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FINITE STATIONARY PHASE EXPANSIONS*

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Abstract. Functions which are moment maps of Hamiltonian actions on symplectic manifolds have the property that their stationary phase expansions have only finitely many nonzero terms and are therefore precise rather than asymptotic. In this paper, we exhibit another type of function which has this property and explain why in terms of equivariant cohomology and the geometry of the spaces involved.

Key words. equivariant cohomology, stationary phase, momentum maps, symplectic geometry

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Let S^{2n} denote the unit sphere in $\mathbb{R}^{2n+1}$, given by

$$S^{2n} = \{(x_1, x_2, \ldots, x_{2n}, z) \mid {x_1}^2 + {x_2}^2 + \cdots + {x_{2n}}^2 + z^2 = 1\},$$

and let dV denote the standard volume form on S^{2n} inherited from $\mathbb{R}^{2n+1}$. For any nonzero $t \in \mathbb{R}$ and any C^∞-function $f : S^{2n} \to \mathbb{R}$, the method of stationary phase (see [3] or [4], for example) tells us that

$$\int_{S^{2n}} e^{tf}\, dV = \sum_{p \in C_f} e^{tf(p)} Q_p(t^{-1}),$$

where Q_p is a power series with coefficients depending on p and where

$$C_f = \{p \in S^{2n} \mid f'(p) = 0\}$$

is the set of critical points of f.

In the particular case that $f : S^{2n} \to \mathbb{R}$ is the "height function"

$$f(x_1, \ldots, x_{2n}, z) = z$$

on S^{2n} (with critical points at $z = \pm 1$), one can compute directly that

$$\int_{S^{2n}} e^{tz}\, dV = c \int_{-1}^{1} e^{tz}(1 - z^2)^{n-1}\, dz = \sum_{z=\pm 1} e^{tz} Q_z(t^{-1}),$$

where c is some real constant and Q_z is a *polynomial* with coefficients depending on z. In other words, the stationary phase expansion of the height function on S^{2n} has only finitely many terms and is therefore precise, rather than asymptotic.

This gives rise to the main question of this paper: *Where did this function come from and why does it have this property?* In the coming sections, we give an explanation for why f has this property in terms of equivariant cohomology and the geometry of S^{2n}.

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1. The Duistermaat-Heckman Theorem. The Duistermaat-Heckman Theorem provides examples of functions with this finite stationary phase property, namely moment maps of Hamiltonian actions. In such examples, the manifolds involved are symplectic and the stationary phase expansions of the moment maps are all single-term expansions, as opposed to the multiple-term expansion associated with the height function on S^{2n} (which is, of course, not symplectic for $n > 1$). In this paper, we show a way to generalize the Atiyah-Bott method of proving the Duistermaat-Heckman Theorem to accomodate the example of the height function on S^{2n}. For simplicity, we describe such a generalization only for the case of an S^1- action, but the methods involved carry over readily to the case of a torus action.

The Atiyah-Bott method of proving the Duistermaat-Heckman Theorem for a $2n$-dimensional symplectic manifold M with a Hamiltonian action by S^1 which has only isolated fixed points is as follows (see [1]). If the symplectic form on M is denoted by ω and $u \in \mathfrak{s}^{1^*}$ is dual to a nonzero $X \in \mathfrak{s}^1$, and if

$$\widetilde{\omega} = \omega + fu$$

is equivariantly closed, then the function $f : M \to \mathbb{R}$ is called a moment map associated with ω. To demonstrate that the moment map f has the finite stationary phase property, we exponentiate $\widetilde{\omega}$:

$$\int_M e^{\widetilde{\omega}} = \int_M e^{fu} e^{\omega},$$

which, when expanded out, equals:

$$\int_M e^{fu} \left(1 + \omega + \frac{\omega^2}{2!} + \cdots + \frac{\omega^n}{n!} \right). \tag{1}$$

Removing terms which do not contain differential forms of degree $2n$ and integrating directly, we are left with the integral of e^{fu} times the symplectic volume form:

$$(1) = \int_M e^{fu} \frac{\omega^n}{n!}.$$

On the other hand, applying the Equivariant Localization Theorem (see [1] or [2], for example) to (1), we find that

$$(1) = \sum_{p \in C_f} \frac{e^{f(p)u}}{c_p u^n},$$

where each $c_p \in \mathbb{R}$ is some constant depending on p and where C_f is again the set of critical points of f.

Combining the above two equations and evaluating both sides at tX (for real $t \neq 0$) shows that the stationary phase expansion for the moment map f has only a single term for each critical point:

$$\int_M e^{tf} \frac{\omega^n}{n!} = \sum_{p \in C_f} \frac{e^{tf(p)}}{c_p t^n}.$$

2. Generalizations. The Atiyah-Bott proof of the Duistermaat-Heckman Theorem leads us to investigate what happens if we exponentiate equivariantly closed forms other than those arising from symplectic forms and moment maps. In particular, let M be a compact smooth (not necessarily symplectic) $2n$-dimensional manifold acted on by S^1 with only isolated fixed points, and let M have a non-degenerate closed S^1-invariant $2k$-form $\alpha \in \Omega^{2k}(M)$. If k and l are positive integers with $2kl = 2n$, then $\alpha^l/l!$ is a volume form on M.

