ASIAN J. MATH.
Vol. 8, No. 4, pp. 639–652, December 2004

EXOTIC STRUCTURES AND THE LIMITATIONS OF CERTAIN ANALYTIC METHODS IN GEOMETRY*

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In this survey we review some results concerning negatively curved exotic structures ($DIFF$ and PL) and its (unexpected) implications on the limitations of some analytic methods in geometry. Among these methods are the harmonic map method and the Ricci flow method.

First in section 1 we mention certain results about the rigidity of negatively curved manifolds. In section 2 and 3 we survey some results concerning the limitations of the harmonic map technique and the natural map technique for negatively curved manifolds. Finally, in section 4, we mention some limitations of the Ricci flow method for pinched negatively curved manifolds.

We are grateful to J-F. Lafont and R. Spatzier for the useful information they provided to us and to E. Gasparim for suggesting some improvements in the text. We are also grateful to the referee for pointing out certain inaccuracies.

1. Negative curvature and rigidity. We begin with a basic question in geometry and topology:

When are two homotopy equivalent manifolds diffeomorphic, PL homeomorphic or homeomorphic?

If both manifolds are closed, hyperbolic and of dimension greater than 2, Mostow's Rigidity Theorem [35] says that they are isometric, in particular diffeomorphic. When both manifolds have strictly negative curvature, results of Eells and Sampson [10], Hartman [22] and Al'ber [1] show that if $f : M_1 \to M_2$ is a homotopy equivalence then it is homotopic to a unique harmonic map (see also next section). Lawson and Yau conjectured that this harmonic map is always a diffeomorphism (see problem 12 of a list of problems presented by Yau in [48]). Farrell and Jones [12] gave counterexamples to this conjecture by proving the following.

THEOREM 1. [12] *If M is a real hyperbolic manifold and Σ is an exotic sphere, then given $\epsilon > 0$, M has a finite covering $\tilde{M}$ such that the connected sum $\tilde{M}\#\Sigma$ is not diffeomorphic to $\tilde{M}$ and admits a Riemannian metric with all sectional curvatures in the interval* $(-1-\epsilon, -1+\epsilon)$.

Since there are no exotic spheres in dimensions < 7 this does not give counterexamples to Lawson-Yau conjecture in dimensions less than 7. (Also note that, for example, there are no exotic 12-dimensional spheres.) Moreover, since the $DIFF$ category is equivalent to the PL category in dimensions less than 7, changing the differentiable structure is equivalent to changing the PL structure. The Theorem above was generalized by Ontaneda in [37] to dimension 6, by changing the PL structure, and this result was extended in [15] to all dimensions greater that five:

*Received February 11, 2004; accepted for publication May 14, 2004.
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THEOREM 2. [15] *For every $n > 5$, there are closed real hyperbolic n-manifolds M such that the following holds. Given $\epsilon > 0$, M has a finite cover $\tilde{M}$ that supports an exotic smoothable PL structure that admits a Riemannian metric with sectional curvatures in the interval* $(-1-\epsilon, -1+\epsilon)$.

This result gives counterexamples to the Lawson-Yau conjecture in all dimensions > 5 since Whitehead showed that a smooth manifold has a unique PL structure. (Recall that two smooth manifolds are PL equivalent if and only if there is a simplicial complex which smoothly triangulates both manifolds.)

The hyperbolic manifolds mentioned in the Theorem above are obtained using methods of Millson and Raghunathan [33], which were based on a earlier work of Millson [32]. These methods provide a large class of examples of hyperbolic manifolds with many non-vanishing cohomology classes.

2. Negative curvature and harmonic maps. Let M and N be two compact Riemannian manifolds. Recall that the energy of a map $f : M \to N$ is defined to be $\frac{1}{2}\int_M |df|^2$. A harmonic map is a map which is a critical point of this functional. It satisfies the equation $\tau(f) = 0$, where $\tau(f)$ is the tension field of f, (see for example [9], p.14.)

Part of the interest in harmonic maps comes from the fact that they are very successful in proving rigidity (and superrigidity) results for non-positively curved Riemannian manifolds. We can mention for example results of Siu [40], Sampson [39], Hernández [24], Corlette [6], Gromov and Schoen [20], Jost and Yau [27], and Mok, Sui and Yeung [34]. All of which are based on the pioneering existence Theorem of Eells and Sampson [10] and the uniqueness Theorem of Hartman [22] and Al'ber [1]. Eells and Sampson proved that given any smooth map $k_0 : M \to N$ between Riemannian manifolds, the heat flow equation, that is, the PDE initial value problem

$$\frac{\partial k_t}{\partial t} = \tau(k_t), \quad k_t\big|_{t=0} = k_0 \tag{1}$$

has a unique solution k_t (for all $t \geq 0$) and that $\lim_{t\to\infty} k_t = k$; cf.[9], pp. 22-24. Here N is assumed to have non-positive curvature. Note that k_t is a homotopy between k_0 and k and that k is a harmonic map. Also, if in addition, M has negative curvature the results of Eells and Sampson together with the results of Hartman [22] and Al'ber [1] show that there is a unique harmonic map homotopic to k_0.

