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HOMOGENEOUS VARIETIES – ZERO-CYCLES OF DEGREE ONE VERSUS RATIONAL POINTS*

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Abstract. Examples of projective homogeneous varieties over the field of Laurent series over p-adic fields which admit zero-cycles of degree one and which do not have rational points are constructed.

Key words. projective homogeneous varieties, zero-cycles, rational points

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Let k be a field and X a quasi-projective variety over k. Let $Z_0(X)$ denote the group of zero-cycles on X and deg : $Z_0(X) \to \mathbb{Z}$ the degree homomorphism which associates to a closed point x of X, the degree $[k(x) : k]$ of its residue field.

One would like to understand which classes $\mathfrak{X}$ of smooth absolutely irreducible varieties (respectively, what classes of fields k) satisfy the property: $X \in \mathfrak{X}$, if X admits a zero-cycle of degree one, then X has a rational point.

Even in the setting of rational varieties, there are examples, due to Colliot-Thélène and Coray [CTC] of conic bundles over the projective line over a p-adic field with a zero-cycle of degree one, which have no rational points. In the next section, we briefly recall from the literature some questions in this direction for homogeneous varieties and their status. In the final section, we show by an example that there exist projective homogeneous varieties over fields with cohomological dimension 3 which admit zero-cycles of degree one, but which have no rational points.

The author thanks the referee for pointing out the generality in Lemma 1.

1. Some open questions. We begin by listing some open questions from the literature concerning homogeneous spaces under linear algebraic groups - existence of zero-cycles of degree one versus rational points.

$Q(HP)$ **(Veĭsfeĭler)[V]** *Let X be a projective homogeneous variety under a connected linear algebraic group defined over a field k. If X has a zero-cycle of degree one, does X have a rational point?*

$Q(PHS)$ **(Serre) ([Se] p 192, [Se1] p 166)** *Let X be a principal homogeneous space for a connected linear algebraic group G defined over k. If X has a zero cycle of degree one, does X have a rational point?*

The following questions combine the above two in a more general setting.

$Q(H)$ **(Colliot-Thélène)[To]** *Let X be a quasi-projective variety over k which is a homogeneous space for a connected linear algebraic group defined over k. If X has a zero-cycle of degree one, does X have a rational point?*

$Q(Hd)$ **(Totaro)[To]** *Let X be a quasi-projective variety over k which is a homogeneous space for a connected linear algebraic group defined over k. If X has a zero-cycle of degree $d > 0$, does X have a closed point of degree dividing d?*

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Totaro mentions that the most reasonable cases of his question are where X is a principal homogeneous space or a projective homogeneous space.

Connectedness. Serre points out in ([Se] p 192, [Se1] p 166) that if the connectedness assumption is dropped in $Q(PHS)$, it has a negative answer in general. We thank P. Gille for bringing to our attention the following example.

We recall that for any subgroup H of a group G with trivial G action, the set $H^1(H,G)$ is the quotient of $Hom(H,G)$ under the equivalence: $f \sim g$ if and only if there exists $y \in G$ such that $f(x) = yg(x)y^{-1}$ for all $x \in H$.

There exists a finite group G equipped with an automorphism f (Coleman automorphisms [HK]) such that f is not inner, but for every Sylow subgroup H of G, $f|_H : H \to G$ is given by $f(x) = y_H x y_H^{-1}$ for some $y_H \in G$. The class of $[f]$ in $H^1(G,G)$ for the trivial action of G on G is non-trivial, but restricted to each p-Sylow subgroup, it is trivial. Since any finite group may be realised as the Galois group of a finite Galois extension, for instance, of a number field, the above example shows that $Q(PHS)$ has a negative answer in general if the connectedness hypothesis is dropped.

Principal homogeneous spaces. The case of $Q(PHS)$ is wide open, and in special cases $Q(PHS)$ is proved to have an affirmative answer. The cases of PGL_n and SO_n are classical; for SO_n the result goes back to a theorem of Springer [Sp]. The case of unitary groups is settled in the affirmative by Eva-Bayer and Lenstra [BL]. A positive answer to $Q(PHS)$ when k is a number field is due to Sansuc ([Sa],§4 Cor. 4.8).

