

SPECTRAL INVARIANTS AND THE LENGTH MINIMIZING PROPERTY OF HAMILTONIAN PATHS*

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Abstract. In this paper we provide a criterion for the quasi-autonomous Hamiltonian path (“Hofer’s geodesic”) on arbitrary closed symplectic manifolds (M, ω) to be length minimizing in its homotopy class in terms of the spectral invariants $\rho(G; 1)$ that the author has recently constructed. As an application, we prove that any autonomous Hamiltonian path on *arbitrary* closed symplectic manifolds is length minimizing in *its homotopy class* with fixed ends, as long as it has no contractible periodic orbits of *period one* and it has a maximum and a minimum that are *generically under-twisted*, and all of its critical points are non-degenerate in the Floer theoretic sense.

Key words. Hofer’s norm, Hamiltonian diffeomorphism, autonomous Hamiltonians, chain level Floer theory, spectral invariants, canonical fundamental Floer cycle, tight Floer cycles

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1. Introduction. In [H1], Hofer introduced an invariant pseudo-norm on the group $\mathcal{H}am(M, \omega)$ of compactly supported Hamiltonian diffeomorphisms of the symplectic manifold (M, ω) by putting

$$\|\phi\| = \inf_{H \mapsto \phi} \|H\| \tag{1.1}$$

where $H \mapsto \phi$ means that $\phi = \phi_H^1$ is the time-one map of Hamilton’s equation

$$\dot{x} = X_H(t, x),$$

and $\|H\|$ is the function defined by

$$\|H\| = \int_0^1 \text{osc } H_t dt = \int_0^1 (\max H_t - \min H_t) dt$$

which is the Finsler length of the path $t \mapsto \phi_H^t$. He [H2] also proved that the path of any compactly supported *autonomous* Hamiltonian on \mathbb{C}^n is length minimizing as long as the corresponding Hamilton’s equation has no non-constant periodic orbit *of period less than or equal to one*. This result has been generalized in [En], [MS] and [Oh3] under the additional hypothesis that the linearized flow at each fixed point is not over-twisted i.e., has no closed trajectory of period less than one. The latter hypothesis was shown to be necessary for any length minimizing geodesics with some regularity condition on the Hamiltonian path [U], [LM]. The following result is the main result from [Oh3] restricted to the autonomous Hamiltonians among other results.

THEOREM I [OH3]. *Let (M, ω) be an arbitrary compact symplectic manifold with-
out boundary. Suppose that G is an autonomous Hamiltonian such that*

- (1) *it has no non-constant contractible periodic orbits “of period less than one”,*

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- (2) *it has a maximum and a minimum that are generically under-twisted*
 (3) *all of its critical points are non-degenerate in the Floer theoretic sense (i.e., the linearized flow of X_G at each critical point has only the zero as a periodic orbit). Then the one parameter group ϕ_G^t is length minimizing in its homotopy class with fixed ends for $0 \leq t \leq 1$.*

A similar result with slightly different assumptions and statements was proven by McDuff-Slimowitz [MS] by a different method around the same time.

There is also a result by Entov [En] for the strongly semi-positive case. With some additional restriction on the manifold (M, ω) , we can remove the condition (3) for which we refer to [Oh7].

As remarked in [MS] before, the apparently weaker condition “of period less than one” than “of period less than or equal to one” does not give rise to a stronger result. This is because once we have proven the length minimizing property under the phrase “of period less than or equal to one”, the improvement under the former phrase in Theorem I follows by an approximation argument as in [Lemma 5.1, Oh3].

We call two Hamiltonians G and F equivalent if there exists a smooth family $\{F^s\}_{0 \leq s \leq 1}$ with $F^0 = G$, $F^1 = F$ such that

$$\phi_{F^s}^1 = \phi_G^1$$

for all $s \in [0, 1]$. We denote $G \sim F$ in that case and say that two Hamiltonian paths ϕ_G^t and ϕ_F^t are homotopic to each other with fixed ends, or just homotopic to each other when there is no danger of confusion.

DEFINITION 1.1. A Hamiltonian H is called *quasi-autonomous* if there exists two points $x_-, x_+ \in M$ such that

$$H(x_-, t) = \min_x H(x, t), \quad H(x_+, t) = \max_x H(x, t)$$

for all $t \in [0, 1]$.

We now recall the Ustilovsky-Lalonde-McDuff’s necessary condition on the stability of geodesics. Ustilovsky [U] and Lalonde-McDuff [LM] proved that for a generic ϕ in the sense that all its fixed points are isolated, any stable geodesic ϕ_t , $0 \leq t \leq 1$ from the identity to ϕ must have at least two fixed points which are under-twisted.

DEFINITION 1.2. Let $H : M \times [0, 1] \rightarrow \mathbb{R}$ be a Hamiltonian which is not necessarily time-periodic and ϕ_H^t be its Hamiltonian flow.

(1) We call a point $p \in M$ a *time T periodic point* if $\phi_H^T(p) = p$. We call $t \in [0, T] \mapsto \phi_H^t(p)$ a *contractible time T -periodic orbit* if it is contractible.

(2) When H has a fixed critical point p over $t \in [0, T]$, we call p *over-twisted* as a time T -periodic orbit if its linearized flow $d\phi_H^t(p); t \in [0, T]$ on $T_p M$ has a closed trajectory of period less than or equal to T . Otherwise we call it *under-twisted*. If in addition the linearized flow has only the origin as the fixed point, then we call the fixed point *generically under-twisted*.

Here we follow the terminology used in [KL] for the “generically under-twisted”. Note that under this definition of the under-twistedness, under-twistedness is C^2 -stable property of the Hamiltonian H .

The following is the main application of the present paper, which improves Theorem I by replacing the phrase “of period less than (or equal to) one” by “of period

one”. This is motivated by a recent result [KL] of Kerman and Lalonde who first studied the length minimizing property of the Hamiltonian paths under the phrase “of period one” instead of “of period less than (or equal to) one” on the symplectically aspherical case, with the same kind of chain level Floer theory as in [Oh3], but specialized to the *symplectically aspherical* case. In this case, the condition (3) is not needed and the phrase “in its homotopy class” can be replaced by “among all paths” as proved in [KL]. This improvement, when combined with our minimality criterion via the spectral invariants and the chain level Floer theory, much clarifies and simplifies the study of minimality question. This has also enabled us to reduce the minimality question to a C^1 -perturbation problem of Hamiltonian functions at least for a *certain class of* symplectic manifolds [Oh7]. We refer readers to [KL] for further explanation on the significance of this improvement.

