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# MODULAR REPRESENTATIONS OF THE GROUP $MQ$ OVER THE RING $K_M$*

PEDRO DOMÍNGUEZ WADE†

**Abstract.** Let $K_m$ be a finite commutative semi-local ring of characteristic $m$, and let $MQ$ be the generalized dicyclic group. Descriptions are given of the simple and projective $K_mMQ$-modules.

**Key words.** finite group, semi-local ring, indecomposable projective module, quasi-simple module.

**AMS subject classifications.** Primary 20C20; Secondary 20C34.

**1. Introduction.** Let $K_m$ be a finite commutative semi-local ring of characteristic $m$ with maximal ideals $(\Pi_i)$ and residue fields $F_{p_i} = K_m/(\Pi_i)$. Let $\prod_{i=1}^{t} p_i^{r_i}$ be the prime factorization of $m$. We denote by $J(K_m)$ to the Jacobson radical of $K_m$. Then $K_m/J(K_m)$ is the direct sum of the ideals $I_j/J(K_m)$ where $I_j = \bigcap_{i\neq j}(\Pi_i)$. Since $(\Pi_j)$ is maximal, $I_j/J(K_m) \cong K_m/(\Pi_j)$ is a field. Thus the direct summand $K_{p_j^{r_j}} = \bigcap_{n=0}^{m} I_j^n$ of $K_m$ which is such that $K_{p_j^{r_j}}/J(K_m)K_{p_j^{r_j}} = I_j/J(K_m)$ is a field, is a local ring of characteristic $p_j^{r_j}$. Assume that $p_1^{r_1}\cdots p_t^{r_t}$ is the prime factorization of the characteristic $m \geq 2$. Then we have

$$K_m = K_{p_1^{r_1}} \oplus \cdots \oplus K_{p_t^{r_t}}.$$

Therefore, if $G$ is a finite group then we have

$$K_mG = K_{p_1^{r_1}}G \oplus \cdots \oplus K_{p_t^{r_t}}G. \tag{1.0.1}$$

From (1.0.1) it follows that the indecomposable projective $K_{p_i^{r_i}}G$-modules are the indecomposable summands of the regular representation.

**1.1. Notations and Definitions.** Throughout the paper $K_m$ is a finite commutative semi-local ring of characteristic $m$ with maximal ideals $(\Pi_i)$ and residue fields $F_{p_i} = K_m/(\Pi_i)$ of characteristic $p_i$, and $K_{p^r}$ denotes a finite commutative local ring of characteristic $p^r$ with maximal ideal $(\Pi)$ and residue field $F_p = K_{p^r}/(\Pi)$. Let G be a finite group, $K_mG$ denotes the group ring of $G$, and $J_m(G)$ denotes the Jacobson radical of this ring. We denote the largest normal $p$-subgroup of $G$ by $O_p(G)$. The factor group $G/O_p(G) = \bar{G}$ is called reduced group modulo $p$.

**2. Indecomposable Projective Modules.** Let $K_{p^r}$ be a finite local ring of characteristic $p^r$ with maximal ideal $(\Pi)$ and residue field $F_p = K_{p^r}/(\Pi)$. As $K_{p^r}$ is Artinian ring and $K_{p^r}G$ is finitely-generated as $K_{p^r}$-module, it is Artinian. Hence the Jacobson radical $J_{p_r}(G)$ is nilpotent ideal. We consider the surjection $K_{p^r}G \longrightarrow F_p\bar{G}$. We denote the kernel of the surjection by $I_p(G) \subseteq J_{p_r}(G)$. Observe that $I_p(G)$ is nilpotent ideal. We have

$$K_{p^r}G/I_p(G) \cong F_p\bar{G}. \tag{2.0.1}$$

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†Department of Mathematics, Matanzas University, Cuba (pedro.dominguez@umcc.cu).

Proposition 2.0.1. *Let $K_{p^r}$ be a finite local ring of characteristic $p^r$ with maximal ideal* $(\Pi)$ *and residue field $F_p = K_{p^r}/(\Pi)$. Then we can write $1 = \hat{e}_1 + \cdots + \hat{e}_n$ in $K_{p^r}G$, where the $\hat{e}_i$ are primitive idempotents such that $\hat{e}_i \equiv \bar{e}_i \mod I_p(G)$ for all $i$, where the $\bar{e}_i$ are primitive idempotents in $F_p\bar{G}$.*

*Proof.* As $F_p$ is Artinian and $F_p\bar{G}$ is a $F_p$-algebra finitely generated as $F_p$-vector space, it is Artinian. Hence can write $1 = \bar{e}_1 + \cdots + \bar{e}_n$ in $F_p\bar{G}$, where the $\bar{e}_i$ are primitive idempotents. Since $F_p\bar{G} \cong K_{p^r}G/I_p(G)$ and $I_p(G)$ is nilpotent we can write $1 = \hat{e}_1 + \cdots + \hat{e}_n$ in $K_{p^r}G$,where the $\hat{e}_i$ are primitive idempotents such that $\hat{e}_i \equiv \bar{e}_i \mod I_p(G)$ for all $i$(See [2] theorem (7.11) ). □

Lemma 2.0.2. *Let $K_{p^r}$ be a finite local ring of characteristic $p^r$ with maximal ideal* $(\Pi)$ *and residue field $F_p = K_{p^r}/(\Pi)$. Let $G$ be a finite group. Then the simple $K_{p^r}G$-modules are precisely the simple $F_p\bar{G}$-modules made into $K_{p^r}G$-modules via the surjection $K_{p^r}G \longrightarrow F_p\bar{G}$.*

*Proof.* If $S$ is a simple $K_{p^r}G$-module, then also $S$ is a simple $F_p\bar{G}$-module, since $K_{p^r}G/I_p(G) \cong F_p\bar{G}$ and $I_p(G)$ annihilates the simple $K_{p^r}G$-modules. □

Recall that if $p$ is a prime, then an element in a finite group is said to be $p$-regular if is has order prime to $p$.

Proposition 2.0.3. *Let $K_{p^r}$ be a finite local ring of characteristic $p^r$ with maximal ideal* $(\Pi)$ *and residue field $F_p = K_{p^r}/(\Pi)$, and let $G$ be a finite group with splitting field $F_p$. Then the number of non-isomorphic simple $K_{p^r}G$-modules equals the number of conjugacy classes of $p$-regular elements of the reduced group $\bar{G}$.*

*Proof.* It well known that the number of non-isomorphic simple $F_p\bar{G}$-modules equals the number of conjugacy classes of $p$-regular elements of $G$ (See [2] theorem 9.11). The result follows by (2.0.2). □

Let $K_{p^r}$ be a finite local ring with maximal ideal $(\Pi)$ and residue field $F_p = K_{p^r}/(\Pi)$ of characteristic $p$ and let $G$ be a finite group with reduced group $\bar{G}$. Consider the ring homomorphism $\epsilon : F_pG \longrightarrow F_p\bar{G}$. The kernel of $\epsilon$ is denoted $IG$. Observe that $IG$ is nilpotent ideal, since $IG \subseteq Rad(F_pG)$.

