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SUPERSYMMETRIES IN CALABI-YAU GEOMETRY*

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Abstract. We introduce a class of Lie superalgebras, called $\mathfrak{spin}_c$ supersymmetry algebras, constructed from spinor representations. The construction is motivated by supersymmetry algebras used by physicists. On a Riemannian manifold, a Kähler manifold, and a hyperkähler manifold respectively, it is known that some natural operators on the space of differential forms generate certain Lie superalgebras. It turns out that they correspond to $\mathfrak{spin}_c(2)$, $\mathfrak{spin}_c(3)$, and $\mathfrak{spin}_c(5)$ supersymmetry algebras respectively. Motivated by Mirror Symmetry Conjecture, we also consider supersymmetries on Calabi-Yau manifolds.

Key words. supersymmetry, Calabi-Yau manifolds

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The supersymmetry (SUSY) algebra [10] is a special kind of Lie superalgebras [6] which involves spinor representations. It was invented by physicists in the seventies to formulate a unified theory for fermions and bosons. A guiding principle in physics is to examine the symmetries of Lagrangians. Quite often a classical Lagrangian can be extended to have supersymmetries. For example, Donaldson theory has been interpreted by Witten [12] as a twisted $N = 2$ supersymmetric quantum field theory which extends the Lagrangian of the classical Yang-Mills theory. The study of this theory has led to Seiberg-Witten theory [13]. Other examples include supersymmetric extensions of nonlinear sigma models. When the source manifold is a Riemann surface, it turns out [1] that an $N = 1$ supersymmetric extension is always possible; when the target Riemannian manifold is Kähler, an $N = 2$ supersymmetric extension is possible; when the target manifold is hyperkähler, an $N = 4$ supersymmetric extension is possible.

It is well-known to physicists that when one considers the large volume limit, the topological sigma model leads to the space of differential forms on the target manifold and differential operators on it. So it is conceivable that supersymmetries in the topological sigma model leads to some supersymmetries among these operators. One is naturally led to the problem of finding a relationship between manifolds with special holonomy groups and the supersymmetry algebras formed by differential operators on them. Another motivation for this problem is to find the analogues of Kähler manifolds etc. in non-commutative geometry. This is discussed in a recent paper by Fröhlich, Grandjean and Recknagel [3]. Their idea is as follows: since the space of exterior forms is a super vector space, the space of linear operators on it is naturally a Lie superalgebra (actually a Poisson superalgebra) under the supercommutators. For manifolds with special holonomy groups, one can find operators which generate finite dimensional Lie (super)algebras.

Actually some examples of this type have been well-known. In Riemannian geometry, Witten considered the following very simple Lie superalgebra in his influential paper on Morse theory [11]: for a Riemannian manifold (M, g), let $d : \Omega^*(M) \to \Omega^*(M)$

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be the exterior differential operator, d^* its formal adjoint. Set $Q_1 = d + d^*$, $Q_2 = \sqrt{-1}(d - d^*)$, $H = \Box = dd^* + d^*d$. Then one has

$$[Q_1, Q_1] = [Q_2, Q_2] = \frac{1}{2}H, [Q_1, Q_2] = 0.$$

In Kähler geometry, the proof of the Hard Lefschetz Theorem for Kähler manifolds, usually attributed to Chern, uses three algebraic operators (see e.g. Griffiths-Harris [4]): L is the exterior product with the Kähler form, Λ its adjoint, H is defined by $H(\alpha) = (m - p)\alpha$ for exterior form α of degree p on a manifold of complex dimension m. Then we have

$$[\Lambda, L] = H, \qquad [H, L] = -2L, \qquad [H, \Lambda] = 2\Lambda.$$

These are the commutation relations of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. The Hodge identities and their consequences ($\Box_\partial = \Box_{\bar\partial} = \frac{1}{2}\Delta$) reveal that the operators L, Λ, H, ∂, $\bar\partial$, ∂^*, $\bar\partial^*$, Δ generates a Lie superalgebra. In hyperkähler geometry, the Lie algebra generated by the multiplications by the three Kähler forms and their adjoints has been studied by Verbitskii [9]. Together with the ∂ and $\bar\partial$ operators for each of the three complex structures and their adjoints, they generate a Lie superalgebra. These observations appear in the paper by Fröhlich *et al* mentioned above. They will be reviewed in §1. Our observation in this paper is that the Lie superalgebras of operators in Riemannian, Kähler and hyperkähler geometries mentioned above can be constructed in a unified way using spinor representations of $\mathfrak{spin}_c(n)$. Hence they will be called $\mathfrak{spin}_c(n)$ *SUSY algebras*. It turns out that for Riemannian geometry, $n = 2$; for Kähler geometry, $n = 3$; for hyperkähler geometry, $n = 5$. A strange coincidence here is that $2 = 1 + 1$, $3 = 2 + 1$, $5 = 4 + 1$, and $1, 2, 4$ correspond respectively to the number N in supersymmetry extensions of nonlinear sigma models in these geometries.

