold but comes equipped instead with a locally integrable system of complex vector fields in the sense of Baouendi and Treves [3].

This article is organised as follows. In §2 and §3 we review the construction of the complex structure on Z and give an elementary proof of the isomorphism (1.1). For the Euclidean case, the algebra of quaternions is basic. Replacing the quaternions by the split quaternions gives the corresponding results in split signature. This is carried out in §4 and §5. Finally, in §6 we discuss how these constructions and transforms fit with existing geometrical results. These include LeBrun and Mason's nonlinear graviton in split signature [9] and John's classical X-ray transform [8].

2. The Euclidean construction. Let us take the complex structure on

$$S^2 = \{(u, v, w) \in \mathbb{R}^3 \text{ s.t. } u^2 + v^2 + w^2 = 1\}$$

to be defined by the action of the unit outward-pointing normal via cross product in $\mathbb{R}^3$. Specifically, the matrix

$$J = \begin{bmatrix} 0 & -w & v \\ w & 0 & -u \\ -v & u & 0 \end{bmatrix}$$

acts by left multiplication on $\mathbb{R}^3$ tangentially to S^2 as rotation through a quarter turn. We shall define the complex structure on $\mathbb{R}^4 \times S^2$ by extending J to an almost complex structure and checking that this extension is integrable.

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The quaternions $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ are characterised as an associative algebra by the relations

$$i^2 = -1 \qquad j^2 = -1 \qquad ij = -ji = k. \tag{2.1}$$

Using $1, i, j, k$ as a basis of $\mathbb{R}^4$, the actions of i, j, k by left multiplication are given by the following matrices.

(2.2)

$$\mathbb{I} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \mathbb{J} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \qquad \mathbb{K} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

We define an almost complex structure on $Z = \mathbb{R}^4 \times S^2$ by the 7×7 matrix

$$\begin{bmatrix} u\mathbb{I} + v\mathbb{J} + w\mathbb{K} & 0 \\ 0 & \begin{bmatrix} 0 & -w & v \\ w & 0 & -u \\ -v & u & 0 \end{bmatrix} \end{bmatrix}. \tag{2.3}$$

Theorem 2.1. *The almost complex structure defined by* (2.3) *is integrable.*

Proof. The construction and resulting theorem are well-known [1, 2]. The following explicit verification, however, is useful in providing an elementary proof of (1.1) in §3 and in providing a model for the corresponding verification in split signature in §4.

It suffices to check integrability in local coördinates. Stereographic projection gives coördinates on the sphere away from a pole. Explicitly, we find that

$$\mathbb{C} \ni z = x + iy \mapsto (u, v, w) = \tfrac{1}{x^2+y^2+1}(2y, 2x, x^2 + y^2 - 1) \tag{2.4}$$

provides a holomorphic coördinate. In fact, by the chain rule,

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \tfrac{1}{(x^2+y^2+1)^2}\left(2i(1+z^2)\frac{\partial}{\partial u} + 2(1-z^2)\frac{\partial}{\partial v} + 4z\frac{\partial}{\partial w}\right)$$

and it is easy to check that this vector is in the $(-i)$-eigenspace of J. A further computation verifies that

$$\begin{bmatrix} 1 \\ z \\ iz \\ -i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -iz \\ i \\ 1 \\ z \end{bmatrix}$$

span the $(-i)$-eigenspace of $u\mathbb{I} + v\mathbb{J} + w\mathbb{K}$. Altogether $T^{0,1}Z$ is spanned by

$$X \equiv \frac{\partial}{\partial p} + z\frac{\partial}{\partial q} + iz\frac{\partial}{\partial r} - i\frac{\partial}{\partial s}, \quad Y \equiv -iz\frac{\partial}{\partial p} + i\frac{\partial}{\partial q} + \frac{\partial}{\partial r} + z\frac{\partial}{\partial s}, \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} \tag{2.5}$$

from which it is clear that $[T^{0,1}, T^{0,1}] \subseteq T^{0,1}$, as required. □

If the opposite complex structure on S^2 is chosen whilst retaining our other choices, then we obtain an alternative almost complex structure: the complex vector field $\partial/\partial\bar{z}$ is replaced by $\partial/\partial z$ in (2.5). This alternative, however, is clearly not integrable.