THEOREM II. *Suppose that G is an autonomous Hamiltonian as in Theorem I except the condition (1) is replaced by*

(1') *it has no non-constant contractible periodic orbits “of period one”*

Then the one parameter group ϕ_G^t is length minimizing in its homotopy class with fixed ends for $0 \leq t \leq 1$.

From now on, we will always assume, unless otherwise said, that the Hamiltonian functions are normalized so that

$$\int_M H_t d\mu = 0.$$

When we use a Hamiltonian which is not normalized, we will explicitly mention it.

Our proof of Theorem II will be again based on the chain level Floer theory from [Oh3,5], but this time incorporating usage of the spectral invariants constructed in [Oh5] a year after the paper [Oh3] appeared. One crucial additional ingredient in this chain level Floer theory that plays an important role in our proof of Theorem II is Kerman-Lalonde’s lemma [Proposition 5.2, KL] (see [KL] or §4 for detailed account of this).

In the present paper, in addition to the proof of Theorem II, using the spectral invariant $\rho(H; 1)$ that was constructed in [Oh5], we provide a much simpler and more elegant scheme than the one used in [Oh3] for the whole study of length minimizing property. In the point of view of this scheme, the phrase “of period one” is more natural than “of period less than equal to one” in the above hypothesis (1) in Theorem I. We note that there has been a general scheme, the so called *energy-capacity inequality*, for the study of length minimizing property used by Lalonde-McDuff [LM]. Our scheme belongs to the category of this general scheme. In this respect, we will state a simple criterion for the length minimizing property of general quasi-autonomous Hamiltonian paths in terms of $\rho(\cdot, 1)$ on arbitrary closed symplectic manifolds. This criterion was implicitly used in [Proposition 5.3, Oh3] without referring to the spectral invariant. A similar criterion was used by Hofer [H2] and Bialy-Polterovich [BP] for the compactly supported Hamiltonians in \mathbb{R}^{2n} . Bialy and Polterovich also predicted existence of similar criterion in general [Remark 1.5, BP]. The present paper confirms their prediction on arbitrary closed symplectic manifolds by using the *selector* $\rho(\cdot; 1)$ in their terminology.

To describe this criterion, we rewrite the Hofer norm into

$$\|H\| = E^-(H) + E^+(H)$$

where E^\pm are the negative and positive parts of the Hofer norms defined by

$$E^-(H) = \int_0^1 -\min H dt$$

$$E^+(H) = \int_0^1 \max H dt.$$

These are called the *negative Hofer-length* and the *positive Hofer-length* of H respectively. We will consider them separately as usual by now. First note

$$E^+(H) = E^-(\overline{H}) \tag{1.3}$$

where \overline{H} is the Hamiltonian generating $(\phi_H^t)^{-1}$ defined by

$$\overline{H}(t, x) = -H(t, \phi_H^t(x)).$$

Therefore we will focus only on the semi-norm E^- .

THEOREM III. *Let $G : [0, 1] \times M \rightarrow \mathbb{R}$ be any quasi-autonomous Hamiltonian that satisfies*

$$\rho(G; 1) = E^-(G) \tag{1.4}$$

Then G is negative Hofer-length minimizing in its homotopy class with fixed ends.

The proof will be based on the general property of $\rho(\cdot; 1)$ that were proved in [Oh5] which we will recall in §3. With this criterion in mind, Theorem II will follow from the homological essentialness of the two critical values of \mathcal{A}_G

$$E^-(G) := \int_0^1 -\min G dt$$

$$E^+(G) := \int_0^1 \max G dt$$

for autonomous Hamiltonian paths of the type as in Theorem II.

THEOREM IV. *Let G be as in Theorem II. Then (1.4) holds, i.e., we have*

$$\rho(G; 1) = E^-(G).$$

In particular the critical value $E^-(G)$ is homologically essential in the Floer theoretic sense. The same holds for \overline{G} .

The proof of this theorem is an adaptation of the proof of Proposition 7.11 (Non-pushing down lemma II) [Oh3] to the current setting. We will clarify the role of spectral invariant $\rho(G; 1)$ here in its proof.

Finally we would like to compare the scheme of [Oh3] and the scheme used in the present paper. Both schemes are based on the mini-max theory via the chain level Floer theory. However while we explicitly use the chain level Floer theory, more specifically use sophisticated moving-around of the Floer semi-infinite cycles via delicate choice of homotopies in [Oh3], these are mostly hidden in the present paper. This is because we have written the paper [Oh5] after [Oh3] which provides construction

of spectral invariants whose general properties already reflect this chain level Floer theory. Furthermore by doing so, we have greatly simplified and clarified the schemes that we use in [Oh3]. One should note that statements of the above theorems do not explicitly involve the Floer theory at all. For example, the Hamiltonian G in Theorem III is not required to be time one-periodic (see the end of §3). But the Floer theory is implicit and subsumed in the definition of the spectral invariant $\rho(\cdot; 1)$ in [Oh5]. This opens up a possibility of suppressing a large part of analytic arguments of the Floer theory in its application to the study of Hofer's geodesics or more generally of the Hamiltonian diffeomorphism group. We refer [Oh6-7] for further applications of spectral invariants to various problems related to the Hamiltonian diffeomorphism group.