The main motivation for this paper comes from Mirror Symmetry Conjecture. On a Calabi-Yau n-manifold M, there are some natural operators on $\Omega^{-*,*}(M)$. Conjecturally, these operators correspond to the operators ∂, $\bar\partial$, ∂^* and $\bar\partial^*$ on a mirror manifold of M. Furthermore, there is an obvious analogue of H. One naturally asks whether there is an element of $\Omega^{-1,1}(M)$ which corresponds to the Kähler form and defines the analogues of the operators L and Λ. For Calabi-Yau hypersurfaces in weighted $(n+1)$-projective spaces, Hübsch and Yau [5] have found such an element in $H^{-1,1}(M)$ and discussed the induced $SL(2, \mathbb{C})$-action on $\oplus_{p=0}^n H^{n-p,p}(M)$. In this paper, we will give conditions for an element in $\Omega^{-1,1}(M)$ to play the role of the Kähler form in the sense that together with the natural differential operators on $\Omega^{-*,*}(M)$, it generates the $Spin_c(3)$ SUSY algebra.

1. ***Spin**$_c$* **supersymmetry algebras.** In this section, we first review the definition of Lie superalgebras and briefly describe supersymmetry algebras. We then give a construction of a Lie superalgebra using spinor representation which we call $Spin_c$ SUSY algebras.

1.1. Lie superalgebras. A $\mathbb{Z}_2$-graded vector space $L = L_0 \oplus L_1$ over a field $\mathbf{k}$ is called *a Lie superalgebra* over $\mathbf{k}$ if there is an even $\mathbf{k}$-linear map $[\cdot,\cdot] : L \otimes L \to L$, such that

$$[a, b] = -(-1)^{|a||b|}[b, a], \quad \text{(super anti-symmetric)}$$
$$[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}[b, [a, c]], \quad \text{(super Jacobi identity)}$$

for homogeneous elements $a, b, c \in L$, where $|\cdot|$ stands for the degree. It follows that L_0 is a Lie algebra, and L_1 is a representation of L_0. Hence, to construct a Lie superalgebra, one can start with a Lie algebra L_0, a representation L_1 of L_0, then carefully specify $[\cdot,\cdot]$ on $L_1 \times L_1$.

EXAMPLE 1.1. Let $V = V_0 \oplus V_1$ be a super vector space (not necessarily finite dimensional). Then $\mathrm{End}(V)$ has an induced $\mathbb{Z}_2$-grading. A linear transformation $\phi \in \mathrm{End}(V)$ is said to be even if $\phi(V_j) \subset V_j$, $j = 0, 1$. It is said to be odd if switch V_0 with V_1. For homogeneous ϕ, $\psi \in \mathrm{End}(V)$, set

$$[\phi, \psi] = \phi\psi - (-1)^{|\phi||\psi|}\psi\phi.$$

Then $(\mathrm{End}(V), [\cdot,\cdot])$ is a Lie superalgebra. Furthermore, if ϕ, ψ and φ are homogeneous elements of $\mathrm{End}(V)$, then we have

$$[\phi, \psi\varphi] = [\phi, \psi]\varphi + (-1)^{||\phi||\psi|}\psi[\phi, \varphi].$$

This means $(\mathrm{End}(V), [\cdot,\cdot])$ is actually a Poisson superalgebra. For us, V will be the space of exterior forms on manifold graded by degrees, and we will consider Lie superalgebras generated by differential operators.

1.2. SUSY algebra. The *supersymmetry algebra* used by physicists in supergravity theory on flat Minkowski spaces has L_0 the Lie algebra of Poincaré group, it is the semi-direct product of the Lie algebra of Lorentz group with its standard representation on the Minkowski space. A set of generators for L_0 are given by L_{ij}'s, which generates the rotations and boosts, and P_k's, which generates the space-time translations. The odd part L_1 are some copies of spinor representations. The bracket of two elements in L_1 are linear combinations of P_k's. For details, see Wess-Bagger [10].

1.3. ***Spin$_c$*** **supersymmetry algebra.** Motivated by Seiberg-Witten theory [13], we introduce the notion of $\mathfrak{spin}_c(n)$ supersymmetry algebras.

Let us first describe a procedure to produce Lie superalgebras. Let $\mathfrak{g}$ be any complex Lie algebra, V^+ a complex representation of $\mathfrak{g}$, and $V^- = (V^+)^*$ the dual representation. The pairing of V^+ with V^- extends to a natural pairing $\langle\cdot,\cdot\rangle$ on $L_1 = V^+ \oplus V^-$. Take $L_0 = \mathfrak{g} \oplus \mathbb{C}$. Let $\mathbb{C} \subset L_0$ acts trivially on V^+ and V^-, hence L_1 is a representation of L_0. Now we define a bracket on $L = L_0 \oplus L_1$. For $a, b \in L_0$, $[a, b]$ is the Lie bracket in L_0. For $a \in L_0$, $b \in L_1$, $[a, b]$ is given by the action of a on b by the representation. For $a \in L_1$, $b \in L_0$, $[a, b] = -[b, a]$. For $a, b \in L_1$, $[a, b] = \langle a, b\rangle \in \mathbb{C} \subset L_0$.