Proposition 2.0.4. *Let $K_{p^r}$ be a finite local ring of characteristic $p^r$ with maximal ideal* $(\Pi)$ *and residue field $F_p = K_{p^r}/(\Pi)$. Let $G$ be a finite group with reduced group $\bar{G}$.*

1. *For each simple $K_{p^r}G$-module $S$ there is an indecomposable projective $F_p\bar{G}$-module $\bar{P}_S = F_p\bar{G}\bar{e}$ with the property that $\bar{P}_S/Rad(\bar{P}_S) \cong S$. Here $\bar{e}$ is a primitive idempotent which $\bar{e}S \neq 0$.*
2. *For each simple $K_{p^r}G$-module $S$ there is an indecomposable projective $F_pG$-module $P_S = F_pGe$ with the property that $P_S/IGP_S \cong \bar{P}_S$. Here $e$ is a primitive idempotent in $F_pG$ such that $eS \neq 0$.*
3. *For each simple $K_{p^r}G$-module $S$ there is an indecomposable projective $K_{p^r}G$-module $\hat{P}_S = K_{p^r}G\hat{e}$ with the property that $\hat{P}_S/(\Pi)\hat{P}_S \cong P_S$ is the projective cover of $S$ as a $F_pG$-module. Here $\hat{e}$ is a primitive idempotent in $K_{p^r}G$ such that $\hat{e}S \neq 0$.*
4. *$\hat{P}_S$ is projective cover of their radical quotient as $K_{p^r}G$-module.*

*Proof.*

1. Let $\bar{e} \in F_p\bar{G}$ be any primitive idempotent such that $\bar{e}S \neq 0$. We define $\bar{P_S} = F_p\bar{G}\bar{e}$. Then $\bar{P_S}$ is projective, and it is indecomposable since $\bar{e}$ is primitive. If $J_p(\bar{G})$ is the Jacobson radical of $F_p\bar{G}$ then we have

$$P_S/Rad(P_S) = F_p\bar{G}\bar{e}/J_p(\bar{G})F_p\bar{G}\bar{e} \cong F_p\bar{G}/J_p(\bar{G})(\bar{e} + J_p(\bar{G})) \cong S.$$

2. Let $\bar{e} \in F_p\bar{G}$ be any primitive idempotent for which $\bar{e}S \neq 0$. Since $F_pG/IG \cong F_p\bar{G}$ and $IG$ is nilpotent there is a primitive idempotent $e \in F_pG$ such that $\bar{e} \equiv e \mod IG$, so that $eS \neq 0$. We define $P_S = F_pGe$. Therefore $P_S$ is indecomposable projective $F_pG$-module, since $e$ is primitive idempotent. Thus we have

$$P_S/IGP_S = F_pGe/IGF_pGe \cong F_pG/IG(e + IG) \cong F_p\bar{G}\bar{e} = \bar{P}_S.$$

3. Consider the surjection of group rings $\theta : K_{p^r}G \longrightarrow F_pG$ with $\ker\theta = (\Pi)G$. Observe that $(\Pi)G \subseteq J_{p^r}(G)$, so $(\Pi)G$ is nilpotent. Therefore if $e \in F_pG$ is any primitive idempotent for which $eS \neq 0$, then there is a primitive idempotent $\hat{e} \in K_{p^r}G$ with the property that $e \equiv \hat{e} \mod (\Pi)G$. Hence $\hat{e}S \neq 0$. We define the indecomposable projective $\hat{P_S} = K_{p^r}G\hat{e}$. Furthermore $\hat{P_S}/(\Pi)\hat{P_S} = K_{p^r}G\hat{e}/(\Pi)K_{p^r}G\hat{e} = K_{p^r}G/(\Pi)G(\hat{e} + (\Pi)G) = F_pGe = P_S$. Now

$$\begin{aligned} P_S/Rad(P_S) &= P_S/J_{p^r}(G)P_S \\ &= F_pGe/J_{p^r}(G)F_pGe \cong F_pG/J_{p^r}(G)(e + J_{p^r}(G)) \cong S. \end{aligned}$$

   Hence the epimorphism $P_S \longrightarrow S$ is essential by Nakayama's lemma (See [2] theorem 7.6), and it is a projective cover.
4. Since $P_S$ is Noetherian as $F_pG$-module, and $\hat{P_S}$ is Noetherian as $K_{p^r}G$-module the result follows by Nakayama's lemma .

□

LEMMA 2.0.5. *Let $K_{p^r}$ be a finite local ring of characteristic $p^r$ with maximal ideal $(\Pi)$ and residue field $F_p = K_{p^r}/(\Pi)$. Let $G$ be a finite group. Let $P$ and $Q$ be projective $K_{p^r}G$-modules. Then $P \cong Q$ as $K_{p^r}G$-modules if and only if $P/(\Pi)P \cong Q/(\Pi)Q$ as $F_pG$-modules.*

*Proof.* If $P/(\Pi)P \cong Q/(\Pi)Q$ as $F_pG$-modules then the radical quotients of $P$ and $Q$ are isomorphic, $P/Rad(P) \cong Q/Rad(Q)$, since $(\Pi)G \subseteq J_{p^r}(G)$. Now $P$ and $Q$ are projective covers of their radical quotients, by Nakayama's lemma, so $P \cong Q$ by uniqueness of projective covers(See [2] proposition 7.8). The converse implication is trivial. □

PROPOSITION 2.0.6. *Let $K_{p^r}$ be a finite local ring of characteristic $p^r$ with maximal ideal $(\Pi)$ and residue field $F_p = K_{p^r}/(\Pi)$. Let $G$ be a finite group.*

1. *Every finitely- generated indecomposable projective $F_pG$-module $P$ is isomorphic to $P_S$ for some simple module $S$.*
2. *Every finitely- generated indecomposable projective $K_{p^r}G$-module $\hat{P}$ is isomorphic to $\hat{P_S}$ for some simple module $S$.*

*Proof.*

1. As $F_pG$ is Artinian ring and $P$ is finitely- generated indecomposable projective, it is Artinian. Hence the radical quotient $P/Rad(P) \cong S$ is a simple $F_pG$-module. By (2.0.4) part (3) we have

$$P/Rad(P) \cong P_S/Rad(P_S) \cong S.$$

As $P$ and $P_S$ are projective covers of their radical quotients, by Nakayama"s lemma, so that $P \cong P_S$ by uniqueness of projective covers(See [2] proposition (7.8)).