Let us also assume that α can be "extended" to an equivariantly closed form

$$\widetilde{\alpha} = \alpha + \alpha_{k-1}u + \cdots + \alpha_1 u^{k-1} + fu^k,$$

where each $\alpha_j \in \Omega^{2j}(M)$ and $f : M \to \mathbb{R}$. (Here as before, $X \in \mathfrak{s}^1$ is nonzero, and $u \in \mathfrak{s}^{1*}$ is its dual.) The form α will always have such an extension if, for example, M has no nontrivial cohomology in dimensions less than $2k$.

Any such $\widetilde{\alpha}$ can be exponentiated and integrated as in the Atiyah-Bott proof of the Duistermaat-Heckman Theorem. This time, however, there are additional complications. We have

$$\int_M e^{\widetilde{\alpha}} = \int_M e^{fu^k} e^{\alpha + \alpha_{k-1}u + \cdots + \alpha_1 u^{k-1}},$$

which, when expanded out, equals:

$$\begin{aligned} &\int_M e^{fu^k}\Big(1 + (\alpha + \alpha_{k-1}u + \cdots + \alpha_1 u^{k-1}) \\ &\qquad\qquad + \frac{(\alpha + \alpha_{k-1}u + \cdots + \alpha_1 u^{k-1})^2}{2!} + \cdots\Big). \end{aligned} \tag{2}$$

As was the case with (1), we can evaluate (2) by applying the Equivariant Localization Theorem to obtain:

$$(2) = \sum_{p \in C_f} \frac{e^{f(p)u^k}}{c_p u^n},$$

where each $c_p \in \mathbb{R}$ is again some constant depending on p.

Removing the terms in (2) which do not contain differential forms of degree $2n$ as we did before does not leave only the volume form this time though. Rather, we have that

$$(2) = \int_M \sum_{i_1 + \cdots + i_j = n} \frac{\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_j} u^{jk-n}}{j!},$$

where $i_1, \ldots, i_j$ are positive integers and with the convention that $\alpha_k = \alpha$. It is not apparent in general what contribution these "cross-terms" $\frac{1}{j!}\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_j}u^{jk-n}$ make to the integral.

3. The constant curvature case. In the particular case where M has constant curvature and where α is the nonzero Euler class of M, we now show how to compute the contribution of the above cross-terms to the integral. These extra restrictions rule out nearly everything but S^{2n}, however, so it is hoped that the methods involved in computing the contribution of the cross-terms will be found applicable in a wider setting, as we discuss in Section 6.

To compute the contribution of the cross-terms, we first introduce some notation. Let M be any compact, smooth $2n$-dimensional manifold with nonzero Euler class and with an action by S^1 which has only isolated fixed points. As before, let $X \in \mathfrak{s}^1$ be nonzero and let $u \in {\mathfrak{s}^1}^*$ be dual to X. Also assume that $\nabla : \Gamma(TM) \to \Gamma(T^*M \otimes TM)$ is an S^1-invariant connection on M with (necessarily S^1-invariant) curvature $R \in \Omega^2(M; \mathrm{End}(TM))$.

The equivariant Euler class of M then (see [2], for example) is given by

$$\widetilde{\chi} = \mathrm{Pfaff}\left(\frac{R + uL}{2\pi}\right) := \chi_n + \chi_{n-1}u + \cdots + \chi_1 u^{n-1} + \chi_0 u^n, \tag{3}$$

where the map L can be described as follows. Any element $Y \in \mathfrak{s}^1$ induces a vector field on M given by

$$\widetilde{Y}(p) = \frac{d}{dt}\big(\exp(-tY) \cdot p\big)\bigg|_{t=0}$$

for any $p \in M$. If $\mathcal{L}$ denotes the Lie derivative, then the map L is defined by

$$L(Y) = \mathcal{L}_{\widetilde{Y}} - \nabla_{\widetilde{Y}}$$

for any $Y \in \mathfrak{s}^1$.

Note that by the definition given in (3), we have that

$$\chi = \chi_n = \mathrm{Pfaff}\left(\frac{R}{2\pi}\right) \tag{4}$$

is the ordinary Euler class of M, and that

$$f = \chi_0 = \mathrm{Pfaff}\left(\frac{L}{2\pi}\right). \tag{5}$$

By the same definition, we also have that $\chi_j \in \Omega^{2j}(M)$ for each $j = 0, 1, \ldots, n$.