LEMMA 1.1. $(L, [\cdot,\cdot])$ *defined above is a Lie superalgebra.*

Proof. It is clear that $[\cdot,\cdot]$ is super-antisymmetric, so we only need to check the super-Jacobi identity. It certainly holds when a, b, c are all from L_0 or L_1. If two of them are in L_0, and the third in L_1, it suffices to check the Jacobi identity for the case of a, $b \in L_0$, $c \in L_1$:

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]].$$

This follows from the fact that L_1 is a representation of L_0. When two of them are from L_1, and the third from L_0, it suffices to check the Jacobi identity for the case of $a \in L_0$, $b, c \in L_1$. Now the required identity is reduced to

$$0 = \langle [a, b], c\rangle + \langle b, [a, c]\rangle. \tag{1}$$

The terms on the right hand side are nonzero only if one of b and c lies in V, the other lies in V^*. Without loss of generality, let $b \in V^*$ and $c \in V$, then (1) is equivalent to the definition of dual representation. $\Box$

The above construction can certainly be carried out for real Lie algebra and real representations. One can also let $\mathbb{C}$ act on V and V^* with opposite weights.

When $\mathfrak{g}$ is the complexified $\mathfrak{so}(n)$, $L_0 = \mathfrak{so}(n, \mathbb{C}) \oplus \mathbb{C}$ is the complexified Lie algebra of $Spin_c(n)$. We take V^+ to be N copies of an irreducible spinor representation, when restricted to $\mathfrak{so}(n)$. The resulting Lie superalgebra is called the N-extended *spin*$_c$ SUSY algebra, and denote it by N-$\mathfrak{spin}_c(n)$. For $N = 1$, it is also called the $\mathfrak{spin}_c(n)$ SUSY algebra. We remark that the N-$\mathfrak{spin}_c(n)$ SUSY algebra can be understood as the non-relativistic version of SUSY algebra, since L_0 consists of $\mathfrak{so}(n, \mathbb{C})$, the Lie algebra of rotation group, and $\mathbb{C}$, which corresponds to the Hamiltonian. Furthermore, the bracket of two odd elements lies in $\mathbb{C}$.

2. Examples. We review some results appeared in Fröchlich *et al* [3] concerning the the supersymmetry algebras in Riemannian, Kähler and hyperkähler geometries. They are examples of $Spin_c$ SUSY algebras defined in last section.

2.1. $Spin_c(2)$ **SUSY algebra and Riemannian geometry.** Denote the generator of $\mathfrak{so}(2) \cong \mathbb{R}$ by T, and the generator of $\mathbb{C}$ by H. The spinor representation S^+ of $SO(2) = S^1$ has weight $1/2$. For $N = 1$, let G^+ be a generator of S^+, and G^- the dual basis for S^-. Then for $\mathfrak{spin}_c(1)$ SUSY algebra, we have

$$\begin{gathered} [T, T] = [H, H] = [T, H] = 0, \\ [T, G^{\pm}] = \pm\frac{1}{2}G^{\pm}, \quad [H, G^{\pm}] = 0, \\ [G^{\pm}, G^{\pm}] = 0, \quad [G^+, G^-] = H. \end{gathered}$$

Let (M, g) be a Riemannian manifold. Define $h : \Omega^*(M) \to \Omega^*(M)$ by $h(\alpha) = (n-p)\alpha$ for $\alpha \in \Omega^p(M)$. Denote by $\Box$ the Laplace operator. Then we have

$$\begin{aligned} &[h, h] = [\Box, \Box] = [h, \Box] = 0, & [\Box, d] = [\Box, d^*] = 0, \\ &[h, d] = d, & [h, d^*] = -d^*, \\ &[d, d] = [d^*, d^*] = 0, & [d, d^*] = \Box. \end{aligned}$$

Then $G^+ \mapsto d$, $G^- \mapsto d^*$, $H \mapsto \Box = dd^* + d^*d$, $T \mapsto \frac{1}{2}h$ defines a representation of $\mathfrak{spin}_c(2)$ SUSY algebra. Let

$$Q_1 = G^+ + G^-, Q_2 = \sqrt{-1}(G^+ - G_-).$$

Then we have

$$[Q_1, Q_1] = [Q_2, Q_2] = 2H, [Q_1, Q_2] = 0.$$

This is the supersymmetry algebra used in Witten [11].

