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SUPERSINGULAR $K3$ SURFACES IN CHARACTERISTIC 2 AS DOUBLE COVERS OF A PROJECTIVE PLANE *

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Abstract. For every supersingular $K3$ surface X in characteristic 2, there exists a homogeneous polynomial G of degree 6 such that X is birational to the purely inseparable double cover of $\mathbb{P}^2$ defined by $w^2 = G$. We present an algorithm to calculate from G a set of generators of the numerical Néron-Severi lattice of X. As an application, we investigate the stratification defined by the Artin invariant on a moduli space of supersingular $K3$ surfaces of degree 2 in characteristic 2.

1. Introduction. We work over an algebraically closed field k of characteristic 2 in Introduction.

In [17], we have shown that every supersingular $K3$ surface X in characteristic 2 is isomorphic to the minimal resolution X_G of a purely inseparable double cover Y_G of $\mathbb{P}^2$ defined by

$$w^2 = G(X_0, X_1, X_2),$$

where G is a homogeneous polynomial of degree 6 such that the singular locus $\mathrm{Sing}(Y_G)$ of Y_G consists of 21 ordinary nodes. Conversely, if Y_G has 21 ordinary nodes as its only singularities, then X_G is a supersingular $K3$ surface. In characteristic 2, we can define the differential dG of a homogeneous polynomial G of degree 6 as a global section of the vector bundle $\Omega^1_{\mathbb{P}^2}(6)$. The condition that $\mathrm{Sing}(Y_G)$ consists of 21 ordinary nodes is equivalent to the condition that the subscheme $Z(dG)$ of $\mathbb{P}^2$ defined by $dG = 0$ is reduced of dimension 0. The homogeneous polynomials of degree 6 satisfying this condition form a Zariski open dense subset $\mathcal{U}_{2,6}$ of $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6))$. The kernel of the linear homomorphism $G \mapsto dG$ is the linear subspace

$$\mathcal{V}_{2,6} := \{\, H^2 \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6)) \;\mid\; H \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)) \,\}$$

of $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6))$. If $G \in \mathcal{U}_{2,6}$, then $G + H^2 \in \mathcal{U}_{2,6}$ holds for any $H^2 \in \mathcal{V}_{2,6}$; that is, $\mathcal{V}_{2,6}$ acts on $\mathcal{U}_{2,6}$ by translation. Let G and G' be polynomials in $\mathcal{U}_{2,6}$. The supersingular $K3$ surfaces X_G and $X_{G'}$ are isomorphic over $\mathbb{P}^2$ if and only if there exist $c \in k^\times$ and $H^2 \in \mathcal{V}_{2,6}$ such that

$$G' = cG + H^2.$$

Therefore we can construct a moduli space $\mathfrak{M}$ of supersingular $K3$ surfaces of degree 2 in characteristic 2 by

$$\mathfrak{M} := \mathbb{P}_*(\mathcal{U}_{2,6}/\mathcal{V}_{2,6})/\, PGL(3, k).$$

The purpose of this paper is to investigate the stratification of $\mathcal{U}_{2,6}$ by the Artin invariant of the supersingular $K3$ surfaces. Our investigation yields an algorithm to calculate a set of generators of the numerical Néron-Severi lattice of X_G from the homogeneous polynomial $G \in \mathcal{U}_{2,6}$.

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Suppose that a polynomial G in $\mathcal{U}_{2,6}$ is given. The singular points of Y_G are mapped bijectively to the points of $Z(dG)$ by the covering morphism. We denote by

$$\phi_G : X_G \to \mathbb{P}^2$$

the composite of the minimal resolution $X_G \to Y_G$ and the covering morphism $Y_G \to \mathbb{P}^2$. The numerical Néron-Severi lattice of the supersingular $K3$ surface X_G is denoted by S_G, which is a hyperbolic lattice of rank 22. Let $H_G \subset X_G$ be the pull-back of a general line of $\mathbb{P}^2$ by ϕ_G. For a point $P \in Z(dG)$, we denote by Γ_P the (-2)-curve on X_G that is contracted to P by ϕ_G. It is obvious that the sublattice S_G^0 of S_G generated by the numerical equivalence classes $[\Gamma_P]$ $(P \in Z(dG))$ and $[H_G]$ is of rank 22, and hence is of finite index in S_G.

Definition 1.1. Let $C \subset \mathbb{P}^2$ be a reduced irreducible plane curve. We say that C is *splitting in* X_G if the proper transform D_C of C in X_G is not reduced. If C is splitting in X_G, then the divisor D_C is written as $2F_C$, where F_C is a reduced irreducible curve on X_G.

Definition 1.2. A pencil $\mathcal{E}$ of cubic curves on $\mathbb{P}^2$ is called a *regular pencil splitting in* X_G if the following hold;

- the base locus of $\mathcal{E}$ consists of distinct 9 points,
- every singular member of $\mathcal{E}$ is an irreducible nodal curve, and
- every member of $\mathcal{E}$ is splitting in X_G.

The correctness of our main algorithm (Algorithm 9.4) is a consequence of the following:

Main Theorem. *Suppose that* $G \in \mathcal{U}_{2,6}$.

(1) *Let* $\mathcal{I}_{Z(dG)} \subset \mathcal{O}_{\mathbb{P}^2}$ *denote the ideal sheaf of* $Z(dG)$. *Then the linear system* $|\mathcal{I}_{Z(dG)}(5)|$ *is of dimension* 2, *and a general member of* $|\mathcal{I}_{Z(dG)}(5)|$ *is reduced, irreducible, and splitting in* X_G.

(2) *A line* $L \subset \mathbb{P}^2$ *is splitting in* X_G *if and only if* $|L \cap Z(dG)| = 5$.

(3) *A smooth conic* $Q \subset \mathbb{P}^2$ *is splitting in* X_G *if and only if* $|Q \cap Z(dG)| = 8$.

(4) *Let* $\mathcal{E}$ *be a regular pencil of cubic curves of* $\mathbb{P}^2$ *splitting in* X_G. *Then the base locus* $\mathrm{Bs}(\mathcal{E})$ *of* $\mathcal{E}$ *is contained in* $Z(dG)$.

(5) *The lattice* S_G *is generated by the sublattice* S_G^0 *and the classes* $[F_C]$, *where* C *runs through the set of splitting curves of the following type:*

- *a general member of the linear system* $|\mathcal{I}_{Z(dG)}(5)|$,
- *a line splitting in* X_G,
- *a smooth conic splitting in* X_G,
- *a member of a regular pencil of cubic curves splitting in* X_G.

Example 1.3. Consider the polynomial

$$G_{\mathrm{DK}} := X_0X_1X_2(X_0^3 + X_1^3 + X_2^3), \tag{1.1}$$

which was discovered by Dolgachev and Kondo in [6]. They showed that every supersingular $K3$ surface in characteristic 2 with Artin invariant 1 is isomorphic to $X_{G_{\mathrm{DK}}}$. The subscheme $Z(dG_{\mathrm{DK}}) \subset \mathbb{P}^2$ consists of the $\mathbb{F}_4$-rational points of $\mathbb{P}^2$. A line $L \subset \mathbb{P}^2$ is splitting in $X_{G_{\mathrm{DK}}}$ if and only if L is $\mathbb{F}_4$-rational. The numerical Néron-Severi lattice of $X_{G_{\mathrm{DK}}}$ is generated by the classes of the (-2)-curves

$$\Gamma_P \quad (P \in \mathbb{P}^2(\mathbb{F}_4)) \quad \text{and} \quad F_L \quad (L \in (\mathbb{P}^2)^\vee(\mathbb{F}_4)).$$

(The classes $[H_{G_{\mathrm{DK}}}]$ and $[F_C]$, where C is a general member of $|\mathcal{I}_{Z(dG_{\mathrm{DK}})}(5)|$, are written as linear combinations of $[\Gamma_P]$ and $[F_L]$.)

EXAMPLE 1.4. Consider the polynomial

$$G := {X_0}^5X_1 + {X_0}^5X_2 + {X_0}^3{X_1}^3 + {X_0}^3{X_1}^2X_2 + {X_0}^3X_1{X_2}^2 + \\ + {X_0}^3{X_2}^3 + {X_0}^2X_1{X_2}^3 + X_0{X_2}^5 + {X_1}^5X_2.$$

We put

$$\begin{aligned} P_0 &:= [\alpha^{13}+\alpha^{11}+\alpha^{10}+\alpha^9+\alpha^7+\alpha^4+\alpha^3+\alpha^2, \\ &\qquad \alpha^{12}+\alpha^{11}+\alpha^9+\alpha^5+\alpha^3+\alpha^2+\alpha,\ 1], \quad \text{and} \\ P_7 &:= [\alpha^{12}+\alpha^{11}+\alpha^{10}+\alpha^7+\alpha^6+\alpha^5+\alpha^4+\alpha, \\ &\qquad \alpha^{13}+\alpha^{11}+\alpha^9+\alpha^5+\alpha^4+\alpha^3+\alpha^2+\alpha,\ 1], \end{aligned}$$

where α is a root of the irreducible polynomial

$$t^{14}+t^{13}+t^{12}+t^8+t^5+t^4+t^3+t^2+1 \ \in\ \mathbb{F}_2[t].$$

The subscheme $Z(dG)$ is reduced of dimension 0 consisting of the points

$$P_\nu := \mathrm{Frob}^\nu(P_0) \quad (\nu = 0,\dots,6) \quad \text{and} \quad P_{7+\nu} := \mathrm{Frob}^\nu(P_7) \quad (\nu = 0,\dots,13),$$

where Frob is the Frobenius morphism $\alpha \mapsto \alpha^2$ over $\mathbb{F}_2$. (We have $\mathrm{Frob}^7(P_0) = P_0$ and $\mathrm{Frob}^{14}(P_7) = P_7$.) There exists a line L that passes through the points P_0, P_1, P_3, P_7, P_{14}. There exists a smooth conic Q that passes through the points P_7, P_8, P_9, P_{11}, P_{14}, P_{15}, P_{16}, P_{18}. The lattice S_G is generated by the classes in S_G^0 and the classes $[F_C]$ associated to a general member of $|\mathcal{I}_{Z(dG)}(5)|$, the splitting lines $\mathrm{Frob}^\nu(L)$ and the splitting smooth conics $\mathrm{Frob}^\nu(Q)$ for $\nu = 0,\dots,6$. (We have $\mathrm{Frob}^7(L) = L$ and $\mathrm{Frob}^7(Q) = Q$.) The Artin invariant of X_G is 4.

EXAMPLE 1.5. Consider the polynomial

$$G := {X_0}^5X_2 + {X_0}^3{X_1}^3 + {X_0}^3{X_2}^3 + X_0X_1{X_2}^4 + {X_1}^5X_2.$$

The subscheme $Z(dG)$ is reduced of dimension 0 consisting of the point $[0,0,1]$ and the Frobenius orbit of the point

$$[\alpha^{19}+\alpha^{18}+\alpha^{16}+\alpha^{15}+\alpha^8+\alpha^3+\alpha^2+\alpha, \\ \alpha^{19}+\alpha^{17}+\alpha^{16}+\alpha^{15}+\alpha^{14}+\alpha^9+\alpha^8+\alpha^7+\alpha^5+\alpha^3+\alpha,\ 1],$$

where α is a root of the irreducible polynomial

$$t^{20}+t^{19}+t^{18}+t^{15}+t^{10}+t^7+t^6+t^4+1 \ \in\ \mathbb{F}_2[t].$$

There are no reduced irreducible plane curves of degree ≤ 3 that are splitting in X_G. Hence S_G is generated by the classes in S_G^0 and the class $[F_C]$ associated to a general member of $|\mathcal{I}_{Z(dG)}(5)|$. Therefore the Artin invariant of X_G is 10. Note that it is a non-trivial problem to find explicit examples of supersingular $K3$ surfaces with big Artin invariant. See [20] and [8, 9].

EXAMPLE 1.6. Consider the polynomial

$$G := {X_0}^5X_1 + {X_0}^3{X_1}^2X_2 + X_0{X_2}^5 + {X_1}^5X_2.$$

We put

$$\begin{aligned} P_0 &:= [\alpha^{13}+\alpha^{12}+\alpha^{10}+\alpha^9+\alpha^8+\alpha^3+\alpha^2,\ \alpha^{13}+\alpha^8+\alpha^2,\ 1], \quad \text{and} \\ P_{14} &:= [\alpha^{13}+\alpha^{12}+\alpha^{11}+\alpha^{10}+\alpha^9+\alpha^8+\alpha^7+\alpha^6+\alpha^2, \\ &\qquad \alpha^{10}+\alpha^9+\alpha^7+\alpha^4,\ 1], \end{aligned}$$

where α is a root of the irreducible polynomial

$$t^{14}+t^{13}+t^{12}+t^8+t^5+t^4+t^3+t^2+1 \ \in \ \mathbb{F}_2[t].$$

The subscheme $Z(dG)$ is reduced of dimension 0. It consists of the points $P_\nu := \mathrm{Frob}^\nu(P_0)$ ($\nu = 0, \dots, 13$) and $P_{14+\nu} := \mathrm{Frob}^\nu(P_{14})$ ($\nu = 0, \dots, 6$). (We have $\mathrm{Frob}^{14}(P_0) = P_0$ and $\mathrm{Frob}^7(P_{14}) = P_{14}$.) We put

$$A := \{P_0, P_1, P_3, P_7, P_8, P_{10}, P_{14}, P_{18}, P_{19}\}.$$

We have $\mathrm{Frob}^7(A) = A$. For each $\nu = 0, \dots, 6$, there exists a regular pencil $\mathcal{E}_\nu$ of cubic curves splitting in X_G such that the base locus $\mathrm{Bs}(\mathcal{E}_\nu)$ is equal to $\mathrm{Frob}^\nu(A)$. The lattice S_G is generated by the classes in S_G^0 and the classes $[F_C]$ associated to a general member of $|\mathcal{I}_{Z(dG)}(5)|$ and the members of $\mathcal{E}_\nu$ for $\nu = 0, \dots, 6$. The Artin invariant of X_G is 7.

The configuration of irreducible curves of degree ≤ 3 splitting in X_G is encoded by the 2-elementary group

$$\mathcal{C}_G^\sim := S_G/S_G^0,$$

which we will regard as a linear code in the $\mathbb{F}_2$-vector space $(S_G^0)^\vee/S_G^0$ of dimension 22, where $(S_G^0)^\vee$ is the dual lattice of S_G^0. Using the basis

$$[\Gamma_P]/2 \quad (P \in Z(dG)) \quad \text{and} \quad [H_G]/2$$

of $(S_G^0)^\vee$, we can identify the $\mathbb{F}_2$-vector space $(S_G^0)^\vee/S_G^0$ with

$$\mathrm{Pow}(Z(dG)) \oplus \mathbb{F}_2,$$

where $\mathrm{Pow}(Z(dG))$ is the power set of $Z(dG)$ equipped with a structure of the $\mathbb{F}_2$-vector space by

$$A + B = (A \cup B) \setminus (A \cap B) \qquad (A, B \subset Z(dG)).$$

We define the code $\mathcal{C}_G \subset \mathrm{Pow}(Z(dG))$ to be the image of $\mathcal{C}_G^\sim$ by the projection $(S_G^0)^\vee/S_G^0 \to \mathrm{Pow}(Z(dG))$. It turns out that we can recover from $\mathcal{C}_G$ the numerical Néron-Severi lattice S_G, and obtain the configuration of curves of degree ≤ 3 splitting in X_G. In particular, we have

$$\text{the Artin invariant of } X_G \ = \ 11 - \dim_{\mathbb{F}_2} \mathcal{C}_G.$$

THEOREM 1.7. *Let* $\mathtt{Z}$ *be a finite set with* $|\mathtt{Z}| = 21$, *and let* $\mathtt{C} \subset \mathrm{Pow}(\mathtt{Z})$ *be a code. There exists a polynomial* $G \in \mathcal{U}_{2,6}$ *such that* $\mathtt{C}$ *is mapped to* $\mathcal{C}_G \subset \mathrm{Pow}(Z(dG))$ *by a certain bijection* $\mathtt{Z} \xrightarrow{\sim} Z(dG)$ *if and only if* $\mathtt{C}$ *satisfies the following conditions;*

(a) $\dim_{\mathbb{F}_2} \mathtt{C} \le 10$,

(b) *the word* $\mathtt{Z} \in \mathrm{Pow}(\mathtt{Z})$ *is contained in* $\mathtt{C}$, *and*

(c) $|A| \in \{0, 5, 8, 9, 12, 13, 16, 21\}$ *for every word* $A \in \mathtt{C}$*.*

We say that two codes $\mathtt{C}$ and $\mathtt{C}'$ in $\mathrm{Pow}(\mathtt{Z})$ are said to be $\mathfrak{S}_{21}$*-equivalent* if there exists a permutation τ of $\mathtt{Z}$ such that $\tau(\mathtt{C}) = \mathtt{C}'$ holds. By computer-aided calculation, we have classified all the $\mathfrak{S}_{21}$-equivalence classes of codes satisfying the conditions (a), (b) and (c) in Theorem 1.7. The list is given in §8.

THEOREM 1.8. *The number* $r(\sigma)$ *of the* $\mathfrak{S}_{21}$*-equivalence classes of codes with dimension* $11 - \sigma$ *satisfying the conditions* (b) *and* (c) *in Theorem 1.7 is given as follows:*

(1.2)

σ	1	2	3	4	5	6	7	8	9	10
$r(\sigma)$	1	3	13	41	58	43	21	8	3	1

.

From the list, we obtain the following facts about the stratification of $\mathcal{U}_{2,6}$ by the Artin invariant. For $\sigma = 1, \dots, 10$, we put

$$\mathcal{U}_\sigma := \{\ G \in \mathcal{U}_{2,6}\ \mid\ \text{the Artin invariant of } X_G \text{ is } \sigma\ \} \quad \text{and} \quad \mathcal{U}_{\le\sigma} := \bigcup_{\sigma' \le \sigma} \mathcal{U}_{\sigma'}.$$

Note that each $\mathcal{U}_{\le\sigma}$ is Zariski closed in $\mathcal{U}_{2,6}$.

COROLLARY 1.9. *The number of the irreducible components of* $\mathcal{U}_\sigma$ *is at least* $r(\sigma)$*, where* $r(\sigma)$ *is given in* (1.2)*.*

COROLLARY 1.10. *The Zariski closed subset* $\mathcal{U}_{\le 9}$ *of* $\mathcal{U}_{2,6}$ *consists of three irreducible hypersurfaces* $\mathcal{U}[33]$*,* $\mathcal{U}[42]$ *and* $\mathcal{U}[51]$*, where* $\mathcal{U}[ab]$ *is the locus of all* $G \in \mathcal{U}_{2,6}$ *that can be written as* $G = G_a G_b + H^2$*, where* G_a*,* G_b *and* H *are homogeneous polynomials of degree* a*,* b *and* 3*, respectively.*

COROLLARY 1.11. *If the Artin invariant of* X_G *is* 1*, then, via a linear automorphism of* $\mathbb{P}^2$*, the covering morphism* $Y_G \to \mathbb{P}^2$ *is isomorphic to the Dolgachev-Kondo surface* $Y_{G_{\mathrm{DK}}} \to \mathbb{P}^2$ *in Example 1.3. In particular, the locus* $\mathcal{U}_1$ *is irreducible, and, in the moduli space* $\mathfrak{M} = \mathbb{P}_*(\mathcal{U}_{2,6}/\mathcal{V}_{2,6})/\,PGL(3,k)$*, the locus of supersingular* $K3$ *surfaces with Artin invariant* 1 *consists of a single point.*

Purely inseparable covers of the projective plane are called *Zariski surfaces*, and their properties have been studied by P. Blass and J. Lang [2]. In particular, an algorithm to calculate the Artin invariant has been established [2, Chapter 2, Proposition 6]. Our algorithm gives us not only the Artin invariant but also a geometric description of generators of the numerical Néron-Severi group.

In [23], C. T. C. Wall classified quartic curves in characteristic 2 by an invariant theory for the quartic form modulo the subspace of perfect squares.

This paper is organized as follows.

As is suggested above, the global section dG of $\Omega^1_{\mathbb{P}^2}(6)$ plays an important role in the study of X_G. In §2, we study global sections of $\Omega^1_{\mathbb{P}^2}(b)$ in general, where b is an integer ≥ 4. The problem that is considered in this section is to characterize the subschemes defined by $s = 0$, where s is a global section of $\Omega^1_{\mathbb{P}^2}(b)$, among reduced 0-dimensional subschemes Z of $\mathbb{P}^2$. A characterization is given in terms of the linear system $|\mathcal{I}_Z(b-1)|$. The results in this section hold in any characteristics.

In §3, we assume that the ground field is of characteristic $p > 0$, and define a global section dG of $\Omega^1_{\mathbb{P}^2}(b)$, where G is a homogeneous polynomial of degree b divisible by

p. We then investigate geometric properties of the purely inseparable cover $Y_G \to \mathbb{P}^2$ defined by $w^p = G$, and the minimal resolution X_G of Y_G. Many results of this section have been already presented in [2].

From §4, we assume that the ground field is of characteristic 2. Let b be an even integer ≥ 4. In §4, we consider the problem to determine whether a given global section of $\Omega^1_{\mathbb{P}^2}(b)$ is written as dG by some homogeneous polynomial G. In §5, we associate to a homogeneous polynomial G a binary linear code $\mathcal{C}_G$ that describes the numerical Néron-Severi lattice of X_G. A notion of *geometrically realizable* $\mathfrak{S}_n$*-equivalence classes of codes* is introduced. In §6, we define a word $w_G(C)$ of $\mathcal{C}_G$ for each curve C splitting in X_G, and study the geometry of splitting curves.

From §7, we put $b = 6$, and study the supersingular $K3$ surfaces X_G in characteristic 2. In §7, we review some known facts about $K3$ surfaces. In §8, the relation between the code $\mathcal{C}_G$ and the configuration of curves splitting in X_G is explained. We present the complete list of geometrically realizable $\mathfrak{S}_{21}$-equivalence classes of codes. Theorems and Corollaries stated above are proved in this section. In §9, we present an algorithm that calculates the code $\mathcal{C}_G$ from a given homogeneous polynomial $G \in \mathcal{U}_{2,6}$, and give concrete examples. Some irreducible components of $\mathcal{U}_\sigma$ are described in detail.

2. Global sections of $\Omega^1_{\mathbb{P}^2}(b)$ in arbitrary characteristic. In this section, we work over an algebraically closed field k of *arbitrary* characteristic.

Let b be an integer ≥ 4. We consider the locally free sheaf

$$\Omega(b) := \Omega^1_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(b)$$

of rank 2 on the projective plane $\mathbb{P}^2$. From the exact sequence

$$0 \to \Omega(b) \to \mathcal{O}_{\mathbb{P}^2}(b-1)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^2}(b) \to 0, \tag{2.1}$$

we obtain

$$n := c_2(\Omega(b)) = b^2 - 3b + 3.$$

For a global section $s \in H^0(\mathbb{P}^2, \Omega(b))$, we denote by $Z(s)$ the subscheme of $\mathbb{P}^2$ defined by $s = 0$, and by $\mathcal{I}_{Z(s)} \subset \mathcal{O}_{\mathbb{P}^2}$ the ideal sheaf of $Z(s)$. If $Z(s)$ is a reduced 0-dimensional scheme, then $Z(s)$ consists of n reduced points.