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2. Preliminaries. Let $\Omega_0(M)$ be the set of contractible loops and $\tilde{\Omega}_0(M)$ be its standard covering space in the Floer theory. Note that the universal covering space of $\Omega_0(M)$ can be described as the set of equivalence classes of the pair (γ, w) where $\gamma \in \Omega_0(M)$ and w is a map from the unit disc $D = D^2$ to M such that $w|_{\partial D} = \gamma$: the equivalence relation to be used is that $[\bar{w}\#w']$ is zero in $\pi_2(M)$. We say that (γ, w) is Γ -equivalent to (γ, w') iff

$$\omega([w'\#\bar{w}]) = 0 \quad \text{and} \quad c_1([w'\#\bar{w}]) = 0 \tag{2.1}$$

where \bar{w} is the map with opposite orientation on the domain and $w'\#\bar{w}$ is the obvious glued sphere. And c_1 denotes the first Chern class of (M, ω) . We denote by $[\gamma, w]$ the Γ -equivalence class of (γ, w) , by $\tilde{\Omega}_0(M)$ the set of Γ -equivalence classes and by $\pi : \tilde{\Omega}_0(M) \rightarrow \Omega_0(M)$ the canonical projection. We also call $\tilde{\Omega}_0(M)$ the Γ -covering space of $\Omega_0(M)$. The unperturbed action functional $\mathcal{A}_0 : \tilde{\Omega}_0(M) \rightarrow \mathbb{R}$ is defined by

$$\mathcal{A}_0([\gamma, w]) = - \int w^* \omega. \tag{2.2}$$

Two Γ -equivalent pairs (γ, w) and (γ, w') have the same action and so the action is well-defined on $\tilde{\Omega}_0(M)$. When a periodic Hamiltonian $H : M \times (\mathbb{R}/\mathbb{Z}) \rightarrow \mathbb{R}$ is given, we consider the functional $\mathcal{A}_H : \tilde{\Omega}(M) \rightarrow \mathbb{R}$ defined by

$$\mathcal{A}_H([\gamma, w]) = - \int w^* \omega - \int H(\gamma(t), t) dt.$$

We would like to note that *under this convention the maximum and minimum are reversed when we compare the action functional \mathcal{A}_G and the (quasi-autonomous) Hamiltonian G* . One should compare our convention with those used in [Po] or [KL] where they use the action functional defined by

$$\mathcal{A}_H([\gamma, w]) = - \int w^* \omega + \int H(\gamma(t), t) dt.$$

Together with their change of the sign in the definition of the Hamiltonian vector field X_H

$$i_{X_H} \omega = -dH,$$

the difference between the two conventions will be cancelled away if one makes the substitution of the Hamiltonian

$$H \longleftrightarrow \tilde{H} : \quad \tilde{H}(t, x) := -H(1-t, x).$$

We denote by $\text{Per}(H)$ the set of periodic orbits of X_H .

DEFINITION 2.1 [ACTION SPECTRUM]. We define the *action spectrum* of H , denoted as $\text{Spec}(H) \subset \mathbb{R}$, by

$$\text{Spec}(H) := \{\mathcal{A}_H(z, w) \in \mathbb{R} \mid [z, w] \in \tilde{\Omega}_0(M), z \in \text{Per}(H)\},$$

i.e., the set of critical values of $\mathcal{A}_H : \tilde{\Omega}(M) \rightarrow \mathbb{R}$. For each given $z \in \text{Per}(H)$, we denote

$$\text{Spec}(H; z) = \{\mathcal{A}_H(z, w) \in \mathbb{R} \mid (z, w) \in \pi^{-1}(z)\}.$$

Note that $\text{Spec}(H; z)$ is a principal homogeneous space modelled by the period group of (M, ω)

$$\Gamma_\omega = \Gamma(M, \omega) := \{\omega(A) \mid A \in \pi_2(M)\}$$

and

$$\text{Spec}(H) = \cup_{z \in \text{Per}(H)} \text{Spec}(H; z).$$

Recall that Γ_ω is either a discrete or a countable dense subset of \mathbb{R} . It is trivial, i.e., $\Gamma_\omega = \{0\}$ in the weakly exact case. The following lemma was proved in [Oh3].

LEMMA 2.2. [LEMMA 2.2, OH3] *Spec(H) is a measure zero subset of \mathbb{R} .*

For given $\phi \in \mathcal{H}am(M, \omega)$, we denote by $H \mapsto \phi$ if $\phi_H^1 = \phi$, and denote

$$\mathcal{H}(\phi) = \{H \mid H \mapsto \phi\}, \quad \mathcal{H}_m(\phi) = \{H \in \mathcal{H}(\phi) \mid H \text{ mean normalized}\}.$$

We say that two Hamiltonians H and K are equivalent if they are connected by one parameter family of Hamiltonians $\{F^s\}_{0 \leq s \leq 1}$ such that $F^s \mapsto \phi$ i.e.,

$$\phi_{F^s}^1 = \phi \tag{2.3}$$

for all $s \in [0, 1]$. We denote by $\tilde{\phi} = [\phi, H] = [H]$ the equivalence class of H . Then the universal covering space $\widetilde{\mathcal{H}am}(M, \omega)$ of $\mathcal{H}am(M, \omega)$ is realized by the set of such equivalence classes.

Let $F, G \mapsto \phi$ and denote

$$f_t = \phi_F^t, g_t = \phi_G^t, \text{ and } h_t = f_t \circ g_t^{-1}.$$

Note that $h = \{h_t\}$ defines a loop in $\mathcal{H}am(M, \omega)$ based at the identity. Suppose $F \sim G$ so there exists a family $\{F^s\}_{0 \leq s \leq 1} \subset \mathcal{H}(\phi)$ with $F_1 = F$ and $F_0 = G$ that satisfies (2.3). In particular h defines a contractible loop.

The following is proved in [Oh4] (see [Sc] for the symplectically aspherical case where the action functional is single-valued. In this case Schwarz [Sc] proved that the normalization works on $\mathcal{H}am(M, \omega)$ not just on $\widetilde{\mathcal{H}am}(M, \omega)$ as long as $F, G \mapsto \phi$, without assuming $F \sim G$).

PROPOSITION 2.3 [THEOREM I, OH4]. *Let $F, G \in \mathcal{H}_m(\phi)$ with $F \sim G$. Then we have*

$$\text{Spec}(G) = \text{Spec}(F)$$

as a subset of \mathbb{R} .

3. Chain level Floer theory and spectral invariants. In this section, we will briefly recall the basic chain level operators in the Floer theory, and the definition and basic properties of $\rho(\cdot, 1)$ from [Oh5].