2. Let $\hat{P}$ be a finitely-generated projective $K_{p^r}G$-module. Since $K_{p^r}G$ is Artinian ring then $\hat{P}$ is Artinian module. Combining part (1) and proposition (2.0.4) part 3 we obtain:

$$\hat{P}/(\Pi)\hat{P} \cong \hat{P_{S_1}}/(\Pi)\hat{P_{S_1}} \oplus \cdots \oplus \hat{P_{S_n}}/(\Pi)\hat{P_{S_n}}.$$

Therefore by (2.0.5) it follows that $\hat{P} \cong \hat{P_{S_1}} \oplus \cdots \oplus \hat{P_{S_n}}$. If we assume that $\hat{P}$ is indecomposable then $n = 1$ and $\hat{P} \cong \hat{P_{S_1}}$.

□

PROPOSITION 2.0.7. *Let $K_{p^r}$ be a local ring of characteristic $p^r$ with maximal ideal $(\Pi)$ and residue field $F_p = K_{p^r}/(\Pi)$ and let $G$ be a finite group with splitting field $F_p$. The number of non-isomorphic finitely-generated indecomposable projective $F_pG$-modules equals the number of conjugacy classes of $p$-regular elements of the reduced group $\bar{G}$.*

*Proof.* Let $P_{S_1}, \ldots, P_{S_n}$ be a complete list of indecomposable projective $F_pG$-modules, then $S_1, \ldots, S_n$ is a complete list of simple $F_pG$-modules by the uniqueness of projective covers. According to the last proposition every finitely- generated indecomposable projective $F_pG$-module is isomorphic to $P_S$ for some simple module $S$. The result follows from proposition (2.0.3). □

PROPOSITION 2.0.8. *Let $K_{p^r}$ be a local ring of characteristic $p^r$ with maximal ideal $(\Pi)$ and residue field $F_p = K_{p^r}/(\Pi)$ and let $G$ be a finite group with splitting field $F_p$. The number of non-isomorphic finitely-generated indecomposable projective $K_{p^r}G$-modules equals the number of conjugacy classes of $\bar{G}$.*

*Proof.* We proceed as in proposition (2.0.7). □

Recall that if the finite group $G$ has a is called be a finite group and let $H$ be a subgroup of $G$ such that $|G : H| = |P|$, where $P$ is a Sylow $p$-subgroup of $G$. We denote the subgroup $O_p(G) \rtimes H$ of $G$ by $G'$. Moreover, $[G/G']$ denotes a set of representatives of left cosets $\{gG'|g \in G\}$.

THEOREM 2.0.9. *Let $K_{p^r}$ be a finite local ring of characteristic $p^r$ with maximal ideal $(\Pi)$ and residue field $F_p = K_{p^r}/(\Pi)$. Let $G$ be a finite group with splitting field $F_p$, containing a subgroup $G'$. Assume that $S_{H_1}, \ldots, S_{H_n}$ is a complete list of non-isomorphic simple $K_{p^r}G'$-modules.*

1. *If $Stab_G(S_{H_i}) = G$ then $S_{H_i}$ is simple $K_{p^r}G$-module.*
2. *If $Stab_G(S_{H_i}) < G$ then $S_{H_i} \uparrow_{G'}^{G}$ is simple $K_{p^r}G$-module.*

*Proof.*

1. Obvious.
2. We show that $End_{F_pG}(S_{H_i} \uparrow_{G'}^{G})$ is a division ring. Suppose $\phi \in End_{F_pG}(S_{H_i} \uparrow_{G'}^{G})$ is a non-zero endomorphism. Therefore $Stab_G(\ker\phi) = G$. It is well know that $S_{H_i} \uparrow_{G'}^{G} = \oplus_{g\in[G/G']} g\otimes S_{H_i}$, where the $F_p$-modules $g\otimes S_{H_i}$ are permuted under the action of $G$ and $Stab_G(g\otimes S_{H_i}) = G'$. Therefore $\ker\phi = 0$, since $\phi$ is non-zero endomorphism. The result follows by Schur's lemma (See [3] theorem (2.1)).

□

Let $K_{p^r}$ be a finite local ring of characteristic $p^r$ with maximal ideal $(\Pi)$ and residue field $F_p = K_{p^r}/(\Pi)$. Let $G$ be a finite group with splitting field $F_p$. Assume that $S$ is a simple $K_{p^r}G$-module. Then the finitely-generated $K_{p^r}G$-module $Q_S = K_{p^r} \otimes S$ is called quasi-simple $K_{p^r}G$-module corresponding to $S$. Observe that $Q_S$ is free as $K_{p^r}$-module and $Rad(Q_S) = (\Pi)Q_S$.

Lemma 2.0.10. *Let $K_{p^r}$ be a finite local ring with maximal ideal $(\Pi)$ and residue field $F_p = K_{p^r}/(\Pi)$ and let $G = K \rtimes H$ where $K$ is a $p$-group and $H$ has order prime to $p$. If $S$ is any simple $K_{p^r}G$-module then $\hat{P}_S = K_{p^r}K \otimes Q_S$.*

*Proof.* Since $F_pH$ is semisimple we may write $F_pH = F_p \oplus U$ for some $F_pH$-module $U$. Thus $\hat{P}_{F_p} = K_{p^r}$ is a projective $K_{p^r}H$-module and may write $K_{p^r}H = K_{p^r} \oplus \hat{U}$ for some projective $K_{p^r}H$-module $\hat{U}$, and now $K_{p^r}G = K_{p^r}H \uparrow_H^G = K_{p^r} \uparrow_H^G \oplus U \uparrow_H^G$. Here $K_{p^r} \uparrow_H^G \cong K_{p^r}P$ as $K_{p^r}G$-module, and so $K_{p^r}P$ is projective, being a summand of $K_{p^r}G$. Therefore $K_{p^r}K \otimes Q_S$ is projective (See [3] proposition 8.4). Now

$$Rad(K_{p^r}K \otimes Q_S) \supseteq I_p(G)K_{p^r}K \otimes I_p(G)Q_S.$$

Therefore

$$\begin{array}{rcl} K_{p^r}K \otimes Q_S/I_p(G)K_{p^r}K \otimes I_p(G)Q_S & = & K_{p^r}K/I_p(G)K_{p^r}K \otimes Q_S/I_p(G)Q_S \\ & \cong & F_p \otimes (F_p \otimes S) \\ & \cong & F_p \otimes S \cong S. \end{array}$$