2.1. Lawson-Yau conjecture. Let $f : M \to N$ be a homotopy equivalence between negatively curved manifolds. As already mentioned in section 1, Lawson and Yau conjectured that the unique harmonic map $\phi : M \to N$ homotopic to f is a diffeomorphism. Theorems 1 and 2 proved that this conjecture is false in dimensions > 5. That is, for every dimension > 5 there are harmonic homotopy equivalences $f : M \to N$ which are not diffeomorphisms. Theorems 1 and 2 already place some limitations to the harmonic maps technique. But there remained the question whether a "topological" Lawson-Yau conjecture could hold:

(*) *Let $\phi : M \to N$ be a harmonic homotopy equivalence between closed negatively curved manifolds. Is ϕ a homeomorphism?.*

A positive answer to this conjecture would give an analytic proof of "Borel's Conjecture" for closed negatively curved manifolds:

BOREL'S CONJECTURE. *Let M and N be homotopy equivalent closed aspherical manifolds. Then M and N are homeomorphic.*

Borel's conjecture has been verified in [13] when one of the manifolds is non-positively curved and dimensions $\neq$ 3,4. The proof uses sophisticated topological methods. On the other hand, a negative answer to **(*)**, would imply that this last result (the proof of Borel's conjecture for closed non-positively curved manifolds in [13]) cannot be obtained, at least directly, using the harmonic maps technique.

REMARK. Conjecture **(*)** was studied in [14] and some partial (negative) results were given.

But the topological Lawson-Yau conjecture **(*)** is also false:

THEOREM 3. [16] *In every dimension $n \geq 6$, there is a pair of closed negatively curved manifolds M^n and N^n and a harmonic homotopy equivalence $\phi : M^n \to N^n$, which is not one-to-one.*

Actually, we can prove a little more:

THEOREM 4. [16] *In every dimension $n \geq 6$, there is a pair of closed negatively curved manifolds M^n and N^n such that the following holds. For any homotopy equivalence $f : M^n \to N^n$, the unique harmonic map $\phi : M^n \to N^n$ homotopic to f is not one-to-one.*

This Theorem can be directly deduced from Theorem 2 and the $C^\infty-$ Hauptvermutung of Scharlemann and Siebenmann [41]. We reproduce this short deduction here since it shows, unexpectedly, how the theory of PL manifolds interweaves with the theory of harmonic maps.

Proof. By Theorem 2 we have

(2.1.1) *In every dimension $n \geq 6$, there is a pair of non-PL-equivalent closed negatively curved manifolds M^n and N^n with $\pi_1(M^n)$ isomorphic to $\pi_1(N^n)$.*

Let M^n and N^n be a pair of manifolds satisfying **(2.1.1)**, and let $\phi : M^n \to N^n$ be the unique harmonic map realizing the isomorphism $\pi_1(M^n) \to \pi_1(N^n)$ induced by the homotopy equivalence f. The Theorem now follows by just applying the following result of M. Scharlemann and L. Siebenmann [41]

(2.1.2) *Smoothly homeomorphic closed manifolds of dimension $\geq$ 6 are PL-homeomorphic.*

REMARK. Smooth homeomorphisms are not necessarily diffeomorphisms. A simple example is given by the smooth homeomorphism $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$.

Thus, the harmonic map ϕ cannot be a homeomorphism because M^n and N^n satisfy **(2.1.1)**. This proves the Theorem.

Recall that Poincaré Conjecture in low dimensional topology asserts that the only simply connected closed 3-dimensional manifold is the 3-sphere (up to homeomorphism), or, equivalently, that every homotopy 3-sphere is homeomorphic to S^3. Now, by using another result of M. Scharlemann (and assuming Poincare's conjecture) we can get a little more:

THEOREM 5. [16] *Assume that every homotopy 3-sphere is homeomorphic to S^3. Then in every dimension $n \geq 6$, there is a pair of closed negatively curved manifolds*

M^n and N^n and a harmonic homotopy equivalence $\phi : M^n \to N^n$, which is not cellular

See [7] for a discussion of cellular maps (which are called cell like maps in that article). Siebenmann [45] showed that a continuous map $f : X \to Y$ between a pair of closed manifolds of dimension ≥ 5 is cellular if and only if it is the limit of homeomorphisms.