For each given generic time-periodic $H : M \times S^1 \rightarrow \mathbb{R}$, we consider the free \mathbb{Q} vector space over

$$\text{Crit}\mathcal{A}_H = \{[z, w] \in \widetilde{\Omega}_0(M) \mid z \in \text{Per}(H)\}. \quad (3.1)$$

To be able to define the Floer boundary operator correctly, we need to complete this vector space downward with respect to the real filtration provided by the action $\mathcal{A}_H([z, w])$ of the element $[z, w]$ of (3.1). More precisely, following [F1], [HS], [Oh3], we introduce

DEFINITION 3.1. (1) We call the formal sum

$$\beta = \sum_{[z, w] \in \text{Crit}\mathcal{A}_H} a_{[z, w]}[z, w], \quad a_{[z, w]} \in \mathbb{Q} \quad (3.2)$$

a *Floer Novikov chain* if there are only finitely many non-zero terms in the expression (3.2) above any given level of the action. We denote by $CF(H)$ the set of Novikov chains. We often simply call them *Floer chains*, especially when we do not need to work on the covering space $\widetilde{\Omega}_0(M)$ as in the weakly exact case.

(2) Two Floer chains α and α' are said to be *homologous* to each other if they satisfy

$$\alpha' = \alpha + \partial\gamma$$

for some Floer chain γ . We call β a *Floer cycle* if $\partial\beta = 0$.

(3) Let β be a Floer chain in $CF(H)$. We define and denote the *level* of the chain β by

$$\lambda_H(\beta) = \max_{[z, w]} \{\mathcal{A}_H([z, w]) \mid a_{[z, w]} \neq 0 \text{ in (3.2)}\} \quad (3.3)$$

if $\beta \neq 0$, and just put $\lambda_H(0) = +\infty$ as usual.

(4) We say that $[z, w]$ is a *generator* of or *contributes* to β and denote

$$[z, w] \in \beta$$

if $a_{[z, w]} \neq 0$.

Let $J = \{J_t\}_{0 \leq t \leq 1}$ be a periodic family of compatible almost complex structures on (M, ω) .

For each given such periodic pair (J, H) , we define the boundary operator

$$\partial : CF(H) \rightarrow CF(H)$$

considering the perturbed Cauchy-Riemann equation

$$\begin{cases} \frac{\partial u}{\partial \tau} + J \left(\frac{\partial u}{\partial t} - X_H(u) \right) = 0 \\ \lim_{\tau \rightarrow -\infty} u(\tau) = z^-, \lim_{\tau \rightarrow \infty} u(\tau) = z^+ \end{cases} \quad (3.4)$$

This equation, when lifted to $\tilde{\Omega}_0(M)$, defines nothing but the *negative* gradient flow of \mathcal{A}_H with respect to the L^2 -metric on $\tilde{\Omega}_0(M)$ induced by the metrics $g_{J_t} := \omega(\cdot, J_t \cdot)$. For each given $[z^-, w^-]$ and $[z^+, w^+]$, we define the moduli space

$$\mathcal{M}_{(J,H)}([z^-, w^-], [z^+, w^+])$$

of solutions u of (3.3) satisfying

$$w^- \# u \sim w^+. \quad (3.5)$$

∂ has degree -1 and satisfies $\partial \circ \partial = 0$.

When we are given a family (j, \mathcal{H}) with $\mathcal{H} = \{H^s\}_{0 \leq s \leq 1}$ and $j = \{J^s\}_{0 \leq s \leq 1}$, the chain homomorphism

$$h_{(j,\mathcal{H})} : CF(H^0) \rightarrow CF(H^1)$$

is defined by the non-autonomous equation

$$\begin{cases} \frac{\partial u}{\partial \tau} + J^{\rho_1(\tau)} \left(\frac{\partial u}{\partial t} - X_{H^{\rho_2(\tau)}}(u) \right) = 0 \\ \lim_{\tau \rightarrow -\infty} u(\tau) = z^-, \lim_{\tau \rightarrow \infty} u(\tau) = z^+ \end{cases} \quad (3.6)$$

where ρ_i , $i = 1, 2$ is functions of the type $\rho : \mathbb{R} \rightarrow [0, 1]$,

$$\rho(\tau) = \begin{cases} 0 & \text{for } \tau \leq -R \\ 1 & \text{for } \tau \geq R \end{cases}$$

$$\rho'(\tau) \geq 0$$

for some $R > 0$. We denote by

$$\mathcal{M}^{(j,\mathcal{H})}([z^-, w^-], [z^+, w^+])$$

or sometimes with j suppressed the set of solutions of (3.6) that satisfy (3.5). The chain map $h_{(j,\mathcal{H})}$ is defined similarly as ∂ using this moduli space instead. $h_{(j,\mathcal{H})}$ has degree 0 and satisfies

$$\partial_{(J^1, H^1)} \circ h_{(j,\mathcal{H})} = h_{(j,\mathcal{H})} \circ \partial_{(J^0, H^0)}.$$

Finally, when we are given a homotopy $(\bar{j}, \bar{\mathcal{H}})$ of homotopies with $\bar{j} = \{j_\kappa\}_{0 \leq \kappa \leq 1}$, $\bar{\mathcal{H}} = \{\mathcal{H}_\kappa\}_{0 \leq \kappa \leq 1}$, consideration of the parameterized version of (3.5) for $0 \leq \kappa \leq 1$ defines the chain homotopy map

$$\tilde{H} : CF(H^0) \rightarrow CF(H^1)$$

which has degree $+1$ and satisfies

$$h_{(j_1, \mathcal{H}_1)} - h_{(j_0, \mathcal{H}_0)} = \partial_{(J^1, H^1)} \circ \tilde{H} + \tilde{H} \circ \partial_{(J^0, H^0)}.$$

Although we will not use this operator explicitly in the present paper, we have recalled them just for completeness' sake.

The following lemma played an important role in controlling the action filtration in the symplectic homology theory of Floer and Hofer [FH]. It has played a particularly fundamental role in [Ch], [Oh1-7] and by now become well-known among the experts and can be proven by a straightforward calculation.