To compute the contribution of the cross-terms in (2) in the case where $\widetilde{\alpha} = \widetilde{\chi}$, we use the following lemma.

Lemma 3.1. *If M has constant curvature, then the coefficients of u in the expansion* (3) *for $\widetilde{\chi}$ satisfy the following relationships:*

$$\chi_k \chi_l = \frac{(k+l)!\ (k+l)!\ (2k)!\ (2l)!}{k!\ k!\ l!\ l!\ (2k+2l)!} f\, \chi_{k+l},$$

for any $k, l \in \{1, \ldots, n\}$ such that $k + l \leq n$.

In the interest of clarity and brevity of exposition, we postpone the proof of Lemma 3.1 to Section 4. We proceed now to the main theorem of this paper.

Theorem 3.2. *If M has constant curvature with Euler class χ given by Equation* (4)*, and if f is the function defined by Equation* (5)*, then the stationary phase expansion of*

$$\int_M e^{tf} \chi$$

relative to the parameter t has only finitely many non-zero terms.

Proof. Proceeding to exponentiate the equivariant Euler class $\widetilde{\chi}$ as we did the forms $\widetilde{\omega}$ and $\widetilde{\alpha}$ above, we find that if M_0 denotes the zero set of the vector field $\widetilde{X}$, then the Equivariant Localization Theorem gives us that

$$\begin{aligned}\int_M e^{\widetilde{\chi}} &= \sum_{p\in M_0} \frac{(2\pi)^n\, e^{\widetilde{\chi}}|_p}{f(p)\,u^n}\\ &= \sum_{p\in M_0} \frac{(2\pi)^n}{f(p)\,u^n} \left(e^{f\,u^n+\chi_1 u^{n-1}+\cdots+\chi_n}\right)\Big|_p\\ &= \sum_{p\in M_0} \frac{(2\pi)^n}{f(p)\,u^n} e^{f(p)\,u^n},\end{aligned}$$

using again that, since the fixed points of the action are isolated, the equivariant Euler class of the normal bundle of a fixed point $p \in M_0$ is simply $\dfrac{f(p)u^n}{(2\pi)^n}$.

The zero set M_0 of $\widetilde{X}$ is equal to the set C_f of critical points of f, however. Since $df = \iota_{\widetilde{X}}\chi_2$, then $C_f \supset M_0$, and since df is S^1-invariant, it must vanish only on entire orbits, so C_f consists only of entire orbits. By assumption, the function f has only isolated fixed points though, so we must have $C_f \subset M_0$ as well.

Now expanding the expression for $e^{\widetilde{\chi}}$, we obtain

$$\begin{aligned}\int_M e^{\widetilde{\chi}} &= \int_M e^{f\,u^n+\chi_1 u^{n-1}+\cdots+\chi_n}\\ &= \int_M e^{f\,u^n}\Big(1+(\chi_1 u^{n-1}+\cdots+\chi_n)+\frac{1}{2!}(\chi_1 u^{n-1}+\\ &\qquad\qquad \cdots+\chi_n)^2+\cdots+\frac{1}{n!}(\chi_1 u^{n-1}+\cdots+\chi_n)^n\Big)\\ &= \int_M e^{f\,u^n}\Big(\chi_n+\frac{u^n}{2!}\sum_{j_1+j_2=n}\chi_{j_1}\chi_{j_2}+\\ &\qquad\qquad \frac{u^{2n}}{3!}\sum_{j_1+j_2+j_3=n}\chi_{j_1}\chi_{j_2}\chi_{j_3}+\cdots+\frac{u^{(n-1)n}}{n!}{\chi_1}^n\Big),\end{aligned}\tag{6}$$

where the indices j_m in the summations all range from 1 to n. The series terminates because since $\chi_j \in \Omega^{2j}(M)^G$, then ${\chi_j}^{n+1}$ vanishes for $j \geq 1$. Also, the terms which do not contain $2n$-forms are removed since they contribute nothing to the integral, as M is $2n$-dimensional.

We can now use Lemma 3.1 to complete the proof of the theorem. Computing directly from the relations in the lemma, we find that for any $j_1, \ldots, j_m \in \{1, \ldots, n\}$ such that $j_1 + \cdots + j_m = n$, we have

$$\chi_{j_1}\cdots\chi_{j_m} = \binom{2j_1}{j_1}\cdots\binom{2j_m}{j_m}\binom{2n}{n}^{-1} f^{m-1}\,\chi_n.$$