Projective homogeneous varieties. The Hasse principle holds for the existence of rational points for projective homogeneous varieties defined over number fields, thanks to a theorem of Harder([H]). Borovoi gives an alternate proof of Harder's theorem, ([B], Cor. 7.5) using the non-abelian H^2 of Springer to study homogeneous varieties. Following Borovoi's proof and using the results of [CGP], one can show that if k is a 2-dimensional strict henselian field, $Q(HP)$ has a positive answer. Borovoi's proof also leads to a positive answer to $Q(HP)$ for number fields. It is good to study $Q(HP)$ in the case of 2-dimensional fields.

The first example where $Q(H)$ has a negative answer is due to Florence [F]. The base field in the examples of Florence may be taken to be $\mathbb{C}((x))((y))$ or a local field or a global field; these fields have virtual cohomological dimension at most 2. The stabiliser of a rational point over the algebraic closure for these homogeneous spaces is a finite group. In the next section, we give examples to show that $Q(HP)$ has a negative answer in general. In particular, the stabiliser of a rational point over the algebraic closure in these examples is a connected group; the base field is the Laurent series field over a p-adic field which has cohomological dimension 3.

2. Examples. In this section, we construct examples to show that $Q(HP)$ has a negative answer in general. These examples are a refinement of an example of an anisotropic rank 2 hermitian form over a division algebra with a unitary involution admitting a nontrivial zero in two coprime degree extensions, given in [PSS]. In the example in [PSS], the variety Y of zeros of the hermitian form over the algebraic closure $\bar{K}$ of the base field K is not homogeneous under the action of the unitary group of the form. This variety over $\bar{K}$ is defined by:

$$Y = \{(X_1, X_2, Y_1, Y_2);\ X_1Y_1^t + X_2Y_2^t = 0,\ X_1, X_2, Y_1, Y_2 \in M_p(\bar{K})\},$$

where p is the degree of the division algebra. We begin with the following lemma:

Lemma 1. *Let $\ell|k$ be a finite extension of p-adic fields with $[\ell : k] > 1$. Then $|\ell^*/\ell^{*p}| > |k^*/k^{*p}|$.*

Proof. For any field M with $char(M) \neq p$, from the Kummer exact sequence, one has an isomorphism $H^1(M,\mu_p) \simeq M^*/M^{*p}$. Let $h^i_M(\mu_p) = |H^i(M,\mu_p)|$ and $\chi_M(\mu_p) = h^0_M(\mu_p)h^2_M(\mu_p)/h^1_M(\mu_p)$. If M is a p-adic field, $h^0_M(\mu_p)$ is p or 1 according as M contains a primitive p^{th} root of unity or not; further, $h^2_M(\mu_p) = p$. In view of ([Se], §5.7, Theorem 5), $\chi_M(\mu_p) = 1/p^N$, where $[M : \mathbb{Q}_p] = N$. Thus $h^1_M(\mu_p) = p^{N+1+\epsilon_M}$, where $\epsilon_M = 1$ or 0 according as M contains a p^{th} root of unity or not. It follows that $h^1_\ell(\mu_p) > h^1_k(\mu_p)$ and $|\ell^*/\ell^{*p}| > |k^*/k^{*p}|$, provided $[\ell : k] > 1$. □

Let k be a p-adic field containing a primitive p^{th} root of unity ξ with $p \geq 5$. Let $K = k((t))$.