LEMMA 3.2. *Let H, K be any Hamiltonian not necessarily non-degenerate and $j = \{J^s\}_{s \in [0,1]}$ be any given homotopy and $\mathcal{H}^{lin} = \{H^s\}_{0 \leq s \leq 1}$ be the linear homotopy $H^s = (1-s)H + sK$. Suppose that (3.5) has a solution satisfying (3.6). Then we have the identity*

$$\begin{aligned} & \mathcal{A}_F([z^+, w^+]) - \mathcal{A}_H([z^-, w^-]) \\ &= - \int \left| \frac{\partial u}{\partial \tau} \right|_{J^{\rho_1(\tau)}}^2 - \int_{-\infty}^{\infty} \rho'_2(\tau) \left(F(t, u(\tau, t)) - H(t, u(\tau, t)) \right) dt d\tau \end{aligned} \quad (3.7)$$

Now we recall the definition and some basic properties of spectral invariant $\rho(H; a)$ from [Oh5]. We refer readers to [Oh5] for the complete discussion on general properties of $\rho(H; a)$.

DEFINITION & THEOREM 3.3 [OH5]. *For any given quantum cohomology class $0 \neq a \in QH^*(M)$, we have a continuous function denoted by*

$$\rho_a = \rho(\cdot; a) : C_m^0([0, 1] \times M) \rightarrow \mathbb{R}$$

such that for two C^1 functions $H \sim K$ we have

$$\rho(H; a) = \rho(K; a) \quad (3.8)$$

for all $a \in QH^*(M)$. Let $\tilde{\phi}, \tilde{\psi} \in \widetilde{Ham}(M, \omega)$ and $a \neq 0 \in QH^*(M)$. We define the map

$$\rho : \widetilde{Ham}(M, \omega) \times QH^*(M) \rightarrow \mathbb{R}$$

by $\rho(\tilde{\phi}; a) := \rho(H; a)$.

Now we focus on the invariant $\rho(\tilde{\phi}; 1)$ for $1 \in QH^*(M)$. We first recall the following quantities

$$E^-(\tilde{\phi}) = \inf_{[\phi, H] = \tilde{\phi}} E^-(H) \quad (3.9)$$

$$E^+(\tilde{\phi}) = \inf_{[\phi, H] = \tilde{\phi}} E^+(H). \quad (3.10)$$

The quantities

$$\rho^\pm(\phi) := \inf_{\pi(\tilde{\phi}) = \phi} E^\pm(\tilde{\phi})$$

then define pseudo-norms on $\mathcal{H}am(M, \omega)$. It is still an open question whether ρ^\pm are non-degenerate.

PROPOSITION 3.4 [THEOREM II, OH5]. *Let (M, ω) be arbitrary closed symplectic manifold. We have*

$$\rho(\tilde{\phi}; 1) \leq E^-(\tilde{\phi}), \quad \rho(\tilde{\phi}^{-1}; 1) \leq E^+(\tilde{\phi}). \quad (3.11)$$

In particular, we have

$$\rho(H; 1) \leq E^-(H), \quad \rho(\overline{H}; 1) \leq E^+(H) \quad (3.12)$$

for any Hamiltonian H .

For the exact case, the inequality (3.12) had been earlier proven in [Oh1,2] in the context of Lagrangian submanifolds and in [Sc] for the Hamiltonian diffeomorphism.

Now the following theorem (Theorem III) is an immediate consequence of Theorem 3.3 and Proposition 3.4.

THEOREM 3.5. *Let $G : [0, 1] \times M \rightarrow \mathbb{R}$ be a quasi-autonomous Hamiltonian. Suppose that G satisfies*

$$\rho(G; 1) = E^-(G). \quad (3.13)$$

Then G is negative Hofer-length minimizing in its homotopy class with fixed ends.

Proof. Let F be any Hamiltonian with $F \sim G$. Then we have a string of equalities and inequality

$$E^-(G) = \rho(G; 1) = \rho(F; 1) \leq E^-(F)$$

from (3.13), (3.8) for $a = 1$, (3.12) respectively. This finishes the proof. \square

So far in this section, we have presumed that the Hamiltonians are time one-periodic. Now we explain how to dispose the periodicity and extend the definition of $\rho(H; a)$ for arbitrary time dependent Hamiltonians $H : [0, 1] \times M \rightarrow \mathbb{R}$. Note that it is obvious that the semi-norms $E^\pm(H)$ and $\|H\|$ are defined without assuming the periodicity. For this purpose, the following lemma from [Oh3] is important. We leave its proof to readers or to [Oh3].

LEMMA 3.6 [LEMMA 5.2, OH3]. *Let H be a given Hamiltonian $H : [0, 1] \times M \rightarrow \mathbb{R}$ and $\phi = \phi_H^1$ be its time-one map. Then we can re-parameterize ϕ_H^t in time so that the re-parameterized Hamiltonian H' satisfies the following properties:*

- (1) $\phi_{H'}^1 = \phi_H^1$
- (2) $H' \equiv 0$ near $t = 0, 1$ and in particular H' is time periodic
- (3) Both $E^\pm(H' - H)$ can be made as small as we want
- (4) If H is quasi-autonomous, then so is H'
- (5) For the Hamiltonians H', H'' generating any two such re-parameterizations of ϕ_H^t , there is canonical one-one correspondences between $\text{Per}(H')$ and $\text{Per}(H'')$, and $\text{Crit } \mathcal{A}_{H'}$ and $\text{Crit } \mathcal{A}_{H''}$ with their actions fixed.

Furthermore this re-parameterization is canonical with the “smallness” in (3) can be chosen uniformly over H depending only on the C^0 -norm of H .

Using this lemma, we can now define $\rho(H; a)$ for arbitrary H by

$$\rho(H; a) := \rho(H'; a)$$

where H' is the Hamiltonian generating the canonical re-parameterization of ϕ_H^t in time provided in Lemma 3.6. It follows from (3.8) that this definition is well-defined because any such re-parameterizations are homotopic to each other with fixed ends. This being said, we will always assume that our Hamiltonians are time one-periodic without mentioning further in the rest of the paper.

4. Construction of fundamental Floer cycles. In this section and the next, we will prove the following result (Theorem IV). This in particular proves homologically essentialness of the critical value

$$E^-(G) = \int_0^1 -\min G dt \quad (4.1)$$

of \mathcal{A}_G .

Note that the hypotheses on G in Theorem IV already makes it regular in the Floer theory and so we can define the Floer complex of G *without doing any perturbation on it*. The proof will use the chain level Floer theory as in [Oh3].