Hence

$$K_{p^r}K \otimes Q_S/Rad(K_{p^r}K \otimes Q_S) \cong S.$$

Combining proposition (2.0.4) and proposition (2.0.6) we conclude that $\hat{P}_S = K_{p^r}K \otimes Q_S$. □

Theorem 2.0.11. *Let $K_{p^r}$ be a finite local ring of characteristic $p^r$ with maximal ideal $(\Pi)$ and residue field $F_p = K_{p^r}/(\Pi)$. Let $G$ be a finite group with splitting field $F_p$, containing a subgroup $G'$.*

1. $\hat{P}_S = \begin{cases} K_{p^r}P \otimes Q_S & \text{if} \quad p \nmid \dim S \\ K_{p^r}O_p(G) \otimes Q_S & \text{otherwise.} \end{cases}$
2. $rank_{K_{p^r}} \hat{P}_S = \dim_{F_p} P_S = \frac{\dim S|P|}{p^\alpha}$, *where $p^\alpha$ is the exact power of $p$ which divides $dimS$.*
3. *The indecomposable projective $K_{p^r}G$-module $\hat{P}_S$ appears as a direct summand of the regular representation, with multiplicity $n_S = \dim S$.*

*Proof.*

1. Let $S_{H_1}, \dots, S_{H_n}$ be a complete list of non-isomorphic simple $K_{p^r}G'$-modules. According to the last lemma we may write
$$K_{p^r}G' = K_{p^r}O_p(G) \otimes Q_{S_{H_1}} \oplus \cdots \oplus K_{p^r}O_p(G) \otimes Q_{S_{H_n}}.$$
Now
$$K_{p^r}G = K_{p^r}G' \uparrow_{G'}^{G} = (O_p(G) \otimes Q_{S_{H_1}}) \uparrow_{G'}^{G} \oplus \cdots \oplus (K_{p^r}O_p(G) \otimes Q_{S_{H_n}}) \uparrow_{G'}^{G} .$$
Notice that
$$\begin{array}{rcc} (K_{p^r}O_p(G) \otimes Q_{S_{F_p}}) \uparrow_{G'}^{G} & = & (K_{p^r}O_p(G) \otimes (K_{p^r} \otimes F_p)) \uparrow_{G'}^{G} \\ & \cong & (K_{p^r}O_p(G) \otimes K_{p^r}) \uparrow_{G'}^{G} \\ & \cong & K_{p^r}O_p(G) \uparrow_{G'}^{G} \\ & \cong & K_{p^r}P. \end{array}$$
Thus $K_{p^r}P$ is projective, being a direct summand of $K_{p^r}G$. We have to check two cases.
   - $Stab_G(S_{H_i}) = G$. In this case $S = S_{H_i}$ is a simple $K_{p^r}G$-module and $p \not| \dim S$. As $K_{p^r}P$ is projective and $Q_S$ is free as $K_{p^r}$-module the $K_{p^r}G$-module $K_{p^r}P \otimes Q_S$ is projective (See [2] proposition 8.4). Now
   $$Rad(K_{p^r}P \otimes Q_S) \supseteq Rad(K_{p^r}P) \otimes Rad(Q_S).$$
   Therefore
   $$\begin{array}{rcc} K_{p^r}P \otimes Q_S/Rad(K_{p^r}P) \otimes Rad(Q_S) & \cong & K_{p^r}P/Rad(K_{p^r}P) \otimes Q_S/Rad(Q_S) \\ & \cong & F_p \otimes S \\ & \cong & S. \end{array}$$
   Since $K_{p^r}P \otimes Q_S$ is Artinian it follows that
   $$K_{p^r}P \otimes Q_S/Rad(K_{p^r}P \otimes Q_S) \cong S.$$
   This shows that $K_{p^r}P \otimes Q_S$ is projective cover of $S$.
   - $Stab_G(S_{H_i}) < G$. By theorem (2.0.9) it follows that $S = S_{H_i} \uparrow_{G'}^{G}$ is a simple $K_{p^r}G$-module and $p \mid \dim S$. Now
   $$\begin{array}{rcc} K_{p^r}O_p(G) \otimes Q_{S_{H_i}} \uparrow_{G'}^{G} & = & \oplus_{g\in[G/G']} g \otimes (K_{p^r}O_p(G) \otimes Q_{S_{H_i}}) \\ & = & K_{p^r}O_p(G)) \otimes (\oplus_{g\in[G/G']} g \otimes Q_{S_{H_i}}) \\ & \cong & K_{p^r}O_p(G) \otimes (Q_{S_{H_i}} \uparrow_{G'}^{G}) \\ & \cong & K_{p^r}O_p(G) \otimes (K_{p^r} \otimes S_{H_i} \uparrow_{G'}^{G}) \\ & \cong & K_{p^r}O_p(G) \otimes Q_S. \end{array}$$
   Thus $K_{p^r}O_p(G) \otimes Q_S$ is projective. We may now proceed as in the previous case.
2. If $p \not| \dim S$ then $rank_{K_{p^r}} \hat{P}_S = \dim_{F_p} P_S = \dim_{F_p} S|P|$ by part (1). We now assume that $p \mid \dim S$. Then $\dim S = \dim S_H \mid G : G' \mid = \dim S_H \mid P : O_p(G) \mid$,where $S_H$ is a simple $K_{p^r}G'$-module. From (1) it follows that
$$\begin{aligned} rank_{K_{p^r}} \hat{P}_S = \dim_{F_p} P_S &= rank_{K_{p^r}} (K_{p^r}O_p(G) \otimes S) \\ &= \dim_{F_p} S|O_p(G)| = \dim_{F_p} S|P| / \mid P : O_p(G) \mid \end{aligned}$$
which complete the proof.

3. Each projective $P_S$ appear as direct summand of the regular representation, with multiplicity equal to the multiplicity of $S$ as a summand of $F_pG/Rad(F_pG)$ (See [2] proposition 7.14). Since $F_p$ is is splitting field of $G$ it follows that $S$ is absolutely simple. Hence $S$ occurs with multiplicity $n_S = \dim S$ as a summand of $F_pG/Rad(F_pG)$ (See [2]proposition 9.2). The number of non-isomorphic indecomposable projective $F_pG$-modules equals the number of non-isomorphic indecomposable projective $K_{p^r}G$-modules. Therefore the assertion follows by part 2.