If we introduce the notation

$$b_m = \frac{1}{(m+1)!}\sum_{j_1+\cdots+j_{m+1}=n}\binom{2j_1}{j_1}\cdots\binom{2j_{m+1}}{j_{m+1}}\binom{2n}{n}^{-1},$$

then, substituting this into the expression above for the integral of the exponential of the equivariant Euler class, we obtain

$$\begin{aligned}\int_M e^{\tilde{\chi}} &= \int_M e^{f\,u^n}(\chi_n + b_1 u^n f\chi_n + \cdots + b_{n-1}u^{(n-1)n}f^{n-1}\chi_n)\\ &= \int_M e^{f\,u^n}\chi + b_1 u^n \int_M e^{f\,u^n} f\,\chi + \cdots + b_{n-1}u^{(n-1)n}\int_M e^{f\,u^n} f^{n-1}\chi.\end{aligned}$$

Combining this with the expression we obtained for the same integral by localization, we have

$$\begin{aligned}\sum_{p\in C_f}\frac{(2\pi)^n}{f(p)\,u^n}e^{f(p)\,u^n} = \int_M e^{f\,u^n}\chi + b_1 u^n\int_M e^{f\,u^n} f\,\chi + \\ \cdots + b_{n-1}u^{(n-1)n}\int_M e^{f\,u^n}f^{n-1}\chi.\end{aligned}$$

We now "evaluate" both sides of this equation at $t^{1/n}X$. Since $u(t^{1/n}X) = t^{1/n}$, this gives

$$\begin{aligned}\sum_{p\in C_f}\frac{(2\pi)^n}{f(p)\,t}e^{f(p)\,t} &= \int_M e^{f\,t}\chi + b_1 t\int_M e^{f\,t} f\,\chi + \cdots + b_{n-1}t^{n-1}\int_M e^{f\,t}f^{n-1}\chi\\ &= \int_M e^{f\,t}\chi + b_1 t\frac{\partial}{\partial t}\left(\int_M e^{f\,t}\chi\right) + \\ &\qquad \cdots + b_{n-1}t^{n-1}\frac{\partial^{n-1}}{\partial t^{n-1}}\left(\int_M e^{f\,t}\chi\right)\\ &= \left(1 + b_1 t\frac{\partial}{\partial t} + \cdots + b_{n-1}t^{n-1}\frac{\partial^{n-1}}{\partial t^{n-1}}\right)\int_M e^{f\,t}\chi.\end{aligned}$$

Now the stationary phase expansion of e^{tf} integrated against the Euler class of M is, as usual, of the form

$$\int_M e^{f\,t}\chi = \sum_{p\in C_f} e^{f(p)\,t}\sum_{j=0}^{\infty} c_{p,j}t^{-j},$$

where the $c_{p,j}\in\mathbb{R}$ are constants depending on p and j. Putting this in for the integral on the right side of the previous equation, we have

$$\begin{aligned}\sum_{p\in C_f}\frac{(2\pi)^n}{f(p)\,t}e^{f(p)\,t} &= \Bigg(1 + b_1 t\frac{\partial}{\partial t} + \\ &\qquad \cdots + b_{n-1}t^{n-1}\frac{\partial^{n-1}}{\partial t^{n-1}}\Bigg)\left(\sum_{p\in C_f} e^{f(p)\,t}\sum_{j=0}^{\infty}c_{p,j}t^{-j}\right)\\ &= \sum_{p\in C_f}\Bigg(1 + b_1 t\frac{\partial}{\partial t} + \\ &\qquad \cdots + b_{n-1}t^{n-1}\frac{\partial^{n-1}}{\partial t^{n-1}}\Bigg)\left(e^{f(p)\,t}\sum_{j=0}^{\infty}c_{p,j}t^{-j}\right).\end{aligned}$$

Comparing coefficients on both sides of the equation, we find that all the constants $c_{p,j}$ must vanish for suitably large j.

Therefore, the stationary phase expansion has only finitely many nonzero terms. After we prove Lemma 3.1 then, the proof of Theorem 3.2 will be complete.

4. Proof of Lemma 3.1. We now compute the Pfaffian in Equation (3) above at a point $p \in M$. Since L is skew-symmetric, then with a suitable choice of orthonormal basis $\partial x^1, \ldots, \partial x^{2n}$ for T_pM, the matrix of L is of the form

$$L = \begin{pmatrix} 0 & \lambda_1 & & & & \\ -\lambda_1 & 0 & & & & \\ & & \ddots & & & \\ & & & & 0 & \lambda_n \\ & & & & -\lambda_n & 0 \end{pmatrix},$$

where some of the λ_i's are possibly equal to zero.