As with Theorem 3, Theorem 5 is a direct consequence of little more general one:

Theorem 6. [16] *Assume that every homotopy 3-sphere is homeomorphic to S^3. Then in every dimension $n \geq 6$, there is a pair of closed negatively curved manifolds M^n and N^n such that the following holds. For any homotopy equivalence $f : M^n \to N^n$, the unique harmonic map $\phi : M^n \to N^n$ homotopic to f is not cellular, i.e. it is not the uniform limit of homeomorphisms.*

Consequently the maps k_t and l_t in the heat flow of $f = k_0$ to $k = k_\infty$ and of $g = l_0$ to $l = l_\infty$ are not one-to-one for all t sufficiently large. Here g is a homotopy inverse to f.

The proof is the same as the one in Theorem 4, but now we use (2.1.2) together with (see [42]):

Assume that every homotopy 3-sphere is homeomorphic to S^3. Then any smooth cellular map $\phi : M^n \to N^n$ of smooth closed n-manifolds (where $n \geq 6$) is smoothly homotopic, through cellular maps, to a smooth homeomorphism.

Remark. In all the Theorems of this subsection we can assume that one of the manifolds is hyperbolic. This follows from Theorem 2.

2.2. Yau's problem 111. Let $f : M_1 \to M_2$ be a homotopy equivalence between negatively curved manifolds and let $h : M_1 \to M_2$ be the unique harmonic map homotopic to f. In the examples provided by the Theorems above, the main obstruction to h being a diffeomorphism or a homeomorphism is that M_1 and M_2 are not PL equivalent, even though they are homotopy equivalent (in fact homeomorphic). We may ask then what happens if this obstruction vanishes, that is, if M_1 and M_2 are diffeomorphic. Can the harmonic map technique be applied in this context to obtain diffeomorphisms or, at least, homeomorphisms? Or, equivalently, if we flow a diffeomorphism (using the heat flow), will the limit be also a diffeomorphism or a homeomorphism? This is considered in Problem 111 of the list compiled by S.-T. Yau in [48]. Here is a restatement of this problem.

Problem 111 of [48]. Let $f : M_1 \to M_2$ be a diffeomorphism between two compact manifolds with negative curvature. If $h : M_1 \to M_2$ is the unique harmonic map which is homotopic to f, is h a homeomorphism?, or equivalently, is h one-to-one?

(This problem had been reposed in [46] as Grand Challenge Problem 3.6.) The answer to the problem was proved to be yes when $\dim M_1 = 2$ by Schoen-Yau [43] and Sampson [38]. But it was proved by Farrell, Ontaneda and Raghunathan [17] that the answer to this question is negative.

Theorem 7. [17] *For every integer $n \geq 6$, there is a diffeomorphism $f : M_1 \to M_2$ between a pair of closed negatively curved n-dimensional Riemannian manifolds such that the unique harmonic map $h : M_1 \to M_2$ homotopic to f is not one-to-one.*

ADDENDUM. *In the Main Theorem, either M_1 or M_2 can be chosen to be a real hyperbolic manifold and the other chosen to have its sectional curvatures pinched within ε of -1; where ε is any preassigned positive number.*

Hence the negative answer given by this Theorem to Problem 111 places more limits to the applicability of the harmonic map technique to rigidity questions.

Theorem 7 evolves from Theorems 1-6 above and follows from Theorem 8 below.

THEOREM 8. [17] *Given an integer $n \geq 6$ and a positive real number ε, there exists a n-dimensional closed connected orientable (real) hyperbolic manifold M and a homeomorphism $g : \mathcal{M} \to M$ with the following properties:*

1. *$\mathcal{M}$ is a negatively curved Riemannian manifold whose sectional curvatures are all in the interval $(-1-\varepsilon, -1+\varepsilon)$.*
2. *M and $\mathcal{M}$ are not PL homeomorphic.*
3. *There is a connected 2-sheeted covering space $\tilde{M} \to M$ such that $\tilde{g} : \tilde{\mathcal{M}} \to \tilde{M}$ is homotopic to a diffeomorphism.*

REMARK. In property 3, $\tilde{\mathcal{M}} \to \mathcal{M}$ denotes the pullback of the covering space $\tilde{M} \to M$ via g, and $\tilde{g}$ is the induced homeomorphism making the diagram

$$\begin{array}{ccc} \tilde{\mathcal{M}} & \xrightarrow{\tilde{g}} & \tilde{M} \\ \downarrow & & \downarrow \\ \mathcal{M} & \xrightarrow{g} & M \end{array}$$

into a Cartesian square. Also, $\tilde{M}$ and $\tilde{\mathcal{M}}$ are given the differential structure and Riemannian metric induced by $\tilde{M} \to M$ and $\tilde{\mathcal{M}} \to \mathcal{M}$, respectively.

The key ingredient in the proof of Theorem 8 is the existence of closed real hyperbolic manifolds with interesting cup product properties. Such manifolds are constructed in section 2 of [17].