The main result of this section is the following:

THEOREM 2.1. *Let Z be a 0-dimensional reduced subscheme of $\mathbb{P}^2$ with the ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_{\mathbb{P}^2}$. Suppose that* $\operatorname{length} \mathcal{O}_Z = n$. *Then the following two conditions are equivalent:*

(i) *There exists a global section s of $\Omega(b)$ such that $Z = Z(s)$.*

(ii) *There exists a pair (C_0, C_1) of members of the linear system $|\mathcal{I}_Z(b-1)|$ such that the scheme-theoretic intersection $C_0 \cap C_1$ is the union of Z and a 0-dimensional subscheme $\Gamma \subset \mathbb{P}^2$ of* $\operatorname{length} \mathcal{O}_\Gamma = b - 2$ *that is contained in a line disjoint from Z.*

If these conditions are satisfied, then the global section s with $Z = Z(s)$ is unique up to multiplicative constants.

Let $[X_0, X_1, X_2]$ be homogeneous coordinates of $\mathbb{P}^2$. We put

$$l_\infty := \{X_2 = 0\}, \quad U := \mathbb{P}^2 \setminus l_\infty,$$

and let (x_0, x_1) be the affine coordinates on U given by

$$x_0 := X_0/X_2 \quad \text{and} \quad x_1 := X_1/X_2.$$

We also regard $[x_0, x_1]$ as homogeneous coordinates of l_∞. Let e_b be the global section of $\mathcal{O}_{\mathbb{P}^2}(b)$ that corresponds to $X_2^b \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(b))$. A section

$$\sigma_0(x_0, x_1) dx_0 \otimes e_b + \sigma_1(x_0, x_1) dx_1 \otimes e_b \tag{2.2}$$

of $\Omega(b)$ on U extends to a global section of $\Omega(b)$ over $\mathbb{P}^2$ if and only if the following holds;

(2.3) the polynomials σ_0, σ_1, and $\sigma_2 := x_0\sigma_0 + x_1\sigma_1$ are of degree $\leq b-1$.

For $i = 0, 1$ and 2, let $\sigma_i^{(b-1)}(x_0, x_1)$ be the homogeneous part of degree $b-1$ of σ_i. Then the condition (2.3) is rephrased as follows;

(2.4) $\deg \sigma_0 < b$, $\deg \sigma_1 < b$, and there exists a homogeneous polynomial $\gamma(x_0, x_1)$ of degree $b-2$ such that $\sigma_0^{(b-1)} = x_1\gamma$ and $\sigma_1^{(b-1)} = -x_0\gamma$.

In particular, we have

$$h^0(\mathbb{P}^2, \Omega(b)) = b^2 - 1.$$

This equality also follows from the exact sequence (2.1).

Remark 2.2. Suppose that a global section s of $\Omega(b)$ is given by (2.2) on U. The subscheme $Z(s)$ of $\mathbb{P}^2$ is defined on U by $\sigma_0 = \sigma_1 = 0$. The intersection $Z(s) \cap l_\infty$ is set-theoretically equal to the common zeros of the homogeneous polynomials $\sigma_0^{(b-1)}$, $\sigma_1^{(b-1)}$ and $\sigma_2^{(b-1)}$ on l_∞. In particular, if $s \in H^0(\mathbb{P}^2, \Omega(b))$ is chosen generally, then $Z(s)$ is reduced of dimension 0.

Let Θ be the sheaf of germs of regular vector fields on $\mathbb{P}^2$, that is, Θ is the dual of $\Omega^1_{\mathbb{P}^2}$. Let e_{-1} be the rational section of $\mathcal{O}_{\mathbb{P}^2}(-1)$ that corresponds to $1/X_2$. The vector space $H^0(\mathbb{P}^2, \Theta(-1))$ is of dimension 3, and is generated by $\theta_0, \theta_1, \theta_2$, where

$$\theta_0|U = \frac{\partial}{\partial x_0} \otimes e_{-1}, \quad \theta_1|U = \frac{\partial}{\partial x_1} \otimes e_{-1}, \quad \theta_2|U = \left(x_0\frac{\partial}{\partial x_0} + x_1\frac{\partial}{\partial x_1}\right) \otimes e_{-1}.$$

Since $c_2(\Theta(-1)) = 1$, every non-zero global section θ of $\Theta(-1)$ has a single reduced zero, which we will denote by $\zeta([\theta])$, where $[\theta] \in \mathbb{P}_*(H^0(\mathbb{P}^2, \Theta(-1)))$ is the one-dimensional linear subspace of $H^0(\mathbb{P}^2, \Theta(-1))$ generated by θ. When θ is given by

$$\theta|U = A\theta_0 + B\theta_1 + C\theta_2 \qquad (A, B, C \in k),$$

then $\zeta([\theta])$ is equal to $[A, B, -C]$ in terms of the homogeneous coordinates $[X_0, X_1, X_2]$. Thus we obtain an isomorphism

$$\zeta : \mathbb{P}_*(H^0(\mathbb{P}^2, \Theta(-1))) \xrightarrow{\sim} \mathbb{P}^2.$$

For a hyperplane $V \subset H^0(\mathbb{P}^2, \Theta(-1))$, we denote by $l_V \subset \mathbb{P}^2$ the line corresponding to V by ζ. For a line $l \subset \mathbb{P}^2$, we denote by $V_l \subset H^0(\mathbb{P}^2, \Theta(-1))$ the hyperplane corresponding to l by ζ.

Remark 2.3. Suppose that a hyperplane V of $H^0(\mathbb{P}^2, \Theta(-1))$ is generated by τ_0 and τ_1. Then there exist affine coordinates (y_0, y_1) on $U_V := \mathbb{P}^2 \setminus l_V$ and a rational section e'_{-1} of $\mathcal{O}_{\mathbb{P}^2}(-1)$ having the pole along l_V such that

$$\tau_0|U_V = \frac{\partial}{\partial y_0} \otimes e'_{-1}, \quad \tau_1|U_V = \frac{\partial}{\partial y_1} \otimes e'_{-1}.$$

A global section s of $\Omega(b)$ defines a linear homomorphism

$$\varphi_s \,:\, H^0(\mathbb{P}^2, \Theta(-1)) \,\to\, H^0(\mathbb{P}^2, \mathcal{I}_{Z(s)}(b-1))$$

via the natural coupling $\Omega^1_{\mathbb{P}^2} \otimes \Theta \to \mathcal{O}_{\mathbb{P}^2}$. Suppose that s is given by (2.2). For $i = 0, 1$ and 2, we put

$$\tilde{\sigma}_i(X_0, X_1, X_2) := X_2^{b-1} \sigma_i(X_0/X_2, X_1/X_2).$$

Then φ_s is given by

$$\varphi_s(\theta_i) = \tilde{\sigma}_i \qquad (i = 0, 1, 2). \tag{2.5}$$

Proposition 2.4. *Let s be a global section of $\Omega(b)$ such that $Z(s)$ is reduced of dimension 0. Then the following hold:*

(1) *The linear homomorphism φ_s is an isomorphism.*

(2) *Let $l \subset \mathbb{P}^2$ be a line such that $l \cap Z(s) = \emptyset$, and let $P_{s,l} \subset |\mathcal{I}_{Z(s)}(b-1)|$ be the pencil corresponding to the hyperplane $V_l \subset H^0(\mathbb{P}^2, \Theta(-1))$ via the isomorphism φ_s. Then the base locus of $P_{s,l}$ is of the form*

$$Z(s) + \Gamma(s, l),$$

where $\Gamma(s, l)$ is a 0-dimensional scheme of length $\mathcal{O}_{\Gamma(s,l)} = b - 2$. *Moreover the ideal sheaf $\mathcal{I}_{\Gamma(s,l)} \subset \mathcal{O}_{\mathbb{P}^2}$ of $\Gamma(s, l)$ contains the ideal sheaf $\mathcal{I}_l$ of the line l.*

Proof. First we show that φ_s is injective. Suppose that there exists a non-zero global section θ of $\Theta(-1)$ such that $\varphi_s(\theta) = 0$. We have affine coordinates (y_0, y_1) on some affine part U' of $\mathbb{P}^2$ such that

$$\theta|U' = \frac{\partial}{\partial y_0} \otimes e'_{-1},$$

where e'_{-1} is a rational section of $\mathcal{O}_{\mathbb{P}^2}(-1)$ that is regular on U'. We express s by

$$s|U' = (\sigma'_0 dy_0 + \sigma'_1 dy_1) \otimes e'_b,$$

where $e'_b := 1/(e'_{-1})^{\otimes b}$. Since $\varphi_s(\theta) = 0$, we have $\sigma'_0 = 0$. Because $Z(s)$ is of dimension 0, $Z(s) \cap U'$ must be empty. Hence σ'_1 is a non-zero constant. Because $b \geq 4$, the line $\mathbb{P}^2 \setminus U'$ at infinity is contained in $Z(s)$ by Remark 2.2, which contradicts the assumption. Therefore φ_s is injective.

Next we prove (2). We choose the homogeneous coordinates $[X_0, X_1, X_2]$ in such a way that l is defined by $X_2 = 0$. The hyperplane V_l of $H^0(\mathbb{P}^2, \Theta(-1))$ is generated by θ_0 and θ_1. Since their images by φ_s are $\tilde{\sigma}_0$ and $\tilde{\sigma}_1$, the pencil $P_{s,l} \subset |\mathcal{I}_{Z(s)}(b-1)|$ is spanned by the curves C_0 and C_1 of degree $b-1$ defined by $\tilde{\sigma}_0 = 0$ and $\tilde{\sigma}_1 = 0$. Since $Z(s) \cap l = \emptyset$ by the assumption, we see from Remark 2.2 that the scheme-theoretic intersection $C_0 \cap C_1 \cap U$ coincides with $Z(s)$, and at least one of C_0 or C_1 does not contain l as an irreducible component. Hence the base locus of $P_{s,l}$ is $Z(s) + \Gamma(s,l)$, where $\Gamma(s,l)$ is a 0-dimensional scheme whose support is contained in l. We have

$$\operatorname{length} \mathcal{O}_{\Gamma(s,l)} = (b-1)^2 - n = b - 2.$$

Note that the support of $\Gamma(s,l)$ is the zeros on l of the homogeneous polynomial γ of degree $b-2$ that has appeared in (2.4). Suppose that s is general. Then γ is a reduced polynomial, and hence $\Gamma(s,l)$ is equal to the reduced scheme defined by $X_2 = \gamma(X_0, X_1) = 0$, because their supports and lengths coincide. In particular, the ideal sheaf $\mathcal{I}_{\Gamma(s,l)}$ of $\Gamma(s,l)$ contains the ideal sheaf $\mathcal{I}_l$ of l. By the specialization argument, we see that $\mathcal{I}_{\Gamma(s,l)}$ contains $\mathcal{I}_l$ for any s such that $Z(s)$ is reduced, of dimension 0 and disjoint from l.

It remains to show that φ_s is surjective. It is enough to show that

$$h^0(\mathbb{P}^2, \mathcal{I}_{Z(s)}(b-1)) = 3.$$

We follow the argument of [10, pp. 712-714]. Let $\pi : S \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at the points of $Z(s)$, and let E be the union of (-1)-curves on S that are contracted by π. We have

$$E^2 = -n, \quad K_S \cong \pi^* \mathcal{O}_{\mathbb{P}^2}(-3) \otimes \mathcal{O}_S(E), \quad \text{and} \quad h^0(S, K_S) = h^1(S, K_S) = 0.$$

Let $L \to S$ be the line bundle corresponding to the invertible sheaf

$$\pi^* \mathcal{O}_{\mathbb{P}^2}(b-1) \otimes \mathcal{O}_S(-E).$$

There exists a natural isomorphism

$$H^0(S, L) \cong H^0(\mathbb{P}^2, \mathcal{I}_{Z(s)}(b-1)). \tag{2.6}$$

From $h^2(S,L) = h^0(S, K_S - L) = 0$ and $\chi(\mathcal{O}_S) = 1$, we obtain from the Riemann-Roch theorem that

$$h^0(S, L) = h^1(S, L) - (b^2 - 7b + 6)/2. \tag{2.7}$$

Let ξ_0 and ξ_1 be the global sections of the line bundle L corresponding to the homogeneous polynomials $\varphi_s(\theta_0) = \tilde{\sigma}_0$ and $\varphi_s(\theta_1) = \tilde{\sigma}_1$ in $H^0(\mathbb{P}^2, \mathcal{I}_{Z(s)}(b-1))$ by the natural isomorphism (2.6). Since $Z(s)$ is reduced, the curves $C_0 = \{\tilde{\sigma}_0 = 0\}$ and $C_1 = \{\tilde{\sigma}_1 = 0\}$ are smooth at each point of $Z(s)$, and they intersect transversely at each point of $Z(s)$. Hence the divisors on S defined by $\xi_0 = 0$ and $\xi_1 = 0$ have no common points on E. Therefore we can construct the Koszul complex

$$0 \to \mathcal{O}_S(K_S - L) \to \mathcal{O}_S(K_S) \oplus \mathcal{O}_S(K_S) \to \mathcal{I}_{\pi^{-1}(\Gamma(s,l))}(K_S + L) \to 0$$

from ξ_0 and ξ_1, where $\mathcal{I}_{\pi^{-1}(\Gamma(s,l))} \subset \mathcal{O}_S$ is the ideal sheaf of $\pi^{-1}(\Gamma(s,l))$. From this complex, we obtain

$$h^1(S, L) = h^0(S, \mathcal{I}_{\pi^{-1}(\Gamma(s,l))}(K_S + L)) = h^0(\mathbb{P}^2, \mathcal{I}_{\Gamma(s,l)}(b-4)). \tag{2.8}$$

Suppose that $b = 4$. Then we have $h^0(\mathbb{P}^2, \mathcal{I}_{\Gamma(s,l)}(b-4)) = 0$, and hence, from (2.6)-(2.8), we obtain $h^0(\mathbb{P}^2, \mathcal{I}_{Z(s)}(b-1)) = 3$.

Suppose that $b \geq 5$. Assume that a general member D of $|\mathcal{I}_{\Gamma(s,l)}(b-4)|$ satisfies $l \not\subset D$. Then the length of the scheme-theoretic intersection of l and D is $b-4$. Since

$\mathcal{I}_D \subset \mathcal{I}_{\Gamma(s,l)}$ and $\mathcal{I}_l \subset \mathcal{I}_{\Gamma(s,l)}$, this contradicts length $\mathcal{O}_{\Gamma(s,l)} = b-2$. Therefore the linear system $|\mathcal{I}_{\Gamma(s,l)}(b-4)|$ possesses l as a fixed component. Since $\mathcal{I}_{\Gamma(s,l)} \supset \mathcal{I}_l$, we have

$$h^0(\mathbb{P}^2, \mathcal{I}_{\Gamma(s,l)}(b-4)) = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(b-5)) = 3 + (b^2 - 7b + 6)/2. \tag{2.9}$$

Combining (2.6)-(2.9), we obtain $h^0(\mathbb{P}^2, \mathcal{I}_{Z(s)}(b-1)) = 3$. □

Remark 2.5. Let $s \in H^0(\mathbb{P}^2, \Omega(b))$ be as in Proposition 2.4. The 2-dimensional linear system $|\mathcal{I}_{Z(s)}(b-1)|$ defines a morphism

$$\Phi_s \;:\; \mathbb{P}^2 \setminus Z(s) \;\to\; \mathbb{P}^*(H^0(\mathbb{P}^2, \mathcal{I}_{Z(s)}(b-1))) \cong (\mathbb{P}^2)^\vee,$$

where the second isomorphism is obtained from the isomorphism φ_s and the dual of ζ. Let $l \in (\mathbb{P}^2)^\vee$ be a general line of $\mathbb{P}^2$. The inverse image of l by Φ_s coincides with $\Gamma(s,l)$. Therefore Φ_s is generically finite of degree $b-2$.

Remark 2.6. Let s, l, V_l and $P_{s,l}$ be as in Proposition 2.4. We have isomorphisms $P_{s,l} \cong \mathbb{P}_*(V_l)$ by φ_s, and $\mathbb{P}_*(V_l) \cong l$ by ζ. By composition, we obtain an isomorphism

$$\psi_{s,l} \;:\; P_{s,l} \;\xrightarrow{\sim}\; l.$$

The restriction of the pencil $P_{s,l}$ to l consists of the fixed part $\Gamma(s,l)$ and one moving point. The isomorphism $\psi_{s,l}$ maps $C \in P_{s,l}$ to the moving point of the divisor $C \cap l$ of l. Indeed, let us fix affine coordinates (x_0, x_1) on $U = \mathbb{P}^2 \setminus l$ as in the proof of Proposition 2.4 so that V_l is generated by θ_0 and θ_1. The isomorphism $\mathbb{P}_*(V_l) \cong l$ is written explicitly as

$$\zeta([\theta_0 + t\theta_1]) = [1, t, 0] \in l.$$

On the other hand, the projective plane curve of degree $b-1$ defined by the homogeneous polynomial

$$\varphi_s(\theta_0 + t\theta_1) = \tilde{\sigma}_0 + t\tilde{\sigma}_1$$

passes through the point $[1, t, 0]$ by (2.4).

Corollary 2.7. *Let s be a global section of $\Omega(b)$ such that $Z(s)$ is reduced of dimension 0. Then the linear system $|\mathcal{I}_{Z(s)}(b-1)|$ is of dimension 2, and its base locus coincides with $Z(s)$. A general member of $|\mathcal{I}_{Z(s)}(b-1)|$ is reduced and irreducible.*

Proof. The last statement follows from the assumption that $Z(s)$ is reduced and from Bertini's theorem applied to the morphism Φ_s in Remark 2.5. □

Proof of Theorem 2.1. The implication from (i) to (ii) has been already proved in Proposition 2.4. Suppose that $|\mathcal{I}_Z(b-1)|$ has the property (ii). We will construct a global section s of $\Omega(b)$ such that $Z = Z(s)$. Let l be the line of $\mathbb{P}^2$ containing the subscheme Γ. We choose homogeneous coordinates $[X_0, X_1, X_2]$ such that l is defined by $X_2 = 0$. Let $\tilde{\sigma}_0(X_0, X_1, X_2) = 0$ and $\tilde{\sigma}_1(X_0, X_1, X_2) = 0$ be the defining equations of C_0 and C_1, respectively. We put

$$\begin{aligned} \sigma_0(x_0, x_1) &:= \tilde{\sigma}_0(x_0, x_1, 1), & \sigma_1(x_0, x_1) &:= \tilde{\sigma}_1(x_0, x_1, 1), \\ \sigma_0^{(b-1)}(x_0, x_1) &:= \tilde{\sigma}_0(x_0, x_1, 0), & \sigma_1^{(b-1)}(x_0, x_1) &:= \tilde{\sigma}_1(x_0, x_1, 0). \end{aligned}$$

Let $\gamma(x_0, x_1)$ be the homogeneous polynomial of degree $b-2$ such that $\gamma = 0$ defines the subscheme Γ on the line l. Since $C_0 \cap C_1$ is scheme-theoretically equal to $Z + \Gamma$, and l is disjoint from Z, the scheme-theoretic intersection $C_0 \cap C_1 \cap l$ coincides with Γ. Hence there exist linearly independent homogeneous linear forms $\lambda_0(x_0, x_1)$ and $\lambda_1(x_0, x_1)$ such that

$$\sigma_0^{(b-1)} = \lambda_0 \gamma, \quad \sigma_1^{(b-1)} = \lambda_1 \gamma.$$

By linear change of coordinates (x_0, x_1), we can assume that $\lambda_0 = x_1$ and $\lambda_1 = -x_0$. Then the section $(\sigma_0 dx_0 + \sigma_1 dx_1) \otimes e_b$ of $\Omega(b)$ on $\mathbb{P}^2 \setminus l$ extends to a global section s of $\Omega(b)$. We have $Z(s) \cap (\mathbb{P}^2 \setminus l) = C_0 \cap C_1 \cap (\mathbb{P}^2 \setminus l) = Z$. Because $l \not\subset Z(s)$, the subscheme $Z(s)$ is of dimension 0. Since the length $n = c_2(\Omega(b))$ of $\mathcal{O}_{Z(s)}$ is equal to that of $\mathcal{O}_Z$, we have $Z = Z(s)$.

Next we prove the uniqueness (up to multiplicative constants) of s satisfying $Z = Z(s)$. Let s' be another global section of $\Omega(b)$ such that $Z(s') = Z$. The morphism

$$\widetilde{\Phi}_Z \ : \ \mathbb{P}^2 \setminus Z \ \to \ \mathbb{P}^*(H^0(\mathbb{P}^2, \mathcal{I}_Z(b-1)))$$

defined by the linear system $|\mathcal{I}_Z(b-1)|$ does not depend on the choice of s. Let $\widetilde{P} \in \mathbb{P}^*(H^0(\mathbb{P}^2, \mathcal{I}_Z(b-1)))$ be a general point. By Remark 2.5, there exist lines l and l' of $\mathbb{P}^2$ such that $\widetilde{\Phi}_Z^{-1}(\widetilde{P})$ is equal to $\Gamma(s, l) = \Gamma(s', l')$. On the other hand, since the length $b-2$ of $\mathcal{O}_{\Gamma(s,l)}$ is ≥ 2 by the assumption $b \geq 4$, the subscheme $\Gamma(s, l)$ determines the line l containing $\Gamma(s, l)$ uniquely. Hence we have $l = l'$, which implies that $\Phi_s = \Phi_{s'}$. Therefore the linear isomorphisms φ_s and $\varphi_{s'}$ are equal up to a multiplicative constant, and hence so are s and s' by (2.5). □

Remark 2.8. If there exists a pair (C_0, C_1) of members of $|\mathcal{I}_Z(b-1)|$ satisfying the condition in Theorem 2.1 (ii), then a *general* pair of members of $|\mathcal{I}_Z(b-1)|$ also satisfies it.

3. Geometric properties of purely inseparable covers of $\mathbb{P}^2$. In this section, we assume that the ground field k is of positive characteristic p. We fix a multiple b of p greater than or equal to 4.

3.1. Definition of $\mathcal{U}_{p,b}$. Let $\mathcal{M}$ and $\mathcal{L}$ be line bundles on $\mathbb{P}^2$ corresponding to the invertible sheaves $\mathcal{O}_{\mathbb{P}^2}(b/p)$ and $\mathcal{O}_{\mathbb{P}^2}(b)$, respectively. We have a canonical isomorphism

$$\mathcal{M}^{\otimes p} \ \xrightarrow{\sim} \ \mathcal{L}. \tag{3.1}$$

Using this isomorphism, we have local trivializations of the line bundle $\mathcal{L}$ such that the transition functions are p-th powers, and hence the usual differentiation of functions defines a linear homomorphism

$$H^0(\mathbb{P}^2, \mathcal{L}) \ \to \ H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2} \otimes \mathcal{L}) = H^0(\mathbb{P}^2, \Omega(b)),$$

which we denote by $G \mapsto dG$. We put

$$\mathcal{V}_{p,b} := \{ \ H^p \in H^0(\mathbb{P}^2, \mathcal{L}) \ \mid \ H \in H^0(\mathbb{P}^2, \mathcal{M}) \ \}.$$

Note that $\mathcal{V}_{p,b}$ is a *linear* subspace of $H^0(\mathbb{P}^2, \mathcal{L})$, because we are in characteristic p. In fact, the kernel of the linear homomorphism $G \mapsto dG$ is equal to $\mathcal{V}_{p,b}$.