Before we describe what R looks like with respect to this basis, we give a definition, using the notation that

$$\varepsilon^{i_1 i_2 \cdots i_{2m}} = \begin{cases} 1 & \text{if } i_1, \ldots, i_{2m} \text{ is an even permutation of } 1, \ldots, 2m \\ -1 & \text{if } i_1, \ldots, i_{2m} \text{ is an odd permutation of } 1, \ldots, 2m \\ 0 & \text{if } i_1, \ldots, i_{2m} \text{ are not all distinct.} \end{cases}$$

Definition 4.1. *Let $A = (a_{ij})$ be a $2m \times 2m$ matrix. We say that A has* Property Φ *if for any integers $i_1, i_2, \ldots, i_{2m}$ between 1 and $2m$,*

$$a_{i_1 i_2} a_{i_3 i_4} \cdots a_{i_{2m-1} i_{2m}} = \varepsilon^{i_1 i_2 \cdots i_{2m}} a_{12} a_{34} \cdots a_{2m-1,2m}.$$

The Pfaffian of a matrix A with Property Φ is equal to

$$\begin{aligned} \mathrm{Pfaff}(A) &= \frac{1}{2^m m!} \sum_{i_1, \ldots, i_{2m}=1}^{2m} \varepsilon^{i_1 i_2 \cdots i_{2m}} a_{i_1 i_2} a_{i_3 i_4} \cdots a_{i_{2m-1} i_{2m}} \\ &= \frac{(2m)!}{2^m m!} a_{12} a_{34} \cdots a_{2m-1,2m}. \end{aligned}$$

We note also that if A has Property Φ, then any minor obtained by removing rows $i_1, \ldots, i_{2k}$ and the corresponding columns $i_1, \ldots, i_{2k}$ has Property Φ as well.

If M has constant curvature, then, relative to the orthonormal basis $\partial x^1, \ldots, \partial x^{2n}$ at p, the matrix for R has Property Φ. This holds because, if M has constant curvature κ, then for any vectors $v_1, v_2, v_3, v_4 \in T_pM$,

$$\langle R(v_1, v_2)v_3, v_4 \rangle = \kappa(\langle v_2, v_3 \rangle \langle v_1, v_4 \rangle - \langle v_1, v_3 \rangle \langle v_2, v_4 \rangle),$$

so the (i, j)-th entry in the matrix for R at p is given by

$$r_{ij} = \kappa \, dx^i dx^j,$$

where $dx^1, \ldots, dx^{2n} \in T_p^*M$ is the dual basis to $\partial x^1, \ldots, \partial x^{2n}$.

Using this property of R, we can now compute $\mathrm{Pfaff}(R+uL)$ at p. To help in writing it out, we introduce the following notation: for any $k \in \{1,\dots,n\}$, we let I_k and J_k represent "complementary" ordered multi-indices, in the sense that

$$\begin{aligned} I_k &= \{i_1,\dots,i_{n-k}\} \text{ with } i_1 < i_2 < \cdots < i_{n-k}, \\ J_k &= \{j_1,\dots,j_k\} \text{ with } j_1 < j_2 < \cdots < j_k, \\ I_k \cap J_k &= \emptyset, \text{ and } I_k \cup J_k = \{1,\dots,n\}. \end{aligned}$$

Also, for any such multi-indices, we let $R^{(J_k)}$ denote the $2k \times 2k$ minor of R obtained by removing all the rows and columns corresponding to those which are occupied by $\lambda_{i_1},\dots,\lambda_{i_{n-k}}$ in L, i.e. rows $2i_1-1,\dots,2i_{n-k}-1$, $2i_1,\dots,2i_{n-k}$ as well as the corresponding columns $2i_1-1,\dots,2i_{n-k}-1$, $2i_1,\dots,2i_{n-k}$.

In this notation then,

$$\begin{aligned} \mathrm{Pfaff}(R+uL) = \mathrm{Pfaff}(R) &+ u\sum_{J_{n-1}} \lambda_{i_1}\mathrm{Pfaff}(R^{(J_{n-1})}) \\ &+u^2\sum_{J_{n-2}} \lambda_{i_1}\lambda_{i_2}\mathrm{Pfaff}(R^{(J_{n-2})}) + \cdots \\ &+u^{n-1}\sum_{J_1} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_{n-1}}\mathrm{Pfaff}(R^{(J_1)}) + u^n\lambda_1\lambda_2\cdots\lambda_n. \end{aligned}$$

Since R has Property Φ, then the $2k \times 2k$ minors $R^{(J_k)}$ have Property Φ as well, meaning that

$$\mathrm{Pfaff}(R^{(J_k)}) = \frac{(2k)!}{2^k k!} r_{j_1 j_2} r_{j_3 j_4} \cdots r_{j_{2k-1} j_{2k}}.$$

Substituting this into the equation above, we obtain

$$\begin{aligned} \mathrm{Pfaff}(R+uL) &= \mathrm{Pfaff}(R) \\ &+u\frac{(2n-2)!}{2^{n-1}(n-1)!}\sum_{J_{n-1}} \lambda_{i_1} r_{2j_1-1,2j_1} r_{2j_2-1,2j_2}\cdots r_{2j_{n-1}-1,2j_{n-1}} \\ &+u^2\frac{(2n-4)!}{2^{n-2}(n-2)!}\sum_{J_{n-2}} \lambda_{i_1}\lambda_{i_2} r_{2j_1-1,2j_1} r_{2j_2-1,2j_2}\cdots r_{2j_{n-2}-1,2j_{n-2}} \\ &+\cdots + u^{n-1}\frac{2!}{2^1 1!}\sum_{J_1} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_{n-1}} r_{2j_1-1,2j_1} + u^n\lambda_1\lambda_2\cdots\lambda_n. \end{aligned}$$