Proof of Theorem 7 assuming Theorem 8. Let $g : \mathcal{M} \to M$ be the homeomorphism given by Theorem 8 relative to n and ε. Set $M_1 = \tilde{\mathcal{M}}, M_2 = \tilde{M}$ and let $f : M_1 \to M_2$ be a diffeomorphism homotopic to $\tilde{g} : \tilde{\mathcal{M}} \to \tilde{M}$ which exists by property 3 of Theorem 8. Let $k : \mathcal{M} \to M$ be the unique harmonic map homotopic to g given by the fundamental existence result of Eells and Sampson [10] and uniqueness by Hartmann [22] and Al'ber [1]. Lifting this homotopy to the covering spaces $\tilde{\mathcal{M}}, \tilde{M}$ gives a smooth map

$$\tilde{k} : \tilde{\mathcal{M}} \to \tilde{M}$$

covering k and homotopic to $\tilde{g}$. Note that $\tilde{k}$ is also a harmonic map as is easily deduced from [8], 2.20 and 2.32. Consequently, $\tilde{k}$ is the harmonic map $h : M_1 \to M_2$ mentioned in the statement of Theorem 7. Also note that if $\tilde{k}$ is univalent, then so is k. Hence it suffices to show that k is *not* univalent. Since k is smooth, k univalent would mean that

$$k : \mathcal{M} \to M$$

is a C^∞-homeomorphism and hence M and $\mathcal{M}$ are PL-homeomorphic by the C^∞-Hauptvermutung proved by Scharlemann and Siebenmann [41]. And this would contradict property 2 of Theorem 8; consequently, k and hence also h are *not* univalent. This proves the Theorem 7 and the part of the Addendum where M_2 is real hyperbolic.

To do the case where M_1 is real hyperbolic; set $M_1 = \tilde{M}, M_2 = \tilde{\mathcal{M}}$ and let f be a diffeomorphism homotopic to $\tilde{g}^{-1}$. The rest of the argument is a before. This concludes the deduction of Theorem 7 from Theorem 8.

Note that, as in Theorems 3-6, crucial use is made here of the Scharlemann-Siebenmann C^∞-Hauptvermutung [41].

Hence the idea of the proof of Theorem 7 can be paraphrased in the following few words. Take a homotopy equivalence $f : M_1 \to M_2$ between homeomorphic negatively curved manifolds, with M_1 not PL-homeomorphic to M_2. Theorem 2 grants the existence of such objects in every dimension > 5. Then, as shown in Theorem 4, the unique harmonic map $h : M_1 \to M_2$ homotopic to f cannot be one-to-one. Suppose that after taking some finite cover $\tilde{f} : \tilde{M}_1 \to \tilde{M}_2$ $\tilde{f}$ becomes homotopic to a diffeomorphism. Let k be the unique harmonic map homotopic to $\tilde{f}$. Then k is homotopic to a diffeomorphism but k is not one-to-one since (even though the PL obstruction now vanishes) the damage is already done: $k = \tilde{h}$. The existence of manifolds admitting such finite covers (in fact double covers) is granted by Theorem 8.

2.3. Cellular harmonic maps. Since a harmonic map (between closed negatively curved manifolds) homotopic to a diffeomorphism is not necessarily a homeomorphism we can ask a deeper question: suppose now that the harmonic map can be approximated by homeomorphisms (or even diffeomorphisms), that is, the harmonic map is cellular. Does this imply that the harmonic map is a diffeomorphism? The following Theorem shows that the answer to this question is also negative, showing even more limitations to the harmonic map technique:

THEOREM 9. [18] *For every integer $m > 10$, there is a harmonic cellular map $h : M_1 \to M_2$, between a pair of closed negatively curved m-dimensional Riemannian manifolds, which is not a diffeomorphism.*

ADDENDUM. *The map h in Theorem 9 can be approximated by diffeomorphisms. Also, either M_1 or M_2 can be chosen to be a real hyperbolic manifold and the other chosen to have its sectional curvatures pinched within ϵ of -1; where ϵ is any preassigned positive number.*

We conjecture that this can be improved to all dimensions ≥ 6. We do not know whether the harmonic map h in the statement of Theorem 9 can ever be a homeomorphism.