Let $\ell|k$ be a degree two extension of k. Then by lemma (1), $|\ell^*/\ell^{*p}| > |k^*/k^{*p}|$ so that the norm map $N_{\ell/k}$ has a non-trivial kernel. Let $\mu \in \ell^*$ be such that $[\mu] \in \ker(N_{\ell/k} : \ell^*/\ell^{*p} \to k^*/k^{*p})$ and $[\mu] \neq 1$ in ℓ^*/ℓ^{*p}. Let $L = \ell((t))$. Let D be the cyclic algebra of degree p over L defined by:

$$X^p = \mu, Y^p = t, XY = \xi YX.$$

The algebra D is clearly a division algebra and is represented by $(\mu)\cup(t) \in H^2(L,\mu_p)$. We have, using the projection formula (*cf.* [CF], Prop. 9(iv), page 107), $\mathrm{cores}_{L|K}((\mu)\cup(t)) = (N_{L/K}(\mu))\cup(t) = 1$ so that by a theorem of Albert (*cf.* [Sc], Theorem 9.5, page 309) the division algebra D supports an involution of second kind. Let τ be an $L|K$ involution on D. Let $\lambda \in k^*$ be such that $\lambda \notin N_{\ell/k}(\ell^*)$. Local class field theory guarantees the existence of such a λ. Let h be the rank 3 hermitian form $\langle 1, -\lambda, t\rangle$ over (D,τ). Then we have the following:

Lemma 2. *The hermitian form h is anisotropic over (D,τ).*

Proof. Let $\Delta = \{a \in D : \mathrm{Nrd}_{D|L}(a) \in l[[t]]\}$ be the unique maximal $\ell[[t]]$-order in D (*cf.* [R], Theorem 12.8, page 137). Every element a of Δ can be written as $a = \pi^n b$, where b is a unit in Δ and π a generator of the unique maximal right ideal in Δ (which is indeed a two sided ideal) and n is a non-negative integer.
Suppose there exist $v_1, v_2, v_3 \in D$, not all zero, such that:

$$v_1\tau(v_1) - \lambda v_2\tau(v_2) + tv_3\tau(v_3) = 0. \tag{1}$$

Without loss of generality, we may assume that each $v_i \in \Delta$. We write $v_i = \pi^{n_i}u_i$, where u_i is a unit in Δ for $i = 1,2,3$ and $n_i \geq 0$. Thus (1) becomes:

$$\pi^{n_1}u_1\tau(\pi^{n_1}u_1) - \lambda\pi^{n_2}u_2\tau(\pi^{n_2}u_2) + t\pi^{n_3}u_3\tau(\pi^{n_3}u_3) = 0. \tag{2}$$

We first consider the case when n_1 is the smallest of all n_i. Let $m_2 = n_2 - n_1$ and $m_3 = n_3 - n_1$; both m_2 and m_3 are non-negative. We can rewrite (2) as

$$u_1\tau(u_1) = \lambda\pi^{m_2}u_2\tau(\pi^{m_2}u_2) - t\pi^{m_3}u_3\tau(\pi^{m_3}u_3). \tag{3}$$

Since $t = \pi^p u_0$ with u_0 a unit in Δ and p odd, the valuation of

$$\lambda\pi^{m_2}u_2\tau(\pi^{m_2}u_2) - t\pi^{m_3}u_3\tau(\pi^{m_3}u_3)$$

is the minimum of $\{2m_2, p+2m_3\}$, and it is zero by (3). This implies that $m_2 = 0$. Thus

$$\lambda u_2 \tau(u_2) = u_1 \tau(u_1) \left(1 + t\tau(u_1)^{-1} u_1^{-1} \pi^{m_3} u_3 \tau(\pi^{m_3} u_3)\right). \tag{4}$$

Set $w = \tau(u_1)^{-1} u_1^{-1} \pi^{m_3} u_3 \tau(\pi^{m_3} u_3)$; the element w is in the maximal order Δ. Taking reduced norm on both sides of (4) we get:

$$\mathrm{Nrd}_D\left(\lambda u_2 \tau(u_2)\right) = \mathrm{Nrd}_D\left(u_1 \tau(u_1)\right) \mathrm{Nrd}_D(1+tw)$$

which gives:

$$\lambda^p N_{L|K}\left(\mathrm{Nrd_D}(u_2)\right) = N_{L|K}\left(\mathrm{Nrd_D}(u_1)\right) \mathrm{Nrd}_D(1+tw).$$

Reading the above equality modulo t, we conclude that $\lambda^p \in N_{\ell|k}(\ell^*)$. Since ℓ is a quadratic extension over k and p is odd, this implies that $\lambda \in N_{\ell|k}(\ell^*)$. But this is a contradiction to the choice of λ and therefore the form h is anisotropic in this case. The case when n_2 is smallest can be treated in a similar manner.