That is, if we now insert the factor of $1/2\pi$ into the Pfaffian and denote the "near-diagonal" element $r_{2k-1,2k}$ of the matrix for R by

$$\rho_k = r_{2k-1,2k},$$

then we obtain an expression for the coefficient χ_k of u^{n-k} in the equivariant Euler class $\widetilde{\chi} = \mathrm{Pfaff}\left(\dfrac{R+uL}{2\pi}\right)$, namely

$$\chi_k = \frac{(2k)!}{(2\pi)^n 2^k k!}\sum_{J_k} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_{n-k}}\rho_{j_1}\rho_{j_2}\cdots\rho_{j_k}.$$

Multiplying two of these coefficients together (and using Property Φ), we find that

$$\chi_k \chi_l = \frac{(k+l)!\ (k+l)!\ (2k)!\ (2l)!}{k!\ k!\ l!\ l!\ (2k+2l)!} \frac{\lambda_1 \lambda_2 \cdots \lambda_n}{(2\pi)^n} \chi_{k+l},$$

for any $k, l \in \{1, \ldots, n\}$ such that $k + l \leq n$. Noting that

$$f(p) = \mathrm{Pfaff}\left(\frac{L|_p}{2\pi}\right) = \frac{\lambda_1 \lambda_2 \cdots \lambda_n}{(2\pi)^n}$$

then completes the proof of the lemma. □

This then completes the proof of Theorem 3.2. □

5. The explicit computation on S^{2n}. As mentioned in the introduction, the original question, posed by Bott, motivating this paper was *why does the height function z on S^{2n} have the property that e^{tz} has a finite stationary phase expansion?* As demonstrated in the previous section, this property follows from the constant curvature of S^{2n}. We compute this explicitly here, so let $S^{2n} \subset \mathbb{R}^{2n+1}$, where $\mathbb{R}^{2n+1}$ has coordinates $x^1, x^2, \ldots, x^{2n}, z$, and

$$S^{2n} = \{(x^1, x^2, \ldots, x^{2n}, z) \in \mathbb{R}^{2n+1} \mid (x^1)^2 + (x^2)^2 + \cdots + (x^{2n})^2 + z^2 = 1\}.$$

Then S^1 acts on S^{2n} by rotating the first n planes in $\mathbb{R}^{2n+1}$ (the k-th plane being the $x^{2k-1}x^{2k}$-plane) with weights $\lambda_1, \lambda_2, \ldots, \lambda_n$. An infinitesimal generator for this action is given by

$$\widetilde{X} = \lambda_1(x^1 \partial x^2 - x^2 \partial x^1) + \cdots + \lambda_n(x^{2n-1} \partial x^{2n} - x^{2n} \partial x^{2n-1}),$$

and we define $u \in \mathfrak{s}^{1^*}$ to be the dual of $X \in \mathfrak{s}^1$. We also define

$$\theta = \frac{1}{2}\left(\frac{1}{\lambda_1}(x^1\, dx^2 - x^2\, dx^1) + \cdots + \frac{1}{\lambda_n}(x^{2n-1}\, dx^{2n} - x^{2n}\, dx^{2n-1})\right).$$

Then θ has the following properties:

$$\begin{aligned}
\iota_{\widetilde{X}} \theta &= \frac{1}{2}(1 - z^2) \\
d\theta &= \frac{1}{\lambda_1} dx^1\, dx^2 + \cdots + \frac{1}{\lambda_n} dx^{2n-1} dx^{2n} \\
\iota_{\widetilde{X}} d\theta &= z\, dz.
\end{aligned}$$

The equivariant Euler class of S^{2n} relative to this action then is given by

$$\begin{aligned}
\widetilde{\chi} = &\frac{\lambda_1 \cdots \lambda_n}{(2\pi)^n} \frac{(2n+1)!!}{n!} \left(\frac{2n}{2n+1} \theta (d\theta)^{n-1} dz + \frac{1}{2n+1} (d\theta)^n z\right) \\
&+ \frac{\lambda_1 \cdots \lambda_n}{(2\pi)^n} \frac{(2n-1)!!}{(n-1)!} \left(\frac{2n-2}{2n-1} \theta (d\theta)^{n-2} dz + \frac{1}{2n-1} (d\theta)^{n-1} z\right)\ u \\
&+ \frac{\lambda_1 \cdots \lambda_n}{(2\pi)^n} \frac{(2n-3)!!}{(n-2)!} \left(\frac{2n-4}{2n-3} \theta (d\theta)^{n-3} dz + \frac{1}{2n-3} (d\theta)^{n-2} z\right)\ u^2 + \\
&\cdots + \frac{\lambda_1 \cdots \lambda_n}{(2\pi)^n} \frac{3!!}{1!} \left(\frac{2}{3} \theta\, dz + \frac{1}{3} d\theta\, z\right)\ u^{n-1} + \frac{\lambda_1 \cdots \lambda_n}{(2\pi)^n} z\ u^n.
\end{aligned}$$

This can be exponentiated as in the previous section to show that the stationary phase expansion of the function e^{tz} (integrated against the Euler class) is finite.