Theorem 9 follows from the next Theorem, which is of independent interest:

THEOREM 10. [18] *For every integer $m > 10$, and $\epsilon > 0$, there are an m-dimensional closed orientable smooth manifold $\mathcal{M}$, and a C^∞ family of Riemannian metrics μ_s, on $\mathcal{M}$, $s \in [0,1]$, such that:*

(i) *μ_1 is hyperbolic.*
(ii) *The sectional curvatures of μ_s, $s \in [0,1]$, are all in interval $(-1-\epsilon, -1+\epsilon)$.*
(iii) *The maps k and l are both not univalent (i.e. not one-to-one) where $k : (\mathcal{M}, \mu_0) \to (\mathcal{M}, \mu_1)$ and $l : (\mathcal{M}, \mu_1) \to (\mathcal{M}, \mu_0)$ are the unique harmonic maps homotopic to $\mathrm{id}_{\mathcal{M}}$.*

The derivation of Theorem 9 from Theorem 10 uses the continuous dependence (in the C^∞-topology) of the harmonic map homotopic to a homotopy equivalence $f : (M, \mu_M) \to (N, \mu_N)$ on the negatively curved Riemannian metrics μ_M and μ_N.

This dependence was proved by Sampson [38], Schoen and Yau [43], and Eells and Lemaire [9]. To derive Theorem 9 from Theorem 10 let $k_t : (\mathcal{M}, \mu_0) \to (\mathcal{M}, \mu_t)$ be the unique harmonic map homotopic to id. Then $k_0 =$id and $k_1 = k$, which is not one-to-one. Since the space of diffeomorphisms is open in the C^k topology ($k \geq 1$) it follows that there is a minimal $t_0 > 0$ such that k_{t_0} is not a diffeomorphism and k_{t_0} can be approximated by the diffeomorphisms k_t, $t < t_0$.

Likewise, as with Theorems 3 and 4, Scharlemann's result [42] also implies another curious relationship between Poincaré Conjecture in low dimensional topology and the existence of a certain type of harmonic map $k : M \to N$ between high dimensional (i.e. dim $M > 10$) closed negatively curved Riemannian manifolds. If Poincaré Conjecture holds, then there exists a harmonic map k which is homotopic to a diffeomorphism but cannot be approximated by homeomorphisms; i.e. is not a cellular map. Explicitly, we have the following addendum to Theorem 10:

Addendum to Theorem 10. *Assuming that the Poincaré Conjecture is true, then the harmonic maps k and l (of Theorem 10) are not cellular. And consequently the maps k_t and l_t in the heat flow of* id $= k_0$ *to* $k = k_\infty$ *and of* id $= l_0$ *to* $l = l_\infty$ *are not univalent for all t sufficiently large.*

The key to the proof of Theorem 10 is the following important result, which is also used in the proofs of the results of section 4 that show some limitations of the Ricci flow method:

Theorem 11 [18]. *Given an integer $m > 10$ and a positive number ϵ, there exist a m-dimensional closed orientable real hyperbolic manifold M and a smooth manifold $\mathcal{M}$ with the following properties:*

(i) *M is homeomorphic to $\mathcal{M}$.*
(ii) *M is not PL homeomorphic to $\mathcal{M}$.*
(iii) *$\mathcal{M}$ admits a Riemannian metric μ, whose sectional curvatures are all in the interval $(-1-\epsilon, -1+\epsilon)$.*
(iv) *There is a finite sheeted cover $p : \bar{\mathcal{M}} \to \mathcal{M}$ and a one-parameter C^∞ family of Riemannian metrics μ_s, on $\bar{\mathcal{M}}$, $s \in [0,1]$, such that $\mu_0 = p^*\mu$ and μ_1 is hyperbolic. The sectional curvatures of μ_s, $s \in [0,1]$, are all in the interval $(-1-\epsilon, -1+\epsilon)$.*

The proof of Theorem 10 assuming Theorem 11 resembles the proof of Theorem 7 (assuming Theorem 8) given before. Again, crucial use is made of the C^∞-Hauptvermutung of Scharlemann-Siebenmann [41].

We outline the proof of Theorem 11. By Theorem 8 there is a pair of homeomorphic but not PL homeomorphic closed negatively curved Riemannian manifolds M and $\mathcal{M}$ satisfying:

1. M is real hyperbolic.
2. $\mathcal{M}$ has a 2-sheeted cover $q : \hat{\mathcal{M}} \to \mathcal{M}$ where $\hat{\mathcal{M}}$ admits a real hyperbolic metric ν.

Let μ be a given negatively curved Riemannian metric on $\mathcal{M}$ and $q^*(\mu)$ be the induced Riemannian metric on $\hat{\mathcal{M}}$. We would like to find a 1-parameter family of negatively curved Riemannian metrics connecting $q^*(\mu)$ to ν. But we don't know how to do this. In fact this is in general an open problem [5, Question 7.1]. However by passing to a large finite sheeted cover $r : \bar{\mathcal{M}} \to \hat{\mathcal{M}}$, we are able to connect $(q \circ r)^*(\mu)$ to the real hyperbolic metric $r^*(\nu)$ by a 1-parameter family of negatively curved Riemannian metrics; this is essentially the content of Theorem 11 in which $p = q \circ r$. To accomplish this, several results about smooth pseudo-isotopies are used; in particular, the main result of [11] concerning the space of stable topological pseudo-isotopies of real hyperbolic manifolds together with the comparison between the spaces of stable smooth and stable topological pseudo-isotopies contained in [4] and [23]. And finally we need Igusa's fundamental result [26] comparing the spaces of pseudo-isotopies and stable pseudo-isotopies. We need that dim $M > 10$ in order to invoke Igusa's result.