□

**3. Indecomposable Projective $K_mG$-modules.** Let $K_m$ be a finite semi-local ring of characteristic $m$ with maximal ideals $(\Pi_i)$ and residue fields $F_{p_i} = K_m/(\Pi_i)(i = 1, \ldots, t)$. Throughout the section $p_1^{r_1} \cdots p_t^{r_t}$ is the prime factorization of the characteristic $m \geq 2$. The decompositions of $K_m$ as a direct sum of local rings:

$$K_m = K_{p_1^{r_1}} \oplus \cdots \oplus K_{p_t^{r_t}}$$

biject with expressions $1 = f_1 + \cdots + f_t$ for the identity of $K_m$ as a sum of orthogonal idempotents, in such a may that $K_{p_i^{r_i}} = K_m f_i$. Here the idempotent $f_i$ is primitive. By (1.0.1) it follows that

$$K_mG = K_{p_1^{r_1}}G \oplus \cdots \oplus K_{p_t^{r_t}}G = K_mGf_1 \oplus \cdots \oplus K_mGf_t, \tag{3.0.2}$$

where $K_{p_i^{r_i}}G = K_mGf_i$.

Remark 3.0.12. Observe that the $f_i$ are central idempotents in $K_mG$.

Theorem 3.0.13. *Let $K_m$ be a finite semi-local ring of characteristic $m$ with maximal ideals $(\Pi_i)$ and residue fields $F_{p_i} = K_m/(\Pi_i)(i = 1, \ldots, t)$. Let $G$ be a finite group.*

1. *The simple $K_mG$-modules are exactly the simple $K_{p_i^{r_i}}G$-modules made into $K_mG$-modules via the surjection $K_mG \longrightarrow K_{p_i^{r_i}}G$.*
2. *For each simple $K_mG$-module $S^{(i)}$ there is an indecomposable projective $K_{p_i^{r_i}}G$-module $\hat{P}_{S^{(i)}} = K_{p_i^{r_i}}G\hat{e}_i$ with the property that $\hat{P}_{S^{(i)}}/Rad(\hat{P}_{S^{(i)}}) \cong S^{(i)}$. Here $\hat{e}_i$ is a primitive idempotent in $K_{p_i^{r_i}}G$ such that $\hat{e}_iS^{(i)} \neq 0$.*
3. *Every finitely-generated indecomposable $K_mG$-module $\hat{P}$ is isomorphic to $\hat{P}_{S^{(i)}}$ for some simple module $S^{(i)}$.*

*Proof.*

1. Let $S^{(i)}$ be a $K_mG$-module. Then $S^{(i)} = S^{(i)}f_1 \oplus \cdots \oplus S^{(i)}f_t$. If $S^{(i)}$ is simple we have $S^{(i)}f_i = S^{(i)}$ for precisely one $i$ and $S^{(i)}f_j = 0$ for $j \neq i$. The result follows.
2. By part (1) the simple $K_mG$-modules are the simple $K_{p_i^{r_i}}G$-modules. The assertion follows from proposition (2.0.6).
3. If $\hat{P}$ is finitely-generated indecomposable $K_mG$-module then there is a unique $i$ such that $\hat{P}f_i = \hat{P}$ and $\hat{P}f_j = 0$ for $j \neq i$. Thus, this assertion also follows by (2.0.6).

□

Notice that the indecomposable projective $K_mG$-module $\hat{P}_{S^{(i)}}$ is not free. Let $G$ be a finite group. We denote the number of conjugacy classes of $p_i$-regular elements of $G$ by $n_{p_i}$, and $[P_i]$ denotes a complete list of indecomposable projective $K_{p_i^{r_i}}G$-modules $P_{S^{(i)}}$ for some simple $K_mG$-module $S^{(i)}$.

THEOREM 3.0.14. *Let $K_m$ be a finite semi-local ring of characteristic $m$ with maximal ideals $(\Pi_i)$ and residue fields $F_{p_i} = K_m/(\Pi_i)(i = 1, \dots, t)$. Let $G$ be a finite group with splitting fields $F_{p_i}$. Then the number of non-isomorphic finitely-generated indecomposable projective $K_mG$-modules is given by $n_m = \sum_{i=1}^{t} n_{p_i}$.*

*Proof.* According to the last theorem $[P_1], \dots, [P_t]$ is a complete list of indecomposable $K_mG$-modules. Since $|[P_i]| = n_{p_i}$ the assertion follows. □

**4. Some subgroups of $MQ$.** Let $MQ = \langle a, b : a^k = b^{ls}, bab^{-1} = a^u, a^{dk} = b^{dls} = e\rangle$ be the finite group, where $k, s$ and $u$ are integers with $k > 1$ and $s \geq 1$ .The positive integer $d$ is a divisor of $u - 1$ and $l$ is the multiplicative order of $u$ modulo $dk$. The group is called "generalized dicyclic group". Let $j = lsq + r', 0 \leq r' < ls$. Observe that for all elements $g = b^j a^i (0 \leq i \leq dk - 1, 0 \leq j \leq bls - 1)$ we have:

$$g = b^j a^i = b^j a^i b^{-j} b^j = a^{u^j i} b^j = a^{u^j i} b^{lsq+r'} = a^{u^j i + kq} b^{r'}.$$

Therefore all element $g$ of $MQ$ can be expressed in the following form: $a^i b^j (0 \leq i \leq dk - 1; 0 \leq j \leq ls - 1)$. Thus the order of the group $MQ$ is $dkls$.

REMARK 4.0.15. Observe that when $u = -1$ and $s = 1$, the group is dihedral or general quaternion group according to $d = 1$ or $d = 2$.

**4.1. Center of the Group.** We denote the center of the group by $Z(MQ)$. Let $d^*$ be the greatest common divisor of $k$ and $\frac{u-1}{d}$. Set $H_z = \langle h_z \in MQ \mid h_z = a^{\frac{k}{d^*}\alpha} b^{l\delta}\rangle$, where $\alpha = 0 \dots, dd^* - 1, \delta = 0, \dots, s - 1$. Then if $h_z = a^{\frac{k}{d^*}\alpha} b^{l\delta} \in H_z$ we have for any element $g = a^i b^j \in MQ$

$$\begin{aligned} h_z g h_z^{-1} &= a^{\frac{k}{d^*}\alpha + u^{l\delta} i} b^j a^{-\frac{k}{d^*}\alpha} \\ &= a^{(1-u^j)\frac{k}{d^*}\alpha + i} b^j \\ &= a^{-(u^j-1)\frac{k}{d^*}\alpha + i} b^j \\ &= a^{-(u-1)(u^{j-1}+u^{j-2}+\dots+1)\frac{k}{d^*}\alpha + i} b^j \\ &= a^i b^j \\ &= g. \end{aligned}$$