2.2. $Spin_c(3)$ **SUSY algebra and Kähler geoemtry.** It is well-known that there is an isomorphism $\mathfrak{so}(3) \cong \mathfrak{su}(2)$. The spinor representation of $\mathfrak{so}(3)$ is given by the natural representation of $\mathfrak{su}(2)$ on $\mathbb{C}^2$. Alternatively, there is an isomorphism $\mathfrak{so}(3, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C})$. The spinor representation extends the standard representation of $\mathfrak{sl}(2, \mathbb{C})$ on $\mathbb{C}^2$ or its dual. In fact, since there is only one irreducible complex two-dimensional representation for $\mathfrak{so}(3)$ up to isomorphisms, any irreducible two-dimensional representation of $\mathfrak{so}(3,\mathbb{C})$ restricted to $\mathfrak{so}(3)$ is the spinor representation. Explicitly, $\mathfrak{so}(3,\mathbb{C})$ has a basis $\{T^i, i = 1, 2, 3\}$, such that

$$[T^i, T^j] = \sqrt{-1}\epsilon^{ij}_k T^k.$$

Let $\{G^{a+}, a = 1, 2\}$ be a basis of the spinor representation of $\mathfrak{so}(3)$, such that

$$T^i(G^{a+}) = \frac{1}{2}\overline{\tau^i_{ab}}G^{b+},$$

where τ^i's are the Pauli matrices. Denote a (suitable) generator of $\mathbb{C}$ by H, we then get the following set of commutation relations for $\mathfrak{spin}_c(3)$ SUSY algebra:

$$\begin{aligned}
&[H, G^{a\pm}] = 0,\ a = 1, 2,\\
&[H, T^i] = 0,\ i = 1, 2, 3,\\
&[G^{a\pm}, G^{b\pm}] = 0,\ a, b = 1, 2,\\
&[G^{a-}, G^{b+}] = \delta^{ab}H,\ a, b = 1, 2,\\
&T^i(G^{a+}) = \frac{1}{2}\overline{\tau^i_{ab}}G^{b+},\ i = 1, 2, 3,\ a = 1, 2,\\
&T^i(G^{a-}) = \frac{1}{2}\tau^i_{ab}G^{b-},\ i = 1, 2, 3,\ a = 1, 2,\\
&[T^i, T^j] = \sqrt{-1}\epsilon^{ij}_k T^k,\ i, j = 1, 2, 3.
\end{aligned}$$

Let (M, g, ω) be a Kähler manifold. As usual, denote by L the operator on exterior forms given by multiplication by the Kähler form ω, Λ its adjoint, and $h = [\Lambda, L]$. Then the identities used in the proof of the Hard Lefschetz Theorem (see e.g. Griffiths-Harris [4], p.p. 111 - 121) are

$$\begin{array}{lll}
[\Lambda, L] = h, & [h, L] = -2L, & [h, \Lambda] = 2\Lambda,\\
[L, \partial] = 0, & [\Lambda, \partial] = \sqrt{-1}\bar\partial^*, & [h, \partial] = -\partial,\\
[L, \sqrt{-1}\bar\partial^*] = \partial, & [\Lambda, \sqrt{-1}\bar\partial^*] = 0, & [h, \sqrt{-1}\bar\partial^*] = \sqrt{-1}\bar\partial^*,\\
[L, \bar\partial] = 0, & [\Lambda, \bar\partial] = -\sqrt{-1}\partial^*, & [h, \bar\partial] = -\bar\partial,\\
[L, \sqrt{-1}\partial^*] = \bar\partial, & [\Lambda, \sqrt{-1}\partial^*] = 0, & [h, \sqrt{-1}\partial^*] = \sqrt{-1}\partial^*,\\
[\partial, \partial] = 0, & [\bar\partial, \bar\partial] = 0, & [\partial, \bar\partial] = 0,\\
[\partial^*, \partial^*] = 0, & [\bar\partial^*, \bar\partial^*] = 0, & [\partial^*, \bar\partial^*] = 0,\\
[\partial, \bar\partial^*] = 0, & [\partial^*, \bar\partial] = 0, & \\
[\partial, \partial^*] = \frac{1}{2}\delta, & [\bar\partial, \bar\partial^*] = \frac{1}{2}\Box, &
\end{array}$$

These identities reveal that the involved operators generate the $\mathfrak{spin}_c(3)$ SUSY algebra. Indeed, L, Λ, h generate the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. They correspond to the basis $\{T^i\}$ after a suitable basis change. ∂ and $\bar\partial^*$ span a spinor representation and they correspond to G^{1+} and G^{2+} respectively, while $\bar\partial$ and ∂^* span the dual representation and correspond to G^{1-} and G^{2-} respectively. Finally, the Laplace operator $\Box$ corresponds to the Hamiltonian H.

2.3. $Spin_c(5)$ **SUSY algebra and hyperkähler geometry.** There is an isomorphism $\mathfrak{so}(5) \cong \mathfrak{sp}(2)$. Therefore, the spinor representation of $\mathfrak{so}(5, \mathbb{C})$ is a complex 4-dimensional vector space with an invariant complex symplectic form.