3. Natural maps and negative curvature. It was pointed out to us by M. Varisco [47] that the limitations of the harmonic map technique obtained by the results of section 2 can also be applied to the *natural maps* defined by G. Besson, G. Courtois and S. Gallot [3].

Given a homotopy equivalence $f : M \to N$ between closed negatively curved manifolds G. Besson, G. Courtois and S. Gallot [3] defined the *natural map* $f^* : M \to N$ associated to f. The map f^* has many interesting geometric and dynamic properties. Like harmonic maps they are also useful for proving rigidity results. For example Mostow's Rigidity Theorem for hyperbolic manifolds [35] follows from the following Theorem:

THEOREM. [3] *Let $f : M \to N$ be a homotopy equivalence between closed negatively curved locally symmetric spaces of dimension ≥ 3. Then (possibly after rescaling) the natural map $f^* : M \to N$ is an isometry.*

But we are interested in the following properties of natural maps:

1. f^* is at least C^1 ([3], p. 635).
2. f^* is homotopic to f ([3], p.634).
3. If f is homotopic to g then $f^* = g^*$.
4. If $\bar{f} : \bar{M} \to \bar{N}$ is a finite cover of $f : M \to N$ then $(\bar{f})^* = \overline{f^*}$.

Property 3 holds because $\partial\tilde{f} = \partial\tilde{g} : \partial\tilde{M} \to \partial\tilde{N}$ (see [3], pp.633-634). Property 4 follows directly from the definition of natural maps.

It may be argued that natural maps are, in some sense, better than harmonic maps. But, as observed by M. Varisco, property 2 above implies that Theorem 4 (with its addendum) also holds for natural maps:

THEOREM 12. *In every dimension $n \geq 6$, there is a pair of closed negatively curved manifolds M^n and N^n such that the following holds. For any homotopy equivalence $f : M^n \to N^n$, the the natural map $f^* : M^n \to N^n$ is not one-to-one.*

Also, a version of Theorem 6 holds for natural maps. Note also that properties 1,2,3,4 imply a version of Theorem 7 for natural maps:

THEOREM 13. *For every integer $n \geq 6$, there is a diffeomorphism $f : M_1 \to M_2$ between a pair of closed negatively curved n-dimensional Riemannian manifolds such that the natural map $f^* : M_1 \to M_2$ is not one-to-one.*

(It is the first time that the statements of Theorems 12 and 13 appear in print. The proofs are similar to the proofs of Theorems 4 and 6, respectively.)

REMARK. We do not know whether versions of Theorems 9 and 10 hold for natural maps. In particular, we do not know if there are natural maps that can be approximated by diffeomorphisms but are not diffeomorphisms. This is an interesting question. To have versions of Theorems 9 and 10 hold for natural maps we would need to show that f^* depends continuously on the metrics of M and N, where we consider

f^* varying in the C^1 topology and the metrics varying in the C^2 topology. (In fact, in our examples the metrics vary in the C^∞ topology.) One way one may try to verify this continuous dependence would be to use the fact that the natural map is defined implicitly by the equation $G(F(y), y) = 0$ in p.636 of [3] (F is the natural map in this equation). The entropy of one of the metrics and the Busemann functions appear in the definition of the function G. Note that the perturbations of the entropy (with respect to the metric) have some regularity (see [28]). We could not find a reference for the regularity of the perturbations of the Busemann functions (with respect to the metric) but the proof of the smoothness of the Busemann functions (for universal covers of closed smooth negatively curved manifolds) that appears in [44] might be useful.

All this can be generalized. The following definition tries to formalize any process (analytic or otherwise) that assigns to every continuous map between closed negatively curved manifolds a *special* map. For manifolds M, N, we denote the space of continuous maps $M \to N$ by $\mathcal{C}(M, N)$.

Definitions. A *special correspondence* Ψ for closed negatively curved manifolds is just a family of maps $\Psi_{M,N} : \mathcal{C}(M, N) \to \mathcal{C}(M, N)$, for each pair of closed negatively curved manifolds M, N. Note that $\Psi_{M,N}$ depends on the metrics of M and N. For $f : M \to N$, we say that Ψf is the Ψ-special map associated to f. We say that Ψ is C^k if Ψf is C^k, for every f. We say that Ψ is a homotopy special correspondence if Ψf is homotopic to f, for every f, and $\Psi f = \Psi\, g$ for every f homotopic to g. If Ψ is C^1 we say that Ψ is continuous if $\Psi_{M,N} f$ depends continuously on the metrics of M and N, for every pair M, N (here we consider Ψf varying in the C^1 topology and the metrics varying in the C^2 topology).