Let $[X_0, X_1, X_2]$ be homogeneous coordinates of $\mathbb{P}^2$, and let U be the affine part $\{X_2 \neq 0\}$ of $\mathbb{P}^2$, on which affine coordinates $x_0 := X_0/X_2$ and $x_1 := X_1/X_2$ are defined. Suppose that a global section G of $\mathcal{L}$ is given by a homogeneous polynomial $G(X_0, X_1, X_2)$ of degree b. Then dG is given by

$$dG|U = \left(\frac{\partial g}{\partial x_0} dx_0 + \frac{\partial g}{\partial x_1} dx_1\right) \otimes e_b,$$

where $g(x_0, x_1) := G(x_0, x_1, 1)$, and e_b is the section of $\mathcal{L}$ corresponding to X_2^b

Definition 3.1. Let G and G' be global sections of $\mathcal{L}$. We write $G \sim G'$ if there exist a non-zero constant c and a global section H of $\mathcal{M}$ such that $G = c\,G' + H^p$.

Remark 3.2. For a homogeneous polynomial $G := \sum_{i+j+k=b} a_{ijk} X_0^i X_1^j X_2^k$ of degree b, we put

$$\bar{G} := \sum_{(i,j,k) \not\equiv (0,0,0) \bmod p} a_{ijk} X_0^i X_1^j X_2^k.$$

Let G and G' be two global sections of $\mathcal{L}$. Then $G \sim G'$ holds if and only if there exists a non-zero constant c such that $\bar{G} = c\,\bar{G}'$.

Let G be a global section of $\mathcal{L}$. Using the isomorphism (3.1), we can define a subscheme Y_G of the total space of the line bundle $\mathcal{M}$ by the equation

$$w^p = G,$$

where w is a fiber coordinate of $\mathcal{M}$. We denote by

$$\pi_G : Y_G \to \mathbb{P}^2$$

the canonical projection, which is a purely inseparable finite morphism of degree p. It is easy to see that, set-theoretically, we have

$$\pi_G^{-1}(Z(dG)) = \mathrm{Sing}(Y_G).$$

Remark 3.3. If $G \sim G'$, then we have $Z(dG) = Z(dG')$, and the schemes Y_G and $Y_{G'}$ are isomorphic over $\mathbb{P}^2$.

Proposition 3.4. *For a global section G of $\mathcal{L}$, the following conditions are equivalent to each other:*

(i) *The subscheme $Z(dG)$ of $\mathbb{P}^2$ is reduced of dimension* 0.

(ii) *For any G' with $G' \sim G$, the curve defined by $G' = 0$ has only ordinary nodes as its singularities.*

(iii) *The surface Y_G has only rational double points of type A_{p-1} as its singularities.*

If G is chosen generally from $H^0(\mathbb{P}^2, \mathcal{L})$, then G satisfies these conditions.

Proof. Let P be an arbitrary point of $\mathbb{P}^2$, and Q the unique point of Y_G such that $\pi_G(Q) = P$. We fix affine coordinates (x_0, x_1) with the origin P on an affine part $U \subset \mathbb{P}^2$. Let G be expressed on U by an inhomogeneous polynomial of x_0 and x_1;

$$G|U = c_{00} + c_{10}x_0 + c_{01}x_1 + c_{20}x_0^2 + c_{11}x_0x_1 + c_{02}x_1^2 + (\text{terms of higher degrees}).$$

Let G' be another global section of $\mathcal{L}$ that is expressed on U by

$$G'|U = c'_{00} + c'_{10}x_0 + c'_{01}x_1 + c'_{20}x_0^2 + c'_{11}x_0x_1 + c'_{02}x_1^2 + (\text{terms of higher degrees}).$$

If $G \sim G'$, there exists a non-zero constant c such that

$$c'_{10} = c\, c_{10}, \quad c'_{01} = c\, c_{01}, \quad \text{and} \quad c'_{11} = c\, c_{11}.$$

If $p > 2$, we also have

$$c'_{20} = c\, c_{20}, \quad \text{and} \quad c'_{02} = c\, c_{02}.$$

Since $Z(dG)$ is defined by

$$\frac{\partial(G|U)}{\partial x_0} = \frac{\partial(G|U)}{\partial x_1} = 0$$

locally around P, we have the following equivalences, from which the equivalence of the conditions (i), (ii) and (iii) follows:

$P \notin Z(dG)$
$\iff c_{10} \neq 0$ or $c_{01} \neq 0$
$\iff$ if $G' \sim G$ and $G'(P) = 0$, then the curve defined by $G' = 0$ is smooth at P
$\iff Y_G$ is smooth at Q;

P is a reduced isolated point of $Z(dG)$
$\iff c_{10} = c_{01} = 0$ and $4c_{20}c_{02} - c_{11}^2 \neq 0$
$\iff$ if $G' \sim G$ and $G'(P) = 0$, then the curve defined by $G' = 0$ is reduced at P and has an ordinary node at P
$\iff Y_G$ has a rational double point of type A_{p-1} at Q.

As was shown above, the locus

$$N_P := \left\{ G \in H^0(\mathbb{P}^2, \mathcal{L}) \;\middle|\; \begin{array}{l} P \in Z(dG), \text{ and} \\ P \text{ is } \textit{not} \text{ a reduced isolated point of } Z(dG) \end{array} \right\}$$

is of codimension 3 in $H^0(\mathbb{P}^2, \mathcal{L})$ for any $P \in \mathbb{P}^2$. Therefore, if $G \in H^0(\mathbb{P}^2, \mathcal{L})$ is general, G is not contained in N_P for any $P \in \mathbb{P}^2$, and hence $Z(dG)$ is reduced of dimension 0. □

Definition 3.5. We denote by $\mathcal{U}_{p,b}$ the Zariski open dense subset of $H^0(\mathbb{P}^2, \mathcal{L})$ consisting of all G satisfying the conditions in Proposition 3.4. Note that, if $G \in \mathcal{U}_{p,b}$ and $G' \sim G$, then $G' \in \mathcal{U}_{p,b}$. For $G \in \mathcal{U}_{p,b}$, we put

$$k^\times G + \mathcal{V}_{p,b} := \{\, c\,G + H^p \mid c \in k^\times,\ H \in H^0(\mathbb{P}^2, \mathcal{M}) \,\} = \{\, G' \in \mathcal{U}_{p,b} \mid G \sim G' \,\}.$$

Remark 3.6. By the linear homomorphism

$$\varphi_{dG} \,:\, H^0(\mathbb{P}^2, \Theta(-1)) \,\to\, H^0(\mathbb{P}^2, \mathcal{I}_{Z(dG)}(b-1))$$

that is an isomorphism for $G \in \mathcal{U}_{p,b}$ in virtue of Proposition 2.4, we see that the 2-dimensional linear system $|\mathcal{I}_{Z(dG)}(b-1)|$ is spanned by the three curves defined by $\partial G/\partial X_0 = 0$, $\partial G/\partial X_1 = 0$ and $\partial G/\partial X_2 = 0$.

3.2. Geometric properties of X_G for $G \in \mathcal{U}_{p,b}$. From now on, we fix a polynomial $G \in \mathcal{U}_{p,b}$. Then $\mathrm{Sing}(Y_G)$ consists of $n = b^2 - 3b + 3$ rational double points of type A_{p-1}. Let

$$\phi_G : X_G \to \mathbb{P}^2$$

denote the composite of the minimal resolution $X_G \to Y_G$ of Y_G and the purely inseparable finite morphism π_G. We denote by $H_G \subset X_G$ the pull-back of a general line of $\mathbb{P}^2$ via ϕ_G.

Proposition 3.7. *The canonical divisor K_G of the nonsingular surface X_G is linearly equivalent to $(b - b/p - 3)H_G$.*

Proof. Let (x_0, x_1) be affine coordinates on an affine part U of $\mathbb{P}^2$ that contains $Z(dG)$, and let $g(x_0, x_1)$ be the inhomogeneous polynomial that corresponds to G on U. On the surface Y_G, we have

$$0 = d(w^p) = \frac{\partial g}{\partial x_0} dx_0 + \frac{\partial g}{\partial x_1} dx_1.$$

The rational 2-form

$$\frac{dw \wedge dx_0}{\partial g / \partial x_1} = -\frac{dw \wedge dx_1}{\partial g / \partial x_0}$$

is therefore regular and nowhere vanishing on the Zariski open dense subset

$$\pi_G^{-1}(U \setminus Z(dG)) = \pi_G^{-1}(U) \setminus \mathrm{Sing}(Y_G)$$

of Y_G. By direct calculation, we can show that this rational 2-form has a zero of order $b - b/p - 3$ along the pull-back $\pi_G^{-1}(l_\infty)$ of the line $l_\infty := \mathbb{P}^2 \setminus U$ at infinity. Since $\mathrm{Sing}(Y_G)$ consists of only rational double points, the canonical divisor of X_G is $(b - b/p - 3)$ times $\phi_G^{-1}(l_\infty)$. ◻

Definition 3.8. We denote by S_G the numerical Néron-Severi lattice of X_G, and by S_G^0 the sublattice of S_G that is generated by the class $[H_G]$, and the classes $[\Gamma_i]$ $(i = 1, \dots, n(p-1))$ of smooth rational curves Γ_i on X_G that are contracted to the singular points of Y_G.

Proposition 3.9. *The quotient group S_G/S_G^0 is a finite elementary p-group.*

Proof. Let C be a reduced irreducible curve on X_G. If $\phi_G(C)$ is a point, then C is one of the curves Γ_i, and hence $[C] \in S_G^0$. Suppose that $\phi_G(C)$ is of dimension 1. Let $D \subset \mathbb{P}^2$ denote the curve $\phi_G(C)$ with the reduced structure, and let $\widetilde{D} \subset X_G$ be the proper transform of D by ϕ_G. Obviously we have $[\widetilde{D}] \in S_G^0$. If the morphism $\phi_G|_C : C \to D$ is birational, then $\widetilde{D} = pC$ holds, because ϕ_G is purely inseparable of degree p over the generic point of D. Hence we have $p[C] \in S_G^0$. If $\phi_G|_C : C \to D$ is of degree > 1, then it must be of degree p and $C = \widetilde{D}$ holds, and hence $[C]$ is contained in S_G^0. ◻

Since $[H_G]$ and $[\Gamma_i]$ $(i = 1, \dots, n(p-1))$ are linearly independent in $S_G^0 \otimes \mathbb{Q}$, we obtain the following:

Corollary 3.10. *The rank of S_G is equal to $n(p-1) + 1$.*

Definition 3.11. A non-singular projective surface X is called *supersingular* (in the sense of Shioda) if the rank of the numerical Néron-Severi lattice of X is equal to the second Betti number $b_2(X)$.

Definition 3.12. A reduced irreducible surface X is called *unirational* if there exists a dominant rational map from $\mathbb{P}^2$ to X.

Proposition 3.13. *The surface X_G is unirational and supersingular.*

Proof. Let $k(x_0, x_1)$ be the rational function field of $\mathbb{P}^2$. Since $\phi_G : X_G \to \mathbb{P}^2$ is purely inseparable of degree p, the function field of X_G is contained in the purely transcendental extension $k(x_0^{1/p}, x_1^{1/p})$ of k. Therefore X_G is unirational. The supersingularity of X_G then follows from [18, Corollary 2]. □

Remark 3.14. Note that the second Betti number $n(p-1)+1$ of X_G is equal to that of a p-th cyclic cover of a *complex* projective plane branched along a nonsingular plane curve of degree b.

4. Global sections of $\Omega(b)$ in characteristic 2. From this section, we assume that $p = 2$. Let b be an even integer ≥ 4.

Let s be a global section of $\Omega(b)$ such that $Z(s)$ is reduced of dimension 0. Recall from Remark 2.5 that the 2-dimensional linear system $|\mathcal{I}_{Z(s)}(b-1)|$ defines a morphism

$$\Phi_s \;:\; \mathbb{P}^2 \setminus Z(s) \;\to\; \mathbb{P}^*(H^0(\mathbb{P}^2, \mathcal{I}_{Z(s)}(b-1))) \cong (\mathbb{P}^2)^\vee.$$

Proposition 4.1. *There exists a polynomial $G \in \mathcal{U}_{2,b}$ such that $s = dG$ holds if and only if the morphism Φ_s is inseparable.*

Proof. Recall that, for a general $l \in (\mathbb{P}^2)^\vee$, the inverse image of l by Φ_s is the divisor $\Gamma(s, l)$ of l with degree $b-2$ defined in Proposition 2.4. Therefore the following three conditions on s are equivalent to each other:

(i) The morphism Φ_s is inseparable.

(ii) For a general line $l \subset \mathbb{P}^2$, there exists a divisor $\Delta(s, l)$ of l with degree $b/2-1$ such that $\Gamma(s, l) = 2\Delta(s, l)$ holds.

(iii) Let (x_0, x_1) be general affine coordinates of $\mathbb{P}^2$, and let s be given on the affine part by $(\sigma_0 dx_0 + \sigma_1 dx_1) \otimes e_b$. Then there exists a homogeneous polynomial $\delta(x_0, x_1)$ of degree $b/2 - 1$ such that $\sigma_0^{(b-1)} = x_1\delta^2$ and $\sigma_1^{(b-1)} = x_0\delta^2$ hold.

Suppose that there exists $G \in \mathcal{U}_{2,b}$ such that $s = dG$. Let (x_0, x_1) be general affine coordinates on an affine part U. Then $G|U$ is written as follows;

$$\gamma_{00}(x_0, x_1)^2 + x_0\gamma_{10}(x_0, x_1)^2 + x_1\gamma_{01}(x_0, x_1)^2 + x_0x_1\gamma_{11}(x_0, x_1)^2,$$

where γ_{00} is an inhomogeneous polynomial of degree $\leq b/2$, and γ_{10}, γ_{01} and γ_{11} are inhomogeneous polynomials of degree $\leq b/2 - 1$. Then $s = dG$ is written on U as

$$((\gamma_{10}^2 + x_1\gamma_{11}^2)dx_0 + (\gamma_{01}^2 + x_0\gamma_{11}^2)dx_1) \otimes e_b.$$

Therefore the homogeneous part of γ_{11} of degree $b/2 - 1$ yields the polynomial δ required in the condition (iii).

Conversely, suppose that the condition (ii) holds. Again we choose affine coordinates (x_0, x_1) of $\mathbb{P}^2$ defined on an affine part $U \subset \mathbb{P}^2$ containing $Z(s)$, and let s be given by $(\sigma_0 dx_0 + \sigma_1 dx_1) \otimes e_b$ on U. Let l be a line defined by

$$x_0 + Ax_1 + B = 0 \quad (A, B \in k).$$

Then the hyperplane $V_l \subset H^0(\mathbb{P}^2, \Theta(-1))$ corresponding to l via ζ is generated by θ_∞ and θ_0, where

$$\theta_\infty|U = \left(A\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1}\right) \otimes e_{-1}, \quad \text{and} \quad \theta_0|U = \left(B\frac{\partial}{\partial x_0} + x_0\frac{\partial}{\partial x_0} + x_1\frac{\partial}{\partial x_1}\right) \otimes e_{-1}.$$

For $u \in k$, we put

$$\theta_u := u\theta_\infty + \theta_0 \quad \in \quad V_l.$$

The zero point $\zeta([\theta_u])$ of θ_u is $(Au + B, u) \in l$. The member C_u of the pencil $P_{s,l} \subset |\mathcal{I}_{Z(s)}(b-1)|$ corresponding to θ_u via the isomorphism φ_s is defined by

$$\varphi_s(\theta_u) = (Au + B)\sigma_0 + u\sigma_1 + (x_0\sigma_0 + x_1\sigma_1) = 0.$$

We put $t := x_1|_l$, which is an affine parameter of the line l. The divisor of l cut out by C_u is defined by the polynomial

$$\varphi_s(\theta_u)(At + B, t) = (u + t)(A\sigma_0(At + B, t) + \sigma_1(At + B, t))$$

of t. Therefore the pencil $\{l \cap C_u\}$ of divisors on l cut out by $P_{s,l}$ has a unique moving point $(Au + B, u)$ corresponding to the factor $u + t$, and the fixed part

$$\Gamma(s, l) = \{A\sigma_0(At + B, t) + \sigma_1(At + B, t) = 0\}.$$

By the assumption, we see that

$$\begin{aligned} &\frac{d}{dt}(A\sigma_0(At + B, t) + \sigma_1(At + B, t)) \\ &= A^2\frac{\partial\sigma_0}{\partial x_0}(At + B, t) + A\Big(\frac{\partial\sigma_0}{\partial x_1} + \frac{\partial\sigma_1}{\partial x_0}\Big)(At + B, t) + \frac{\partial\sigma_1}{\partial x_1}(At + B, t) \end{aligned}$$

is zero for generic (and hence all) A, B and t. Therefore we have

$$\frac{\partial\sigma_0}{\partial x_0} \equiv 0, \quad \frac{\partial\sigma_0}{\partial x_1} \equiv \frac{\partial\sigma_1}{\partial x_0}, \quad \frac{\partial\sigma_1}{\partial x_1} \equiv 0.$$

This implies that there exist polynomials α, β and γ such that

$$\sigma_0 = \alpha^2 + x_1\gamma^2, \quad \sigma_1 = \beta^2 + x_0\gamma^2.$$

We put

$$g := x_0\alpha^2 + x_1\beta^2 + x_0x_1\gamma^2,$$

and let G be the homogeneous polynomial of degree b obtained from g by homogenization. Since $\partial g/\partial x_0 = \sigma_0$ and $\partial g/\partial x_1 = \sigma_1$, we have $dG = s$. □

5. Codes arising from purely inseparable double covers of $\mathbb{P}^2$. We assume that $p = 2$ and that b is an even integer ≥ 4.

Remark on notation. From this section, we use typewriter fonts $\mathtt{Z}$, $\mathtt{S}^0_{\mathtt{Z}}$, $\mathtt{C}$, $\mathtt{S}_{\mathtt{Z}}(\mathtt{C})$, $\mathtt{h}$, $\mathtt{e}_{\mathtt{P}}$ and $\mathtt{P} \in \mathtt{Z}$ in the situation where we are dealing with abstract codes and lattices in order to distinguish them from the corresponding objects $Z(dG)$, S^0_G, $\mathcal{C}_G$, S_G, $[H_G]$, $[\Gamma_P]$ and $P \in Z(dG)$ of geometric origin.

5.1. The discriminant group of a lattice. In this subsection, we review the theory of discriminant groups of lattices due to Nikulin [12].

A *lattice* is a free $\mathbb{Z}$-module of finite rank with a non-degenerate symmetric bilinear form

$$\Lambda \times \Lambda \;\to\; \mathbb{Z}$$

denoted by $(u, v) \mapsto uv$. A lattice Λ is said to be *even* if $u^2 \in 2\mathbb{Z}$ holds for every $u \in \Lambda$. For a lattice Λ, let $\Lambda^\vee$ denote the $\mathbb{Z}$-module $\mathrm{Hom}(\Lambda, \mathbb{Z})$. We have a natural injective homomorphism $\Lambda \hookrightarrow \Lambda^\vee$, whose cokernel

$$\mathrm{DG}(\Lambda) := \Lambda^\vee/\Lambda$$

is called the *discriminant group* of Λ. The order of $\mathrm{DG}(\Lambda)$ is equal, up to sign, to the discriminant $\operatorname{disc}\Lambda$ of Λ. We denote by

$$\mathrm{pr}_\Lambda \,:\, \Lambda^\vee \;\to\; \mathrm{DG}(\Lambda)$$

the natural projection. We have a $\mathbb{Q}$-valued symmetric bilinear form on $\Lambda^\vee$ that extends the symmetric bilinear form on Λ. Hence a symmetric bilinear form

$$b_\Lambda \,:\, \mathrm{DG}(\Lambda) \times \mathrm{DG}(\Lambda) \;\to\; \mathbb{Q}/\mathbb{Z}$$

is defined. When Λ is an even lattice, the quadratic form $u \mapsto u^2$ on $\Lambda^\vee$ induces a quadratic form

$$q_\Lambda \,:\, \mathrm{DG}(\Lambda) \;\to\; \mathbb{Q}/2\mathbb{Z}$$

on $\mathrm{DG}(\Lambda)$ that relates to b_Λ by

$$b_\Lambda(u, v) = \frac{1}{2}\big(\, q_\Lambda(u + v) - q_\Lambda(u) - q_\Lambda(v)\,\big).$$

DEFINITION 5.1. For a subgroup H of $\mathrm{DG}(\Lambda)$, we put

$$H^\perp := \{\, u \in \mathrm{DG}(\Lambda) \;\mid\; b_\Lambda(u, v) = 0 \quad \text{for all} \quad v \in H \,\}.$$

A subgroup H of $\mathrm{DG}(\Lambda)$ is called *b-isotropic* if H is contained in $H^\perp$. When Λ is even, we say that H is *q-isotropic* if $q_\Lambda(u) = 0$ holds for every $u \in H$.

An *overlattice* of Λ is a submodule Λ' of $\Lambda^\vee$ such that Λ' contains Λ and that the $\mathbb{Q}$-valued symmetric bilinear form of $\Lambda^\vee$ takes values in $\mathbb{Z}$ on Λ'. Let Λ'' be a lattice, and suppose that there exists an injective isometry $\Lambda \hookrightarrow \Lambda''$ such that Λ''/Λ is finite. Then we have a canonical injection $\Lambda'' \hookrightarrow \Lambda^\vee$, and Λ'' can be regarded as an overlattice of Λ. When Λ' is an overlattice of Λ, we have a sequence

$$\Lambda \;\subset\; \Lambda' \;\subset\; (\Lambda')^\vee \;\subset\; \Lambda^\vee$$

of submodules of $\Lambda^\vee$ such that $[\Lambda' : \Lambda] = [\Lambda^\vee : (\Lambda')^\vee]$.

PROPOSITION 5.2 (Nikulin [12]). *Let Λ be a lattice.*

(1) *The correspondence*

$$\Lambda' \mapsto H_{\Lambda'} := \mathrm{pr}_\Lambda(\Lambda'), \quad H \mapsto \Lambda'_H := \mathrm{pr}_\Lambda^{-1}(H)$$

gives rise to a bijection between the set of overlattices of Λ and the set of b-isotropic subgroups of $\mathrm{DG}(\Lambda)$. We have $\Lambda'_H/\Lambda = H$ and $(\Lambda'_H)^\vee/\Lambda = H^\perp$. In particular, the discriminant group $\mathrm{DG}(\Lambda'_H)$ is isomorphic to $H^\perp/H$.

(2) *Suppose that Λ is even. Then the above correspondence yields a bijection between the set of even overlattices of Λ and the set of q-isotropic subgroups of $\mathrm{DG}(\Lambda)$.*

5.2. Certain hyperbolic 2-elementary lattices and associated codes.

Definition 5.3. A lattice Λ is called *hyperbolic* if the signature of the real quadratic form on $\Lambda \otimes \mathbb{R}$ is $(1, \operatorname{rank} \Lambda - 1)$.

Definition 5.4. A lattice Λ is called 2-*elementary* if the finite abelian group $\mathrm{DG}(\Lambda)$ is 2-elementary, that is, if $\mathrm{DG}(\Lambda)$ is an $\mathbb{F}_2$-vector space of dimension $\log_2 |\operatorname{disc} \Lambda|$.

A 2-elementary lattice Λ is called *of type I* if $u^2 \in \mathbb{Z}$ holds for every $u \in \Lambda^\vee$, that is, if $b_\Lambda(x, x) = 0$ holds for every $x \in \mathrm{DG}(\Lambda)$.