Therefore we have:

$$H_z \subseteq Z(MQ). \tag{4.1.1}$$

Let $z = a^{i'} b^{j'}$ be an element of $Z(MQ)$ and let $g = a^i b^j$ be any element of $MQ$. Then we have:

$$zgz^{-1} = a^{(1-u^j)i' + u^{j'} i} b^j = a^i b^j = g. \tag{4.1.2}$$

From (4.1.2) we obtain:

$$a^{(1-u^j)i' + (u^{j'}-1)i} = e \tag{4.1.3}$$

where $e$ is the identity of $MQ$. From (4.1.3) it follows that:

$$(u^{j'} - 1)i - (u^j - 1)i' \equiv 0 \mod dk. \tag{4.1.4}$$

The congruence (4.1.4) is true if $i' \equiv 0 \mod k/d^*$ and $j' \equiv 0 \mod l$. In fact we have:

$$\begin{array}{rcll} (u^{j'}-1)i-(u^j-1)i' & \equiv & -(u^j-1)i' & \mod dk \\ & \equiv & -(u-1)(u^{j-1}+u^{j-2}+\cdots+1)i' & \mod dk \\ & \equiv & 0 & \mod dk \end{array}.$$

Therefore we obtain:

$$Z(MQ) \subseteq H_z. \tag{4.1.5}$$

Combining (4.1.1) and (4.1.5) we obtain

$$H_z = Z(MQ).$$

Thus the order of the center is $dd^*s$.

**4.2. Commutator Group.** We will denote the commutator subgroup of $MQ$ by $MQ'$. Then

$$\langle a^{u-1}\rangle \subseteq MQ' \tag{4.2.1}$$

since $bab^{-1}a^{-1} = a^{u-1}$. In order to prove the reverse inclusion, we note that for any commutator $a^ib^ja^{-i}b^{-j}$ we have:

$$a^ib^ja^{-i}b^{-j} = a^{(1-u^j)i} = a^{-i(u-1)(u^{j-1}+\cdots+1)}.$$

Therefore we obtain:

$$MQ' \subseteq \langle a^{u-1}\rangle. \tag{4.2.2}$$

Combining (4.2.1) and (4.2.2) leads to

$$MQ' = \langle a^{u-1}\rangle.$$

The commutator quotient group $\frac{MQ}{MQ'}$ has order $dd^*ls$, since $|MQ'| = k/d^\star$.

**4.3. Largest Normal $p$-subgroup.** Let $MQ$ be the generalized dicyclic group where $d = p^{r_1}\bar{d}, k = p^{r_2}\bar{k}$ and $s = p^{r_4}\bar{s}$, with $\bar{d}, \bar{k}$ and $\bar{s}$ relatively prime to $p$. We denote the largest normal $p$-subgroup of $MQ$ by $O_p(MQ)$. Let $\tau$ be the multiplicative order of $u$ modulo $\overline{dk}$. We denote the least common multiple of $\tau$ and $\bar{l}$ by $n$. Set $H_o = \langle h_o \in MQ \mid h_o = a^{\overline{dk}\rho_1}b^{n\bar{s}\rho_2}\rangle$,where $\rho_1 = 0,\ldots,p^{r_1+r_2}-1, \rho_2 = 0,\ldots,\frac{l}{n}p^{r_4}-1$. Thus, if $h_o = a^{\overline{dk}\rho_1}b^{n\bar{s}\rho_2} \in H_o$ we have for any element $g = a^ib^j \in MQ$

$$\begin{array}{rcl} gh_og^{-1} & = & a^{i+u^j\overline{dk}\rho_1}b^{n\bar{s}\rho_2}a^{-i} \\ & = & a^{i(1-u^{n\bar{s}\rho_2})+u^j\overline{dk}\rho_1}b^{n\bar{s}\rho_2}. \end{array}$$

Since $u^n \equiv 1 \mod \overline{dk}$ it follows that

$$a^{-i(u^n-1)[(u^n)^{\bar{s}\rho_1-1}+\cdots+1]+u^j\overline{dk}\rho_1}b^{n\bar{s}\rho_2} = a^{\overline{dk}[-i(\frac{u^n-1}{\overline{dk}})((u^n)^{\bar{s}\rho_2-1}+\cdots+1)+u^j\rho_1]}b^{n\bar{s}\rho_2}.$$

Hence $gh_og^{-1} \in H_o$, so $H_o$ is a normal $p$-subgroup of $MQ$. Therefore we have

$$H_o \leq O_p(MQ). \tag{4.3.1}$$

Let $h = a^{\alpha}b^{\beta}$ be an element of $O_p(MQ)$, and let $g = a^i b^j$ be any element of $MQ$. Then we have

$$ghg^{-1} = a^i b^j a^{\alpha} b^{\beta} b^{-j} a^{-i} = a^{i(1-u^{\beta})+u^j\alpha} b^{\beta}.$$

From (4.3.1) it follows that $\langle a^{\bar{d}\bar{k}} \rangle \leq O_p(MQ)$. Therefore $ghg^{-1} \in O_p(MQ)$ if $\alpha \equiv 0 \mod \bar{d}\bar{k}$ and $\beta \equiv 0 \mod n$. Hence

$$O_p(MQ) \leq H_o. \tag{4.3.2}$$

From (4.3.2) we conclude that $O_p(MQ) = H_o$, since in every finite group there is a unique largest normal $p$-subgroup.

Theorem 4.3.1. *Let $MQ$ be the generalized dicyclic group. Then $MQ$ contains a subgroup $MQ' = O_p(MQ) \rtimes H$ with $\mid G : H \mid = \mid P \mid$. Here $P$ is a Sylow $p$-subgroup.*