3. The Euclidean Penrose transform. The aim of this section is to provide an elementary proof of (1.1). It is well-known that $H^1(\mathbb{CP}_1, \Omega^1) = \mathbb{C}$. Indeed, if we cover $S^2 = \mathbb{CP}_1$ by two coördinate patches with local coördinate z as in §2 and $\zeta = 1/z$ near the pole, then

$$\Upsilon = \frac{d\bar{z} \otimes dz}{(1 + z\bar{z})^2} = \frac{d\bar{\zeta} \otimes d\zeta}{(1 + \zeta\bar{\zeta})^2} \tag{3.1}$$

is well-defined on the whole of $\mathbb{CP}_1$ and generates $H^1(\mathbb{CP}_1, \Omega^1)$. The basis (2.5) of $T^{0,1}Z$ allows us to express the $\bar{\partial}$-complex $\Lambda^{0,\bullet}$ quite explicitly. Let us denote by dX, dY, $d\bar{z}$ the dual basis of $\Lambda^{0,1}$. Then

(3.2)
$$\begin{array}{ccccl}
f & \in & \Lambda^{0,0} & & \\
\downarrow & & \downarrow & & \\
(Xf)\,dX + (Yf)\,dY + \frac{\partial}{\partial\bar{z}} f\,d\bar{z} & \in & \Lambda^{0,1} & \ni & a\,dX + b\,dY + c\,d\bar{z} \\
 & & \downarrow & & \qquad\downarrow \\
 & & \Lambda^{0,2} & \ni & (Xb - Ya)\,dX \wedge dY \\
 & & \downarrow & & \quad + (Xc - \frac{\partial}{\partial\bar{z}} a)\,dX \wedge d\bar{z} \\
 & & \vdots & & \qquad + (Yc - \frac{\partial}{\partial\bar{z}} b)\,dY \wedge d\bar{z}
\end{array}$$

The fibration $\tau : Z = \mathbb{R}^4 \times \mathbb{CP}_1 \to \mathbb{R}^4$ has holomorphic fibres. Equivalently, the intersection of $T^{0,1}$ on Z with the kernel of $d\tau$ is a 1-dimensional sub-bundle, which we shall denote $T^{0,1}_\tau$. Dual to the inclusion $T^{0,1}_\tau \hookrightarrow T^{0,1}$, we obtain a short exact sequence of vector bundles

$$0 \to K \to \Lambda^{0,1} \to \Lambda^{0,1}_\tau \to 0, \tag{3.3}$$

where K is the kernel of $\Lambda^{0,1} \to \Lambda^{0,1}_\tau$. With respect to our preferred basis, it is clear that K is spanned by dX and dY. Recall that L denotes the pull-back of Ω^1 on $\mathbb{CP}_1$ to Z. As a holomorphic bundle on Z, the $\bar{\partial}$-operator gives a well-defined complex $\Lambda^{0,\bullet} \otimes L$, so that the first cohomology of $\Gamma(Z, \Lambda^{0,\bullet} \otimes L)$ is the left hand side of (1.1). In combination with the exact sequence (3.3), we obtain a diagram

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & K \otimes L & & K \wedge K \otimes L & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
0 & \to & \Lambda^{0,0} \otimes L & \xrightarrow{\bar{\partial}} & \Lambda^{0,1} \otimes L & \xrightarrow{\bar{\partial}} & \Lambda^{0,2} \otimes L & \xrightarrow{\bar{\partial}} & \Lambda^{0,3} \otimes L & \to & 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & \Lambda^{0,1}_\tau \otimes L & & \Lambda^{0,1}_\tau \otimes K \otimes L & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array} \tag{3.4}$$

with exact columns.

Lemma 3.1. *The composition*

$$\Gamma(Z, L) \xrightarrow{\bar{\partial}} \Gamma(Z, \Lambda^{0,1} \otimes L) \to \Gamma(Z, \Lambda^{0,1}_\tau \otimes L) \tag{3.5}$$

is injective. Its cokernel may be canonically identified with the smooth functions on $\mathbb{R}^4$. *The composition*

$$\Gamma(Z, K \otimes L) \to \Gamma(Z, \Lambda^{0,1} \otimes L) \xrightarrow{\bar{\partial}} \Gamma(Z, \Lambda^{0,2} \otimes L) \to \Gamma(Z, \Lambda^{0,1}_\tau \otimes K \otimes L) \tag{3.6}$$

is an isomorphism.