For the proof of Theorem IV, we need to unravel the definition of $\rho(G; 1)$ from [Oh5] in general for arbitrary Hamiltonians G . First for generic (one periodic) Hamiltonians G , we consider the Floer homology class dual to the quantum cohomology class $1 \in H^*(M) \subset QH^*(M)$, which we denote by 1^b following the notation of [Oh5] and call the *semi-infinite* fundamental class of M . Then according to [Definition 5.2 & Theorem 5.5, Oh5], we have

$$\rho(G; 1) = \inf\{\lambda_G(\gamma) \mid \gamma \in \ker \partial_G \subset CF(G) \text{ with } [\gamma] = 1^b\}. \quad (4.2)$$

Then ρ is extended to arbitrary Hamiltonians by continuity in C^0 -topology. Therefore to prove (4.1), we need to construct cycles γ with $[\gamma] = 1^b$ whose level $\lambda_G(\gamma)$ become arbitrarily close to $E^-(G)$. In fact, this was one of the most crucial observations exploited in [Oh3], without being phrased in terms of the invariant $\rho(G; 1)$ because at the time of writing of [Oh3] construction of spectral invariants in the level of [Oh5] was not carried out yet.

Instead this point was expressed in terms of the existence theorem of certain Floer's continuity equation over the linear homotopy (see [Proposition 5.3, Oh3]). Then the author proved the existence result by proving homological essentialness of the critical value

$$E^-(G) = \int_0^1 -\min G dt.$$

The proof relies on a construction of 'effective' fundamental Floer cycles dual to the quantum cohomology class 1. In [Oh3], for a suitably chosen Morse function f and for sufficiently small ϵ , we transferred the fundamental Morse cycle of ϵf

$$\alpha_{\epsilon f} := \sum_i^\ell a_{[p_i, w_{p_i}]} [p_i, w_{p_i}] \quad (4.3)$$

to a Floer cycle of G over the *adiabatic* homotopy along a piecewise linear path

$$\epsilon f \mapsto \epsilon_0 G^{\epsilon_0} \mapsto G \quad (4.4)$$

where $w_p : D^2 \mapsto M$ denote the constant disc $w_p \equiv p$, and proved the following two facts (see Proposition 7.11 [Oh3]):

(1) the transferred cycle has the level $E^-(G)$ and

(2) the cycle cannot be pushed further down under the Cauchy-Riemann flow under the hypotheses as in Theorem I [Oh3] stated in the introduction, not just for autonomous but for general quasi-autonomous Hamiltonians. Now we are ready to introduce the following fundamental concept of homological essentialness in the chain level theory, which is already implicitly present in the series of Non-pushing down lemmas in [Oh3]. As we pointed out in [Oh3,5], this concept is the heart of the matter in the chain level theory. In the terminology of [Oh5], the level of any *tight* Floer Novikov cycle of G lies in the *essential spectrum* $\text{spec } G \subset \text{Spec } G$ i.e., realizes the value $\rho(G; a)$ for some $a \in QH^*(M; \mathbb{Q})$.

DEFINITION 4.1. We call a Floer cycle $\alpha \in CF(H)$ *tight* if it satisfies the following non-pushing down property under the Cauchy-Riemann flow (3.4): for any Floer cycle $\alpha' \in CF(H)$ homologous to α (in the sense of Definition 3.1 (2)), it satisfies

$$\lambda_H(\alpha') \geq \lambda_H(\alpha). \quad (4.5)$$

Now we will attempt to construct a *tight* fundamental Floer cycle of G whose level is precisely $E^-(G)$. As a first step, we will construct a fundamental cycle of G whose level is $E^-(G)$ but which may not be tight in general. We choose a Morse function f such that f has the unique global minimum point x^- and

$$f(x^-) = 0, \quad f(x^-) < f(x_j) \quad (4.6)$$

for all other critical points x_j . Then we choose a fundamental Morse cycle

$$\alpha = \alpha_{\epsilon f} = [x^-, w_{x^-}] + \sum_j a_j [x_j, w_{x_j}]$$

as in [Oh3] where $x_j \in \text{Crit}_{2n}(-f)$. Recall that the *positive* Morse gradient flow of ϵf corresponds to the *negative* gradient flow of $\mathcal{A}_{\epsilon f}$ in our convention.

Considering Floer's homotopy map $h_{\mathcal{L}}$ over the linear path

$$\mathcal{L} : s \mapsto (1-s)\epsilon f + sH$$

for sufficiently small $\epsilon > 0$, we transfer the above fundamental Morse cycle α and define a fundamental Floer cycle of H by

$$\alpha_H := h_{\mathcal{L}}(\alpha) \in CF(H). \quad (4.7)$$

We call this particular cycle *the canonical fundamental Floer cycle* of H . Recently Kerman and Lalonde [KL] proved the following important property of this fundamental cycle. Partly for the reader's convenience and since [KL] only deals with the aspherical case and our setting is slightly different from [KL], we give a complete proof here adapting that of [Proposition 5.2, KL] to our setting of Floer Novikov cycles.

PROPOSITION 4.2 (COMPARE WITH [PROPOSITION 5.2, KL]). *Suppose that H is a generic one-periodic Hamiltonian such that H_t has the unique non-degenerate global minimum x^- which is fixed and under-twisted for all $t \in [0, 1]$. Suppose that $f : M \rightarrow \mathbb{R}$ is a Morse function such that f has the unique global minimum point x^- and $f(x^-) = 0$. Then the canonical fundamental cycle has the expression*

$$\alpha_H = [x^-, w_{x^-}] + \beta \in CF(H) \quad (4.8)$$

for some Floer Novikov chain $\beta \in CF(H)$ with the inequality

$$\lambda_H(\beta) < \lambda_H([x^-, w_{x^-}]) = \int_0^1 -H(t, x^-) dt. \quad (4.9)$$

In particular its level satisfies

$$\begin{aligned} \lambda_H(\alpha_H) &= \lambda_H([x^-, w_{x^-}]) \\ &= \int_0^1 -H(t, x^-) dt = \int_0^1 -\min H dt. \end{aligned} \quad (4.10)$$

The proof is based on the following simple fact (see the proof of [Proposition 5.2, KL]). Again we would like to call reader's attention on the signs due to the different convention we are using from [KL]. Other than that, we follow the notations from [KL] in this lemma. To make sure that the different conventions used in [KL] and here do not cause any problem, we here provide details of the proof of this lemma.