A Riemannian manifold (M, g) is called hyperkähler if there are three complex structures J_1, J_2, J_3, such that $J_1J_2 = -J_2J_1 = J_3$, and each of them makes (M, g) a Kähler manifold. For $j = 1, 2, 3$, let ω_j be the Kähler form for J_j. Let L_j be the multiplication by ω_j, and Λ_j its adjoint. Let $H : \Lambda^*(M) \to \Lambda^*(M)$ be defined as above by $H(\alpha) = (2m - p)\alpha$, $\alpha \in \Omega^p(M)$. For $i \neq j$, set $K_{ij} = [L_i, \Lambda_j]$. The meaning of operators K_{jk} is as follows. Let $\alpha \in \Omega^{p,q}(M)$ for J_1 then $K_{32}(\alpha) = \sqrt{-1}(q - p)\alpha$. Similarly for K_{12} and K_{31}. It is straightforward to check that [9]:

$$[L_i, L_j] = [\Lambda_i, \Lambda_j] = 0, [L_i, \Lambda_i] = -H; \tag{2}$$

$$[H, L_i] = -2L_i, [H, \Lambda_j] = 2\Lambda_j, \tag{3}$$

$$K_{ij} = -K_{ji}, [K_{ij}, K_{jk}] = 2K_{ik}, [H_{ij}, K] = 0, \tag{4}$$

$$[K_{ij}, L_j] = 2L_i, [K_{ij}, \Lambda_j] = 2\Lambda_j, \tag{5}$$

$$[K_{ij}, L_k] = [K_{ij}, \Lambda_k] = 0,\ k \neq i, j. \tag{6}$$

As shown by Verbitskii [9], these are the translation relations of the Lie algebra $\mathfrak{so}(4, 1)$. One can also consider the commutation relations of the above algebraic operator with the natural differential differential operators $\partial_j = \frac{1}{2}(d - \sqrt{-1}J_jd)$ and $\bar{\partial}_j$. Fröchlich *et al* [3] showed that these differential operators, together with the above $\mathfrak{so}(4, 1)$ Lie algebra, generate the $Spin_c(5)$ SUSY algebra.

3. Supersymmetries in Calabi-Yau geometry. For a Calabi-Yau n-manifold M, besides those operators on Dolbeault complex $\Omega^{*,*}(M)$, we are also interested in operators on $\Omega^{-*,*}(M) = \oplus_{p,q\geq 0}\Gamma(M, \Lambda^pTM \otimes \Lambda^q\overline{T}^*M)$. There is a canonical way to obtain an operator on $\Omega^{-*,*}(M)$ from an operator on $\Omega^{*,*}(M)$. Indeed, let Ω be a nontrivial holomorphic volume for on M, then Ω defines an isomorphism $\flat : \Omega^{-p,q}(M) \to \Omega^{n-p,q}(M)$. Let $\sharp$ denote the inverse of $\flat$. For any linear operator ϕ on $\Omega^{*,*}(M)$, define $\tilde{\phi}$ on $\Omega^{-*,*}(M)$ by $\tilde{\phi}(\alpha) = \phi(\alpha^\flat)^\sharp$ for $\alpha \in \Omega^{-*,*}(M)$. Such a construction first appeared in Tian [7] and Todorov [8]. It induces a homomorphism $\mathrm{End}(\Omega^{*,*}(M)) \to \mathrm{End}(\Omega^{-*,*}(M))$, which is a morphism of graded Poisson superalgebras. Under this homomorphism, the $Spin_c(3)$ SUSY algebra formed by $L, \Lambda, H, \partial, \bar{\partial}, \partial^*, \bar{\partial}^*, \Box$ gives rise to a $Spin_c(3)$ SUSY algebra formed by the corresponding operators on $\Omega^{-*,*}(M)$. However, mirror symmetry requires $\Omega^{-p,q}(M)$ to have degree (p, q). In this grading, $\tilde{L}$ has degree $(-1, 1)$. So this $Spin_c(3)$ SUSY algebra on $\Omega^{-*,*}(M)$ does not correspond to the $Spin_c(3)$ SUSY algebra on the Dolbeault complex of its mirror manifold. The right approach is to find an element in $\Omega^{-1,1}(M)$ which will play the role of the Kähler form, in the sense that the exterior multiplication by this element will generate the right $Spin_c(3)$ SUSY algebra together with its adjoint operator and $\tilde{\partial}, \tilde{\bar{\partial}}$, etc. We will give some sufficient conditions for an element in $\Omega^{-1,1}(M)$ to have this property.