Proof. The bundle Ω^1 on $\mathbb{CP}_1$ has transition functions determined by $dz = -d\zeta/\zeta^2$. It follows that there is a (non-canonical) isomorphism $\Omega^1 \cong \mathcal{O}(-2)$ where $\mathcal{O}(1)$ is the usual hyperplane section bundle on $\mathbb{CP}_1$. It is well-known that $H^0(Z, \mathcal{O}(-2)) = 0$. This is responsible for the injectivity of the composition (3.5). Indeed, the explicit local description (3.2) of the $\bar{\partial}$-complex on Z remains valid when tensored with L and (3.5) is simply $f \mapsto \frac{\partial}{\partial \bar{z}} f \, d\bar{z}$ in the z-coördinate patch and $f \mapsto \frac{\partial}{\partial \bar{\zeta}} f \, d\bar{\zeta}$ in the ζ-coördinate patch just as is the $\bar{\partial}$-operator on $\mathbb{CP}_1$. The upshot is that the kernel and cokernel of the composition (3.5) are just smoothly parameterised versions of the usual Dolbeault cohomology on $\mathbb{CP}_1$. The isomorphism $H^1(\mathbb{CP}_1, \mathcal{O}(-2)) \cong \mathbb{C}$ implies that the cokernel of (3.5) may be identified with smooth functions on $\mathbb{R}^4$ and the canonical isomorphism $H^1(\mathbb{CP}_1, \Omega^1) = \mathbb{C}$ implies that it is canonically so. Specifically, we may take

$$f = f(p, q, r, s) \longmapsto f\Upsilon = \frac{f \, d\bar{z} \otimes dz}{(1 + z\bar{z})^2} = \frac{f \, d\bar{\zeta} \otimes d\zeta}{(1 + \zeta\bar{\zeta})^2} \tag{3.7}$$

in accordance with (3.1).

The statements concerning the composition (3.6) follow similarly. Firstly, let us check that K is isomorphic to the pull-back of $\mathcal{O}(1) \oplus \mathcal{O}(1)$ on $\mathbb{CP}_1$. In Theorem 2.1, we found the basis (2.5) for $T^{0,1}$ in the z-coördinate patch. The ζ-coördinate patch is provided by

$$\mathbb{C} \ni \zeta = \xi + i\eta \mapsto (u, v, w) = \tfrac{1}{\xi^2+\eta^2+1}(-2\eta, 2\xi, 1 - \xi^2 - \eta^2),$$

easily computing from (2.4) that $z\zeta = 1$ on the overlap. It is readily verified that

$$\mathcal{X} \equiv \zeta\frac{\partial}{\partial p} + \frac{\partial}{\partial q} + i\frac{\partial}{\partial r} - i\zeta\frac{\partial}{\partial s}, \quad \mathcal{Y} \equiv -i\frac{\partial}{\partial p} + i\zeta\frac{\partial}{\partial q} + \zeta\frac{\partial}{\partial r} + \frac{\partial}{\partial s}, \quad \text{and} \quad \frac{\partial}{\partial \bar{\zeta}} \tag{3.8}$$

gives a basis for $T^{0,1}$ in the ζ-coördinate patch. Noticing that $\mathcal{X} = \zeta X$ and $\mathcal{Y} = \zeta Y$, our check is complete. It follows immediately that $K \otimes L$ on Z is isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ on each fibre of τ. Since $H^0(\mathbb{CP}_1, \mathcal{O}(-1))$ and $H^1(\mathbb{CP}_1, \mathcal{O}(-1))$ both vanish, the proof is complete. $\square$

Now consider the diagram (3.4) and $\omega \in \Gamma(Z, \Lambda^{0,1}_\tau \otimes L)$. The isomorphism of Lemma 3.1 yields a preferred lift of ω to $\tilde{\omega} \in \Gamma(Z, \Lambda^{0,1} \otimes L)$. It is the one whose image under the composition

$$\Gamma(Z, \Lambda^{0,1} \otimes L) \xrightarrow{\bar{\partial}} \Gamma(Z, \Lambda^{0,2} \otimes L) \to \Gamma(Z, \Lambda^{0,1}_\tau \otimes K \otimes L)$$

vanishes. We immediately obtain a complex

$$0 \to \Gamma(Z, \Lambda^{0,0} \otimes L) \to \Gamma(Z, \Lambda^{0,1}_\tau \otimes L) \xrightarrow{\mathcal{D}} \Gamma(Z, K \wedge K \otimes L) \to \Gamma(Z, \Lambda^{0,3} \otimes L) \to 0$$