*Proof.* Assume that $d = \bar{d}p^{r_1}, k = \bar{k}p^{r_2}, l = \bar{l}p^{r_3}$ and $s = \bar{s}p^{r_4}$, where $\bar{d}, \bar{k}, \bar{l}$ and $\bar{s}$ are prime to $p$. Set $H = \{g \in MQ \mid g = a^{ip^{r_1+r_2}} b^{jp^{r_1+r_3+r_4}}, i = 0, \dots, \bar{d}\bar{k} - 1; j = 0, \dots, \bar{l}\bar{s} - 1\}$. Let $g' = a^{i'p^{r_1+r_2}} b^{j'p^{r_1+r_3+r_4}}$ and $g'' = a^{i''p^{r_1+r_2}} b^{j''p^{r_1+r_3+r_4}}$ be two any elements of $H$. Assume that $j' + j'' = \bar{l}\bar{s}q + \bar{r}, 0 \leq \bar{r} < \bar{l}\bar{s}$. We have:

$$\begin{aligned} g'g'' &= (a^{i'p^{r_1+r_2}} b^{j'p^{r_1+r_3+r_4}})(a^{i''p^{r_1+r_2}} b^{j''p^{r_1+r_3+r_4}}) \\ &= a^{i'p^{r_1+r_2} + i''u^{j'p^{r_1+r_3+r_4}} p^{r_1+r_2}} b^{(j'+j'')p^{r_1+r_3+r_4}} \\ &= a^{(i'+i''u^{p^{r_1+r_3+r_4}})p^{r_1+r_2}} b^{p^{r_1+r_3+r_4}(\bar{l}\bar{s}q+\bar{r})} \\ &= a^{(i'+i''u^{p^{r_1+r_3+r_4}})p^{r_1+r_2}} b^{lsqp^{r_1}+\bar{r}p^{r_1+r_3+r_4}} \\ &= a^{p^{r_1+r_2}(i'+u^{p^{r_1+r_3+r_4}} i'')} a^{p^{r_1}qk} b^{\bar{r}p^{r_1+r_3+r_4}} \\ &= a^{(i'+i''u^{p^{r_1+r_3+r_4}}+q\bar{k})p^{r_1+r_2}} b^{\bar{r}p^{r_1+r_3+r_4}} \in H. \end{aligned} \tag{4.3.3}$$

From (4.3.3) it follows that $H \leq MQ$, since $MQ$ is finite group. We claim that $\mid H \mid = \bar{d}\bar{k}\bar{l}\bar{s}$. Since $O_p(MQ) \bigcap H = \{e\}$, the result follows. □

Remark 4.3.2. Let $MQ$ be the generalized dicyclic group. We assume that $d = p^{r_1}\bar{d}, k = p^{r_2}\bar{k}, l = p^{r_3}\bar{l}$ and $s = p^{r_4}\bar{s}$, where $\bar{d}, \bar{k}, \bar{l}$ and $\bar{s}$ are prime to $p$. We denote for $\bar{d}_j$ all positive divisors of $\bar{d}\bar{k}$. Let $d_j^*$ be the multiplicative order of $u$ modulo $\bar{d}_j$. On the set of the primitive $\bar{d}_j$-th roots of unity we define the following equivalence relation:

$$\varepsilon \equiv \varepsilon' \quad \text{if and only if} \quad \varepsilon^{u^{i-1}} = \varepsilon' \quad \text{for some i}(1 \leq i \leq d_j^*).$$

The number of equivalent classes is given by $\frac{\varphi(\bar{d}_j)}{d_j^*}$. We denote a set of representatives of these equivalent classes by $A_j = \{\varepsilon_{1j}, \dots, \varepsilon_{\frac{\varphi(\bar{d}_j}{d_j^*} j}\}$. Set $B_n = \{\omega_h \in F_p \mid \omega_h^{ls} = \varepsilon_{nj}, \varepsilon_{nj} \in A_j\}$. On the set $B_n$ we define the following equivalent relation:

$$\omega_h \equiv \omega_{h'} \quad \text{if and only if} \quad (\omega_h \omega_{h'}^{-1})^{d_j^*} = 1.$$

In this case the number of equivalent classes is $\frac{\bar{l}\bar{s}}{\bar{d}_j^*}$, where $\bar{d}_j^* = \frac{d_j^*}{p\alpha}$ and $p^{\alpha}$ is the exact power of $p$ which divides $d_j^*$. We denote a set of representatives of these equivalent classes by $B_{nj} = \{\omega_{1n}, \dots, \omega_{\frac{\bar{l}\bar{s}}{\bar{d}_j^*} n}\}$.

Theorem 4.3.3. *Let $K_{p^r}$ be a finite local ring of characteristic $p^r$ with maximal ideal $(\Pi)$ and residue field $F_p = K_{p^r}/(\Pi)$. Let $MQ$ be the generalized dicyclic group with splitting field $F_p$. Assume that $S$ is a $F_p$-vector space of dimension $d_j^*$ with basis $X = \{a_1, \ldots, a_{d_j^*}\}$ and an action of $MQ$ given as follows*

$$a(a_i) = \varepsilon_{nj}^{u^{i-1}} a_i, b(a_1) = \omega_{hn} a_{d_j^*}, b(a_i) = \omega_{hn} a_{i-1} (2 \leq i \leq d_j^*) \tag{4.3.4}$$

*where $\varepsilon_{nj} \in A_j$ and $\omega_{hn} \in B_{nj}$.*

1. *$S$ is absolutely simple $K_{p^r}MQ$-module.*
2. *The number of non-isomorphic indecomposable projective $K_{p^r}MQ$-modules is given by*

$$\sum_{j=1}^{\beta} \frac{\varphi(\bar{d}_j)}{d_j^* \bar{d}_j^*} \bar{l}\bar{s}$$

*where $\beta$ equals the number of positive divisors of $\overline{dk}$.*

*Proof.*

1. We may check that is indeed a representation of $MQ$ by verifying that $a^k(x) = b^{ls}(x), a^{dk}(x) = b^{dls}(x) = x, bab^{-1}(x) = a^u(x)$ for all $x \in S$, which is immediate. Let us now show that $S$ is simple $F_pMQ$-module. We will do this by showing that $End_{F_pMQ}(S)$ is a division ring. Suppose $\theta : S \longrightarrow S$ is a singular endomorphism. Then $0 \neq \ker \theta$ contains a basis $Y \subseteq X$, since $a(x) \in \ker \theta$ for all $x \in ker\theta$. Since the element of $X$ are permuted by $b$ we have $X = Y$,i.e. $\ker \theta = S$. The assertion follows by Schur"s lemma. The simple module $S$ is called simple $F_pMQ$-module corresponding to $\bar{d}_j$.
2. Let $S$ and $S'$ be two $FQ$-vector spaces of dimension $d_j^*$ with basis $X = \{a_1, \ldots, a_{d_j^*}\}$ and $X' = \{b_1, \ldots, b_{d_j^*}\}$, respectively, and an action of $MQ$ given by

$$a(a_i) = \varepsilon_{nj}^{u^{i-1}} a_i, b(a_1) = \omega_{hn} a_{d_j^*}, b(a_i) = \omega_{hn} a_{i-1} (2 \leq i \leq d_j^*)$$

and

$$a(b_i) = \varepsilon_{n'j}^{u^{i-1}} b_i, b(b_1) = \omega_{h'n'} b_{d_j^*}, b(b_i) = \omega_{h'n'} b_{i-1} (2 \leq i \leq d_j^*)$$

where $\varepsilon_{nj}, \varepsilon_{n'j} \in A_j$ and $\omega_{hn}, \omega_{h'n'} \in B_{nj}$. Now $S$ and $S'$ are simple $F_pMQ$-modules corresponding to $\bar{d}_j$ by part (1). Assume that $\varepsilon_{nj} \neq \varepsilon_{n'j}$. Let $\phi$ be any element of $Hom_{F_pMQ}(S, S')$, and let $a_i$ be an element of $X$. Then we have

$$\phi(a(a_i)) = \phi(\varepsilon_{nj}^{u^{i-1}} a_i) = \varepsilon_{nj}^{u^{i-1}} \phi(a_i) = a\phi(a_i). \tag{4.3.5}$$