Let $\mathtt{Z}$ be a finite set. (See Remark on notation.) We identify the $\mathbb{F}_2$-vector space $\mathbb{F}_2^{\mathtt{Z}}$ of functions from $\mathtt{Z}$ to $\mathbb{F}_2$ with the power set $\operatorname{Pow}(\mathtt{Z})$ of $\mathtt{Z}$ by

$$v \in \mathbb{F}_2^{\mathtt{Z}} \quad \mapsto \quad v^{-1}(1) \subset \mathtt{Z}.$$

A structure of the $\mathbb{F}_2$-vector space on $\operatorname{Pow}(\mathtt{Z})$ is therefore defined by

$$A + B = (A \cup B) \setminus (A \cap B) \qquad (A, B \subset \mathtt{Z}).$$

An element of $\operatorname{Pow}(\mathtt{Z})$ is called a *word*. For a word $A \subset \mathtt{Z}$, the cardinality $|A|$ is called the *weight* of A.

We consider an even hyperbolic 2-elementary lattice

$$\mathtt{S}^0_{\mathtt{Z}} := \bigoplus_{\mathtt{P} \in \mathtt{Z}} \mathbb{Z} \mathtt{e}_{\mathtt{P}} \oplus \mathbb{Z} \mathtt{h}$$

with the symmetric bilinear form given by

$$\mathtt{e}_{\mathtt{P}} \mathtt{e}_{\mathtt{Q}} = \begin{cases} -2 & \text{if } \mathtt{P} = \mathtt{Q} \\ 0 & \text{if } \mathtt{P} \neq \mathtt{Q} \end{cases}, \quad \mathtt{e}_{\mathtt{P}} \mathtt{h} = 0, \quad \mathtt{h}^2 = 2.$$

Then we have

$$(\mathtt{S}^0_{\mathtt{Z}})^\vee = \bigoplus_{\mathtt{P} \in \mathtt{Z}} \mathbb{Z}\,(\mathtt{e}_{\mathtt{P}}/2) \oplus \mathbb{Z}\,(\mathtt{h}/2) \quad \subset \quad \mathtt{S}^0_{\mathtt{Z}} \otimes \mathbb{Q}.$$

The discriminant group $\mathrm{DG}(\mathtt{S}^0_{\mathtt{Z}})$ is therefore naturally identified with

$$\mathbb{F}_2^{\mathtt{Z}} \oplus \mathbb{F}_2 = \operatorname{Pow}(\mathtt{Z}) \oplus \mathbb{F}_2$$

in such a way that a vector

$$\sum (a_{\mathtt{P}}/2)\, \mathtt{e}_{\mathtt{P}} + (b/2)\, \mathtt{h} \qquad (a_{\mathtt{P}}, b \in \mathbb{Z})$$

of $(\mathtt{S}^0_{\mathtt{Z}})^\vee$ corresponds to

$$(A, b \bmod 2) \in \operatorname{Pow}(\mathtt{Z}) \oplus \mathbb{F}_2, \qquad \text{where } A = \{\mathtt{P} \in \mathtt{Z} \mid a_{\mathtt{P}} \equiv 1 \bmod 2\}.$$

Hence we can consider subgroups of $\mathrm{DG}(\mathtt{S}^0_{\mathtt{Z}})$ as binary linear codes in $\operatorname{Pow}(\mathtt{Z}) \oplus \mathbb{F}_2$. Under this identification, the symmetric bilinear form $b_{\mathtt{S}^0_{\mathtt{Z}}}$ on $\mathrm{DG}(\mathtt{S}^0_{\mathtt{Z}})$ is given by

$$((A, \alpha), (A', \alpha')) \mapsto \begin{cases} (-|A \cap A'| + 1)/2 \bmod \mathbb{Z} & \text{if } \alpha = \alpha' = 1, \\ -|A \cap A'|/2 \bmod \mathbb{Z} & \text{otherwise,} \end{cases}$$

and the quadratic form $q_{\mathtt{S}^0_{\mathtt{Z}}}$ on $\mathrm{DG}(\mathtt{S}^0_{\mathtt{Z}})$ is given by

$$(A,\alpha)\mapsto\begin{cases}(-|A|+1)/2 \bmod 2\mathbb{Z} & \text{if } \alpha=1,\\ -|A|/2 \bmod 2\mathbb{Z} & \text{if } \alpha=0.\end{cases}$$

Therefore, from Proposition 5.2, we obtain the following:

COROLLARY 5.5. *Let* $\widetilde{\mathtt{C}}$ *be a code in* $\mathrm{Pow}(\mathtt{Z})\oplus\mathbb{F}_2$*, which is considered as a subgroup of* $\mathrm{DG}(\mathtt{S}^0_{\mathtt{Z}})$ *by the identification above.*

(1) *If the submodule* $\mathrm{pr}^{-1}_{\mathtt{S}^0_{\mathtt{Z}}}(\widetilde{\mathtt{C}})$ *of* $(\mathtt{S}^0_{\mathtt{Z}})^\vee$ *corresponding to* $\widetilde{\mathtt{C}}$ *is an overlattice of* $\mathtt{S}^0_{\mathtt{Z}}$*, then the following holds;*

$$(5.1)\qquad |A| \bmod 2\equiv\alpha \qquad \textit{for every } (A,\alpha)\in\widetilde{\mathtt{C}}.$$

(2) *The submodule* $\mathrm{pr}^{-1}_{\mathtt{S}^0_{\mathtt{Z}}}(\widetilde{\mathtt{C}})$ *is an even overlattice of* $\mathtt{S}^0_{\mathtt{Z}}$ *if and only if every* $(A,\alpha)\in\widetilde{\mathtt{C}}$ *satisfies*

$$|A|\equiv\begin{cases}0 \bmod 4 & \textit{if } \alpha=0,\\ 1 \bmod 4 & \textit{if } \alpha=1.\end{cases}$$

We denote by

$$\rho_{\mathtt{Z}}\,:\,\mathrm{Pow}(\mathtt{Z})\oplus\mathbb{F}_2\,\to\,\mathrm{Pow}(\mathtt{Z})$$

the projection onto the first factor.

DEFINITION 5.6. Let $\mathtt{C}$ be an arbitrary code in $\mathrm{Pow}(\mathtt{Z})$. We put

$$\mathtt{C}^\sim:=\{\,(A,\alpha)\in\mathrm{Pow}(\mathtt{Z})\oplus\mathbb{F}_2\;\mid\;A\in\mathtt{C}\quad\text{and}\quad|A| \bmod 2=\alpha\,\},$$

and call it the *lift* of $\mathtt{C}$. It is obvious that $\mathtt{C}^\sim$ is a linear subspace of $\mathrm{Pow}(\mathtt{Z})\oplus\mathbb{F}_2$, that $\dim\mathtt{C}^\sim$ is equal to $\dim\mathtt{C}$, and that $\mathtt{C}^\sim$ is the unique code satisfying (5.1) and $\rho_{\mathtt{Z}}(\mathtt{C}^\sim)=\mathtt{C}$.

We denote by $\mathtt{S}_{\mathtt{Z}}(\mathtt{C})$ the submodule $\mathrm{pr}^{-1}_{\mathtt{S}^0_{\mathtt{Z}}}(\mathtt{C}^\sim)$ of $(\mathtt{S}^0_{\mathtt{Z}})^\vee$.

If the submodule $\mathtt{S}_{\mathtt{Z}}(\mathtt{C})$ of $(\mathtt{S}^0_{\mathtt{Z}})^\vee$ is an overlattice of $\mathtt{S}^0_{\mathtt{Z}}$, then we have

$$(5.2)\qquad |\operatorname{disc}(\mathtt{S}_{\mathtt{Z}}(\mathtt{C}))|=2^{n+1}/|\mathtt{C}|^2.$$

Moreover the lattice $\mathtt{S}_{\mathtt{Z}}(\mathtt{C})$ is hyperbolic and 2-elementary, because so is $\mathtt{S}^0_{\mathtt{Z}}$. From Proposition 5.2, we obtain the following:

PROPOSITION 5.7. *The submodule* $\mathtt{S}_{\mathtt{Z}}(\mathtt{C})$ *of* $(\mathtt{S}^0_{\mathtt{Z}})^\vee$ *is an even overlattice of* $\mathtt{S}^0_{\mathtt{Z}}$ *if and only if* $|A|\equiv 0$ *or* $1 \bmod 4$ *holds for every* $A\in\mathtt{C}$.

PROPOSITION 5.8. *Suppose that* $n=|\mathtt{Z}|$ *is odd, and that* $\mathtt{S}_{\mathtt{Z}}(\mathtt{C})$ *is an overlattice of* $\mathtt{S}^0_{\mathtt{Z}}$*. If* $\mathtt{C}$ *contains the word* $\mathtt{Z}$*, then the* 2*-elementary lattice* $\mathtt{S}_{\mathtt{Z}}(\mathtt{C})$ *is of type I.*

Proof. Suppose that $\mathtt{C}$ contains $\mathtt{Z}$. Then $\mathtt{C}^\sim$ contains $(\mathtt{Z},1)$ because $|\mathtt{Z}|$ is odd. If $(A,\alpha)\in(\mathtt{C}^\sim)^\perp$, then

$$b_{\mathtt{S}^0_{\mathtt{Z}}}((\mathtt{Z},1),(A,\alpha))=(-|A|+\alpha)/2=0\quad\text{in}\quad\mathbb{Q}/\mathbb{Z},$$

and hence

$$b_{\mathtt{S}^0_{\mathtt{Z}}}((A,\alpha),(A,\alpha))=(-|A|+\alpha)/2=0.$$

If $u\in(\mathtt{S}_{\mathtt{Z}}(\mathtt{C}))^\vee$, then $u \bmod \mathtt{S}^0_{\mathtt{Z}}\in\mathrm{DG}(\mathtt{S}^0_{\mathtt{Z}})$ is contained in $(\mathtt{C}^\sim)^\perp$, and therefore $u^2\in\mathbb{Z}$ holds. Hence $\mathtt{S}_{\mathtt{Z}}(\mathtt{C})$ is of type I. □

5.3. The lattice S_G and the associated code. We fix a polynomial $G \in \mathcal{U}_{2,b}$. Then $\mathrm{Sing}(Y_G)$ consists of $n = b^2 - 3b + 3$ ordinary nodes that are mapped bijectively to the points of $Z(dG)$.

DEFINITION 5.9. For a point $P \in Z(dG)$, we denote by Γ_P the (-2)-curve on X_G that is contracted to P by $\phi_G : X_G \to \mathbb{P}^2$.

In the numerical Néron-Severi lattice S_G of X_G, we have

$$[\Gamma_P][\Gamma_Q] = \begin{cases} -2 & \text{if } P = Q \\ 0 & \text{if } P \neq Q \end{cases}, \quad [\Gamma_P][H_G] = 0, \quad [H_G]^2 = 2.$$

By sending $\mathtt{e}_P$ to $[\Gamma_P]$ and $\mathtt{h}$ to $[H_G]$, we obtain an isomorphism

$$\mathtt{S}^0_{Z(dG)} \;\cong\; S^0_G. \tag{5.3}$$

Hence $\mathrm{DG}(S^0_G)$ is identified with $\mathrm{Pow}(Z(dG)) \oplus \mathbb{F}_2$. Since S_G/S^0_G is finite by Proposition 3.9, we can regard S_G as an overlattice of S^0_G.

DEFINITION 5.10. We put

$$\begin{aligned} \widetilde{\mathcal{C}}_G &:= S_G/S^0_G \;\subset\; \mathrm{DG}(S^0_G) = \mathrm{Pow}(Z(dG)) \oplus \mathbb{F}_2, \quad \text{and} \\ \mathcal{C}_G &:= \rho_{Z(dG)}(\widetilde{\mathcal{C}}_G) \;\subset\; \mathrm{Pow}(Z(dG)). \end{aligned}$$

Note that $\widetilde{\mathcal{C}}_G$ is the lift $\mathcal{C}_G^{\sim}$ of $\mathcal{C}_G$, and that the overlattice $\mathtt{S}_{Z(dG)}(\mathcal{C}_G) = \mathrm{pr}^{-1}_{\mathtt{S}^0_{Z(dG)}}(\widetilde{\mathcal{C}}_G)$ of $\mathtt{S}^0_{Z(dG)}$ corresponding to $\mathcal{C}_G$ is identified with the overlattice S_G of S^0_G by the isomorphism (5.3).

PROPOSITION 5.11. (1) *Suppose that $b/2$ is odd. Then $|A| \equiv 0$ or $1 \bmod 4$ for every $A \in \mathcal{C}_G$.* (2) *Suppose that $b/2$ is even. Then $|A| \equiv 0$ or $3 \bmod 4$ for every $A \in \mathcal{C}_G$.*

Proof. Let K_G be the canonical divisor of X_G. By Proposition 3.7, we have $[K_G] = (b/2 - 3)\,[H_G]$ in S_G. Let A be a word in $\mathcal{C}_G$. Suppose that $|A|$ is even. Then we have $(A, 0) \in \widetilde{\mathcal{C}}_G$, and hence the vector

$$v := \frac{1}{2} \sum_{P \in A} [\Gamma_P]$$

of $(S^0_G)^\vee$ is contained in S_G. Since $v^2 = -|A|/2$ and $v \cdot [K_G] = 0$, we have

$$(v^2 - v \cdot [K_G])/2 = -|A|/4,$$

which is an integer by the the Riemann-Roch theorem. Therefore $|A| \equiv 0 \bmod 4$ holds. Suppose that $|A|$ is odd. Then we have $(A, 1) \in \widetilde{\mathcal{C}}_G$, and hence

$$w := \frac{1}{2}\Big(\sum_{P \in A} [\Gamma_P] \;+\; [H_G] \Big)$$

is contained in S_G. From

$$(w^2 - w \cdot [K_G])/2 = (7 - |A| - b)/4 \;\in\; \mathbb{Z},$$

we have $|A| + b \equiv 3 \bmod 4$. $\square$

5.4. Geometric realizability of an abstract code. Let $\mathtt{Z}$ be a finite set with

$$|\mathtt{Z}| = n = b^2 - 3b + 3.$$

The symmetric group $\mathfrak{S}_n$ acts on $\mathtt{Z}$ and $\mathrm{Pow}(\mathtt{Z})$.

DEFINITION 5.12. Two codes $\mathtt{C}$ and $\mathtt{C}'$ in $\mathrm{Pow}(\mathtt{Z})$ are said to be $\mathfrak{S}_n$-*equivalent* if there exists $\tau \in \mathfrak{S}_n$ such that $\tau(\mathtt{C}) = \mathtt{C}'$. We denote by $[\mathtt{C}]$ the $\mathfrak{S}_n$-equivalence class of codes containing the code $\mathtt{C} \subset \mathrm{Pow}(\mathtt{Z})$.

DEFINITION 5.13. Let $\mathtt{C}$ be a code in $\mathrm{Pow}(\mathtt{Z})$, and let $[\mathtt{C}]$ be the $\mathfrak{S}_n$-equivalence class of codes containing $\mathtt{C}$. We say that $[\mathtt{C}]$ is *geometrically realizable* if there exist $G \in \mathcal{U}_{2,b}$ and a bijection $\mathtt{Z} \xrightarrow{\sim} Z(dG)$ that maps $\mathtt{C} \subset \mathrm{Pow}(\mathtt{Z})$ to $\mathcal{C}_G \subset \mathrm{Pow}(Z(dG))$.

DEFINITION 5.14. Let $[\mathtt{C}]$ and $[\mathtt{C}']$ be two $\mathfrak{S}_n$-equivalence classes of codes in $\mathrm{Pow}(\mathtt{Z})$. We write $[\mathtt{C}] < [\mathtt{C}']$ if there exist representatives $\mathtt{C} \in [\mathtt{C}]$ and $\mathtt{C}' \in [\mathtt{C}']$ such that $\mathtt{C} \subsetneq \mathtt{C}'$.

Let $[\mathtt{C}]$ be a geometrically realizable class of codes. We put

$$\mathcal{U}_{2,b,[\mathtt{C}]} := \{\, G \in \mathcal{U}_{2,b} \ \mid\ \mathtt{C} \cong \mathcal{C}_G \quad \text{by some bijection } \mathtt{Z} \cong Z(dG) \,\}, \quad \text{and}$$

$$\mathcal{U}_{2,b,\geq[\mathtt{C}]} := \bigsqcup_{[\mathtt{C}'] \geq [\mathtt{C}]} \mathcal{U}_{2,b,[\mathtt{C}']}.$$

THEOREM 5.15. *For every* $[\mathtt{C}]$*, the locus* $\mathcal{U}_{2,b,\geq[\mathtt{C}]}$ *is Zariski closed in* $\mathcal{U}_{2,b}$*.*

Proof. Let $\widetilde{\mathcal{U}}_{2,b} \to \mathcal{U}_{2,b}$ be the étale covering of degree $n!$ over $\mathcal{U}_{2,b}$ such that each point of $\widetilde{\mathcal{U}}_{2,b}$ over $G \in \mathcal{U}_{2,b}$ is a pair (G, τ_G), where τ_G is a bijection from $\mathtt{Z}$ to $Z(dG)$. For a word $A \in \mathrm{Pow}(\mathtt{Z})$, we put

$$\widetilde{\mathcal{U}}_A := \{\, (G, \tau_G) \in \widetilde{\mathcal{U}}_{2,b} \ \mid\ \tau_G(A) \in \mathcal{C}_G \,\}.$$

Since the specialization homomorphism of numerical Néron-Severi lattices is injective for a smooth family of projective varieties, the locus $\widetilde{\mathcal{U}}_A$ is Zariski closed in $\widetilde{\mathcal{U}}_{2,b}$. For a geometrically realizable class $[\mathtt{C}]$, the closed subset

$$\bigcup_{\mathtt{C} \in [\mathtt{C}]} \Big(\bigcap_{A \in \mathtt{C}} \widetilde{\mathcal{U}}_A \Big)$$

of $\widetilde{\mathcal{U}}_{2,b}$ is invariant under the $\mathfrak{S}_n$-action on $\widetilde{\mathcal{U}}_{2,b}$ over $\mathcal{U}_{2,b}$, and is the pull-back of the locus $\mathcal{U}_{2,b,\geq[\mathtt{C}]}$. Therefore $\mathcal{U}_{2,b,\geq[\mathtt{C}]}$ is closed in $\mathcal{U}_{2,b}$. □

COROLLARY 5.16. *For every geometrically realizable class* $[\mathtt{C}]$ *of codes, the locus* $\mathcal{U}_{2,b,[\mathtt{C}]}$ *is locally Zariski closed in* $\mathcal{U}_{2,b}$*.*

Remark 5.17. The étale covering $\widetilde{\mathcal{U}}_{2,b} \to \mathcal{U}_{2,b}$ that has appeared in the proof of Theorem 5.15 is constructed as follows. Let $\mathcal{Z} \to \mathcal{U}_{2,b}$ be the universal family

$$\{\, (P, G) \in \mathbb{P}^2 \times \mathcal{U}_{2,b} \ \mid\ P \in Z(dG) \,\} \quad \to \quad \mathcal{U}_{2,b}$$

of $Z(dG)$, which is an étale covering of degree n. We fix a base point $G_0 \in \mathcal{U}_{2,b}$, and let

$$\mu \,:\, \pi_1(\mathcal{U}_{2,b}, G_0) \,\to\, \mathrm{Aut}(Z(dG_0)) \cong \mathfrak{S}_n$$

be the monodromy action of the algebraic fundamental group of $\mathcal{U}_{2,b}$ on the set $Z(dG_0)$. Let $\widetilde{\mathcal{Z}} \to \mathcal{U}_{2,b}$ be the Galois closure of $\mathcal{Z} \to \mathcal{U}_{2,b}$, which is an étale cover of degree equal to the cardinality of $\mathrm{Im}\,\mu$. Then $\widetilde{\mathcal{U}}_{2,b}$ is a disjoint union of $[\mathfrak{S}_n : \mathrm{Im}\,\mu]$ copies of $\widetilde{\mathcal{Z}}$.

5.5. An algorithm for making lists of codes. In this subsection, we describe an algorithm that will be used in §9, when we make the complete list of geometrically realizable classes of codes for supersingular $K3$ surfaces in characteristic 2.

Let $\mathtt{Z}$ be a finite set with $|\mathtt{Z}| = n$. Suppose that we are given a subset WT of $\{0, 1, 2, \dots, n\}$.

PROBLEM 5.18. Make the complete list L_k ($k = 1, \dots, n$) of the $\mathfrak{S}_n$-equivalence classes $[\mathtt{C}]$ of codes $\mathtt{C} \subset \mathrm{Pow}(\mathtt{Z})$ with the following properties;

(a) $\dim \mathtt{C} = k$,
(b) $\mathtt{Z} \in \mathtt{C}$, and
(c) $|A| \in \mathrm{WT}$ for every $A \in \mathtt{C}$.

First we fix some notation and terminologies. For a code $\mathtt{C} \subset \mathrm{Pow}(\mathtt{Z})$, we put

$$\mathrm{wtenum}(\mathtt{C}) := \sum_{A \in \mathtt{C}} x^{|A|},$$

where x is a formal variable. Let $\mathbf{A} = (A_0, \dots, A_{k-1})$ be a sequence of words $A_i \in \mathrm{Pow}(\mathtt{Z})$. We denote by $\langle \mathbf{A} \rangle \subset \mathrm{Pow}(\mathtt{Z})$ the code generated by $A_0, \dots, A_{k-1}$. A sequence $\mathbf{A}$ of length k is called *linearly independent* if $\dim\langle \mathbf{A} \rangle = k$. We put

$$\mathrm{wt}(\mathbf{A}) := (|A_0|, \dots, |A_{k-1}|).$$

For another word $A \in \mathrm{Pow}(\mathtt{Z})$, we write

$$(\mathbf{A}, A) := (A_0, \dots, A_{k-1}, A).$$

For $\tau \in \mathfrak{S}_n$, we put

$$\tau(\mathbf{A}) := (\tau(A_0), \dots, \tau(A_{k-1})).$$

We define a sequence $\tilde{\omega}(\mathbf{A})$ of length 2^k by the following:

- If $\mathbf{A} = (A_0)$, then $\tilde{\omega}(\mathbf{A}) := (\mathtt{Z}, A_0)$.
- Suppose that $k > 1$. We put $\mathbf{A}' := (A_0, \dots, A_{k-2})$, and let the sequence $\tilde{\omega}(\mathbf{A}')$ be $(B_1, \dots, B_{2^{k-1}})$. Then we define

$$\tilde{\omega}(\mathbf{A}) := (B_1, \dots, B_{2^{k-1}}, B_1 \cap A_{k-1}, \dots, B_{2^{k-1}} \cap A_{k-1}).$$

We then define a sequence $\omega(\mathbf{A})$ of non-negative integers by

$$\omega(\mathbf{A}) := \mathrm{wt}(\tilde{\omega}(\mathbf{A})).$$

Suppose that we are given $\omega(\mathbf{A})$. Then, for any subsets I and J of $\{0, 1, \dots, k-1\}$, the cardinality

$$|\bigcap_{i \in I} A_i \cap \bigcap_{j \in J} (\mathtt{Z} \setminus A_j)|$$

can be obtained from $\omega(\mathbf{A})$. Therefore, for two sequences $\mathbf{A}$ and $\mathbf{A}'$, there exists $\tau \in \mathfrak{S}_n$ such that $\tau(\mathbf{A}) = \mathbf{A}'$ if and only if $\omega(\mathbf{A}) = \omega(\mathbf{A}')$ holds. In particular, we have the following:

PROPOSITION 5.19. *Let $\mathbf{A}$ be a sequence of words, and let $[\mathtt{C}']$ be an $\mathfrak{S}_n$-equivalence class of codes containing $\mathtt{C}'$. Then $[\langle \mathbf{A} \rangle] \leqq [\mathtt{C}']$ holds if and only if there exists a sequence $\mathbf{A}'$ of words of $\mathtt{C}'$ such that $\omega(\mathbf{A}) = \omega(\mathbf{A}')$.*

The following subroutine determines whether two codes $\langle \mathbf{A} \rangle$ and $\langle \mathbf{A}' \rangle$ given by sequences $\mathbf{A}$ and $\mathbf{A}'$ are $\mathfrak{S}_n$-equivalent or not.