3.1. A method of obtaining $Spin_c(3)$ SUSY operator algebra. Let $(\mathcal{A}, \wedge)$ be a graded commutative $\mathbb{C}$-algebra with a Hermitian metric. Suppose that we have two operators $\mathfrak{d}_1$ and $\mathfrak{d}_2$, together with their adjoints $\mathfrak{d}_1^*$ and $\mathfrak{d}_2^*$, such that $(\mathcal{A}, \wedge, \mathfrak{d}_1, \sqrt{-1}\mathfrak{d}_2^*, [\cdot \bullet \cdot]_{\sqrt{-1}\mathfrak{d}_2^*})$ and $(\mathcal{A}, \wedge, \mathfrak{d}_2, -\sqrt{-1}\mathfrak{d}_1^*, [\cdot \bullet \cdot]_{-\sqrt{-1}\mathfrak{d}_1^*})$ are DGBV algebras (for definition, see e.g. Cao-Zhou [2]). This assumption implies many identities, in particular,

$$\begin{aligned} [\mathfrak{d}_j, \mathfrak{d}_j] &= [\mathfrak{d}_j^*, \mathfrak{d}_j^*] = 0, \\ [\mathfrak{d}_1, \mathfrak{d}_2^*] &= [\mathfrak{d}_2, \mathfrak{d}_1^*] = 0. \end{aligned}$$

Denote by $h : \mathcal{A} \to \mathcal{A}$ the operator defined by $h(\alpha) = (n-p)\alpha$ for $\alpha \in \mathcal{A}_p$, and a fixed number n. For $j = 1, 2$, since $\mathfrak{d}_j$ and $\mathfrak{d}_j^*$ have degree 1 and -1 respectively, we have

$$[h, \mathfrak{d}_j] = -\mathfrak{d}_j, [h, \mathfrak{d}_j^*] = \mathfrak{d}_j^*.$$

For an element $\omega \in \mathcal{A}_2$, let $L : \mathcal{A} \to \mathcal{A}$ be the multiplication by ω, and $\Lambda : \mathcal{A} \to \mathcal{A}$ its adjoint. We make the following assumptions:

$$[\Lambda, L] = h, \tag{7}$$

$$\mathfrak{d}_1 \omega = 0, \tag{8}$$

$$[\Lambda, \mathfrak{d}_1] = -\sqrt{-1}\mathfrak{d}_2^*. \tag{9}$$

From (7), we get

$$[h, L] = -2L, \qquad\qquad [h, \Lambda] = 2\Lambda.$$

That is, $\{L, \Lambda, h\}$ span a Lie algebra isomorphic to $\mathfrak{sl}(2)$. Note that (8) is equivalent to $[L, \mathfrak{d}_1] = 0$. From (9), we have

$$\begin{aligned} [L, \mathfrak{d}_2^*] &= \sqrt{-1}[L, [\Lambda, \mathfrak{d}_1]] = \sqrt{-1}[[L, \Lambda], \mathfrak{d}_1] + \sqrt{-1}[\Lambda, [L, \mathfrak{d}_1]] \\ &= -\sqrt{-1}[h, \mathfrak{d}_1] = \sqrt{-1}\mathfrak{d}_1. \end{aligned}$$

We also need $[\Lambda, \mathfrak{d}_2^*] = 0$. By (9), this is equivalent to $[\Lambda, [\Lambda, \sqrt{-1}\mathfrak{d}_1]] = 0$, or by taking the adjoint, to $[L, [L, \sqrt{-1}\mathfrak{d}_1^*]] = 0$. Recall that

$$[\alpha \bullet \beta]_{\sqrt{-1}\mathfrak{d}_1^*} = (-1)^{|\alpha|}\sqrt{-1}(\mathfrak{d}_1^*(\alpha \wedge \beta) - (\mathfrak{d}_1^*\alpha) \wedge \beta - (-1)^{|\alpha|}\alpha \wedge \mathfrak{d}_1^*\beta),$$

and

$$[(\alpha \wedge \beta) \bullet \gamma]_{\sqrt{1}\mathfrak{d}_1^*} = \alpha \wedge [\beta \bullet \gamma]_{\sqrt{1}\mathfrak{d}_1^*} + (-1)^{|\beta|(|\gamma|-1)}[\alpha \bullet \gamma]_{\sqrt{-1}\mathfrak{d}_1^*} \wedge \beta.$$

