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VOLUME MINIMIZATION AND ESTIMATES FOR CERTAIN ISOTROPIC SUBMANIFOLDS IN COMPLEX PROJECTIVE SPACES*

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Abstract. In this note we show the following result using the integral-geometric formula of R. Howard: Consider the totally geodesic $\mathbb{R}P^{2m}$ in $\mathbb{C}P^n$. Then it minimizes volume among the isotropic submanifolds in the same $\mathbb{Z}/2$ homology class in $\mathbb{C}P^n$ (but not among all submanifolds in this $\mathbb{Z}/2$ homology class). Also the totally geodesic $\mathbb{R}P^{2m-1}$ minimizes volume in its Hamiltonian deformation class in $\mathbb{C}P^n$. As a corollary we'll give estimates for volumes of Lagrangian submanifolds in complete intersections in $\mathbb{C}P^n$.

Key words. Isotropic submanifolds, Lagrangian submanifolds, volume minimization

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1. Introduction.

On a Kähler n-fold M there is a class of *isotropic* submanifolds. Those are submanifolds of M on which the Kähler form ω of M vanishes. The maximal dimension of such a submanifold is n (the middle dimension) in which case it is called *Lagrangian*.

In this papers we'll exhibit global volume-minimizing properties among isotropic competitors for certain submanifolds of the complex projective space. In general global volume-minimizing properties of minimal/Hamiltonian stationary Lagrangian/isotropic submanifolds in Kähler (particularly Kähler-Einstein) manifolds are still poorly understood. In dimesion 2 there is a result of Schoen-Wolfson [ScW] (extended to isotropic case by Qiu in [Qiu]) which shows existence of Lagrangian cycles minimizing area among Lagrangians in a given homology class. Still it is not clear whether a *given* minimal Lagrangian has any global volume-minimizing properties.

The only instance where we have a clear cut answer to global volume-minimizing problem is Special Lagrangian submanifolds which are homologically volume-mimizing in Calabi-Yau manifolds [HaL]. In Kähler-Einstein manifolds of negative scalar curvature, besides geodesics on Riemann surfaces of negative curvature, we have some examples [Lee] of minimal Lagrangian submanifolds which are homotopically volume-minimizing. The author has a program for studying homotopy volume-minimizing properties for Lagrangians in Kähler-Einstein manifolds of negative scalar curvature [Gold1], but so far there are no satisfactory results.

In positive curvature case there is a result of Givental-Kleiner-Oh which states that the canonical totally geodesic $\mathbb{R}P^n$ in $\mathbb{C}P^n$ minimizes volume in its Hamiltonian deformation class, [Giv]. The proof uses integral geometry and Floer homology to study intersections for Hamiltonian deformations of $\mathbb{R}P^n$. Those arguments can be generalized to products of Lagrangians in a product of symmetric Kähler manifolds, [IOS]. There is a related conjecture due to Oh that the Clifford torus minimizes volume in its Hamiltonian deformation class in $\mathbb{C}P^n$, [Oh]. Some progress towards this was obtained in [Gold2]. Also general lower bounds for volumes of Lagrangians in a given Hamiltonian deformation class in $\mathbb{C}^n$ were obtained in [Vit].

In this note we extend and improve the result of Givental-Kleiner-Oh to isotropic totally geodesic $\mathbb{R}P^k$ sitting canonically in $\mathbb{C}P^n$. Our main result is the following

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theorem:

THEOREM 1. *Consider the totally geodesic $\mathbb{R}P^{2m}$ in $\mathbb{C}P^n$. Then it minimizes volume among the isotropic submanifolds in the same $\mathbb{Z}/2$ homology class in $\mathbb{C}P^n$ (but not among all submanifolds in this $\mathbb{Z}/2$ homology class). Also consider the totally geodesic $\mathbb{R}P^{2m-1}$ in $\mathbb{C}P^n$. Then it minimizes volume in its Hamiltonian deformation class.*

A corollary of this is:

COROLLARY 1. *Let $f_1, \ldots, f_k$ be real homogeneous polynomials of odd degree in $n+1$ variables with $2m+k=n$. Let N be the zero locus of f_i in $\mathbb{C}P^n$ and L be their real locus. Then $vol(L) \leq \Pi deg(f_i) vol(\mathbb{R}P^{2m})$ and if L' is a Lagrangian submanifold of N homologous mod 2 to L in N then $vol(L') \geq vol(\mathbb{R}P^{2m})$.*

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2. A formula from integral geometry. In this section we establish a formula from integral geometry for volumes of isotropic submanifolds of $\mathbb{C}P^n$ following the exposition in R. Howard [How].