Ψ is cover-invariant if $\Psi\, \bar{f} = \overline{\Psi f}$ for every finite cover $\bar{f} : \bar{M} \to \bar{N}$ of any $f : M \to N$. Then we have:

a. If Ψ is C^1, then versions of Theorems 3 - 6 hold for Ψ-special maps.
b. If Ψ is a C^1, cover-invariant homotopy special correspondence, then versions of Theorems 3 - 7 hold for Ψ-special maps.
c. If, in addition, Ψ is continuous, then versions of Theorems 3 - 10 hold for Ψ-special maps.

4. Ricci flow and pinched negative curvature. Until now we have dealt with processes that produce some special type of map, e.g harmonic maps or natural maps. Now we discuss some processes that produce a special type of metrics: Einstein metrics, that is, metrics of constant Ricci curvature. As argued in the introduction of Besse's book "Einstein Manifolds" [2], Einstein metrics are ideal in the sense that they are not as general as metrics of constant scalar curvature, and they are not as restrictive as metrics of constant sectional curvatures. Note that every metric of constant sectional curvature is an Einstein metric. In particular every hyperbolic manifold is an Einstein manifold (i.e a Riemannian manifold with a complete Einstein metric). In dimension three "constant Ricci curvature" is equivalent to "constant sectional curvature"; hence every 3-dimensional Einstein manifold is a space-form.

The most well known method for obtaining Einstein metrics is the Ricci flow method introduced by Hamilton in his seminal paper [21]. Starting with an arbitrary smooth Riemannian metric h on a closed smooth n-dimensional manifold M^n, he considered the evolution equation

$$\frac{\partial}{\partial t}\, h = \frac{2}{n}\, r\, h \,-\, Ric$$

where $r = \int R\, d\mu / \int d\mu$ is the average scalar curvature (R is the scalar curvature) and Ric is the Ricci curvature tensor of h. Hamilton then spectacularly illustrated the success of this method by proving, when $n = 3$, that if the initial Riemannian metric has strictly positive Ricci curvature it evolves through time to a positively curved Einstein metric h_∞ on M^3. And, because $n = 3$, (M^3, h_∞) is a spherical space-form; i.e. its universal cover is the round sphere. Following Hamilton's approach G. Huisken [25], C. Margerin [30] and S. Nishikawa [36] proved that, for every n, Riemannian n-manifolds whose sectional curvatures are pinched close to +1 (the pinching constant depending only on the dimension) can be deformed, through the Ricci flow, to a spherical-space form.

Ten years after Hamilton's results appeared, R. Ye [50] studied the Ricci flow when the initial Riemannian metric h is negatively curved and proved that a negatively curved Einstein metric is strongly stable; that is, the Ricci flow starting near such a Riemannian metric h converges (in the C^∞ topology) to a Riemannian metric isometric to h, up to scaling. (We introduce the notation $h \equiv h'$ for two Riemannian metrics that are isometric up to scaling.) In [50] R. Ye also proved that sufficiently pinched to -1 manifolds can be deformed, through the Ricci flow, to hyperbolic manifolds, but the pinching constant in his Theorem depends on other quantities (e.g the diameter or the volume). Ye's paper was motivated by the problem on whether the Ricci flow can be used to deform every sufficiently pinched to -1 Riemannian metric to an Einstein metric (the pinching constant depending only on the dimension). His paper partially implements a scheme proposed by Min-Oo [31].

We say the the Ricci flow for a negatively curved Riemannian metric h *converges smoothly* if the Ricci flow, starting at h, is defined for all t and converges (in the C^∞ topology) to a well defined negatively curved (Einstein) metric. The next Theorem shows the existence of pinched negatively curved metrics for which the Ricci flow does not converge smoothly.

THEOREM 14. [19] *Given $n > 10$ and $\epsilon > 0$ there is a closed smooth n-dimensional manifold N such that*

(i) N admits a hyperbolic metric

(ii) N admits a Riemannian metric h with sectional curvatures in $[-1-\epsilon, -1+\epsilon]$ for which the Ricci flow does not converge smoothly.

The key ingredients in the proof of Theorem 14 is Theorem 11 and the fact that the Ricci flow satisfies the following properties:

1. Hyperbolic metrics are fixed by the Ricci flow.
2. The Ricci flow preserves isometries.
3. The limit of the Ricci flow (in case it exists) is cover invariant.
4. The Ricci flow depends continuously on initial data.

REMARKS. **1.** By "cover invariant" we mean the following: let g be a metric on M and $p : \bar{M} \to M$ a cover. If g_t is the Ricci flow starting at g and converging to g_∞, then the Ricci flow starting at p^*g converges to p^*g_∞.
2. Property 4 does not state that the *limit* of the Ricci flow is continuous on initial data.