LEMMA 4.3. *Let H and f as in Proposition 4.4. Then for all sufficiently small $\epsilon > 0$, the function G^H defined by*

$$G^H(t, x) = H(t, x^-) + \epsilon f$$

satisfies

$$\begin{aligned} G^H(t, x^-) &= H(t, x^-) \\ G^H(t, x) &\leq H(t, x) \end{aligned} \tag{4.11}$$

for all (t, x) and equality holds only at x^- .

Proof. Since H_t has the fixed non-degenerate minimum at x^- for all $t \in [0, 1]$, it follows from a parameterized version of the Morse lemma that there exists a local coordinates (U, y_1, \dots, y_{2n}) at x^- such that

$$H(t, x) = H(t, x^-) + \sum_{i,j=1}^{2n} a_{ij}(t, x) y_i y_j$$

with $a_{ij}(t, x^-)$ is a positive definite matrices for each $t \in [0, 1]$ which depend smoothly on t . On the coordinate neighborhood U , we have

$$\begin{aligned} H(t, x) - G^H(t, x) &= H(t, x) - (H(t, x^-) + \epsilon f(x)) \\ &= \sum_{i,j=1}^{2n} a_{ij}(t, x) y_i y_j - \epsilon f(x). \end{aligned} \tag{4.12}$$

Since f has the non-degenerate minimum point at x^- and $f(x^-) = 0$, it follows from (4.12) that for any sufficiently small $\epsilon > 0$, we have

$$H(t, x) - G^H(t, x) \geq 0 \quad \text{for all } (t, x) \in [0, 1] \times U \tag{4.13}$$

and *equality only* at $x = x^-$, if we choose sufficiently small U . On the other hand, since x^- is the *unique* fixed non-degenerate global minimum of H , there exists $\delta > 0$ such

$$H(t, x) - H(t, x^-) \geq \delta$$

for all $(t, x) \in [0, 1] \times (M \setminus U)$. If we choose ϵ so that $\epsilon \max f \leq \frac{1}{2}\delta$, we also have

$$H(t, x) - G^H(t, x) \geq \frac{1}{2}\delta \quad \text{for all } (t, x) \in [0, 1] \times (M \setminus U). \tag{4.14}$$

Combining (4.13) and (4.14), we have finished the proof. \square

Proof of Proposition 4.2. Since x^- is an under-twisted fixed minimum of both H and f , we have the Conley-Zehnder index

$$\mu_H([x^-, w_{x^-}]) = \mu_{\epsilon f}([x^-, w_{x^-}]) (= -n)$$

and so the moduli space $\mathcal{M}^{\mathcal{L}}([x^-, w_{x^-}], [x^-, w_{x^-}])$ has dimension zero. Let $u \in \mathcal{M}^{\mathcal{L}}([x^-, w_{x^-}], [x^-, w_{x^-}])$.

We note that the Floer continuity equation (3.6) for the linear homotopy

$$\mathcal{L} : s \rightarrow (1-s)\epsilon f + sH$$

is unchanged even if we replace the homotopy by the homotopy

$$\mathcal{L}' : s \rightarrow (1-s)G^H + sH.$$

This is because the added term $H(t, x^-)$ in G^H to ϵf does not depend on $x \in M$ and so

$$X_{\epsilon f} \equiv X_{G^H}.$$

Therefore u is also a solution for the continuity equation (3.6) under the linear homotopy \mathcal{L}' . Using this, we derive the identity

$$\begin{aligned} \int \left| \frac{\partial u}{\partial \tau} \right|_{J^{\rho_1(\tau)}}^2 dt d\tau &= \mathcal{A}_{G^H}([x^-, w_{x^-}]) - \mathcal{A}_H([x^-, w_{x^-}]) \\ &\quad - \int_{-\infty}^{\infty} \rho'(\tau) \left(H(t, u(\tau, t)) dt d\tau - G^H(t, u(\tau, t)) \right) dt d\tau \end{aligned} \quad (4.15)$$

from (3.7). Since we have

$$\mathcal{A}_H([x^-, w_{x^-}]) = \mathcal{A}_{G^H}([x^-, w_{x^-}]) = \int_0^1 -\min H dt \quad (4.16)$$

and $G^H \leq H$, the right hand side of (4.15) is non-positive. Therefore we derive that $\mathcal{M}^{\mathcal{L}}([x^-, w_{x^-}], [x^-, w_{x^-}])$ consists only of the constant solution $u \equiv x^-$. This in particular gives rise to the matrix coefficient of $h_{\mathcal{L}}$ satisfying

$$\langle [x^-, w_{x^-}], h_{\mathcal{L}}([x^-, w_{x^-}]) \rangle = \#(\mathcal{M}^{\mathcal{L}}([x^-, w_{x^-}], [x^-, w_{x^-}])) = 1.$$

Now consider any other generator of α_H

$$[z, w] \in \alpha_H \quad \text{with } [z, w] \neq [x^-, w_{x^-}].$$

By the definitions of $h_{\mathcal{L}}$ and α_H , there is a generator $[x, w_x] \in \alpha$ such that

$$\mathcal{M}^{\mathcal{L}}([x, w_x], [z, w]) \neq \emptyset. \quad (4.17)$$

Then for any $u \in \mathcal{M}^{\mathcal{L}}([x, w_x], [z, w])$, we have the identity from (3.7)

$$\begin{aligned} \mathcal{A}_H([z, w]) - \mathcal{A}_{G^H}([x, w_x]) &= - \int \left| \frac{\partial u}{\partial \tau} \right|_{J^{\rho_1(\tau)}}^2 dt d\tau \\ &\quad - \int_{-\infty}^{\infty} \rho'(\tau) \left(H(t, u(\tau, t)) - G^H(t, u(\tau, t)) \right) dt d\tau. \end{aligned}$$

Since $- \int \left| \frac{\partial u}{\partial \tau} \right|_{J^{\rho_1(\tau)}}^2 \leq 0$, and $G^H \leq H$, we have

$$\mathcal{A}_H([z, w]) \leq \mathcal{A}_{G^H}([x, w_x]) \quad (4.18)$$

with equality holding only when u is stationary. There are two cases to consider, one for the case of $x = x^-$ and the other for $x = x_j$ for $x_j \neq x^-$ for $[x_j, w_{x_j}] \in \alpha$.