SUBROUTINE 5.20. First we calculate $\dim\langle \mathbf{A} \rangle$ and $\dim\langle \mathbf{A}' \rangle$. If they differ, then $\langle \mathbf{A} \rangle$ and $\langle \mathbf{A}' \rangle$ are not $\mathfrak{S}_n$-equivalent. Otherwise, we calculate the weight enumerators $\mathrm{wtenum}(\langle \mathbf{A} \rangle)$ and $\mathrm{wtenum}(\langle \mathbf{A}' \rangle)$. If they differ, then $\langle \mathbf{A} \rangle$ and $\langle \mathbf{A}' \rangle$ are not $\mathfrak{S}_n$-equivalent. Otherwise, we calculate $\omega(\mathbf{A})$, and search for a sequence $\mathbf{A}''$ of words of $\langle \mathbf{A}' \rangle$ such that $\omega(\mathbf{A}) = \omega(\mathbf{A}'')$. Note that, if $\mathbf{A}''$ satisfies $\omega(\mathbf{A}) = \omega(\mathbf{A}'')$, then $\dim\langle \mathbf{A}'' \rangle = \dim\langle \mathbf{A} \rangle = \dim\langle \mathbf{A}' \rangle$ holds and hence $\langle \mathbf{A}'' \rangle$ coincides with $\langle \mathbf{A}' \rangle$. The codes $\langle \mathbf{A} \rangle$ and $\langle \mathbf{A}' \rangle$ are $\mathfrak{S}_n$-equivalent if and only if such a sequence $\mathbf{A}''$ is found.

We label the elements of $\mathtt{Z}$ as $\{\mathtt{P}_0, \dots, \mathtt{P}_{n-1}\}$, and represent a word A of $\mathrm{Pow}(\mathtt{Z})$ by a bit vector

$$v(A) := [\alpha_0, \dots, \alpha_{n-1}],$$

where $\alpha_i = 0$ (resp. $\alpha_i = 1$) if $\mathtt{P}_i \notin A$ (resp. $\mathtt{P}_i \in A$). For a column bit vector $\mathbf{b} = {}^T[\beta_0, \dots, \beta_{k-1}]$, we put

$$\mu(\mathbf{b}) := 2^{k-1}\beta_0 + 2^{k-2}\beta_1 + \cdots + 2\beta_{k-2} + \beta_{k-1} \quad \in \quad \mathbb{Z}_{\geq 0}.$$

A sequence $\mathbf{A} = (A_0, \dots, A_{k-1})$ is called $\mathfrak{S}_n$*-increasing* if the column vectors of the $k \times n$ matrix

$$\begin{bmatrix} v(A_0) \\ \vdots \\ v(A_{k-1}) \end{bmatrix} = [\, \mathbf{b}_0, \dots, \mathbf{b}_{n-1} \,]$$

yield an increasing sequence $\mu(\mathbf{b}_0) \leq \cdots \leq \mu(\mathbf{b}_{n-1})$. The following proposition is obvious from the definition:

PROPOSITION 5.21. (1) *If $\mathbf{A} = (A_0, \dots, A_{k-1})$ is $\mathfrak{S}_n$-increasing, then the subsequence $(A_0, \dots, A_{m-1})$ of $\mathbf{A}$ is also $\mathfrak{S}_n$-increasing for any $m \leq k$,*

(2) *For any sequence $\mathbf{A} = (A_0, \dots, A_{k-1})$, there exists $\tau \in \mathfrak{S}_n$ such that $\tau(\mathbf{A})$ is $\mathfrak{S}_n$-increasing.*

(3) *Suppose that $\mathbf{A} = (A_0, \dots, A_{k-1})$ is $\mathfrak{S}_n$-increasing, and let $A \in \mathrm{Pow}(\mathtt{Z})$ be an arbitrary word. Then there exists $\tau \in \mathfrak{S}_n$ such that $\tau(\mathbf{A})$ coincides with $\mathbf{A}$ and that $(\mathbf{A}, \tau(A))$ is $\mathfrak{S}_n$-increasing.*

EXAMPLE 5.22. The sequence given by the first three row vectors of the matrix M below is $\mathfrak{S}_7$-increasing, while the sequence of length 4 given by all the row vectors of M is not $\mathfrak{S}_7$-increasing. By applying transpositions $\mathtt{P}_3 \leftrightarrow \mathtt{P}_4$ and $\mathtt{P}_5 \leftrightarrow \mathtt{P}_6$ to M, we obtain the matrix M', which yields the $\mathfrak{S}_7$-increasing sequence of length 4.

$$M := \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}, \qquad M' := \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Let $[\mathtt{C}]$ be an $\mathfrak{S}_n$-equivalence class satisfying the conditions (a), (b) and (c) in Problem 5.18. Then there exists a sequence $\mathbf{A} = (A_0, \dots, A_{k-1})$ of length k with the following properties;

- $\mathbf{A}$ is linearly independent, and $\langle \mathbf{A} \rangle \in [\mathtt{C}]$,

- $\mathbf{A}$ is $\mathfrak{S}_n$-increasing,
- $A_0 = \mathtt{Z}$, and $|A_i| \le n/2$ for $i = 1, \dots, k-1$.

Indeed, we have a linearly independent sequence $\mathbf{A}' = (A'_0, \dots, A'_{k-1})$ that is a basis of a code $\mathtt{C} \in [\mathtt{C}]$ with $A'_0 = \mathtt{Z}$. If there is a word A'_i $(i > 0)$ with $|A'_i| > n/2$, then we replace A'_i by $\mathtt{Z} + A'_i$ so that we can assume $|A'_i| \le n/2$ for $i = 1, \dots, k-1$. By applying a suitable permutation $\tau \in \mathfrak{S}_n$, the sequence $\mathbf{A} := \tau(\mathbf{A}')$ becomes $\mathfrak{S}_n$-increasing, which is a basis of the code $\tau(\mathtt{C})$ in the class $[\mathtt{C}]$.

Definition 5.23. A sequence $\mathbf{A}$ with these properties is called a *standard basis* of the $\mathfrak{S}_n$-equivalence class $[\mathtt{C}]$.

The complete list L_k that we want to make will be given as a set

$$\mathbf{L}_k = \{\mathbf{A}^{(1)}, \dots, \mathbf{A}^{(N)}\}$$

of standard bases of length k.

Proposition 5.24. *Suppose that the complete list L_k $(k \ge 1)$ has been given as a set $\mathbf{L}_k$ of standard bases of length k. Then Algorithm 5.25 below produces a set $\mathbf{L}_{k+1}$ of standard bases of length $k+1$ that gives the complete list L_{k+1}.*

Algorithm 5.25. *Step 1.* For each basis $\mathbf{A}^{(i)} \in \mathbf{L}_k$, we make the list $\mathcal{A}^{(i)}$ of words $A \in \mathrm{Pow}(\mathtt{Z})$ with the following properties;

(i) $|A| \le n/2$,
(ii) $(\mathbf{A}^{(i)}, A)$ is $\mathfrak{S}_n$-increasing, and
(iii) for any $B \in \langle \mathbf{A}^{(i)} \rangle$, $|B + A| \ne 0$ and $|B + A| \in \mathrm{WT}$.

In other words, $\mathcal{A}^{(i)}$ is the list of all $A \in \mathrm{Pow}(\mathtt{Z})$ such that $(\mathbf{A}^{(i)}, A)$ is a standard basis of an $\mathfrak{S}_n$-equivalence class of $(k+1)$-dimensional codes satisfying the conditions (b) and (c) in Problem 5.18.
Step 2. Set $\mathbf{L}_{k+1}$ to be an empty set.
Step 3. For each pair of $\mathbf{A}^{(i)} \in \mathbf{L}_k$ and $A \in \mathcal{A}^{(i)}$, we check whether there exists $\mathbf{A}' \in \mathbf{L}_{k+1}$ such that $\langle \mathbf{A}' \rangle$ and $\langle (\mathbf{A}^{(i)}, A) \rangle$ are $\mathfrak{S}_n$-equivalent by using Subroutine 5.20. If there are no such $\mathbf{A}'$, then we put $(\mathbf{A}^{(i)}, A)$ in $\mathbf{L}_{k+1}$.

Proof. It is obvious that, if $\mathbf{A} \in \mathbf{L}_{k+1}$, then $\langle \mathbf{A} \rangle$ is a $(k+1)$-dimensional code satisfying (b) and (c). It is also obvious that, if $\mathbf{A}$ and $\mathbf{A}'$ are distinct standard bases in $\mathbf{L}_{k+1}$, then $\langle \mathbf{A} \rangle$ and $\langle \mathbf{A}' \rangle$ are not $\mathfrak{S}_n$-equivalent. Therefore it is enough to show that, for an arbitrary $(k+1)$-dimensional code $\mathtt{C}$ satisfying (b) and (c), there exists an element of $\mathbf{L}_{k+1}$ that is a standard basis of $[\mathtt{C}]$.

Let $\mathbf{A} = (A_0, \dots, A_k)$ be a standard basis of $[\mathtt{C}]$. We put $\mathbf{A}' := (A_0, \dots, A_{k-1})$. Then $\langle \mathbf{A}' \rangle$ is a k-dimensional code satisfying (b) and (c). Hence there exists a standard basis $\mathbf{A}^{(i)} \in \mathbf{L}_k$ of the $\mathfrak{S}_n$-equivalence class $[\langle \mathbf{A}' \rangle]$. Let $\tau \in \mathfrak{S}_n$ be an element that maps the code $\langle \mathbf{A}' \rangle$ to $\langle \mathbf{A}^{(i)} \rangle$. We have

$$\langle (\mathbf{A}^{(i)}, \tau(A_k)) \rangle = \tau(\langle (\mathbf{A}', A_k) \rangle) = \tau(\langle \mathbf{A} \rangle) \in [\mathtt{C}].$$

Because $\mathbf{A}^{(i)}$ is $\mathfrak{S}_n$-increasing, there exists $\sigma \in \mathfrak{S}_n$ such that $\sigma(\mathbf{A}^{(i)}) = \mathbf{A}^{(i)}$ and that

$$\sigma((\mathbf{A}^{(i)}, \tau(A_k))) = (\mathbf{A}^{(i)}, \sigma\tau(A_k))$$

is $\mathfrak{S}_n$-increasing. Note that the sequence $(\mathbf{A}^{(i)}, \sigma\tau(A_k))$ is linearly independent, because the code $\langle (\mathbf{A}^{(i)}, \sigma\tau(A_k)) \rangle = \sigma\tau(\langle \mathbf{A} \rangle)$ is of dimension $k+1$. Note also that

$|\sigma\tau(A_k)| = |A_k| \leq n/2$, because $\mathbf{A} = (\mathbf{A}', A_k)$ is a standard basis. Therefore $(\mathbf{A}^{(i)}, \sigma\tau(A_k))$ is a standard basis of the $\mathfrak{S}_n$-equivalence class

$$[\langle(\mathbf{A}^{(i)}, \sigma\tau(A_k))\rangle] = [\sigma\tau(\langle\mathbf{A}\rangle)] = [\mathtt{C}].$$

In other words, the word $\sigma\tau(A_k)$ appears in $\mathcal{A}^{(i)}$. Therefore we have a hoped-for standard basis in $\mathbf{L}_{k+1}$. □

Starting with $\mathbf{L}_1 = \{(\mathtt{Z})\}$, we can make the lists $\mathbf{L}_k$ inductively.

Remark 5.26. By Proposition 5.19, we can make the list of pairs $\mathbf{A} \in \mathbf{L}_k$ and $\mathbf{A}' \in \mathbf{L}_{k'}$ such that $[\langle\mathbf{A}\rangle] < [\langle\mathbf{A}'\rangle]$.

6. Geometry of splitting curves. In this section, we assume $p = 2$, and fix a polynomial $G \in \mathcal{U}_{2,b}$, where b is an even integer ≥ 4.

6.1. Definition of splitting curves and associated code words. Let $C \subset \mathbb{P}^2$ be a reduced irreducible curve, and let D_C be the proper transform of C in X_G. Since $\phi_G : X_G \to \mathbb{P}^2$ is purely inseparable of degree 2, either one of the following holds;

(i) D_C is reduced and irreducible, or

(ii) $D_C = 2F_C$, where F_C is a reduced irreducible curve on X_G birational to C via ϕ_G.

DEFINITION 6.1. We say that a reduced irreducible plane curve $C \subset \mathbb{P}^2$ is *splitting in* X_G if (ii) above holds. A reduced (but not necessarily irreducible) curve is said to be *splitting in* X_G if every irreducible component of C is splitting in X_G.

DEFINITION 6.2. Let $C \subset \mathbb{P}^2$ be a reduced curve splitting in X_G. We denote by F_C the reduced divisor of X_G such that $2F_C$ is the proper transform of C in X_G, and by $w_G(C) \in \mathcal{C}_G$ the image of the numerical equivalence class $[F_C] \in S_G$ by

$$S_G \longrightarrow S_G/S_G^0 = \widetilde{\mathcal{C}}_G \overset{\rho_{Z(dG)}}{\longrightarrow} \mathcal{C}_G.$$

Let $C \subset \mathbb{P}^2$ be a reduced curve splitting in X_G. For a point $P \in Z(dG)$, let $m_P(C)$ denote the multiplicity of C at P. Then we have

$$[F_C] = \frac{1}{2}\Big(-\sum_{P \in Z(dG)} m_P(C)[\Gamma_P] \;+\; (\deg C)\,[H_G]\Big) \tag{6.1}$$

in S_G. Hence we have

$$w_G(C) = \{\, P \in Z(dG) \;\mid\; m_P(C) \equiv 1 \bmod 2 \,\}. \tag{6.2}$$

Suppose that C is a union $C_1 \cup C_2$ of two splitting curves C_1 and C_2 that have no common irreducible components. From (6.2), we have

$$w_G(C_1 \cup C_2) = w_G(C_1) + w_G(C_2). \tag{6.3}$$

6.2. A general member of the linear system $|\mathcal{I}_{Z(dG)}(b-1)|$.

PROPOSITION 6.3. *A general member C of $|\mathcal{I}_{Z(dG)}(b-1)|$ is splitting in X_G.*

Proof. Recall that C is reduced and irreducible by Corollary 2.7. By Proposition 2.4, there exist an affine part U of $\mathbb{P}^2$ containing $Z(dG)$ and affine coordinates (x_0, x_1) on U such that C is defined by

$$\varphi_{dG}(\theta_0) = 0,$$

where $\theta_0 \in H^0(\mathbb{P}^2, \Theta(-1))$ is given by $\theta_0|U = \partial/\partial x_0 \otimes e_{-1}$. If G is written on U in terms of (x_0, x_1) as

$$g(x_0, x_1) = \gamma_{00}(x_0, x_1)^2 + x_0\gamma_{10}(x_0, x_1)^2 + x_1\gamma_{01}(x_0, x_1)^2 + x_0x_1\gamma_{11}(x_0, x_1)^2,$$

then C is defined by

$$\gamma_{10}^2 + x_1\gamma_{11}^2 = 0,$$

and $Z(dG)$ is defined by

$$\gamma_{10}^2 + x_1\gamma_{11}^2 = \gamma_{01}^2 + x_0\gamma_{11}^2 = 0.$$

Note that $\gamma_{11}|_C$ is not zero, because $Z(dG)$ is reduced. Hence we obtain

$$g|_C = (\gamma_{00}^2 + x_1\gamma_{01}^2)|_C = \Big(\gamma_{00} + \frac{\gamma_{10}}{\gamma_{11}}\gamma_{01}\Big)^2\Big|_C.$$

We put $\delta_C := (\gamma_{00} + \gamma_{10}\gamma_{01}/\gamma_{11})|_C$. The inverse image in X_G of the generic point of C is therefore isomorphic to

$$\operatorname{Spec} k(C)[w]/(w + \delta_C)^2,$$

which is not reduced. Therefore C is splitting in X_G. □

COROLLARY 6.4. *The code $\mathcal{C}_G \subset \operatorname{Pow}(Z(dG))$ contains the word $Z(dG)$.*

Proof. Because $Z(dG)$ is reduced, a general member C of $|\mathcal{I}_{Z(dG)}(b-1)|$ is smooth at each point of $Z(dG)$. Therefore we have $w_G(C) = Z(dG)$ by (6.2). □

COROLLARY 6.5. *The lattice S_G is a 2-elementary hyperbolic lattice of type I. It is even if and only if $b/2$ is odd.*

Proof. The fact that S_G is 2-elementary and hyperbolic follows from the fact that S_G is an overlattice of S_G^0. Because $Z(dG) \in \mathcal{C}_G$, the lattice S_G is of type I by Proposition 5.8. (Note that $n = |Z(dG)|$ is odd.) Suppose that $b/2$ is odd. Then S_G is even by Propositions 5.7 and 5.11. Suppose that $b/2$ is even. Then $|Z(dG)| \equiv 3 \bmod 4$. Because $Z(dG) \in \mathcal{C}_G$, the lattice $S_G \cong \mathtt{S}_{Z(dG)}(\mathcal{C}_G)$ is not even by Proposition 5.7. □

6.3. Splitting curves with mild singularities. Let $C \subset \mathbb{P}^2$ be a reduced (not necessarily irreducible) curve, and P a point of C. Let (ξ, η) be a formal parameter system of $\mathbb{P}^2$ at P.

DEFINITION 6.6. Let (a, b) be a pair of integers such that $a > b > 1$ and that a and b are prime to each other. We say that P is a *cusp of C of type* (a, b) if C is defined by $\xi^a + \eta^b = 0$ locally at P under a suitable choice of (ξ, η). A cusp of type $(3, 2)$ is called an *ordinary cusp*. Note that, if P is a cusp of type (a, b), then C is locally irreducible at P.

DEFINITION 6.7. Let m be a positive integer. We say that P is a *tacnode of C with tangent multiplicity* m if C is defined by $\eta(\eta + \xi^m) = 0$ locally at P under a suitable choice of (ξ, η). A tacnode with tangent multiplicity 1 is called an *ordinary node*.

PROPOSITION 6.8. *Let $C \subset \mathbb{P}^2$ be a reduced curve splitting in X_G, and let P be a point of C.*

(1) *Suppose that $P \in C$ is a cusp of type (a, b). Then $P \in Z(dG)$ if and only if* $a + b \equiv 0 \mod 2$.

(2) *Suppose that $P \in C$ is a tacnode with tangent multiplicity m. Then $P \in Z(dG)$ if and only if* $m \equiv 1 \mod 2$.

Proof. Let (ξ, η) be a formal parameter system of $\mathbb{P}^2$ at P. We fix a global section $e_{b/2}$ of the line bundle $\mathcal{M} \cong \mathcal{O}_{\mathbb{P}^2}(b/2)$ that is not zero at P. The global section G of $\mathcal{L} = \mathcal{M}^{\otimes 2}$ is given by

$$\gamma(\xi, \eta) \cdot e_{b/2}^{\otimes 2}$$

locally at P, where $\gamma(\xi, \eta)$ is a formal power series of ξ and η, which we write as

$$\gamma(\xi, \eta) = \sum c_{ij} \xi^i \eta^j \qquad (c_{ij} \in k).$$

The subscheme $Z(dG)$ is defined by

$$\frac{\partial \gamma}{\partial \xi} = \frac{\partial \gamma}{\partial \eta} = 0$$

locally at P.

(1) We choose (ξ, η) in such a way that C is defined by $\xi^a + \eta^b = 0$ locally at P. Then

$$t \mapsto (\xi, \eta) = (t^b, t^a)$$

is a normalization of C at P. Since C is splitting in X_G, the formal power series $\gamma(t^b, t^a)$ has a square root in the ring $k[[t]]$ of formal power series of t. Suppose that $a + b$ is even. Then both a and b are odd, because a and b are prime to each other. Looking at the coefficients of t^a and t^b in $\gamma(t^b, t^a)$, we obtain $c_{10} = c_{01} = 0$. Hence $P \in Z(dG)$. Suppose that $a + b$ is odd. Looking at the coefficient of t^{a+b} in $\gamma(t^b, t^a)$, we obtain $c_{11} = 0$. If $P \in Z(dG)$, then $c_{11} = 0$ implies that $Z(dG)$ would fail to be reduced of dimension 0 at P. Hence $P \notin Z(dG)$.

(2) We choose (ξ, η) in such a way that C is defined by $\eta(\eta + \xi^m) = 0$ locally at P. Since C is splitting in X_G, both $\gamma(t, 0)$ and $\gamma(t, t^m)$ have square roots in $k[[t]]$. From $\sqrt{\gamma(t, 0)} \in k[[t]]$, we obtain $c_{10} = 0$. Suppose that m is odd. Then we also obtain $c_{m0} = 0$ from $\sqrt{\gamma(t, 0)} \in k[[t]]$. Looking at the coefficient of t^m in $\gamma(t, t^m)$, we have $c_{m0} + c_{01} = 0$. Therefore we have $P \in Z(dG)$. Suppose that m is even. Then we obtain $c_{m+1,0} = 0$ from $\sqrt{\gamma(t, 0)} \in k[[t]]$. Looking at the coefficient of t^{m+1} in $\gamma(t, t^m)$, we have $c_{m+1,0} + c_{11} = 0$. Therefore $c_{11} = 0$ follows and hence $P \notin Z(dG)$. □

COROLLARY 6.9. *Let $C \subset \mathbb{P}^2$ be a reduced curve splitting in X_G. If $P \in C$ is an ordinary node, then $P \in Z(dG)$. If $P \in C$ is an ordinary cusp, then $P \notin Z(dG)$.*

PROPOSITION 6.10. *Let $C \subset \mathbb{P}^2$ be a reduced irreducible curve splitting in X_G. Suppose that C has ordinary nodes and ordinary cusps as its only singularities. Then the morphism $\phi_G|_{F_C} : F_C \to C$ is the normalization of C.*

Proof. Suppose that $P \in C$ is an ordinary node. Then $P \in Z(dG)$ by Corollary 6.9. The curve F_C intersects Γ_P at distinct two points, and F_C is smooth at each of these points.

Suppose that $P \in C$ is an ordinary cusp. Since $P \notin Z(dG)$ by Corollary 6.9, there exists a unique point Q of X_G such that $\phi_G(Q) = P$. We choose a formal parameter system (ξ, η) of $\mathbb{P}^2$ at P so that C is defined by $\xi^3 + \eta^2 = 0$ locally at P, and let $\gamma(\xi, \eta)$ be the formal power series introduced in the proof of Proposition 6.8. Then X_G is defined by

$$w^2 = \gamma(\xi, \eta)$$

locally at Q, where w is a fiber coordinate of $\mathcal{M}$. Since $\sqrt{\gamma(t^2, t^3)} \in k[[t]]$, we have

$$\frac{\partial \gamma}{\partial \eta}(0, 0) = c_{01} = 0.$$

Therefore the pair $(w - w(Q), \eta)$ is a formal parameter system of X_G at Q. Moreover, we have $c_{10} \neq 0$ because $P \notin Z(dG)$. We put

$$\beta(t) := \sqrt{\gamma(t^2, t^3)} = b_0 + b_1 t + \dots .$$

The curve F_C is given by $w = \beta(t)$ and $\eta = t^3$ at Q. Since $c_{10} \neq 0$, we have $b_1 \neq 0$, which implies that F_C is smooth at Q. $\square$

PROPOSITION 6.11. *Let C be a reduced (possibly reducible) curve of degree d that is splitting in X_G. Suppose that C has only ordinary nodes and ordinary cusps as its singularities. Then we have*

$$|w_G(C)| = d(b - d) + 4\kappa, \tag{6.4}$$

where κ is the number of ordinary cusps on C.