Let $\phi(a_i) = \alpha_1 b_1 + \cdots + \alpha_{d_j^*} b_{d_j^*}$ be the unique expression of $\phi(a_i)$ as a $F_p$-linear combination of vectors in $X'$. The equality (4.3.5) is true if $\alpha_i = 0$ by assumption, so that $Hom_{F_pMQ}(S, S') = 0$. Hence $S \ncong S'$ by Schur"s lemma. We now assume $\omega_{hn} \neq \omega_{h'n}$. This case is analogous to the previous one. In fact, the equality $\phi(b(a_i)) = b\phi(a_i)$ is true for if $\phi$ is zero morphism. The number of non-isomorphic absolutely simple $F_pMQ$-modules corresponding to $\bar{d}_j$ is given by $\frac{\varphi(\bar{d}_j}{d_j^*}$, since $|A_j| = \frac{\varphi(\bar{d}_j}{d_j^*}$ and $|B_{nj}| = \frac{\bar{l}\bar{s}}{d_j^*}$. Therefore the number of these non-isomorphic simple $F_pMQ$-modules is given as follows

$$N_p = \sum_{j=1}^{\beta} \frac{\varphi(\bar{d}_j)}{d_j^* \bar{d}_j^*} \bar{l}\bar{s}.$$

Combining (2.0.11) and (4.3.1) we obtain

$$rak_{K_{p^r}} \hat{P}_S = \frac{d_j^* p^{r_1+r_2+r_3+r_4}}{\frac{d_j^*}{\bar{d}_j^*}} = \frac{\bar{d}_j^* d_j^* p^{r_1+r_2+r_3+r_4}}{d_j^*}.$$

As $F_p$ is a splitting field of $MQ$ each indecomposable projective $K_{p^r}MQ$-module $\hat{P}_S$ appears as direct summand of the regular representation with multiplicity equal to $d_j^*$ by theorem (2.0.1) part (3). We will complete the proof showing that $\hat{P}_{S_1}, \ldots, \hat{P}_{S_{N_p}}$ is a complete list of non-isomorphic indecomposable projective $K_{p^r}MQ$-modules. In fact, we have

$$\sum_{j=1}^{\beta} \frac{\varphi(\bar{d}_j)}{d_j^* \bar{d}_j^*} \bar{l}\bar{s} \frac{\bar{d}_j^* d_j^{*2} p^{r_1+r_2+r_3+r_4}}{d_j^*} = \sum_{j=1}^{\beta} \varphi(\bar{d}_j) p^{r_1+r_2} ls = \bar{d}\bar{k} p^{r_1+r_2} ls = dkls$$
$$= |MQ|,$$

which is what we need to prove.

□

REMARK 4.3.4. Let $MQ$ be the generalized dicyclic group. We assume that $d = p_i^{r_1}\bar{d}_i, k = p^{r_2}\bar{k}_i, l = p_i^{r_3}\bar{l}_i$ and $s = p_i^{r_4}\bar{s}_i$, where $\bar{d}_i, \bar{k}_i, \bar{l}_i$ and $\bar{s}_i$ are prime to $p$. We denote for $\bar{d}_{ij}(j = 1, \ldots, \beta_i)$ all positive divisors of $\bar{d}_i\bar{k}_i$. Let $d_{ij}^*$ be the multiplicative order of $u$ modulo $\bar{d}_{ij}$. Preceding exactly as in (4.3.2) we obtain $A_{ij} = \{\varepsilon_{1j}^i, \ldots, \varepsilon_{\frac{\varphi(\bar{d}_j}{d_j^*}j}^i\}$ and $B_{nj}^i = \{\omega_{1n}^i, \ldots, \omega_{\frac{\bar{l}\bar{s}}{d_j^*}n}^i\}$.

THEOREM 4.3.5. *Let $K_m$ be a finite local ring of characteristic $m$ with maximal ideals $(\Pi_i)$ and residue fields $F_{p_i} = K_m/(\Pi_i)$ of characteristic $p_i$. Let $MQ$ be the generalized dicyclic group with splitting fields $F_{p_i}$ . Assume that $S^{(i)}$ is a $F_p$-vector space of dimension $d_{ij}^*$ with basis $X = \{v_1, \ldots, v_{d_{ij}^*}\}$ and an action of $MQ$ given as follows*

$$a(v_\chi) = \varepsilon_{nj}^{iu^{\chi-1}} v_\chi, b(v_1) = \omega_{hn}^i v_{d_{ij}^*}, b(v_\chi) = \omega_{hn}^i v_{\chi-1} (2 \leq \chi \leq d_{ij}^*) \quad (4.3.6)$$

*where $\varepsilon_{nj}^i \in A_{ij}$ and $\omega_{hn}^i \in B_{nj}^i$.*

1. *$S^{(i)}$ is absolutely simple $K_mMQ$-module.*
2. *The number of non-isomorphic indecomposable projective $K_mMQ$-modules is given by*

$$\sum_{i=1}^{t} \sum_{j=1}^{\beta_i} \frac{\varphi(\bar{d}_{ij})}{d_{ij}^* \bar{d}_{ij}^*} \bar{l}_i \bar{s}_i.$$

*Here $\bar{d}_{ij}^* = d_{ij}^*/p^{\alpha_i}$, where $p^{\alpha_i}$ is the exact power of $p$ which divides $d_{ij}^*$.*

*Proof.*

1. By theorem (4.3.3) part (1), $S^{(i)}$ is absolutely simple $K_{p_i^{r_i}}MQ$-module. The result follows from theorem (3.0.12).
2. By theorem (4.3.3) $n_{p_i} = \sum_{j=1}^{\beta_i} \frac{\varphi(\bar{d_{ij}})}{d_{ij}^* \bar{d_{ij}^*}} \bar{l}_i \bar{s}_i$. We may now apply theorem (3.0.14).

□

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