Another, perhaps more direct, way to show this is by using a "tail" for the equivariant Euler class other than the usual one obtained from the equivariant curvature. From the above expression for the Euler class χ (the de Rham $2n$-form term in the equivariant Euler class) on S^{2n}, we obtain

$$\iota_{\widetilde{X}}\chi = \frac{\lambda_1\cdots\lambda_n}{(2\pi)^n}\frac{(2n-1)!!}{(n-1)!}(d\theta)^{n-1}\,dz.$$

Using that

$$d\left(\theta\,(d\theta)^{n-2}\,dz\right) = (d\theta)^{n-1}\,dz,$$

and continuing in a similar fashion with forms of lower de Rham degree, we obtain another equivariant extension of the Euler class:

$$\begin{aligned}
\widetilde{\chi} = {} & \frac{\lambda_1\cdots\lambda_n}{(2\pi)^n}\frac{(2n+1)!!}{n!}\left(\frac{2n}{2n+1}\theta(d\theta)^{n-1}dz + \frac{1}{2n+1}(d\theta)^n z\right) + \\
& \frac{\lambda_1\cdots\lambda_n}{(2\pi)^n}\frac{(2n-1)!!}{(n-1)!}\theta\,(d\theta)^{n-2}\,dz\ u + \\
& \frac{\lambda_1\cdots\lambda_n}{2\ (2\pi)^n}\frac{(2n-1)!!}{(n-1)!}\theta\,(d\theta)^{n-3}\,(1-z^2)\,dz\ u^2 + \\
& \frac{\lambda_1\cdots\lambda_n}{2^2\ (2\pi)^n}\frac{(2n-1)!!}{(n-1)!}\theta\,(d\theta)^{n-4}\,(1-z^2)^2\,dz\ u^3 + \\
& \cdots + \ \frac{\lambda_1\cdots\lambda_n}{2^{n-2}\ (2\pi)^n}\frac{(2n-1)!!}{(n-1)!}\theta\,(d\theta)^0\,(1-z^2)^{n-2}\,dz\ u^{n-1} + \\
& \frac{\lambda_1\cdots\lambda_n}{2^{n-1}\ (2\pi)^n}\frac{(2n-1)!!}{(n-1)!}\left(\int_0^z(1-\zeta^2)^{n-1}\,d\zeta\right)\ u^n.
\end{aligned}$$

This extension has the advantage that, since there is a dz in each middle term, we can obtain directly an equivariant extension for the class $e^{uz}\chi$:

$$\begin{aligned}
\widetilde{(e^{uz}\chi)} = {} & \frac{e^{uz}\lambda_1\cdots\lambda_n}{(2\pi)^n}\frac{(2n+1)!!}{n!}\left(\frac{2n}{2n+1}\theta(d\theta)^{n-1}dz + \frac{1}{2n+1}(d\theta)^n z\right) + \\
& \frac{e^{uz}\lambda_1\cdots\lambda_n}{(2\pi)^n}\frac{(2n-1)!!}{(n-1)!}\theta\,(d\theta)^{n-2}\,dz\ u + \\
& \frac{e^{uz}\lambda_1\cdots\lambda_n}{2\ (2\pi)^n}\frac{(2n-1)!!}{(n-1)!}\theta\,(d\theta)^{n-3}\,(1-z^2)\,dz\ u^2 + \\
& \frac{e^{uz}\lambda_1\cdots\lambda_n}{2^2\ (2\pi)^n}\frac{(2n-1)!!}{(n-1)!}\theta\,(d\theta)^{n-4}\,(1-z^2)^2\,dz\ u^3 + \\
& \quad\vdots \\
& + \frac{e^{uz}\lambda_1\cdots\lambda_n}{2^{n-2}\ (2\pi)^n}\frac{(2n-1)!!}{(n-1)!}\theta\,(d\theta)^0\,(1-z^2)^{n-2}\,dz\ u^{n-1} + \\
& \frac{\lambda_1\cdots\lambda_n}{2^{n-1}\ (2\pi)^n}\frac{(2n-1)!!}{(n-1)!}\left(\int_0^z e^{u\zeta}(1-\zeta^2)^{n-1}\,d\zeta\right)\ u^n.
\end{aligned}$$

Relative to this action, the equivariant Euler class of the normal bundle to the fixed point at $z=1$ is $(2\pi)^{-n}\lambda_1\cdots\lambda_n\ u^n$, and at $z=-1$ it is $-(2\pi)^{-n}\lambda_1\cdots\lambda_n\ u^n$.