Proof. Let $N(C)$ denote the set of ordinary nodes of C. By (6.1), (6.2) and Corollary 6.9, the assumption on the singularities of C implies that

$$w_G(C) = \{ P \in C \cap Z(dG) \mid C \text{ is smooth at } P \}, \tag{6.5}$$

$$C \cap Z(dG) = w_G(C) \sqcup N(C), \quad \text{and} \tag{6.6}$$

$$[F_C] = \frac{1}{2}(- \sum_{P \in w_G(C)} [\Gamma_P] - 2 \sum_{P \in N(C)} [\Gamma_P] \; + \; d\,[H_G]). \tag{6.7}$$

We prove (6.4) by induction on the number of irreducible components of C. Suppose that C is irreducible. Since F_C is the normalization of C by Proposition 6.10, the geometric genus of C is given by

$$\frac{1}{2}(d-1)(d-2) - \kappa - |N(C)| = \frac{1}{2} F_C(F_C + K_G) + 1, \tag{6.8}$$

where K_G is the canonical divisor of X_G. By Proposition 3.7 and (6.5), (6.7), we obtain (6.4). Suppose that C is a union of two splitting curves C_1 and C_2 that have no common irreducible components. Let d_i be the degree of C_i, and κ_i the number of ordinary cusps of C_i. We have $d = d_1 + d_2$ and $\kappa = \kappa_1 + \kappa_2$. By the induction hypothesis, we have $|w_G(C_i)| = d_i(b - d_i) + 4\kappa_i$ for $i = 1, 2$. By (6.3), we have

$$|w_G(C)| = |w_G(C_1)| + |w_G(C_2)| - 2|w_G(C_1) \cap w_G(C_2)|. \tag{6.9}$$

Suppose that $P \in w_G(C_1) \cap w_G(C_2)$. Then $P \in C_1 \cap C_2$ by (6.5). Suppose that $P \in C_1 \cap C_2$. Then P is an ordinary node of C and hence is contained in $Z(dG)$ by Corollary 6.9. Therefore P is contained in $w_G(C_1) \cap w_G(C_2)$ by (6.5). Thus we obtain

$$w_G(C_1) \cap w_G(C_2) = C_1 \cap C_2,$$

which implies $|w_G(C_1) \cap w_G(C_2)| = d_1 d_2$. Putting this into (6.9) and using the induction hypothesis, we obtain (6.4). □

Remark 6.12. Let $G \in \mathcal{U}_{2,b}$ be chosen generally. Then a general member of the linear system $|\mathcal{I}_{Z(dG)}(b-1)|$ has $(b-2)^2/4$ ordinary cusps as its only singularities. Indeed, we choose homogeneous coordinates $[X_0, X_1, X_2]$ generally so that the member C of $|\mathcal{I}_{Z(dG)}(b-1)|$ defined by $\partial G/\partial X_2 = 0$ is general. We write G as

$$X_0^2\Gamma_{00}^2 + X_1^2\Gamma_{11}^2 + X_2^2\Gamma_{22}^2 + X_0X_1\Gamma_{01}^2 + X_1X_2\Gamma_{12}^2 + X_2X_0\Gamma_{20}^2,$$

where Γ_{ij} are homogeneous polynomials of degree $(b-2)/2$. Then C is defined by

$$X_1\Gamma_{12}^2 + X_0\Gamma_{20}^2 = 0.$$

Since G and $[X_0, X_1, X_2]$ are general, the homogeneous polynomials Γ_{12} and Γ_{20} are also general. Hence $\mathrm{Sing}(C)$ consists of $(b-2)^2/4$ ordinary cusps located at the intersection points of the curves defined by $\Gamma_{12} = 0$ and $\Gamma_{20} = 0$. The equality (6.4) becomes

$$n = b - 1 + (b-2)^2$$

in this case. The linear system $|\mathcal{I}_{Z(dG)}(b-1)|$ gives a generalization of Serre's example [11, Chapter 3, Section 10, Exercise 10.7] of linear systems of plane curves with moving singularities in positive characteristics.

6.4. Splitting curves with only ordinary nodes.

PROPOSITION 6.13. *Let G_C and G_D be homogeneous polynomials defining plane curves C and D such that* $\deg G_C + \deg G_D = b$. *Suppose that $G_C G_D$ is a polynomial contained in $k^\times G + \mathcal{V}_{2,b}$. Then the following hold;*

(i) *C and D are reduced and have no common irreducible components,*
(ii) *$C \cup D$ has only ordinary nodes as its singularities,*
(iii) *C and D are splitting in X_G, and*
(iv) *$w_G(C) = w_G(D) = C \cap D$.*

Proof. The assertions (i) and (ii) follow from Proposition 3.4. The assertion (iii) is obvious because $X_{G_C G_D}$ is isomorphic to X_G over $\mathbb{P}^2$. By Corollary 6.9, we have $C \cap D \subset Z(dG)$. Since C and D are smooth at each point of $C \cap D$, we have $C \cap D \subset w_G(C)$ and $C \cap D \subset w_G(D)$ by (6.2). From Proposition 6.11, we have

$$|w_G(C)| = |w_G(D)| = \deg C \cdot \deg D = |C \cap D|.$$

Therefore (iv) holds. □

The converse of Proposition 6.13 is also true:

PROPOSITION 6.14. *Let C be a curve defined by $G_C = 0$. Suppose that C is reduced, has only ordinary nodes as its singularities, and is splitting in X_G. Then there exists a homogeneous polynomial G_D of degree $b - \deg G_C$ such that $G_C G_D$ is contained in $k^\times G + \mathcal{V}_{2,b}$.*

Proof. First note that the degree of G_C is $\leq b$ by Proposition 6.11. Let $N(C)$ denote the set of ordinary nodes of C, and let $\nu : \widetilde{C} \to C$ be the normalization of C, that is, $\widetilde{C}$ is the disjoint union of normalizations of irreducible components of C.

For $P \in N(C)$, let P_1 and P_2 denote the points of $\widetilde{C}$ that are mapped to P by ν. Consider the following commutative diagram:

$$\begin{array}{ccccc} H^0(\mathbb{P}^2, \mathcal{M}) & \xrightarrow{\text{res}} & H^0(C, \mathcal{M}|_C) & \xrightarrow{\nu^*_{\mathcal{M}}} & H^0(\widetilde{C}, \nu^*\mathcal{M}|_C) \\ \downarrow & & \downarrow & & \downarrow \\ H^0(\mathbb{P}^2, \mathcal{L}) & \xrightarrow{\text{res}} & H^0(C, \mathcal{L}|_C) & \xrightarrow{\nu^*_{\mathcal{L}}} & H^0(\widetilde{C}, \nu^*\mathcal{L}|_C), \end{array}$$

where the left horizontal arrows are restrictions, the right horizontal arrows are the pull-backs by ν, and the vertical arrows are the squaring map $f \mapsto f^2$. For each $P \in N(C)$, we have canonical isomorphisms of 1-dimensional vector spaces

$$\nu^*\mathcal{M}|_C \otimes k(P_1) \cong \nu^*\mathcal{M}|_C \otimes k(P_2), \quad \nu^*\mathcal{L}|_C \otimes k(P_1) \cong \nu^*\mathcal{L}|_C \otimes k(P_2), \tag{6.10}$$

where $k(P_i)$ is the residue field of $\mathcal{O}_{\widetilde{C}}$ at $P_i \in \widetilde{C}$. The homomorphisms $\nu^*_{\mathcal{M}}$ and $\nu^*_{\mathcal{L}}$ are injective, and their images coincide with the spaces of all sections f that satisfy $f(P_1) = f(P_2)$ for every $P \in N(C)$, where $f(P_1)$ and $f(P_2)$ are compared by the canonical isomorphisms (6.10). Consider the images $g \in H^0(C, \mathcal{L}|_C)$ and $\tilde{g} \in H^0(\widetilde{C}, \nu^*\mathcal{L}|_C)$ of $G \in H^0(\mathbb{P}^2, \mathcal{L})$. We have

$$\tilde{g}(P_1) = \tilde{g}(P_2) \qquad \text{for any } P \in N(C). \tag{6.11}$$

Because C is splitting in X_G, there exists a global section $\tilde{h} \in H^0(\widetilde{C}, \nu^*\mathcal{M}|_C)$ such that $\tilde{h}^2 = \tilde{g}$. By (6.11), we have $\tilde{h}(P_1) = \tilde{h}(P_2)$ for each $P \in N(C)$. Hence there exists $h \in H^0(C, \mathcal{M}|_C)$ such that $\nu^*_{\mathcal{M}}(h) = \tilde{h}$. Then we have $g = h^2$ because $\nu^*_{\mathcal{L}}$ is injective. Since the restriction homomorphism $H^0(\mathbb{P}^2, \mathcal{M}) \to H^0(C, \mathcal{M}|_C)$ is surjective, there exists $H \in H^0(\mathbb{P}^2, \mathcal{M})$ such that $(G + H^2)|_C = 0$. Then the polynomial $G + H^2$ is divisible by G_C. □

6.5. Splitting lines and splitting smooth conics.

PROPOSITION 6.15. (1) *Let $L \subset \mathbb{P}^2$ be a line. If $|L \cap Z(dG)| > (b-2)/2$, then L is splitting in X_G.* (2) *Let $Q \subset \mathbb{P}^2$ be a smooth conic. If $|Q \cap Z(dG)| > b - 1$, then Q is splitting in X_G.*

Proof. (1) We choose a general line $l_\infty \subset \mathbb{P}^2$, and fix affine coordinates (x_0, x_1) on $U := \mathbb{P}^2 \setminus l_\infty$ such that L is defined by $x_1 = 0$. Let us consider x_0 as an affine parameter of L. We express G on U by

$$\gamma_{00}(x_0, x_1)^2 + x_0\gamma_{10}(x_0, x_1)^2 + x_1\gamma_{01}(x_0, x_1)^2 + x_0x_1\gamma_{11}(x_0, x_1)^2. \tag{6.12}$$

Then $L \cap Z(dG)$ is defined on L by

$$\gamma_{10}(x_0, 0)^2 = \gamma_{01}(x_0, 0)^2 + x_0\gamma_{11}(x_0, 0)^2 = 0.$$

Note that the degree of γ_{10} is at most $(b-2)/2$. Hence the assumption $|L \cap Z(dG)| > (b-2)/2$ implies that $\gamma_{10}(x_0, 0)$ is constantly equal to zero. Therefore $\gamma_{10}(x_0, x_1)$ can be written as $x_1\delta_{10}(x_0, x_1)$. Then G is equal to

$$\gamma_{00}^2 + x_1(x_0x_1\delta_{10}^2 + \gamma_{01}^2 + x_0\gamma_{11}^2)$$

on U. Hence L is splitting in X_G.

(2) Let l_∞ be a general tangent line to Q, and let (x_0, x_1) be affine coordinates on $U = \mathbb{P}^2 \setminus l_\infty$ such that Q is defined by $x_1 + x_0^2 = 0$. We consider x_0 as an affine

parameter of Q. Again we write G on U as in (6.12). Then $Q \cap Z(dG)$ is defined on Q by

$$\gamma_{10}(x_0, x_0^2)^2 + x_0^2\gamma_{11}(x_0, x_0^2)^2 = \gamma_{01}(x_0, x_0^2)^2 + x_0\gamma_{11}(x_0, x_0^2)^2 = 0.$$

Since the degrees of γ_{10} and γ_{11} are at most $(b-2)/2$, the number of the roots of

$$\gamma_{10}(x_0, x_0^2)^2 + x_0^2\gamma_{11}(x_0, x_0^2)^2 = (\gamma_{10}(x_0, x_0^2) + x_0\gamma_{11}(x_0, x_0^2))^2$$

is at most $b-1$. Consequently the assumption $|Q \cap Z(dG)| > b-1$ implies that $(\gamma_{10} + x_0\gamma_{11})|_Q = 0$. Then $G|_Q$ is written as

$$\gamma_{00}(x_0, x_0^2)^2 + x_0^2\gamma_{01}(x_0, x_0^2)^2,$$

which is the square of $(\gamma_{00} + x_0\gamma_{01})|_Q$. Therefore Q is splitting. □

Corollary 6.16. (1) *If $L \subset \mathbb{P}^2$ is a line, then $|L \cap Z(dG)|$ is either $\leq (b-2)/2$ or $b-1$.* (2) *If $Q \subset \mathbb{P}^2$ is a smooth conic, then $|Q \cap Z(dG)|$ is either $\leq b-1$ or $2(b-2)$.*

Example 6.17. Let $q = 2^\nu$ be a power of 2. We put $b := q+2$, and consider the homogeneous polynomial

$$G_{\mathrm{DK},q} = X_0X_1X_2(X_0^{q-1} + X_1^{q-1} + X_2^{q-1})$$

of degree b, which is a generalization of Dolgachev-Kondo's polynomial (1.1) of degree 6. It is easy to see that $Z(dG_{\mathrm{DK},q})$ consists of all $\mathbb{F}_q$-rational points of $\mathbb{P}^2$. Because $n = b^2 - 3b + 3 = q^2 + q + 1$ is equal to the cardinality of $\mathbb{P}^2(\mathbb{F}_q)$, the polynomial $G_{\mathrm{DK},q}$ is a member of $\mathcal{U}_{2,b}$. Every $\mathbb{F}_q$-rational line contains $q+1 = b-1$ points of $Z(dG_{\mathrm{DK},q})$, and hence is splitting in $X_{G_{\mathrm{DK},q}}$.

7. Known facts about $K3$ surfaces.

7.1. The Artin-Rudakov-Shafarevich theory. Let p be an arbitrary prime integer, and X a supersingular $K3$ surface in characteristic p. Artin [1] showed that the discriminant of the numerical Néron-Severi lattice NS_X of X is equal to $-p^{2\sigma}$, where σ is a positive integer ≤ 10. This integer σ is called the *Artin invariant* of X.

Proposition 7.1 (Artin [1], Rudakov-Shafarevich [14], Shioda [19]). *For any pair (p, σ) of a prime integer p and a positive integer $\sigma \leq 10$, there exists a supersingular $K3$ surface in characteristic p with Artin invariant σ.*

For an integer σ with $1 \leq \sigma \leq 10$, let $\Lambda_{2,\sigma}$ denote the lattice with the following properties;

(RS1) even, hyperbolic, and of rank 22,
(RS2) 2-elementary of type I, and
(RS3) $\operatorname{disc}\Lambda_{2,\sigma} = -2^{2\sigma}$.

Proposition 7.2 (Rudakov-Shafarevich [15]). *The conditions* (RS1)-(RS3) *determine the lattice $\Lambda_{2,\sigma}$ uniquely up to isomorphisms.*

Proposition 7.3 (Rudakov-Shafarevich [15]). *Let X be a supersingular $K3$ surface in characteristic 2 with Artin invariant σ. Then the lattice NS_X is isomorphic to $\Lambda_{2,\sigma}$. More precisely, let $v \in \Lambda_{2,\sigma}$ be a vector with $v^2 > 0$. Then there exists an isometry ϕ from $\Lambda_{2,\sigma}$ to NS_X such that $\phi(v)$ is the class $[H]$ of a nef line bundle H of X.*

7.2. **$K3$ surfaces as sextic double planes.** Let T be a negative definite even lattice. A vector $v \in T$ is called a *root* if $v^2 = -2$. We put

$$\mathrm{Roots}(T) := \{\ v \in T\ \mid\ v^2 = -2\ \}.$$

It is well-known that $\mathrm{Roots}(T)$ forms a root system of type ADE ([3, 7]).

DEFINITION 7.4. A pair (X, H) of a $K3$ surface X and a line bundle H of X with $H^2 = 2$ and $|H| \neq \emptyset$ is called a *sextic double plane* if the complete linear system $|H|$ is fixed component free. If (X, H) is a sextic double plane, then $|H|$ defines a generically finite morphism

$$\Phi_{|H|}\ :\ X\ \to\ \mathbb{P}^2$$

of degree 2.

For a sextic double plane (X, H), we denote by

$$X\ \to\ Y_{|H|}\ \to\ \mathbb{P}^2$$

the Stein factorization of $\Phi_{|H|}$. The normal $K3$ surface $Y_{|H|}$ has only rational double points as its singularities. We denote by $R(X, H)$ the ADE-type of the singular points of $Y_{|H|}$, that is, $R(X, H)$ is the type of the ADE-configuration of (-2)-curves that are contracted by $X \to Y_{|H|}$.

Remark 7.5. Let (X, H) be a sextic double plane. We have

$$Y_{|H|} := \mathbf{Spec}\,\Phi_{|H|*}\mathcal{O}_X\ \cong\ \mathrm{Proj}\left(\bigoplus_{m=0}^{\infty} H^0(X, H^{\otimes m})\right). \tag{7.1}$$

Indeed, let s be a non-zero element of $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, and let s_X be the global section $\Phi^*_{|H|}(s)$ of H. We put $U := \{s \neq 0\} \subset \mathbb{P}^2$. Then the module $\Gamma(U, \Phi_{|H|*}\mathcal{O}_X)$ of sections of $\mathcal{O}_X$ over $\Phi^{-1}_{|H|}(U) = \{s_X \neq 0\} \subset X$ is canonically isomorphic to the degree 0 part of the graded ring

$$\bigoplus_{m=0}^{\infty} H^0(X, H^{\otimes m})\left[\frac{1}{s_X}\right].$$

Hence the isomorphism (7.1) holds.

The graded ring $\oplus_{m=0}^{\infty} H^0(X, H^{\otimes m})$ is generated by elements X_0, X_1, X_2 of degree 1 and an element w of degree 3, and the relations are generated by

$$w^2 + C(X_0, X_1, X_2)w + G(X_0, X_1, X_2) = 0. \tag{7.2}$$

where C and G are homogeneous polynomials of degree 3 and 6, respectively. Hence $Y_{|H|}$ is defined by (7.2) in the weighted projective space $\mathbb{P}(3, 1, 1, 1)$.

PROPOSITION 7.6 (Urabe [22], Nikulin [13]). *Let X be a $K3$ surface and H a line bundle on X with $H^2 = 2$.*

(1) *The pair (X, H) is a sextic double plane if and only if H is nef and the set $\{u \in NS_X \mid u^2 = 0, u \cdot [H] = 1\}$ is empty.*

(2) *Suppose that (X, H) is a sextic double plane. Then $R(X, H)$ coincides with the ADE-type of the root system* $\mathrm{Roots}([H]^\perp)$*, where $[H]^\perp \subset NS_X$ is the orthogonal*

complement of $[H]$ *in* NS_X*. More precisely, the classes of* (-2)*-curves contracted by* $X \to Y_{|H|}$ *form a simple root system of* $\mathrm{Roots}([H]^{\perp})$.

Proposition 7.6 is true in any characteristic. Indeed, the proof of Proposition 1.7 in [22] can be transplanted in any characteristic except for the use of the Kawamata-Vieweg vanishing theorem, which can be replaced by Proposition 0.1 in [13].

7.3. Purely inseparable sextic double planes. The following is obvious:

Proposition 7.7. *If* G *is a polynomial in* $\mathcal{U}_{2,6}$*, then* (X_G, H_G) *is a sextic double plane, and* $R(X_G, H_G) = 21A_1$ *holds.*

Conversely, we have the following:

Proposition 7.8 ([17]). *Let* (X, H) *be a sextic double plane. If* $R(X, H) = 21A_1$*, then* $p = 2$ *and the morphism* $\Phi_{|H|} : X \to \mathbb{P}^2$ *is purely inseparable.*

Let (X, H) be a sextic double plane such that $R(X, H) = 21A_1$. Then there exists a homogeneous polynomial $G(X_0, X_1, X_2)$ of degree 6 such that $Y_{|H|}$ is defined by $w^2 = G$. Since $Y_{|H|}$ has rational double points of type $21A_1$ as its only singularities, we have $G \in \mathcal{U}_{2,6}$.

Corollary 7.9. *If* (X, H) *is a sextic double plane with* $R(X, H) = 21A_1$*, then there exists* $G \in \mathcal{U}_{2,6}$ *such that* $X = X_G$*,* $H = H_G$*,* $Y_{|H|} = Y_G$ *and* $\Phi_{|H|} = \phi_G$.

8. The list of geometrically realizable classes of codes. In this section, we study the case where $p = 2$ and $b = 6$.

8.1. A characterization of geometrically realizable classes of codes.

Theorem 8.1. *Let* $\mathtt{Z}$ *be a set with* $|\mathtt{Z}| = 21$*, and let* $\mathtt{C} \subset \mathrm{Pow}(\mathtt{Z})$ *be a code. The* $\mathfrak{S}_{21}$*-equivalence class* $[\mathtt{C}]$ *containing* $\mathtt{C}$ *is geometrically realizable if and only if the following hold:*
(a) $\dim \mathtt{C} \le 10$,
(b) $\mathtt{Z} \in \mathtt{C}$, *and*
(c) $|A| \in \{0, 5, 8, 9, 12, 13, 16, 21\}$ *for any* $A \in \mathtt{C}$.

Proof. Suppose that $[\mathtt{C}]$ is geometrically realizable, and let G be a polynomial in $\mathcal{U}_{2,6}$ such that $\mathtt{C} \cong \mathcal{C}_G$ by some bijection $\mathtt{Z} \cong Z(dG)$. We have

$$|\operatorname{disc} S_G| = 2^{22 - 2\dim \widetilde{\mathcal{C}}_G} = 2^{22 - 2\dim \mathtt{C}}.$$

Since the Artin invariant of X_G is positive, we have $\dim \mathtt{C} \le 10$. By Corollary 6.4, we have $Z(dG) \in \mathcal{C}_G$, and hence $\mathtt{Z} \in \mathtt{C}$. By Proposition 5.11, $|A| \bmod 4$ is either 0 or 1 for any $A \in \mathcal{C}_G$. Therefore, in order to show that $\mathtt{C}$ satisfies (c), it is enough to show that $|A| \notin \{1, 4, 17, 20\}$ for any $A \in \mathcal{C}_G$. Suppose that there is an element $A \in \mathcal{C}_G$ with $|A| = 1$. Then there exists $P \in Z(dG)$ such that $(\{P\}, 1)$ is contained in the lift $\widetilde{\mathcal{C}}_G = \mathcal{C}_G^{\sim}$ of $\mathcal{C}_G$. Hence the vector

$$v := \frac{1}{2}(-[\Gamma_P] \ + \ [H_G])$$

is contained in S_G. Because $v \cdot [H_G] = 1$ and $v^2 = 0$, we see from Proposition 7.6 that (X_G, H_G) is not a sextic double plane, which is absurd. Suppose that there is a word $A \in \mathcal{C}_G$ with $|A| = 4$. Then $(A, 0)$ is a word in the lift $\mathcal{C}_G^{\sim}$ of $\mathcal{C}_G$. Hence the vector

$$v := \frac{1}{2}\Big(\sum_{P \in A} [\Gamma_P]\Big)$$

is contained in S_G. Because $v \cdot [H_G] = 0$ and $v^2 = -2$, the vector v is an element of $\mathrm{Roots}([H_G]^\perp)$. However, we see from Proposition 7.6 that every vector in $\mathrm{Roots}([H_G]^\perp)$ is written as a linear combination of $[\Gamma_P]$ ($P \in Z(dG)$) with *integer* coefficients. Thus we get a contradiction. Suppose that there is a word $A \in \mathcal{C}_G$ with $|A| = 17$ or 20. Then $Z(dG) + A \in \mathcal{C}_G$ is of weight 4 or 1, which is impossible as has been shown above. Therefore the code $\mathtt{C}$ satisfies (a), (b) and (c).