In our case the group $SU(n+1)$ acts on $\mathbb{C}P^n$ with a stabilizer $K \simeq U(n)$. Thus we view $\mathbb{C}P^n = SU(n+1)/K$ and the Fubini-Study metric is induced from the bi-invariant metric on $SU(n+1)$. Let P^{2m} be an isotropic submanifold of $\mathbb{C}P^n$ of dimension $2m$ and let Q be a linear $\mathbb{C}P^{n-m} \subset \mathbb{C}P^n$. For a point $p \in P$ and $q \in Q$ we define an angle $\sigma(p,q)$ between the tangent planes T_pP and T_qQ as follows: First we choose some elements g and h in $SU(n+1)$ which move p and q respectively to the same point $r \in \mathbb{C}P^n$. Now the tangent planes g_*T_pP and h_*T_qQ are in the same tangent space $T_r\mathbb{C}P^n$ and we can define an angle between them as follows: take an orthonormal basis $u_1 \ldots u_{2m}$ for g_*T_pP and an orthonormal basis $v_1 \ldots v_{2n-2m}$ for h_*T_qQ and define

$$\sigma(g_*T_pP, h_*T_qQ) = |u_1 \wedge \ldots \wedge v_{2n-2m}|.$$

The later quantity $\sigma(g_*T_pP, h_*T_qQ)$ depends on the choices g and h we made. To mend this we'll need to average this out by the stabilizer group K of the point r. Thus we define:

$$\sigma(p,q) = \int_K \sigma(g_*T_pP, k_*h_*T_qQ)dk.$$

Since $SU(n+1)$ acts transitively on the Grassmanian of isotropic planes and the complex planes in $\mathbb{C}P^n$ we conclude that this angle is a constant depending just on m and n:

$$\sigma(p,q) = C_{m,n}.$$

There is a following general formula due to R. Howard [How]:

$$\int_{SU(n+1)} \#(P \bigcap gQ)dg = \int_{P\times Q} \sigma(p,q)dpdq = C_{m,n} vol(P) vol(Q).$$

Here $\#(P \bigcap gQ)$ is the number of intersection points of P with gQ, which is finite for a generic $g \in SU(n+1)$. To use the formula we need to have some control over the intersection pattern of P and gQ. We have the following lemma:

LEMMA 1. *Let P be the totally geodesic $\mathbb{R}P^{2m} \subset \mathbb{C}P^n$, let $Q = \mathbb{C}P^{n-m} \subset \mathbb{C}P^n$. Let $g \in SU(n+1)$ s.t. P and gQ intersect transversally. Then $\#(P \bigcap gQ) = 1$. Also let $f_1, \ldots, f_k$ be real homogeneous polynomials in $n+1$ variables with $2m+k=n$ and let P' be their real locus. If P' is transversal to gQ then $\#(P' \bigcap gQ) \leq \Pi deg(f_i)$.*

Proof. For the first claim we have gQ is given by an $(n-m+1)$-plane $H \subset \mathbb{C}^{n+1}$ and hence it is a zero locus of m linear equations on $\mathbb{C}^{n+1}$. Hence $(P \bigcap gQ)$ is cut out by $2m$ linear equations in $\mathbb{R}P^{2m}$.
For the second claim we note that as before $gQ \bigcap \mathbb{R}P^n$ is the zero locus of $2m$ linear polymonials $h_1, \ldots, h_{2m}$ on $\mathbb{R}P^n$. Moreover P' is a zero locus of $f_1, \ldots, f_{n-2m}$ on $\mathbb{R}P^n$. For generic $g \in SU(n+1)$ we'll have that gQ and P' intersect transversally in $\mathbb{R}P^n$. By Bezout's theorem (see [GH], p. 670) the common zero locus of $h_1, \ldots, h_{2m}$ and $f_1, \ldots, f_{n-2m}$ is $\mathbb{C}P^n$ is $\Pi deg(f_i)$ points. Now $P' \bigcap gQ$ is a part of this locus, hence $\#(P' \bigcap gQ) \leq \Pi deg(f_i)$.

3. Proof of the volume minimization. Now we can prove the result stated in the Introduction:

THEOREM 1. *Consider the totally geodesic $\mathbb{R}P^{2m}$ in $\mathbb{C}P^n$. Then it minimizes volume among the isotropic submanifolds in the same $\mathbb{Z}/2$ homology class in $\mathbb{C}P^n$ (but not among all submanifolds in this $\mathbb{Z}/2$ homology class). Also consider the totally geodesic $\mathbb{R}P^{2m-1}$ in $\mathbb{C}P^n$. Then it minimizes volume in its Hamiltonian deformation class.*

Proof. Let P be an isotropic submanifold homologous to $\mathbb{R}P^{2m}$ mod 2 and let $Q = \mathbb{C}P^{n-m}$. By Lemma 1 the intersection number mod 2 of P and gQ is 1. Hence the formula in the previous section tells that

$$C_{m,n} vol(P) vol(Q) = \int_{SU(n+1)} \#(P \bigcap gQ) dg \geq vol(SU(n+1))$$

and

$$C_{m,n} vol(\mathbb{R}P^{2m}) vol(Q) = \int_{SU(n+1)} \#(\mathbb{R}P^{2m} \bigcap gQ) dg = vol(SU(n+1))$$

and this proves the first part. We also note that that $\mathbb{C}P^1$ is homologous to $\mathbb{R}P^2$ mod 2 in $\mathbb{C}P^n$ but