So we have

$$\begin{aligned} [L, [L, \sqrt{-1}\mathfrak{d}_1^*]]\alpha &= (L^2\sqrt{-1}\mathfrak{d}_1^* - 2L\mathfrak{d}_1^*L + \sqrt{-1}\mathfrak{d}_1^*L^2)\alpha \\ &= \sqrt{-1}\mathfrak{d}_1^*(\omega^2 \wedge \alpha) - 2\omega \wedge \sqrt{-1}\mathfrak{d}_1^*(\omega \wedge \alpha) + \omega^2 \wedge \sqrt{-1}\mathfrak{d}_1^*(\alpha) \\ &= (\sqrt{-1}\mathfrak{d}_1^*(\omega^2 \wedge \alpha) - (\sqrt{-1}\mathfrak{d}_1^*\omega^2) \wedge \alpha - \omega^2 \wedge \sqrt{-1}\mathfrak{d}_1^*\alpha) \\ &\quad -2\omega \wedge (\sqrt{-1}\mathfrak{d}_1^*(\omega \wedge \alpha) - (\sqrt{-1}\mathfrak{d}_1^*\omega) \wedge \alpha - \omega \wedge \sqrt{-1}\mathfrak{d}_1^*\alpha) \\ &\quad +(\sqrt{-1}\mathfrak{d}_1^*\omega^2 - 2\omega \wedge \sqrt{-1}\mathfrak{d}_1^*\omega) \wedge \alpha \\ &= [\omega^2 \bullet \alpha]_{\sqrt{-1}\mathfrak{d}_1^*} - 2\omega \wedge [\omega \bullet \alpha]_{\sqrt{-1}\mathfrak{d}_1^*} + [\omega \bullet \omega]_{\sqrt{-1}\mathfrak{d}_1^*} \wedge \alpha \\ &= [\omega \bullet \omega]_{\sqrt{-1}\mathfrak{d}_1^*} \wedge \alpha. \end{aligned}$$

Therefore, if

$$[\omega \bullet \omega]_{\sqrt{-1}\mathfrak{d}_1^*} = 0, \tag{10}$$

then we have

$$[\Lambda, \mathfrak{d}_2^*] = 0.$$

On the other hand, condition (10) is equivalent to

$$\mathfrak{d}_2\omega = 0. \tag{11}$$

This follows from

$$\begin{aligned}
[\omega \bullet \omega]_{\sqrt{-1}\mathfrak{d}_1^*} &= \sqrt{-1}\mathfrak{d}_1^*\omega^2 - 2\omega \wedge \sqrt{-1}\mathfrak{d}_1^*\omega = [\Lambda, \mathfrak{d}_2]\omega^2 - 2\omega \wedge [\Lambda, \mathfrak{d}_2]\omega \\
&= \Lambda\mathfrak{d}_2\omega^2 - \mathfrak{d}_2\Lambda\omega^2 - 2\omega \wedge \Lambda\mathfrak{d}_2\omega + 2\omega \wedge \mathfrak{d}_2\Lambda\omega \\
&= 2\Lambda(\omega \wedge \mathfrak{d}_2\omega) - (2n-2)\mathfrak{d}_2\omega - 2L\Lambda\mathfrak{d}_2\omega \\
&= 2\Lambda L\mathfrak{d}_2\omega - 2L\Lambda\mathfrak{d}_2\omega - (2n-2)\mathfrak{d}_2\omega \\
&= 2h\mathfrak{d}_2\omega - 2(n-1)\mathfrak{d}_2\omega = -4\mathfrak{d}_2\omega.
\end{aligned}$$

Here we have used the facts $\Lambda\omega = n$, $\Lambda\omega^2 = (2n-2)\omega$, and $\sqrt{-1}\mathfrak{d}_1^* = [\Lambda, \mathfrak{d}_2]$.

Therefore, under conditions (7), (8), (9) and (11), $\{\mathfrak{d}_1, \mathfrak{d}_2^*\}$ span a spinor representation. Taking their adjoints, one obtains the identities for $\{\mathfrak{d}_2, \mathfrak{d}_1^*\}$ to be the dual representation. If in addition

$$[\mathfrak{d}_1, \mathfrak{d}_2] = 0,$$

then we have $[\mathfrak{d}_1^*, \mathfrak{d}_2^*] = 0$. Furthermore,

$$\begin{aligned}
[\mathfrak{d}_2, \mathfrak{d}_2^*] &= \sqrt{-1}[\mathfrak{d}_2, [\Lambda, \mathfrak{d}_1]] = \sqrt{-1}[[\mathfrak{d}_2, \Lambda], \mathfrak{d}_1] + \sqrt{-1}[\Lambda, [\mathfrak{d}_2, \mathfrak{d}_1]] \\
&= [\mathfrak{d}_1^*, \mathfrak{d}_1] = [\mathfrak{d}_1, \mathfrak{d}_1^*].
\end{aligned}$$

To summarize, we have proved the following

Lemma 3.1. *Let $(\mathcal{A}, \wedge)$ be a graded commutative $\mathbb{C}$-algebra with a Hermitian metric and two supercommuting linear operators $\mathfrak{d}_1$ and $\mathfrak{d}_2$ of degree 1, such that $(\mathcal{A}, \wedge, \mathfrak{d}_1, \sqrt{-1}\mathfrak{d}_2^*, [\cdot \bullet \cdot]_{\sqrt{-1}\mathfrak{d}_2^*})$ and $(\mathcal{A}, \wedge, \mathfrak{d}_2, -\sqrt{-1}\mathfrak{d}_1^*, [\cdot \bullet \cdot]_{-\sqrt{-1}\mathfrak{d}_1^*})$ are two DGBV algebras. Assume that $\omega \in \mathcal{A}_2$ is an element such that*

$$[\Lambda, L] = h, \qquad \mathfrak{d}_1\omega = 0, \qquad \mathfrak{d}_2\omega = 0, \qquad \mathfrak{d}_2^* = \sqrt{-1}[\Lambda, \mathfrak{d}_1],$$

where Λ, L and h are defined as above. Then the operators $L, \Lambda, h, \mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_1^*, \mathfrak{d}_2^*, \Box = [\mathfrak{d}_1, \mathfrak{d}_1^*]$ generate the $Spin_c(3)$ SUSY algebra under supercommutators.