Applying the Equivariant Localization Theorem to the equivariantly closed form $\widetilde{(e^{uz}\chi)}$ then, we have

$$\begin{aligned}\int_{S^{2n}} e^{uz}\chi &= \frac{(2n-1)!!}{2^{n-1}\,(n-1)!}\left(\int_0^1 e^{u\zeta}(1-\zeta^2)^{n-1}\,d\zeta\right) - \\ &\qquad \frac{(2n-1)!!}{2^{n-1}\,(n-1)!}\left(\int_0^{-1} e^{u\zeta}(1-\zeta^2)^{n-1}\,d\zeta\right) \\ &= \frac{(2n-1)!!}{2^{n-1}\,(n-1)!}\left(\int_{-1}^1 e^{u\zeta}(1-\zeta^2)^{n-1}\,d\zeta\right)\end{aligned}$$

Evaluating both sides on tX (and also evaluating the integrals by multiplying out the factor $(1-\zeta^2)^{n-1}$ and integrating term by term), we obtain an explicit expression for the stationary phase expansion of e^{tz} integrated against the Euler class of S^{2n}.

This equivariant extension then demonstrates the finite length of the stationary phase expansion of e^{tz}, but it can also be used to demonstrate the fact that the stationary phase expansion of the function

$$f(z) = \frac{\lambda_1\cdots\lambda_n}{2^{n-1}(2\pi)^n}\frac{(2n-1)!!}{(n-1)!}\left(\int_0^z (1-\zeta^2)^{n-1}\,d\zeta\right)$$

has only one term. For this, one can follow the Atiyah-Bott procedure and exponentiate the above equivariant extension of the equivariant Euler class. Since each term in this extension except for the de Rham $2n$-form and de Rham 0-form terms contains a factor of dz, all the "cross-terms" introduced by the exponentiation vanish. In other words, using this extension for $\widetilde{\chi}$, we have that

$$\int_{S^{2n}} e^{\widetilde{\chi}} = \int_{S^{2n}} e^{u^n f}\big(1+(\widetilde{\chi}-u^n f)\big).$$

Applying the Equivariant Localization Theorem then yields

$$\int_{S^{2n}} e^{u^n f}\chi = \frac{(2\pi)^n}{\lambda_1\cdots\lambda_n\; u^n}(e^{u^n f(1)} - e^{u^n f(-1)}).$$

Evaluating both sides at $t^{1/n}X$, we obtain a one-term stationary phase expansion for e^{tf} relative to the Euler class.

How to generalize this, however, is not at present apparent. One possibility is that nontrivial cohomology in dimensions lower than the form to be exponentiated is an obstruction to finding an extension with the property that all terms except the highest and lowest degree de Rham terms are divisible by some particular 1-form.

6. Conclusion. We have now given a geometric reason for the finiteness of the stationary phase expansion of the height function on S^{2n}. While we have not explicitly produced a wide class of examples of functions which are not moment maps but which have the finite stationary phase expansion property, it is hoped that the techniques involved in demonstrating the finiteness of the stationary phase expansion of the height function on S^{2n} can be adapted to produce a wider class of functions with finite stationary phase expansions.

In particular, a key aspect of the method used to prove Lemma 3.1 is that the computation showing the relationships among the terms of the equivariant Euler class has been done entirely locally. In verifying the relationships at each point individually rather than globally all at once, matrices for L and R could be written down and the computation could be carried out.

It is not apparent at this point what the most general conditions on M are that bring about similar relationships among the terms of the equivariant Euler class. We have shown that constant curvature is sufficient, but it may be a more stringent condition than is necessary. Perhaps the method of proof used here can be adapted to be more widely applicable.

We also do not have a geometric explanation yet for the finiteness of the stationary phase expansion of the function

$$f(z) = \frac{\lambda_1 \cdots \lambda_n}{2^{n-1}(2\pi)^n} \frac{(2n-1)!!}{(n-1)!} \left(\int_0^z (1-\zeta^2)^{n-1} \, d\zeta \right).$$

As was mentioned, this function has a single-term stationary phase expansion, and there should be a geometric explanation for this fact as well.

In addition, if there are other functions which have finite stationary phase expansions and which arise as the 0-form part of an equivariant extension of a nondegenerate $2k$-form, then there is another, larger, question: *do such equivariant extensions of nondegenerate $2k$-foms act similarly to symplectic forms and moment maps in other ways as well?* An affirmative answer to this question would lead to higher-dimensional analogues of sympletic forms and moment maps.

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