Suppose that $\mathtt{C}$ satisfies (a), (b) and (c). We put

$$\sigma := 11 - \dim \mathtt{C}.$$

By Proposition 5.7 and the property (c), the submodule $\mathtt{S}_\mathtt{Z}(\mathtt{C}) = \mathrm{pr}^{-1}_{\mathtt{S}^0_\mathtt{Z}}(\mathtt{C}^\sim)$ of $(\mathtt{S}^0_\mathtt{Z})^\vee$ corresponding to the lift $\mathtt{C}^\sim \subset \mathrm{DG}(\mathtt{S}^0_\mathtt{Z})$ of $\mathtt{C}$ is an even overlattice of $\mathtt{S}^0_\mathtt{Z}$.

Claim 8.2. The even overlattice $\mathtt{S}_\mathtt{Z}(\mathtt{C})$ of $\mathtt{S}^0_\mathtt{Z}$ is isomorphic to $\Lambda_{2,\sigma}$.

Proof of Claim 8.2. By Proposition 7.2, it is enough to show that $\mathtt{S}_\mathtt{Z}(\mathtt{C})$ satisfies the conditions (RS1), (RS2) and (RS3). It is obvious that $\mathtt{S}_\mathtt{Z}(\mathtt{C})$ is 2-elementary and hyperbolic. By Proposition 5.8, the property (b) implies that $\mathtt{S}_\mathtt{Z}(\mathtt{C})$ is of type I. By (5.2), we have $|\operatorname{disc}(\mathtt{S}_\mathtt{Z}(\mathtt{C}))| = 2^{2\sigma}$. □

By Proposition 7.1, there exists a supersingular $K3$ surface X in characteristic 2 with Artin invariant σ. In $\mathtt{S}_\mathtt{Z}(\mathtt{C})$, we have the vector $\mathtt{h}$ with $\mathtt{h}^2 = 2$. By Proposition 7.3, there exists an isometry

$$\phi : \mathtt{S}_\mathtt{Z}(\mathtt{C}) \xrightarrow{\sim} NS_X$$

such that $\phi(\mathtt{h})$ is the class $[H]$ of a nef line bundle H on X with $H^2 = 2$.

Claim 8.3. The pair (X, H) is a sextic double plane with $R(X, H) = 21A_1$.

Proof of Claim 8.3. By Proposition 7.6 and the isometry ϕ, it is enough to show that the set

$$\{\, u \in \mathtt{S}_\mathtt{Z}(\mathtt{C}) \mid u^2 = 0,\ u\mathtt{h} = 1 \,\} \tag{8.1}$$

is empty, and that the ADE-type of the root system $\mathrm{Roots}(\mathtt{h}^\perp)$ is $21A_1$, where $\mathtt{h}^\perp$ is the orthogonal complement of $\mathtt{h}$ in $\mathtt{S}_\mathtt{Z}(\mathtt{C})$. Suppose that a vector

$$u = \frac{1}{2}\Big(\sum_{\mathtt{P}\in\mathtt{Z}} a_\mathtt{P}\, \mathtt{e}_\mathtt{P} + b\mathtt{h}\Big) \qquad (a_\mathtt{P} \in \mathbb{Z},\ \ b \in \mathbb{Z})$$

of $\mathtt{S}_\mathtt{Z}(\mathtt{C})$ is contained in the set (8.1). Because $u\mathtt{h} = 1$, we have $b = 1$. Because $u^2 = 0$, we have $\sum a_\mathtt{P}^2 = 1$. Hence u is of the form $(\mathtt{h} \pm \mathtt{e}_\mathtt{P})/2$. Its image in $\mathtt{C}^\sim$ by the natural projection $\mathtt{S}_\mathtt{Z}(\mathtt{C}) \to \mathtt{S}_\mathtt{Z}(\mathtt{C})/\mathtt{S}^0_\mathtt{Z}$ therefore yields an element $(\{\mathtt{P}\}, 1) \in \mathrm{Pow}(\mathtt{Z}) \oplus \mathbb{F}_2$. This contradicts the property (c). Let

$$r = \frac{1}{2}\Big(\sum_{\mathtt{P}\in\mathtt{Z}} a_\mathtt{P}\, \mathtt{e}_\mathtt{P} + b\mathtt{h}\Big) \qquad (a_\mathtt{P} \in \mathbb{Z},\ \ b \in \mathbb{Z})$$

be a root of $\mathtt{h}^\perp$. Because $r\mathtt{h} = 0$, we have $b = 0$. Because $u^2 = -2$, we have $\sum a_\mathtt{P}^2 = 4$. Hence r is either

$$\pm \mathtt{e}_\mathtt{P}, \quad \text{or} \quad \frac{1}{2}\sum_{\mathtt{P}\in A}(\pm \mathtt{e}_\mathtt{P}) \ \text{ with } \ |A| = 4.$$

By the property (c) of $\mathtt{C}$, the latter cannot occur. Hence $\mathrm{Roots}(\mathtt{h}^{\perp})$ is equal to $\{\pm \mathtt{e}_{\mathtt{P}} \mid \mathtt{P} \in \mathtt{Z}\}$, and its ADE-type is $21A_1$. □

By Corollary 7.9, there exists $G \in \mathcal{U}_{2,6}$ such that $X = X_G$, $H = H_G$ and $\Phi_{|H|} = \phi_G$. Note that the isometry

$$\phi : \mathtt{S}_{\mathtt{Z}}(\mathtt{C}) \xrightarrow{\sim} NS_X \cong S_G$$

maps $\mathrm{Roots}(\mathtt{h}^{\perp})$ to $\mathrm{Roots}([H_G]^{\perp})$ bijectively. Composing the isometry ϕ with reflections with respect to some $\mathtt{e}_{\mathtt{P}}$ if necessary, we can assume that ϕ maps each $\mathtt{e}_{\mathtt{P}}$ ($\mathtt{P} \in \mathtt{Z}$) to $[\Gamma_P]$ for some $P \in Z(dG)$. The correspondence $\mathtt{e}_{\mathtt{P}} \mapsto \Gamma_P$ gives us a bijection $\mathtt{Z} \cong Z(dG)$ that induces $\mathtt{C} \cong \mathcal{C}_G$. Hence the class $[\mathtt{C}]$ is geometrically realizable. □

8.2. From the code to the configuration of splitting curves. In this subsection, we fix a polynomial $G \in \mathcal{U}_{2,6}$ and show how to read from $\mathcal{C}_G$ the configuration of plane curves of degree ≤ 3 splitting in X_G.

DEFINITION 8.4. For a word $A \in \mathrm{Pow}(Z(dG))$ with $|A| \in \{5, 8, 9\}$, we put

$$\deg A := \begin{cases} 1 & \text{if } |A| = 5, \\ 2 & \text{if } |A| = 8, \\ 3 & \text{if } |A| = 9. \end{cases}$$

We say that a word A of $\mathcal{C}_G$ is *reducible in* $\mathcal{C}_G$ if there exist words A_1 and A_2 of $\mathcal{C}_G$ with $|A_1|, |A_2| \in \{5, 8, 9\}$ such that $A = A_1 + A_2$ and $\deg A = \deg A_1 + \deg A_2$ hold. We say that A is *irreducible in* $\mathcal{C}_G$ if A is not reducible in $\mathcal{C}_G$.

A word of $\mathcal{C}_G$ with weight 5 is always irreducible in $\mathcal{C}_G$.

PROPOSITION 8.5. *The correspondence $L \mapsto L \cap Z(dG)$ gives a bijection from the set of lines $L \subset \mathbb{P}^2$ splitting in X_G to the set of words $A \in \mathcal{C}_G$ of weight* 5.

Proof. Suppose that a line L is splitting in X_G. Then we have $w_G(L) = L \cap Z(dG)$ by (6.2) and $|w_G(L)| = 5$ by Proposition 6.11.

Conversely suppose that a word $A \in \mathcal{C}_G$ with $|A| = 5$ is given. A line L satisfying $L \cap Z(dG) = A$ is, if exists, obviously unique. Because $(A, 1)$ is a word in the lift $\mathcal{C}_G^{\sim}$ of $\mathcal{C}_G$, we have a vector

$$u := \frac{1}{2}(-\sum_{P \in A} [\Gamma_P] \ + \ [H_G])$$

in S_G. Because $u^2 = -2$ and $u \cdot [H_G] = 1$, the class u is represented by an effective divisor D of X_G. Since $DH_G = 1$, there exists a reduced irreducible component D_0 of D such that $\phi_G : X_G \to \mathbb{P}^2$ induces a birational morphism from D_0 to a line $L \subset \mathbb{P}^2$. Moreover $D - D_0$ is a linear combination of the curves Γ_P with non-negative integer coefficients. Since the proper transform of L in X_G is $2D_0$, the line L is splitting in X_G, and $F_L = D_0$ holds. Since $u - [D_0]$ is in S_G^0, we have

$$(A, 1) = u \mod S_G^0 = [D_0] \mod S_G^0 = [F_L] \mod S_G^0.$$

Therefore we obtain $A = w_G(L) = L \cap Z(dG)$. □

Remark 8.6. Let L_1 and L_2 be distinct splitting lines. By Corollary 6.9, we see that $w_G(L_1) \cap w_G(L_2)$ consists of one point, which is the intersection point of L_1 and L_2, and the word $w_G(L_1 \cup L_2) = w_G(L_1) + w_G(L_2)$ is of weight 8.

Remark 8.7. Let L_1, L_2 and L_3 be distinct splitting lines. The word

$$w_G(L_1 \cup L_2 \cup L_3) = w_G(L_1) + w_G(L_2) + w_G(L_3)$$

is of weight 9 if $L_1 \cup L_2 \cup L_3$ has only ordinary nodes as its singularities, while this word is of weight 13 if $L_1 \cap L_2 \cap L_3$ is non-empty.

PROPOSITION 8.8. *The correspondence $Q \mapsto Q \cap Z(dG)$ gives a bijection from the set of smooth conics $Q \subset \mathbb{P}^2$ splitting in X_G to the set of words $A \in \mathcal{C}_G$ of weight 8 irreducible in $\mathcal{C}_G$.*

Proof. Suppose that a smooth conic Q is splitting in X_G. Then the word $w_G(Q) = Q \cap Z(dG)$ of $\mathcal{C}_G$ is of weight 8 by Proposition 6.11. If $w_G(Q)$ were reducible in $\mathcal{C}_G$, then $Q \cap Z(dG)$ would be written as $A_1 + A_2$, where A_1 and A_2 are words of $\mathcal{C}_G$ with weight 5. By Proposition 8.5, the points in A_i $(i = 1, 2)$ are collinear, and hence Q would contain two sets of four collinear points, which contradicts the assumption that Q is smooth. Hence the word $w_G(Q)$ is irreducible in $\mathcal{C}_G$.

Suppose that $A \in \mathcal{C}_G$ is a word of weight 8 that is irreducible in $\mathcal{C}_G$. Since $(A, 0) \in \mathcal{C}_G^{\sim}$, the vector

$$u := \frac{1}{2}(-\sum_{P \in A} [\Gamma_P] \ + \ 2\,[H_G])$$

of $(S_G^0)^\vee$ is contained in S_G. Because $u^2 = -2$ and $u \cdot [H_G] = 2$, the vector u is the class of an effective divisor D on X_G. Let D_0 be the union of irreducible components of D whose image by ϕ_G are of dimension 1. Since $u - [D_0]$ is a linear combination of the classes $[\Gamma_P]$ with non-negative integer coefficients, we have

$$[D_0] \bmod S_G^0 = (A, 0) \qquad \text{in } \mathcal{C}_G^{\sim}.$$

Because $D_0 H_G = 2$, the plane curve $\phi_G(D_0)$ with the reduced structure is either a line or a conic. Suppose that $\phi_G(D_0)$ is a line L. If L is not splitting in X_G, then the morphism $\phi_G|_{D_0} : D_0 \to L$ is of degree 2, while if L is splitting, then D_0 is $2F_L$. In either case, D_0 is the proper transform of L and hence $[D_0]$ is contained in S_G^0. This is absurd because $A \neq 0$. Therefore $\phi_G(D_0)$ is a conic Q. Since $\phi_G|_{D_0} : D_0 \to Q$ is of degree 1, the conic Q is splitting, and $D_0 = F_Q$ holds. From (8.2), we have $A = w_G(Q)$. If Q is a union of two lines L_1 and L_2, then both L_1 and L_2 are splitting and $A = w_G(L_1) + w_G(L_2)$ holds from (6.3), which contradicts the irreducibility of the word A in $\mathcal{C}_G$. (See Remark 8.6.) Therefore Q is a smooth conic. Because $w_G(Q) = Q \cap Z(dG)$ by (6.2), we obtain $A = Q \cap Z(dG)$. □

Remark 8.9. Let L be a splitting line, and Q a splitting smooth conic. If L intersects Q transversely, then $w_G(L) \cap w_G(Q)$ consists of the two intersection points of L and Q, and $w_G(L \cup Q) = w_G(L) + w_G(Q)$ is of weight 9. If L is tangent to Q, then $w_G(L) \cap w_G(Q)$ is empty, and $w_G(L \cup Q) = w_G(L) + w_G(Q)$ is of weight 13.

Remark 8.10. Let Q_1 and Q_2 be distinct splitting smooth conics. Let us investigate the intersection of Q_1 and Q_2. Because

$$|w_G(Q_1 \cup Q_2)| = |w_G(Q_1) + w_G(Q_2)| = 16 - 2|w_G(Q_1) \cap w_G(Q_2)|$$

is in $\{0, 5, 8, 9, 12, 13, 16, 21\}$ by Theorem 8.1, $|w_G(Q_1) \cap w_G(Q_2)|$ is 4, 2 or 0.

Suppose that $|w_G(Q_1) \cap w_G(Q_2)| = 4$. Then Q_1 and Q_2 intersect transversely. Let G_{Q_1} and G_{Q_2} be homogeneous polynomials of degree 2 defining Q_1 and Q_2, respectively. Since $Q_1 \cup Q_2$ is a splitting curve with only ordinary nodes, Proposition 6.14 implies that there exists a homogeneous polynomial G_{Q_3} of degree 2 such that $G_{Q_1}G_{Q_2}G_{Q_3}$ is a member of $k^\times G + \mathcal{V}_{2,6}$. Then the conic Q_3 defined by $G_{Q_3} = 0$ is splitting in X_G, and $w_G(Q_3) = w_G(Q_1) + w_G(Q_2)$ holds.

Suppose that $|w_G(Q_1) \cap w_G(Q_2)| = 2$. By Proposition 6.8, we have the following two possibilities of intersection of Q_1 and Q_2;

- transverse at two points, and with multiplicity 2 at one point, or
- transverse at one point, and with multiplicity 3 at one point.

Suppose that $|w_G(Q_1) \cap w_G(Q_2)| = 0$. Then Q_1 and Q_2 intersect either with multiplicity 2 at two points, or with multiplicity 4 at one point.

Corollary 8.11. *A word $A \in \mathcal{C}_G$ of weight 8 or 9 is irreducible in $\mathcal{C}_G$ if and only if no three points of A are collinear.*

Proof. Suppose that A is reducible in $\mathcal{C}_G$. Then A is written as $A_1 + A_2$, where A_1 and A_2 are words of $\mathcal{C}_G$ such that $(|A|, |A_1|, |A_2|)$ is either $(8, 5, 5)$ or $(9, 5, 8)$. Note that $A \cap A_1 = A_1 \setminus (A_1 \cap A_2)$ is of weight ≥ 3, because $|A_1 \cap A_2| = (|A_1| + |A_2| - |A|)/2$ is ≤ 2. Since the points of A_1 are collinear by Proposition 8.5, three points of A are collinear. Suppose that three points of A are on a line L. By Proposition 6.15, the line L is splitting in X_G. We put $A' := A + w_G(L) \in \mathcal{C}_G$. The weight

$$|A'| = |A| + 5 - 2|A \cap w_G(L)|$$

of A' is among the set $\{0, 5, 8, 9, 12, 13, 16, 21\}$ by Theorem 8.1. Because $w_G(L) = L \cap Z(dG)$ and $A \subset Z(dG)$, we have $A \cap w_G(L) = A \cap L$ and hence $|A \cap w_G(L)|$ is ≥ 3. Therefore the triple $(|A|, |A \cap w_G(L)|, |A'|)$ is either $(8, 4, 5)$ or $(9, 3, 8)$. In either case, $A = A' + w_G(L)$ is reducible in $\mathcal{C}_G$. □

Definition 8.12. A pencil $\mathcal{E} = \{E_t\}$ of cubic curves $E_t \subset \mathbb{P}^2$ is called *regular* if the base locus $\mathrm{Bs}(\mathcal{E})$ of $\mathcal{E}$ consists of distinct 9 points and every singular member of $\mathcal{E}$ is an irreducible nodal curve.

Note that a general member of a regular pencil $\mathcal{E}$ of cubic curves is smooth. Indeed, a general member of $\mathcal{E}$ is reduced because $|\mathrm{Bs}(\mathcal{E})| = 9$. If a general member of $\mathcal{E}$ is singular, then it must have an ordinary cusp ([21, 16]), and hence any singular member cannot be an irreducible nodal curve.

Lemma 8.13. *Let $\mathcal{E}$ be a regular pencil of cubic curves.*
(1) *The pencil $\mathcal{E}$ coincides with $|\mathcal{I}_{\mathrm{Bs}(\mathcal{E})}(3)|$.*
(2) *There are no three collinear points in $\mathrm{Bs}(\mathcal{E})$.*

Proof. In order to prove (1), it is enough to show that $\dim |\mathcal{I}_{\mathrm{Bs}(\mathcal{E})}(3)| \leq 1$. If $\dim |\mathcal{I}_{\mathrm{Bs}(\mathcal{E})}(3)| > 1$, then there would be eight points in $\mathrm{Bs}(\mathcal{E})$ on a conic, or five points in $\mathrm{Bs}(\mathcal{E})$ on a line. (See for example [10, p.715].) In either case, we get a contradiction to Bézout's theorem. Suppose that there exists a subset of $\mathrm{Bs}(\mathcal{E})$ of weight 3 that is on a line L. We put $B' := \mathrm{Bs}(\mathcal{E}) \cap L$, and let $\mathcal{I}_{B' \subset L} \subset \mathcal{O}_L$ be the ideal sheaf of B' on L. From the exact sequence

$$H^0(\mathbb{P}^2, \mathcal{I}_{\mathrm{Bs}(\mathcal{E}) \setminus B'}(2)) \;\to\; H^0(\mathbb{P}^2, \mathcal{I}_{\mathrm{Bs}(\mathcal{E})}(3)) \;\to\; H^0(L, \mathcal{I}_{B' \subset L}(3)),$$

we see that a union of L and a conic is a member of $\mathcal{E} = |\mathcal{I}_{\mathrm{Bs}(\mathcal{E})}(3)|$, which contradicts the regularity of $\mathcal{E}$. □

Definition 8.14. A pencil $\mathcal{E}$ of cubic curves is called *splitting in X_G* if every member of $\mathcal{E}$ is reduced and splitting in X_G.

PROPOSITION 8.15. *The correspondence $\mathcal{E} \mapsto \mathrm{Bs}(\mathcal{E})$ gives a bijection from the set of regular pencils of cubic curves splitting in X_G to the set of irreducible words $A \in \mathcal{C}_G$ of weight 9. The inverse map is given by $A \mapsto |\mathcal{I}_A(3)|$.*

Proof. Let $\mathcal{E}$ be a regular pencil of cubic curves splitting in X_G, and let E and E' be members of $\mathcal{E}$ that span $\mathcal{E}$. Each of E and E' is a smooth or irreducible nodal cubic curve splitting in X_G. Let E^o and E'^o be the smooth parts of E and E', respectively. Then we have

$$w_G(E) = E^o \cap Z(dG) \quad \text{and} \quad w_G(E') = E'^o \cap Z(dG) \tag{8.2}$$

by (6.2), and

$$|w_G(E)| = |w_G(E')| = 9 \tag{8.3}$$

by Proposition 6.11. On the other hand, the base locus $\mathrm{Bs}(\mathcal{E})$ of $\mathcal{E}$ is equal to $E^o \cap E'^o$, and is contained in the set of ordinary nodes of the reducible splitting curve $E \cup E'$. Hence

$$\mathrm{Bs}(\mathcal{E}) = E^o \cap E'^o \subset Z(dG) \tag{8.4}$$

holds by Corollary 6.9. Comparing (8.2), (8.3) and (8.4), we obtain

$$w_G(E) = w_G(E') = \mathrm{Bs}(\mathcal{E}).$$

In particular, $\mathrm{Bs}(\mathcal{E})$ is a word in $\mathcal{C}_G$. From Lemma 8.13 and Corollary 8.11, the word $\mathrm{Bs}(\mathcal{E})$ is irreducible in $\mathcal{C}_G$.

Suppose that an irreducible word A of $\mathcal{C}_G$ with weight 9 is given. A splitting regular pencil $\mathcal{E}$ with $\mathrm{Bs}(\mathcal{E}) = A$ is, if exists, equal to $|\mathcal{I}_A(3)|$ by Lemma 8.13, and hence is unique. Let us prove the existence of such a pencil $\mathcal{E}$. Since $(A, 1) \in \mathcal{C}_G^{\sim}$, we have a vector

$$u := \frac{1}{2}(-\sum_{P \in A} [\Gamma_P] \ + \ 3\,[H_G])$$

in S_G. Because $u^2 = 0$ and $u \cdot [H_G] = 3$, the vector u is the class of an effective divisor D on X_G. Let D_0 be the union of irreducible components of D whose image by ϕ_G are of dimension 1. Because $D - D_0$ is a sum of the curves Γ_P with non-negative integer coefficients, we have

$$[D_0] \bmod S_G^0 = (A, 1) \qquad \text{in } \mathcal{C}_G^{\sim}.$$

Because $D_0 H_G = 3$, there are three possibilities;

- there exists a splitting line L such that $D_0 = 3F_L$,
- there exist distinct lines L and L' such that L is splitting and that D_0 is the union of F_L and the proper transform of L', or
- there exists a reduced cubic curve E splitting in X_G such that $D_0 = F_E$.