Now we consider the case when n_3 is the smallest among all n_i's. Let $r_1 = n_1 - n_3$ and $r_2 = n_2 - n_3$. The integers r_1 and r_2 are nonnegative and (2) becomes:

$$\pi^{r_1} u_1 \tau(\pi^{r_1} u_1) - \lambda \pi^{r_2} u_2 \tau(\pi^{r_2} u_2) = -t u_3 \tau(u_3). \tag{5}$$

Suppose $r_1 \neq r_2$. The valuation of $\pi^{r_1} u_1 \tau(\pi^{r_1} u_1) - \lambda \pi^{r_2} u_2 \tau(\pi^{r_2} u_2)$ is the minimum of $\{2r_1, 2r_2\}$, which is even, while the valuation of $t u_3 \tau(u_3)$ is p which is odd leading to a contradiction. Therefore $r_1 = r_2$ and we have

$$u_1 \tau(u_1) = \lambda u_2 \tau(u_2) - t\pi^{-r_1} u_3 \tau(\pi^{-r_1} u_3). \tag{6}$$

If $p < 2r_1$, $u_3 \tau(u_3) = t^{-1} \pi^{r_1} \left(-u_1 \tau(u_1) + \lambda u_2 \tau(u_2)\right) \tau(\pi^{r_1})$ with the valuation of right hand side positive, leading to a contradiction. Thus $p > 2r_1$. Taking reduced norm on both sides of (6) and then reading modulo t we conclude, as before that $\lambda \in N_{\ell|k}(\ell^*)$, which is a contradiction. Thus the hermitian form h is anisotropic. □

THEOREM 3. *Let X be the variety of rank one (rank over D) zero subspaces of h over (D, τ). Then, X is a projective variety which is a homogeneous space under the action of $SU(h)$. The variety X admits a zero-cycle of degree one, but has no K-rational point.*

Proof. Let $B = M_3(D)$ and τ_h the involution on B adjoint to h. Under the Morita equivalence, a right ideal J_1 in B with $dim_D(J_1) = 3$ and $\tau_h(J_1).J_1 = 0$ corresponds to a rank one (over D) zero subspace V_1 for h. The ideal J_2 of B with $dim_D(J_2) = 6$ and $\tau_h(J_1).J_2 = 0$ is determined by J_1 and corresponds under Morita equivalence to the orthogonal complement $V_1^{\perp}$ of V_1 in h. Thus the variety of flags of right ideals $J_1 \subset J_2$ in B such that $dim_D(J_1) = 3$, $dim_D(J_2) = 6$ and $\tau_h(J_1).J_2 = 0$ is isomorphic to the variety X of rank one (over D) zero subspaces of h. In particular, the variety X is a projective homogeneous variety under the action of $SU(h)$ (*cf.* [MT], §2.4.2).