Usually we have the freedom in interchanging the roles of $\mathfrak{d}_1$ and $\mathfrak{d}_2$: $\mathfrak{d}_1' = \mathfrak{d}_2$, $\mathfrak{d}_2' = -\mathfrak{d}_1$. The fourth condition in the above lemma is symmetric with this switching of operators: it is equivalent to $\mathfrak{d}_1^* = -\sqrt{-1}[\Lambda, \mathfrak{d}_2]$. A consequence of this condition is $\mathfrak{d}_2\alpha = [\omega \bullet \alpha]_{-\sqrt{-1}\mathfrak{d}_1^*}$ and $\mathfrak{d}_1\alpha = [\omega \bullet \alpha]_{\sqrt{-1}\mathfrak{d}_2^*}$.

3.2. Applications to Calabi-Yau manifolds. On a Calabi-Yau n-manifold M, we can consider

$$\Omega^{-*,*}(M) = \Gamma(M, \Lambda^* TM \otimes \Lambda^* \overline{T}^* M).$$

Besides the operator $\bar{\partial}$ which has degree $(0,1)$, Tian [7] defined an operator Δ of degree $(-1,0)$ as follows (see also Todorov [8]): contraction of poly-vector fields with the holomorphic n-form Ω defines an isomorphism $\flat : \Omega^{-*,*}(M) \to \Omega^{n-*,*}(M)$. Denote by $\sharp$ its inverse. Then $\Delta\alpha = (\partial\alpha^\flat)^\sharp$. As noticed by the second author in [14], Theorem 2.3, $\mathfrak{d}_1 = \bar{\partial}$ and $\mathfrak{d}_2 = \Delta^*$ satisfy the condition of Lemma 3.1. Also Tian-Todorov Lemma implies that $[\cdot \bullet \cdot]_\Delta$ is just the Schouten-Nijenhuis bracket on $\Omega^{-*,*}(M)$. Therefore, we have the following

Theorem 3.1. *Let M be a closed Calabi-Yau n-manifold. Suppose that there is an element $\widetilde{\omega} \in \Omega^{-1,1}(M)$, such that*

$$[\tilde{\Lambda}, \tilde{L}] = \tilde{h}, \qquad \bar{\partial}\widetilde{\omega} = 0, \qquad \Delta^*\widetilde{\omega} = 0, \qquad \Delta = \sqrt{-1}[\tilde{\Lambda}, \bar{\partial}],$$

where $\tilde{L}$ is given by multiplication by $\widetilde{\omega}$, $\tilde{\Lambda}$ its adjoint and $\tilde{h}$ is defined by $\tilde{h}(\alpha) = (n-(p+q))\alpha$ for $\alpha \in \Omega^{-p,q}(M)$. Then the operators $\tilde{L}, \tilde{\Lambda}, \tilde{h}, \bar{\partial}, \Delta, \bar{\partial}^*, \Delta^*, \widetilde{\Box} = [\bar{\partial}, \bar{\partial}^*]$ generate the $Spin_c(3)$ SUSY algebra under supercommutators.

Definition 3.1. A representation of $\mathfrak{sl}(2,\mathbb{R})$ on a bigraded vector space is said to be *of Lefschetz type* if X acts by an operator of bidegree $(-1,-1)$, Y acts by an operator of bidegree $(1,1)$, and H acts by an operator of bidegree $(0,0)$, where X, Y, H are generators of $\mathfrak{sl}(2,\mathbb{R})$ such that

$$[X, Y] = H, \qquad [H, X] = 2X, \qquad [H, Y] = -2Y.$$

Corollary 3.1. *Under the condition of Theorem 3.1, there is an action of $\mathfrak{sl}(2,\mathbb{R})$ on $H^{-*,*}(M)$ of Lefschetz type.*

Remark 3.1. For Calabi-Yau hypersurfaces M in weighted projective $(n+1)$-spaces, Hübsch and Yau [5] have found an action by $SL(2,\mathbb{C})$ on $H^{n-*,*}(M) \cong H^{-*,*}(M)$. They constructed on such manifolds an element ϑ which corresponds to the Kähler class. It is interesting to see whether their construction yields an element $\widetilde{\omega}$ satisfying the conditions in Theorem 3.1.

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