where $\mathcal{D}$ is defined by $\mathcal{D}\omega = \bar{\partial}\tilde{\omega}$. By diagram chasing, the cohomology of this complex is $H^*(Z, \mathcal{O}(L))$. To establish the isomorphism (1.1), we should compute $\mathcal{D}(f\Upsilon)$ for $f\Upsilon$ as in (3.7). To do this, it is convenient to write

$$P \equiv \frac{\partial}{\partial p} - i\frac{\partial}{\partial s} \quad Q \equiv \frac{\partial}{\partial q} + i\frac{\partial}{\partial r} \quad R \equiv i\frac{\partial}{\partial q} + \frac{\partial}{\partial r} \quad S \equiv -i\frac{\partial}{\partial p} + \frac{\partial}{\partial s} \tag{3.9}$$

so that

$$X = P + zQ \qquad \mathcal{X} = \zeta P + Q \qquad Y = R + zS \qquad \mathcal{Y} = \zeta R + S.$$

Now consider $\widetilde{f\Upsilon} \in \Gamma(Z, \Lambda^{0,1} \otimes L)$ defined by

$$\widetilde{f\Upsilon} = \begin{cases} \dfrac{(\bar z\, Pf - Qf)\, dX \otimes dz}{1+z\bar z} + \dfrac{(\bar z\, Rf - Sf)\, dY \otimes dz}{1+z\bar z} + \dfrac{f\, d\bar z \otimes dz}{(1+z\bar z)^2} \\[2ex] \dfrac{(\bar\zeta\, Qf - Pf)\, d\mathcal{X} \otimes d\zeta}{1+\zeta\bar\zeta} + \dfrac{(\bar\zeta\, Sf - RF)\, d\mathcal{Y} \otimes d\zeta}{1+\zeta\bar\zeta} + \dfrac{f\, d\bar\zeta \otimes d\zeta}{(1+\zeta\bar\zeta)^2} \end{cases}$$

in the two coördinate patches. It is easily verified that this expression is well-defined and from (3.2), we may compute that

$$\bar\partial(\widetilde{f\Upsilon}) = \left(X\frac{(\bar z\, Rf - Sf)}{1+z\bar z} - Y\frac{(\bar z\, Pf - Qf)}{1+z\bar z} \right) dX \wedge dY \otimes dz,$$

which has vanishing image in $\Gamma(Z, \Lambda_\tau^{0,1} \otimes K \otimes L)$. It follows that $\widetilde{f\Upsilon}$ is the desired lifting and $\mathcal{D}(f\Upsilon) = \bar\partial(\widetilde{f\Upsilon})$. Finally, we compute

$$\begin{aligned} X\frac{(\bar z\, Rf - Sf)}{1+z\bar z} - Y\frac{(\bar z\, Pf - Qf)}{1+z\bar z} &= \frac{\big((P+zQ)(\bar z\, R - S) - (R+zS)(\bar z\, P - Q)\big)f}{1+z\bar z} \\ &= (QR - PS)f \\ &= i\left(\frac{\partial^2}{\partial p^2} + \frac{\partial^2}{\partial q^2} + \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial s^2}\right) f \end{aligned}$$

and (1.1) is proved.

4. The split construction. The split quaternions $\mathbb{R}+\mathbb{R}i+\mathbb{R}j+\mathbb{R}k$ are obtained by changing a sign in the defining relations (2.1). Specifically, we may require that

$$i^2 = -1 \qquad j^2 = 1 \qquad ij = -ji = k$$

and then the actions of i, j, k by left multiplication are given by

$$\mathbb{I} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \mathbb{J} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \qquad \mathbb{K} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

instead of (2.2). According to the Hurwitz Theorem [6], the split quaternions are the only other 4-dimensional real normed algebra.