For the first case, *since we assume* $[z, w] \neq [x^-, w_{x^-}]$, u cannot be constant and so the strict inequality holds in (4.18), i.e.,

$$\mathcal{A}_H([z, w]) < \mathcal{A}_{G^H}([x^-, w_{x^-}]). \quad (4.19)$$

For the second case, we have the inequality

$$\mathcal{A}_H([z, w]) \leq \mathcal{A}_{G^H}([x_j, w_{x_j}]) \quad (4.20)$$

for some $x_j \neq x^-$ with $[x_j, w_{x_j}] \in \alpha$. We note that (4.6) is equivalent to

$$\mathcal{A}_{G^H}([x_j, w_{x_j}]) < \mathcal{A}_{G^H}([x^-, w_{x^-}]).$$

This together with (4.20) again give rise to (4.19). On the other hand we also have

$$\mathcal{A}_{G^H}([x^-, w_{x^-}]) = \mathcal{A}_H([x^-, w_{x^-}])$$

because $G^H(t, x^-) = H(t, x^-)$ from (4.11). Altogether, we have proved

$$\mathcal{A}_H([z, w]) < \mathcal{A}_H([x^-, w_{x^-}]) = \int_0^1 -H(t, x^-) dt$$

for any $[z, w] \in \alpha_H$ with $[z, w] \neq [x^-, w_{x^-}]$. This finishes the proof of (4.9). \square

REMARK 4.4. Note that G^H does not necessarily satisfy the normalization condition. This causes no problem because the proof of Proposition 4.4 does not require normalization condition.

5. The case of autonomous Hamiltonians. In this section, we will restrict to the case of autonomous Hamiltonians G as in Theorem II and prove the following theorem.

THEOREM 5.1. *Suppose that G is autonomous as in Theorem II. Then the canonical fundamental cycle is tight in the sense of Definition 4.3, i.e., α_G satisfies non-pushing down property: for any Floer Novikov cycle $\alpha \in CF(G)$ homologous to α_G , we have*

$$\lambda_G(\alpha) \geq \lambda_G(\alpha_G). \quad (5.1)$$

In particular, we have

$$\rho(G; 1) = \lambda_G(\alpha_G) = \int_0^1 -\min G = E^-(G). \quad (5.2)$$

Proof. The proof is an adaptation of the proof of Proposition 7.11 (Non-pushing down lemma II) [Oh3]. Note that the conditions in Theorem II in particular imply that G is regular in the sense of the Floer theory.

Suppose that α is homologous to α_G , i.e.,

$$\alpha = \alpha_G + \partial_G(\gamma) \quad (5.3)$$

for some Floer Novikov chain $\gamma \in CF(G)$. When G is autonomous and $J \equiv J_0$ is t -independent, there is no non-stationary t -independent trajectory of \mathcal{A}_G landing at $[x^-, w_{x^-}]$ because any such trajectory comes from the negative Morse gradient flow of G but x^- is the minimum point of G . Therefore any non-stationary Floer trajectory u landing at $[x^-, w_{x^-}]$ must be t -dependent. Because of the assumption that G has

no non-constant contractible periodic orbits of period one, any critical points of \mathcal{A}_G has the form

$$[x, w] \quad \text{with } x \in \text{Crit } G.$$

Let u be a trajectory starting at $[x, w]$, $x \in \text{Crit } G$ with

$$\mu([x, w]) - \mu([x^-, w_{x^-}]) = 1, \quad (5.4)$$

and denote by $\mathcal{M}_{(J_0, G)}([x, w], [x^-, w_{x^-}])$ the corresponding Floer moduli space of connecting trajectories. The general index formula shows

$$\mu([x, w]) = \mu([x, w_x]) - 2c_1([w]). \quad (5.5)$$

(See [Appendix, Oh7] for the proof of the index formula in our conventions of the Hamiltonian vector field and the canonical symplectic form on the classical phase space.)

We consider two cases separately: the cases of $c_1([w]) = 0$ or $c_1([w]) \neq 0$. If $c_1([w]) \neq 0$, we derive from (5.4), (5.5) that $x \neq x^-$. This implies that any such trajectory must come with (locally) free S^1 -action, i.e., the moduli space

$$\widehat{\mathcal{M}}_{(J_0, G)}([x, w], [x^-, w_{x^-}]) = \mathcal{M}_{(J_0, G)}([x, w], [x^-, w_{x^-}]) / \mathbb{R}$$

and its stable map compactification have a locally free S^1 -action *without fixed points*. Therefore after a S^1 -invariant perturbation Ξ via considering the quotient Kuranishi structure [FO $\widehat{\mathcal{N}}$] on the quotient space $\widehat{\mathcal{M}}_{(J_0, G)}([x, w], [x^-, w_{x^-}]) / S^1$, the corresponding perturbed moduli space $\widehat{\mathcal{M}}_{(J_0, G)}([x, w], [x^-, w_{x^-}]; \Xi)$ becomes empty for a S^1 -equivariant perturbation Ξ . This is because the quotient Kuranishi structure has the virtual dimension -1 by the assumption (5.4). We refer to [FHS], [FO $\widehat{\mathcal{N}}$] or [LT] for more explanation on this S^1 -invariant regularization process. Now consider the case $c_1([w]) = 0$. First note that (5.4) and (5.5) imply that $x \neq x^-$. On the other hand, if $x \neq x^-$, the same argument as above shows that the perturbed moduli space becomes empty.

It now follows that there is no trajectory of index 1 that land at $[x^-, w_{x^-}]$ after the S^1 -invariant regularization. Therefore $\partial_G(\gamma)$ cannot kill the term $[x^-, w_{x^-}]$ in (5.3) away from the cycle

$$\alpha_G = [x^-, w_{x^-}] + \beta$$

in (4.9), and hence we have

$$\lambda_G(\alpha) \geq \lambda_G([x^-, w_{x^-}]) \quad (5.6)$$

by the definition of the level λ_G . Combining (4.10) and (5.6), we have finished the proof (5.1). \square

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