$$vol(\mathbb{C}P^1) < vol(\mathbb{R}P^2).$$

The second assertion will follow from the first one. Consider $\mathbb{C}^{n+1}$ and a unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$. We have a natural circle action on S^{2n+1} (multiplication by unit complex numbers). Let the vector field u be the generator of this action. We have a 1-form α on S^{2n+1},

$$\alpha(v) = u \cdot v.$$

Also $d\alpha = 2\omega$ where ω is the Kähler form of $\mathbb{C}^{n+1}$. The kernel of α is the *horizontal distribution.* We have a Hopf map $\rho : S^{2n+1} \mapsto \mathbb{C}P^n$. We have $\mathbb{R}P^{2m-1} \subset \mathbb{C}P^n$ and $S^{2m-1} \subset S^{2n+1}$ which is a horizontal double cover of $\mathbb{R}P^{2m-1}$.
Let f be a (time-dependent) Hamiltonian function on $\mathbb{C}P^n$. Then we can lift it to a Hamiltonian function on $\mathbb{C}^{n+1} - (0)$ and its Hamiltonian vector field H_f is horizontal on S^{2n+1}. Consider now the vector field

$$w = -2f \cdot u + H_f.$$

The vector field w is S^1-invariant. We also have:

PROPOSITION 1. *The Lie derivative* $L_w\alpha = 0$.

Proof. We have

$$L_w\alpha = d(i_w\alpha) + i_w d\alpha = -2df + 2df.$$

Let now Φ_t be the time t flow of w on S^{2n+1} and let Ξ_t be the Hamiltonian flow of f on $\mathbb{C}P^n$. Then $\Phi_t(S^{2m-1})$ is horizontal and isotropic and it is a double cover of $\Xi_t(\mathbb{R}P^{2m-1})$. Hence

$$vol(\Phi_t(S^{2m-1})) = 2vol(\Xi_t(\mathbb{R}P^{2m-1})).$$

Let $S_t = \Phi_t(S^{2m-1})$. We build a suspension ΣS_t of S_t in $S^{2n+3} \subset \mathbb{C}^{n+2}$,

$$\Sigma S_t = \big((\sin\theta \cdot x, \cos\theta) \in \mathbb{C}^{n+2} = \mathbb{C}^{n+1} \oplus \mathbb{C} | 0 \le \theta \le \pi \ , \ x \in S_t\big).$$

One immediately verifies that ΣS_t is horizontal and it is a double cover of an isotropic submanifold L_t (with a conical singularity) of $\mathbb{C}P^{n+1}$ with $L_0 = \mathbb{R}P^{2m}$. Also one readily checks that

$$vol(\Sigma S_t) = vol(S_t) \cdot \int_{\theta=0}^{\pi} \sin^{2m-1}\theta \ d\theta.$$

Hence

$$2vol(L_t) = vol(\Sigma S_t) = 2vol(\Xi_t(\mathbb{R}P^{2m-1})) \cdot \int_{\theta=0}^{\pi} \sin^{2m-1}\theta \ d\theta.$$

Now the first part of our theorem implies that $vol(L_t) \ge vol(L_0)$. Hence we conclude that $vol(\Xi_t(\mathbb{R}P^{2m-1})) \ge vol(\mathbb{R}P^{2m-1})$. □

REMARK. One notes from the proof that for $\mathbb{R}P^{2m-1}$ it would be suffient to use exact deformations by isotropic immersions of $\mathbb{R}P^{2m-1}$. A family L_t of isotropic immersions of $\mathbb{R}P^{2m-1}$ is called *exact* if the 1-form $i_v\omega$ is exact when restricted to each element of the family. Here v is the deformation vector field and ω is the symplectic form. Thus embeddedness is not important for the conclusion of the theorem.
The theorem has the following corollary:

COROLLARY 1. *Let $f_1, \dots, f_k$ be real homogeneous polynomials of odd degree in $n+1$ variables with $2m + k = n$. Let N be the zero locus of f_i in $\mathbb{C}P^n$ and L be their real locus. Then $vol(L) \le \Pi deg(f_i) vol(\mathbb{R}P^{2m})$ and if L' is a Lagrangian submanifold of N homologous mod 2 to L in N then $vol(L') \ge vol(\mathbb{R}P^{2m})$.*

Proof. We note that N is a complex $2m$-fold and L is its Lagrangian submanifold. Since the degrees of f_i are odd, we have by adjunction formula that L and $\mathbb{R}P^{2m}$ represent the same homology class in $H_{2m}(\mathbb{R}P^n, \mathbb{Z}/2)$. Let Q be a linear $\mathbb{C}P^{n-m}$ in $\mathbb{C}P^n$ and $g \in SU(n+1)$. The intersection munber mod 2 of gQ with L' is 1. We have that

$$C_{m,n} vol(\mathbb{R}P^{2m}) vol(Q) = \int_{SU(n+1)} 1 dg$$

$$C_{m,n} vol(L') vol(Q) = \int_{SU(n+1)} \#(L' \bigcap gQ) dg.$$

Also using Lemma 1:

$$C_{m,n} vol(L) vol(Q) = \int_{SU(n+1)} \#(L \bigcap gQ) dg \leq \Pi deg(f_i) vol(SU(n+1))$$

and our claims follow. □

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