We may attempt to define a complex structure on $Z = \mathbb{R}^4 \times S^2$ by mimicking the construction in §2. More precisely, the matrix $u\mathbb{I} + iv\mathbb{J} + iw\mathbb{K}$ has $(-i)$-eigenspace spanned by

$$\begin{bmatrix} 1 \\ z \\ -z \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -z \\ 1 \\ 1 \\ z \end{bmatrix}$$

and it is clear that the resulting complex sub-bundle $T^{0,1}$ of $\mathbb{C}TZ$ spanned by

$$X \equiv \frac{\partial}{\partial p} + z\frac{\partial}{\partial q} - z\frac{\partial}{\partial r} + \frac{\partial}{\partial s}, \quad Y \equiv -z\frac{\partial}{\partial p} + \frac{\partial}{\partial q} + \frac{\partial}{\partial r} + z\frac{\partial}{\partial s}, \quad \text{and} \quad \frac{\partial}{\partial \bar z} \tag{4.1}$$

is involutive in the sense that $[T^{0,1}, T^{0,1}] \subseteq T^{0,1}$. The only difference is that $T^{0,1} \cap \overline{T^{0,1}}$ is no longer zero. More precisely,

$$\det \begin{bmatrix} 1 & -z & 1 & -\bar z \\ z & 1 & \bar z & 1 \\ -z & 1 & -\bar z & 1 \\ 1 & z & 1 & \bar z \end{bmatrix} = 16y^2$$

and so $T^{0,1}$ defines a complex structure only outside the hypersurface $\Sigma = \{y = 0\}$. Nevertheless, we have proved

THEOREM 4.1. *There is a natural involutive structure on* $Z = \mathbb{R}^4 \times S^2$,

where an involutive structure is simply defined as a complex distribution closed under Lie bracket. In fact, this distribution satisfies the stronger condition of being locally integrable in the sense of [3]: the functions z, $zp - q + r + zs$, $p + zq + zr - s$ are annihilated by $T^{0,1}$ and have differentials that are everywhere linearly independent in the z-coördinate patch, a similar statement being valid in the ζ-coördinate patch.

5. The split Penrose transform. An involutive structure is all that is needed to define an analogue of the Dolbeault complex starting with $\Lambda^{0,1} \equiv (T^{0,1})^*$. Indeed, closure under Lie bracket is precisely what is needed to ensure that $\bar\partial^2 = 0$. The corresponding cohomology is not a sheaf cohomology because the $\bar\partial$-complex is not exact on the sheaf level. Nevertheless,

$$H^1_{\bar\partial}(Z, L) \xrightarrow{\simeq} \{\text{smooth } f : Z \to \mathbb{C} \text{ s.t. } \left(\frac{\partial^2}{\partial p^2} + \frac{\partial^2}{\partial q^2} - \frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial s^2}\right) f = 0\}, \tag{5.1}$$

where the left-hand side is defined as the first cohomology of the complex

$$0 \to \Gamma(Z, \Lambda^{0,0} \otimes L) \xrightarrow{\bar\partial} \Gamma(Z, \Lambda^{0,1} \otimes L) \xrightarrow{\bar\partial} \Gamma(Z, \Lambda^{0,2} \otimes L) \xrightarrow{\bar\partial} \Gamma(Z, \Lambda^{0,3} \otimes L) \to 0.$$

The proof of this follows exactly the reasoning in §3. The only difference is that, instead of (3.9), we should take

$$P \equiv \frac{\partial}{\partial p} + \frac{\partial}{\partial s} \quad Q \equiv \frac{\partial}{\partial q} - \frac{\partial}{\partial r} \quad R \equiv \frac{\partial}{\partial q} + \frac{\partial}{\partial r} \quad S \equiv -\frac{\partial}{\partial p} + \frac{\partial}{\partial s}$$

so that $X = P + zQ$ and $Y = R + zS$ in accordance with (4.1). This only affects the final computation:

$$(QR - PS) = \frac{\partial^2}{\partial p^2} + \frac{\partial^2}{\partial q^2} - \frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial s^2}$$

and (5.1) is proved.

6. Further discussion. The Euclidean twistor construction extends to the conformal compactification S^4 of $\mathbb{R}^4$ as discussed in [1]. Explicitly, there is a submersion

$$\tau : \mathbb{CP}_3 \to S^4 \subset \mathbb{R}^5 \tag{6.1}$$

given by

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \longmapsto \frac{1}{|z_1|^2+|z_2|^2+|z_3|^2+|z_4|^2} \begin{bmatrix} z_1\bar z_4 + z_4\bar z_1 + z_2\bar z_3 + z_3\bar z_2 \\ i(z_1\bar z_4 - z_4\bar z_1 - z_2\bar z_3 + z_3\bar z_2) \\ i(z_1\bar z_3 - z_3\bar z_1 + z_2\bar z_4 - z_4\bar z_2) \\ z_1\bar z_3 + z_3\bar z_1 - z_2\bar z_4 - z_4\bar z_2 \\ z_1\bar z_1 + z_2\bar z_2 - z_3\bar z_3 - z_4\bar z_4 \end{bmatrix}$$