As we did in section 3, this can also be generalized. Before we give a definition trying to formalize a general process for obtaining Einstein metrics on ϵ-pinched to -1 Riemannian manifolds, we establish some notation. $\mathcal{M}_P$ will denote the space of all Riemannian metrics on a smooth manifold P. For $\epsilon > 0$, let $\mathcal{M}_P^\epsilon$ denote the space of

ϵ-pinched to -1 Riemannian metrics on P. Also, $\mathcal{E}_P \subset \mathcal{M}_P$ will denote the space of negatively curved Einstein metrics on P. Recall that $\mathcal{E}_P/\equiv$ is discrete, see [2], p.357.

Definition. Let $\epsilon > 0$ and n be a positive integer. An *Einstein correspondence* $\Phi : \mathcal{M}^\epsilon \to \mathcal{E}$ *for* n*-dimensional manifolds* is a family of maps $\Phi_P : \mathcal{M}^\epsilon_P \to \mathcal{E}_P$, for every n-dimensional manifold P for which $\mathcal{M}^\epsilon_P$ is not empty. We say that Φ is *cover-invariant* if $\Phi(p^*g) = p^*(\Phi(g))$ for every finite cover $p : P \to Q$ and $g \in \mathcal{M}^\epsilon_Q$, for which Φ_Q is defined.

We say that Φ is *continuous* if each $\Phi_P : \mathcal{M}^\epsilon_P \to \mathcal{E}_P$ is continuous. Here we consider $\mathcal{M}^\epsilon_P$ with the C^∞ topology and $\mathcal{E}_P$ with the C^2 topology.

Let $h,\ h' \in \mathcal{M}_P$. Write $h \equiv_0 h'$ provided (P, h) is isometric to (P, h'), up to scaling, via an isometry homotopic to id_P. Notice that the fibers of $\mathcal{E}_P/\equiv_0\ \to\ \mathcal{E}_P/\equiv$ are discrete; and hence $\mathcal{E}_P/\equiv_0$ is also discrete.

Theorem 15. [19] *Suppose that there are $\epsilon > 0$ and $n > 10$ for which there exists a cover-invariant Einstein correspondence Φ. Then there is a closed n-dimensional Riemannian manifold N, with metric $h \in \mathcal{M}^\epsilon_N$, for which the Einstein metric $\Phi(h)$ is unreachable by the Ricci flow starting at h.*

The proof of Theorem 15 is similar to the proof of Theorem 14, see [19].

Theorem 16. [19] *Suppose that there are $\epsilon > 0$ and $n \geq 6$ for which there exists an Einstein correspondence Φ. Then there is a closed n-dimensional manifold N that admits, at least, two non-isometric (even after scaling) negatively curved Einstein metrics. Moreover, one metric can be chosen to be hyperbolic.*

This Theorem is easily deduced from Theorem 8. We reproduce the proof:

Proof. From Theorem 8 we have the following.

There are closed connected smooth manifolds M_0, M_1, N, of dimension n, Riemannian metrics g_0, g_1 on M_0 and M_1, respectively, and smooth two-sheeted covers $p_0 : N \to M_0$, $p_1 : N \to M_1$ such that:

(1) M_0 and M_1 are homeomorphic but not PL-homeomorphic.
(2) g_0 is hyperbolic
(3) g_1 has sectional curvatures in $[-1-\epsilon, -1+\epsilon]$.

Now, note that the metrics g_1 and $\Phi(g_1)$ are not hyperbolic, otherwise, by Mostow's Rigidity Theorem, M_0 would be diffeomorphic to M_1, which contradicts (1) above. Hence $p_1^*(\Phi(g_1))$ is not hyperbolic either, while $p_0^*(g_0)$ is hyperbolic. Then the two non-isometric negatively curved Einstein metrics on N are $p_0^*(g_0)$ and $p_1^*(\Phi(g_1))$. This proves Theorem 16.

The general form of the following Theorem was suggested to us by Rugang Ye.

Theorem 17. [19] *A cover-invariant Einstein correspondence cannot be continuous.*

Remark. Note that we are not assuming that Φ fixes hyperbolic metrics. If we assumed that Φ(*hyperbolic metric*) = (*hyperbolic metric*), the Theorem then would be easily deduced as before.

Since Ricci flow and elliptic deformation are cover-invariant continuous (analytic) processes, it follows from Theorem 17 that they cannot be used, at least directly, to find Einstein metrics on ϵ-pinched to -1 Riemannian manifolds.

Dedication. This article is respectfully dedicated to the memory of Armand Borel whose conjecture that a closed aspherical manifold is determined (up to homeomorphism) by its fundamental group was one motivation for the research surveyed here.

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