The algebra $D_{K(\sqrt{\lambda})}$ is a division algebra and $h_{K(\sqrt{\lambda})} \simeq \langle 1, -1, t\rangle$ is isotropic over $\left(D_{K(\sqrt{\lambda})}, \tau_{K(\sqrt{\lambda})}\right)$ with a rank one isotropic subspace (over $\left(D_{K(\sqrt{\lambda})}\right)$). Hence $X\left(K\left(\sqrt{\lambda}\right)\right) \neq \phi$. Let $M = K(t^{1/p})$. Then $[M : K] = p, D_M$ is split and h_M is Morita equivalent to a $3p$-dimensional hermitian form $\tilde{h}_M$ over $LM|M$ (*cf.* [BP], section 1.3) Under this equivalence, every rank r zero space of h_M corresponds to a rank rp zero space of the hermitian form $\tilde{h}_M$. To the hermitian form $\tilde{h}_M$ is associated a quadratic form q_M (*cf.* [MH], page 114) and every rank r zero of $\tilde{h}_M$ corresponds to a rank $2r$ zero of q_M. Since M is a Laurent series field over a p-adic field, every 9-dimensional quadratic form over M is isotropic (*cf.* [P], Theorem 2.2, Chapter 5) and every 5-dimensional hermitian form over $LM|M$ is isotropic, (*cf.* [MH], page 114). Since $p \geq 5, \tilde{h}_M$ has a totally isotropic subspace of rank at least p over LM; hence h_M has a rank one isotropic subspace over D_M and $X(M) \neq \phi$. Thus X admits a zero-cycle of degree one. By Lemma(2), the form h has no nontrivial zero over K and X has no K-rational point. □

We shall now describe the parabolic subgroup defining the stabiliser of a rational point in $X(\overline{K})$.

Let $\triangle$ be the set of simple roots with respect to a pair (T, B) for $G_{\overline{k}} = SU(h)_{\overline{k}}$ for a choice of a maximal torus T defined over k and a Borel subgroup B over $\overline{k}$ containing T. Let $S = \triangle \backslash \{p, 2p\}$ where the ordering on vertices of $\triangle$ are as in ([T], Table I, page 53). From the description of a parabolic subgroup of $SU(h)$ belonging to the conjugacy class associated to S as the stabiliser of the flag of right ideals $J_1 \subset J_2$ in B such that $dim_D(J_1) = 3$, $dim_D(J_2) = 6$ and $\tau_h(J_1).J_2 = 0$ (*cf.* [MT], §2.4.2), it follows that X is precisely the variety of parabolic subgroups of $SU(h)$ defined by the conjugacy class associated to S.

We have the following Tits indices for $SU(h)_{K(\sqrt{\lambda})}$ and $SU(h)_M$ (Witt index of h over M is $p + r$).

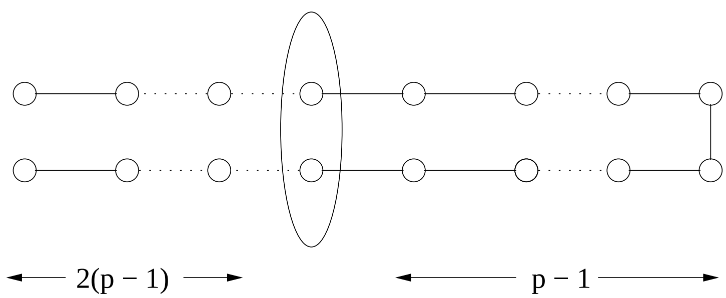

FIG. 2.1. *Tits index for* $SU(h)_{K(\sqrt{\lambda})}$

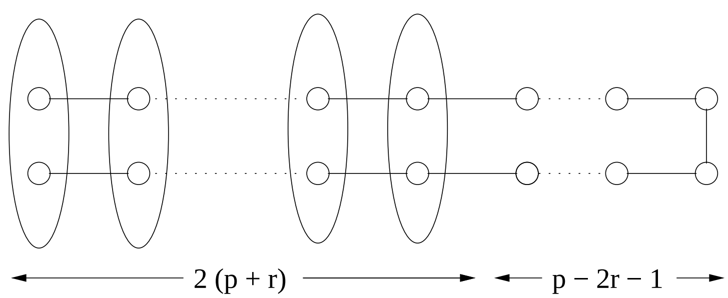

FIG. 2.2. *Tits index for* $SU(h)_M$

REMARK. One may replace the Laurent series fields over p-adic fields in the above examples by the rational function field in one variable over p-adic fields with p sufficiently large. One needs to use results of [HV] and [PS] stating that quadratic forms over such fields in sufficiently many variables have a nontrivial zero.

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