whose fibres are lines in $\mathbb{CP}_3$. In particular, this submersion has holomorphic fibres isomorphic to $\mathbb{CP}_1$. Stereographic projection $\mathbb{R}^4 \hookrightarrow S^4$ is conformal for the round metric on the sphere and it may be readily verified that

$$\begin{array}{ccccc} Z = \mathbb{R}^4 \times \mathbb{CP}_1 & \cong & \tau^{-1}(\mathbb{R}^4) & \subset & \mathbb{CP}_3 \\ \downarrow\tau & & \downarrow\tau & & \downarrow\tau \\ \mathbb{R}^4 & = & \mathbb{R}^4 & \hookrightarrow & S^4 \end{array}$$

gives the complex structure on Z defined in §2. The Penrose transform for the twistor fibration (6.1) states that, for any open $U \subseteq S^4$, there is a natural isomorphism [7]

$$H^1(\tau^{-1}(U), \mathcal{O}(-2)) \xrightarrow{\simeq} \{\text{smooth } \phi \text{ on } U \text{ s.t. } \Box\phi = 0\}, \tag{6.2}$$

where $\Box$ is the conformal Laplacian. Without going into detail, on $\mathbb{R}^4$ the conformal Laplacian coincides with the usual Laplacian and so §3 provides a proof of (6.2) in case $U = \mathbb{R}^4$. In fact, the argument in §3 is manifestly local and so proves (6.2) for any open $U \subseteq \mathbb{R}^4$ or, indeed, $U \subseteq S^4$.

Surprisingly, the involutive structure constructed in §4 also arises from the complex structure on $\mathbb{CP}_3$ as follows. Consider the submersion

$$\begin{array}{ccc} \sigma : \mathbb{CP}_3 \setminus \mathbb{RP}_3 & \to & \mathrm{Gr}_2(\mathbb{R}^4) \\ \cup & & \cup \\ [z] & \longmapsto & [i\, dz \wedge d\bar{z}] \end{array}$$

noting that $i\, dz \wedge d\bar{z}$ is a simple real 2-form. Equivalently,

$$[x + iy] \overset{\sigma}{\longmapsto} \mathrm{span}_{\mathbb{R}}\{x, y\},$$

where $x, y \in \mathbb{R}^4$ are linearly independent. This mapping is ill-defined across $\mathbb{RP}_3$: each point in $\mathrm{Gr}_2(\mathbb{R}^4) \subset \mathrm{Gr}_2(\mathbb{C}^4)$ corresponds to a line in $\mathbb{CP}_3$ but, in contrast to the Euclidean case, these lines no longer foliate $\mathbb{CP}_3$. Instead, they can intersect along $\mathbb{RP}_3 \subset \mathbb{CP}_3$. The remedy is to pass to the real blow-up $\pi : F \to \mathbb{CP}_3$ of $\mathbb{CP}_3$ along $\mathbb{RP}_3$. We obtain a perfectly good foliation and it is shown in [5] that the complex structure on $\mathbb{CP}_3 \setminus \mathbb{RP}_3$ smoothly extends to F as an involutive structure. In summary, we obtain

$$\begin{array}{ccc} \tau : F & \to & \mathrm{Gr}_2(\mathbb{R}^4) \\ \downarrow \pi & & \\ \mathbb{CP}_3 & & \end{array}$$

where F has an involutive structure and the fibres of τ are intrinsically isomorphic to $\mathbb{CP}_1$ as holomorphic submanifolds. It is this involutive structure that was defined in §4 using the split quaternions. Specifically, if

$$\mathbb{R}^4 \cong \{\text{real } 2 \times 2 \text{ matrices}\} \hookrightarrow \mathrm{Gr}_2(\mathbb{R}^4)$$

is a standard affine coördinate patch, then it may be readily verified that

$$\begin{array}{ccccc} Z = \mathbb{R}^4 \times \mathbb{CP}_1 & \cong & \tau^{-1}(\mathbb{R}^4) & \subset & F \\ \downarrow \tau & & \downarrow \tau & & \downarrow \tau \\ \mathbb{R}^4 & = & \mathbb{R}^4 & \hookrightarrow & \mathrm{Gr}_2(\mathbb{R}^4) \end{array}$$

gives the involutive structure on Z defined in §4. Indeed, the split quaternions are isomorphic to the algebra of real 2×2 matrices [6] and this is nicely consistent with the construction in §4. The exceptional variety $\Sigma \equiv \pi^{-1}(\mathbb{RP}_2)$ is given by $\{y = 0\}$ in the local coördinates of §4. Each fibre of σ consists of two hemispheres separated by an equator and it is these equators that are themselves separated in the blow-up F.

The Penrose transform for the fibration $\tau : F \to \mathrm{Gr}_2(\mathbb{R}^4)$ is due to T.N Bailey and the author. It was sketched in [4]. The argument presented in §5 gives an explicit proof of the most basic isomorphism. More generally, for any open $U \subseteq \mathrm{Gr}_2(\mathbb{R}^4)$, there is a canonical isomorphism

$$H^1_{\bar{\partial}}(\tau^{-1}(U), \widetilde{V}(-2)) \xrightarrow{\simeq} \{\text{smooth } \phi \text{ on } U \text{ s.t. } \Box\phi = 0\}, \tag{6.3}$$

where $\Box$ denotes the ultrahyperbolic wave operator (acting between appropriate line bundles on $\mathrm{Gr}_2(\mathbb{R}^4)$ as detailed in [4]) and $\widetilde{V}(-2)$ denotes the pull-back $\pi^*\mathcal{O}(-2)$ on F tensored with the tautological locally constant bundle corresponding to the exceptional variety Σ (also detailed in [4]). Once the appropriate bundles are set up, the proof of (6.3) is local and the reasoning in §5 is essentially complete.

The following theorem is due to C.R. Graham and the author [4].

THEOREM 6.1. *There is an exact sequence*

$$0 \to \Gamma(\mathbb{CP}_3, \mathcal{O}(-2)) \to \Gamma(\mathbb{RP}_3, \mathcal{E}(-2)) \to H^1_{\bar\partial}(F, \widetilde{V}(-2)) \to H^1(\mathbb{CP}_3, \mathcal{O}(-2)) \to 0,$$

where $\mathcal{E}(-2)$ denotes the sheaf of germs of smooth sections of the line bundle $\mathcal{O}(-2)$ restricted to $\mathbb{RP}_3$.

In fact, the statement given in [4] applies to a general complex manifold equipped with a holomorphic vector bundle and blown up along a general real-analytic totally real submanifold. In our particular case, the vanishing of the global cohomology on $\mathbb{CP}_3$ immediately yields an isomorphism

$$\Gamma(\mathbb{RP}_3, \mathcal{E}(-2)) \xrightarrow{\simeq} H^1_{\bar\partial}(F, \widetilde{V}(-2))$$

and, in combination with (6.3), we obtain

$$\Gamma(\mathbb{RP}_3, \mathcal{E}(-2)) \xrightarrow{\simeq} \{\text{smooth } \phi \text{ on } \mathrm{Gr}_2(\mathbb{R}^4) \text{ s.t. } \Box\phi = 0\}.$$

This is a compactified version of John's classical description [8] of solutions of the ultrahyperbolic wave equation on $\mathbb{R}^4$.

In Riemannian signature, the basic twistor construction given in §2 extends to all self-dual four-manifolds. This was first shown by Penrose [10] in the complexified setting but formulated directly in terms of an integrable almost complex structure by Atiyah, Hitchin, and Singer [2]. Given the analogous construction of an involutive structure for the flat split signature metric in §4, one should expect that this generalise immediately to self-dual split signature metrics. This is indeed the case, as shown by LeBrun and Mason [9, Proposition 7.1]. In fact, these authors go on to show that the involutive structure always blows down to a complex structure (in fact, one might have to pass to a double cover to make good sense of this). As detailed in [9], geometric consequences follow from the holomorphic rigidity of $\mathbb{CP}_3$. In particular, LeBrun and Mason conclude that $\mathrm{Gr}_2(\mathbb{R}^4)$ admits no self-dual conformal deformations whereas its double cover $\mathrm{Gr}_2^+(\mathbb{R}^4)$, the Grassmannian of oriented 2-planes, is quite flexible in this regard and may be deformed precisely by deforming the standard embedding $\mathbb{RP}_3 \hookrightarrow \mathbb{CP}_3$ through smooth embeddings.

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