In the first or the second case, we have $(A, 1) = [F_L] \bmod S_G^0$, and hence $|A| = |w_G(L)| = 5$, which contradicts the assumption. Therefore there exists a reduced splitting cubic curve E such that $D_0 = F_E$. In particular, we have $A = w_G(E)$. If E were reducible, then the word A would be also reducible in $\mathcal{C}_G$ by Remarks 8.7 and 8.9. Hence E is irreducible. If E had an ordinary cusp, then $A = w_G(E)$ would be of weight 13 by Proposition 6.11. Therefore E is a smooth or irreducible nodal cubic curve. Let

G_E be a homogeneous polynomial of degree 3 such that E is defined by $G_E = 0$. By Proposition 6.14, there exists another homogeneous cubic polynomial $G_{E'}$ such that $G_E G_{E'} \in k^\times G + \mathcal{V}_{2,6}$. For $t \in k$, we put

$$G_{E_t} := G_{E'} + tG_E.$$

Then we have

$$G_E G_{E_t} \in k^\times G + \mathcal{V}_{2,6}$$

for any $t \in k$. Let E_t denote the cubic curve defined by $G_{E_t} = 0$, and let $\mathcal{E}$ be the pencil $\{E_t \mid t \in k \cup \{\infty\}\}$. By Proposition 6.13, every member E_t is a reduced curve with only ordinary nodes as its singularities, and is splitting in X_G. Moreover, the cubic curves E and E_t intersect transversely and

$$w_G(E) = w_G(E_t) = E \cap E_t = \mathrm{Bs}(\mathcal{E}).$$

Hence $\mathcal{E}$ is a pencil splitting in X_G such that $\mathrm{Bs}(\mathcal{E}) = A$. If a member E_{t_0} of $\mathcal{E}$ were reducible, then the word $A = w_G(E_{t_0})$ would also be reducible in $\mathcal{C}_G$. Hence $\mathcal{E}$ is regular. □

Corollary 8.16. *The word* $\mathrm{Bs}(\mathcal{E})$ *of* $\mathcal{C}_G$ *corresponding to a regular splitting pencil* $\mathcal{E}$ *of cubic curves is equal to* $w_G(E)$*, where* E *is an arbitrary member of* $\mathcal{E}$.

Corollary 8.17. *Let* $A \in \mathcal{C}_G$ *be an irreducible word of weight* 9*. If the* 2*-dimensional vector space* $H^0(\mathbb{P}^2, \mathcal{I}_A(3))$ *is generated by* G_E *and* $G_{E'}$*, then the homogeneous polynomial* $G_E G_{E'}$ *of degree* 6 *is contained in* $k^\times G + \mathcal{V}_{2,6}$.

Remark 8.18. It is known that a regular pencil $\mathcal{E}$ of cubic curves has exactly 12 singular members $\{E_1, \dots, E_{12}\}$. Suppose that the regular pencil $\mathcal{E}$ is splitting in X_G. The ordinary node P_i of a singular member E_i is a point of $Z(dG)$ by Corollary 6.9. By assigning P_i to the singular member E_i, we obtain a bijection

$$\{E_1, \dots, E_{12}\} \;\cong\; Z(dG) \setminus \mathrm{Bs}(\mathcal{E}).$$

Remark 8.19. The decomposition of a reducible word $A \in \mathcal{C}_G$ of weight 9 into a sum of irreducible words is *not* unique. For example, let G_1 and G_1' be general homogeneous polynomials of degree 1, and let G_2 and G_2' be general homogeneous polynomials of degree 2. Then $G := G_1 G_1' G_2 G_2'$ is contained in $\mathcal{U}_{2,6}$. (See Example 9.9.) The lines $L := \{G_1 = 0\}$, $L' := \{G_1' = 0\}$ and the smooth conics $Q := \{G_2 = 0\}$, $Q' := \{G_2' = 0\}$ are splitting in X_G by Proposition 6.13. We have two decompositions of the word

$$w_G(L) + w_G(Q) = w_G(L') + w_G(Q')$$

of weight 9, which is equal to $w_G(E)$, where E is an arbitrary member of the splitting (non-regular) pencil of cubic curves spanned by $L \cup Q$ and $L' \cup Q'$.

Remark 8.20. Let $\mathcal{E}$ be a regular splitting pencil of cubic curves.

Let L be a splitting line. Because

$$|\mathrm{Bs}(\mathcal{E}) + w_G(L)| = 14 - 2|\mathrm{Bs}(\mathcal{E}) \cap w_G(L)|,$$

the weight of $\mathrm{Bs}(\mathcal{E}) \cap w_G(L)$ is either 1 or 3. By Corollary 8.11, $|\mathrm{Bs}(\mathcal{E}) \cap w_G(L)|$ cannot be 3. Let E_t be a general member of $\mathcal{E}$. Suppose that E_t intersects L transversely at a point P. Then P is an ordinary node of the reducible splitting curve $E_t \cup L$, and hence $P \in Z(dG)$ by Corollary 6.9. In particular, P is contained in $\mathrm{Bs}(\mathcal{E}) \cap w_G(L)$. Therefore the restriction $\mathcal{E}|L$ of $\mathcal{E}$ to L consists of one fixed point and a moving non-reduced point of multiplicity 2.

Let Q be a smooth splitting conic. Then $|\mathrm{Bs}(\mathcal{E}) \cap w_G(Q)|$ is either 2 or 4 or 6. Suppose that $|\mathrm{Bs}(\mathcal{E}) \cap w_G(Q)| = 6$, and let P be a point of $w_G(Q) \setminus (\mathrm{Bs}(\mathcal{E}) \cap w_G(Q))$. There exists a member E_P of $\mathcal{E}$ that has an ordinary node at P by Remark 8.18. Then Q must be contained in E_P, which contradicts the regularity of $\mathcal{E}$. Hence $|\mathrm{Bs}(\mathcal{E}) \cap w_G(Q)|$ is 2 or 4. When $|\mathrm{Bs}(\mathcal{E}) \cap w_G(Q)| = 2$ (resp. 4), the restriction $\mathcal{E}|Q$ of $\mathcal{E}$ to Q consists of two (resp. four) fixed points and moving non-reduced points of total multiplicity 4 (resp. 2).

Remark 8.21. Let $A \in \mathcal{C}_G$ be a word of weight 13. Then one of the following holds:

(i) There are three splitting lines L_1, L_2, L_3 meeting at a point such that $A = w_G(L_1) + w_G(L_2) + w_G(L_3)$.

(ii) There are a splitting line L and a splitting smooth conic Q such that L is tangent to Q and that $A = w_G(L) + w_G(Q)$.

(iii) There exists a cuspidal cubic curve C splitting in X_G such that $A = w_G(C)$.

We put $G_Q := X_0^2 + X_1X_2$, and let G_4 be a general homogeneous polynomial of degree 4. Then G_QG_4 is a polynomial in $\mathcal{U}_{2,6}$, and the smooth conic Q defined by $G_Q = 0$ is splitting in $X_{G_QG_4}$. Let C be the cubic curve defined by $\partial G_4/\partial X_0 = 0$. It is easy to see that C has one ordinary cusp as its only singularities, and is splitting in $X_{G_QG_4}$. Moreover, the word $w_{G_QG_4}(C)$ coincides with $Z(dG_QG_4) \setminus w_{G_QG_4}(Q)$.

Since $\mathcal{C}_G$ is generated by $Z(dG) \in \mathcal{C}_G$ and irreducible words of weight 5, 8 and 9, we obtain the following:

COROLLARY 8.22. *The lattice S_G is generated by the following vectors;*
- $[H_G]$ *and* $[\Gamma_P]$ $(P \in Z(dG))$,
- $[F_C]$, *where C is a general member of* $|\mathcal{I}_{Z(dG)}(5)|$,
- $[F_L]$, *where L runs through the set of splitting lines,*
- $[F_Q]$, *where Q runs through the set of splitting smooth conics,*
- $[F_E]$, *where E runs through the set of members of regular splitting pencils of cubic curves.*

Main Theorem in Introduction has now been proved by Propositions 6.3, 8.5, 8.8, 8.15 and Corollary 8.22.

8.3. The list. Using Theorem 8.1 and Algorithm 5.25, we make the complete list of geometrically realizable classes of codes. In the list below, the following data are recorded.
- σ: The Artin invariant $11 - \dim \mathtt{C}$ of the corresponding supersingular $K3$ surfaces. For each σ, the number $r(\sigma)$ of geometrically realizable classes with Artin invariant σ is also given.
- `std`: A standard basis of the $\mathfrak{S}_{21}$-equivalence class $[\mathtt{C}]$. (See Definition 5.23.) A word is expressed by a bit vector, and a bit vector $[\alpha_0, \dots, \alpha_{20}]$ is expressed by the integer $2^{20}\alpha_0 + \cdots + 2\alpha_{19} + \alpha_{20}$. Since $[1, \dots, 1] = 2^{21} - 1$ corresponding to the word `Z` is always in standard bases by definition, it is omitted.
- `l`: The number of words of weight 5; that is, the number of splitting lines.
- `q`: The number of irreducible words of weight 8; that is, the number of splitting smooth conics.

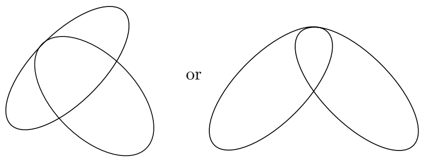

FIG. 8.1. *The configurations of smooth conics for* `qq`

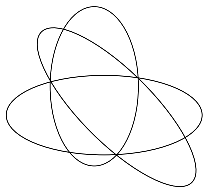

FIG. 8.2. *The configuration of smooth conics for* `tq1`

- `e`: The number of irreducible words of weight 9; that is, the number of splitting regular pencils of cubic curves.

There are several pairs of classes of codes with identical $(\sigma, \mathtt{l}, \mathtt{q}, \mathtt{e})$. (For example, the classes No.134 - No.136. See Examples 9.5 and 9.10.) By trial and error, we have found that the following added data are sufficient to distinguish all the geometrically realizable classes of codes.

- `tl`: The number of triples $\{L_1, L_2, L_3\}$ of splitting lines such that $L_1 \cap L_2 \cap L_3$ consists of one point; that is, the number of triples $\{A_1, A_2, A_3\}$ of distinct words of weight 5 satisfying $|A_1 \cap A_2 \cap A_3| = 1$.
- `lq`: The number of pairs (L, Q) of a splitting line L and a splitting smooth conic Q such that L is tangent to Q; that is, the number of pairs (A, B) of words such that $|A| = 5$, $|B| = 8$, B is irreducible, and $A \cap B = \emptyset$.
- `qq`: The number of pairs $\{Q, Q'\}$ of splitting smooth conics such that there exist exactly two points of $Q \cap Q'$ at which Q and Q' intersect with odd intersection multiplicity; that is, the number of pairs $\{A, A'\}$ of irreducible words of weight 8 such that $|A \cap A'| = 2$. See Figure 8.1.
- `tq1`: The number of triples $\{Q_1, Q_2, Q_3\}$ of smooth splitting conics with the configuration as in Figure 8.3; that is, the number of triples $\{A_1, A_2, A_3\}$ of irreducible words of weight 8 such that $|A_i \cap A_j| = 4$ for each $i \neq j$ and $|A_1 \cap A_2 \cap A_3| = 3$.
- `tq2`: The number of triples $\{Q_1, Q_2, Q_3\}$ of smooth splitting conics such that, for each i, j with $i \neq j$, there exist exactly two points of $Q_i \cap Q_j$ at which Q_i and Q_j intersect with odd intersection multiplicity; that is, the number of triples $\{A_1, A_2, A_3\}$ of irreducible words of weight 8 such that $|A_i \cap A_j| = 2$ for $i \neq j$. See Figure 8.3.

The complete list of geometrically realizable classes of codes

No.	σ	std	l	q	e	tl	lq	qq	tq1	tq2

$\sigma = 10$. $r(10) = 1$.

No.	σ	std	l	q	e	tl	lq	qq	tq1	tq2
0	10		0	0	0	0	0	0,	0,	0

$\sigma = 9$. $r(9) = 3$.

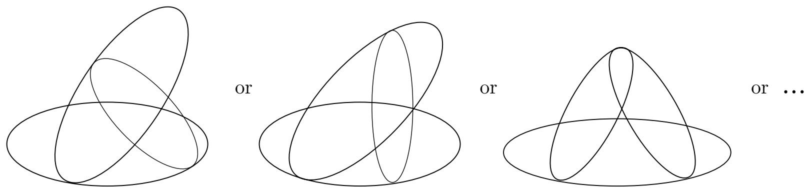

Fig. 8.3. *The configurations of smooth conics for* `tq2`

1	9	31	1	0	0	0	0	0,	0,	0
2	9	255	0	1	0	0	0	0,	0,	0
3	9	511	0	0	1	0	0	0,	0,	0

$\sigma = 8$. $r(8) = 8$.

4	8	31, 481	2	0	0	0	0	0,	0,	0
5	8	31, 8160	1	2	0	0	2	0,	0,	0
6	8	31, 2019	1	1	0	0	0	0,	0,	0
7	8	31, 8161	1	0	2	0	0	0,	0,	0
8	8	255, 3855	0	3	0	0	0	0,	0,	0
9	8	255, 16131	0	2	1	0	0	1,	0,	0
10	8	255, 7951	0	1	2	0	0	0,	0,	0
11	8	511, 32263	0	0	3	0	0	0,	0,	0

$\sigma = 7$. $r(7) = 21$.

12	7	31, 8160, 481	3	1	0	1	3	0,	0,	0
13	7	31, 2019, 2301	3	0	0	0	0	0,	0,	0
14	7	31, 8160, 516193	2	2	0	0	2	0,	0,	0
15	7	31, 2019, 6244	2	2	0	0	0	0,	0,	0
16	7	31, 8161, 253987	2	1	1	0	0	0,	0,	0
17	7	31, 8160, 123360	1	6	0	0	6	0,	0,	0
18	7	31, 8160, 25059	1	4	0	0	2	2,	0,	0
19	7	31, 2019, 63533	1	3	0	0	0	3,	0,	1
20	7	31, 2019, 14565	1	3	0	0	0	0,	0,	0
21	7	31, 8160, 123361	1	2	4	0	2	0,	0,	0
22	7	31, 8161, 25062	1	2	2	0	0	1,	0,	0
23	7	31, 8161, 254178	1	1	4	0	0	0,	0,	0
24	7	255, 3855, 13107	0	7	0	0	0	0,	0,	0
25	7	255, 3855, 28951	0	6	1	0	0	3,	4,	0
26	7	255, 3855, 62211	0	5	2	0	0	4,	0,	0
27	7	255, 3855, 127249	0	4	3	0	0	3,	0,	0
28	7	255, 16131, 115471	0	3	4	0	0	3,	0,	1
29	7	255, 3855, 29491	0	3	4	0	0	0,	0,	0
30	7	255, 16131, 50973	0	2	5	0	0	1,	0,	0
31	7	255, 7951, 123187	0	1	6	0	0	0,	0,	0

32	7	511, 32263, 233016	0	0	7	0	0	0,	0,	0

$\sigma = 6$. $\quad r(6) = 43$.

33	6	31, 8160, 123360, 1966081	5	0	0	10	0	0,	0,	0
34	6	31, 8160, 25059, 28385	4	1	0	1	3	0,	0,	0
35	6	31, 2019, 6244, 8637	4	1	0	0	0	0,	0,	0
36	6	31, 8160, 25059, 105991	3	5	0	1	7	0,	0,	0
37	6	31, 8160, 25059, 26215	3	5	0	1	3	4,	0,	0
38	6	31, 8161, 253987, 319591	3	3	1	0	0	0,	1,	0
39	6	31, 8160, 25059, 238049	3	3	0	1	3	0,	0,	0
40	6	31, 8160, 25059, 42497	3	3	0	0	2	1,	0,	0
41	6	31, 8160, 516193, 582560	2	6	0	0	6	0,	0,	0
42	6	31, 8160, 25059, 100324	2	6	0	0	4	6,	0,	0
43	6	31, 8160, 25059, 44583	2	6	0	0	2	6,	2,	2
44	6	31, 2019, 63533, 68551	2	6	0	0	0	12,	0,	8
45	6	31, 2019, 6244, 27049	2	6	0	0	0	0,	0,	0
46	6	31, 8160, 25059, 492257	2	4	2	0	2	2,	0,	0
47	6	31, 8161, 253987, 271302	2	4	2	0	0	5,	0,	2
48	6	31, 8161, 253987, 288708	2	4	2	0	0	2,	0,	0
49	6	31, 8160, 123360, 419424	1	14	0	0	14	0,	0,	0
50	6	31, 8160, 25059, 241184	1	10	0	0	6	12,	16,	0
51	6	31, 8160, 25059, 124512	1	10	0	0	6	12,	0,	0
52	6	31, 8160, 25059, 492069	1	8	0	0	2	12,	4,	4
53	6	31, 8160, 25059, 42605	1	8	0	0	2	6,	0,	0
54	6	31, 8160, 123360, 419425	1	6	8	0	6	0,	0,	0
55	6	31, 8160, 25059, 99948	1	6	4	0	2	8,	0,	4
56	6	31, 8160, 25059, 238119	1	6	4	0	2	8,	0,	0
57	6	31, 8161, 25062, 99051	1	6	2	0	0	9,	0,	4
58	6	31, 8161, 25062, 42602	1	6	2	0	0	3,	4,	0
59	6	31, 8160, 25059, 239201	1	4	8	0	2	2,	0,	0
60	6	31, 8161, 25062, 229998	1	4	6	0	0	6,	0,	4
61	6	31, 8161, 25062, 501288	1	4	6	0	0	3,	0,	0
62	6	255, 3855, 13107, 21845	0	15	0	0	0	0,	0,	0
63	6	255, 3855, 28951, 46881	0	13	2	0	0	12,	32,	0
64	6	255, 3855, 28951, 492145	0	11	4	0	0	16,	16,	0
65	6	255, 3855, 62211, 208947	0	9	6	0	0	18,	0,	6
66	6	255, 3855, 28951, 233577	0	9	6	0	0	15,	8,	3
67	6	255, 3855, 13107, 116021	0	9	6	0	0	12,	0,	0
68	6	255, 3855, 127249, 405606	0	7	8	0	0	12,	0,	4
69	6	255, 3855, 28951, 111147	0	7	8	0	0	9,	4,	3
70	6	255, 3855, 13107, 54613	0	7	8	0	0	0,	0,	0
71	6	255, 16131, 115471, 412723	0	5	10	0	0	10,	0,	10
72	6	255, 3855, 127249, 144998	0	5	10	0	0	7,	0,	3

73	6	255, 3855, 62211, 79157	0	5	10	0	0	4,	0,	0
74	6	255, 16131, 115471, 396597	0	3	12	0	0	3,	0,	1
75	6	255, 3855, 29491, 230741	0	3	12	0	0	0,	0,	0

$\sigma = 5$. $r(5) = 58$.

76	5	31, 8160, 25059, 238049, 3618	6	0	0	10	0	0,	0,	0
77	5	31, 2019, 6244, 8637, 19179	6	0	0	0	0	0,	0,	0
78	5	31, 8160, 25059, 105991, 26232	5	8	0	10	8	0,	0,	0
79	5	31, 8160, 25059, 105991, 147041	5	4	0	2	8	0,	0,	0
80	5	31, 8160, 25059, 42605, 26781	5	4	0	1	3	3,	0,	0
81	5	31, 8161, 253987, 288708, 894990	4	7	2	0	0	0,	8,	0
82	5	31, 8160, 25059, 238119, 25661	4	7	0	1	7	4,	6,	0
83	5	31, 8160, 25059, 42605, 98704	4	7	0	1	5	8,	3,	0
84	5	31, 8160, 25059, 492069, 534498	4	7	0	0	4	10,	4,	4
85	5	31, 8160, 25059, 105991, 394851	3	13	0	1	15	24,	0,	0
86	5	31, 8160, 25059, 105991, 42605	3	13	0	1	15	0,	0,	0
87	5	31, 8160, 25059, 238119, 377379	3	13	0	1	11	28,	32,	8
88	5	31, 8160, 25059, 105991, 434281	3	13	0	1	7	32,	16,	24
89	5	31, 8160, 25059, 42605, 2724	3	13	0	1	3	12,	0,	0
90	5	31, 8161, 253987, 271302, 901198	3	9	3	0	0	27,	3,	27
91	5	31, 8160, 25059, 42605, 100414	3	9	2	0	2	13,	6,	6
92	5	31, 8160, 25059, 238119, 49277	3	9	1	0	4	17,	5,	7
93	5	31, 8160, 25059, 105991, 140901	3	9	0	1	7	8,	0,	0
94	5	31, 8160, 25059, 238119, 1736	3	9	0	1	3	18,	4,	6
95	5	31, 8160, 25059, 492069, 106180	3	9	0	0	6	15,	4,	6
96	5	31, 8160, 25059, 124512, 951009	3	9	0	0	6	9,	0,	0
97	5	31, 8160, 25059, 238119, 1869504	2	14	0	0	8	36,	22,	18
98	5	31, 8160, 25059, 492069, 1615373	2	14	0	0	4	42,	24,	32
99	5	31, 8160, 25059, 42605, 101942	2	14	0	0	4	30,	24,	16
100	5	31, 8160, 25059, 241184, 370273	2	10	4	0	6	12,	16,	0
101	5	31, 8160, 25059, 492069, 101592	2	10	4	0	4	24,	4,	20
102	5	31, 8160, 25059, 238119, 884843	2	10	4	0	4	18,	0,	0
103	5	31, 8160, 25059, 238119, 888353	2	10	4	0	2	24,	6,	18
104	5	31, 8161, 253987, 288708, 622825	2	10	4	0	0	30,	0,	32
105	5	31, 8161, 253987, 288708, 796873	2	10	4	0	0	24,	0,	16
106	5	31, 8161, 253987, 288708, 567406	2	10	4	0	0	12,	16,	0
107	5	31, 8160, 123360, 419424, 699040	1	30	0	0	30	0,	0,	0
108	5	31, 8160, 25059, 124512, 494240	1	22	0	0	14	56,	128,	0
109	5	31, 8160, 25059, 124512, 396941	1	18	0	0	6	60,	48,	32
110	5	31, 8160, 25059, 124512, 166317	1	18	0	0	6	54,	68,	24
111	5	31, 8160, 25059, 124512, 43685	1	18	0	0	6	36,	0,	0
112	5	31, 8160, 123360, 419424, 699041	1	14	16	0	14	0,	0,	0
113	5	31, 8160, 25059, 238119, 828508	1	14	8	0	6	40,	32,	24

114	5	31, 8160, 25059, 238119, 372292	1	14	8	0	6	40,	0,	16
115	5	31, 8160, 25059, 492069, 124520	1	14	4	0	2	48,	16,	44
116	5	31, 8160, 25059, 238119, 885801	1	14	4	0	2	42,	20,	28
117	5	31, 8160, 25059, 42605, 101044	1	14	4	0	2	24,	32,	12
118	5	31, 8160, 25059, 124512, 436897	1	10	16	0	6	12,	0,	0
119	5	31, 8160, 25059, 238119, 296165	1	10	12	0	2	26,	4,	20
120	5	31, 8160, 25059, 42605, 477857	1	10	12	0	2	20,	0,	12
121	5	31, 8161, 25062, 99051, 427305	1	10	10	0	0	30,	0,	30
122	5	31, 8161, 25062, 99051, 173347	1	10	10	0	0	24,	8,	18
123	5	255, 3855, 28951, 492145, 538402	0	25	6	0	0	60,	240,	0
124	5	255, 3855, 28951, 492145, 564498	0	21	10	0	0	66,	128,	14
125	5	255, 3855, 28951, 492145, 558755	0	21	10	0	0	60,	80,	0
126	5	255, 3855, 28951, 492145, 110650	0	17	14	0	0	58,	48,	30
127	5	255, 3855, 28951, 492145, 623923	0	17	14	0	0	52,	48,	24
128	5	255, 3855, 28951, 233577, 893570	0	13	18	0	0	42,	16,	34
129	5	255, 3855, 13107, 116021, 415508	0	13	18	0	0	42,	0,	30
130	5	255, 3855, 28951, 492145, 570411	0	13	18	0	0	36,	16,	24
131	5	255, 3855, 28951, 111147, 398693	0	9	22	0	0	24,	4,	28
132	5	255, 3855, 127249, 144998, 284986	0	9	22	0	0	24,	0,	20
133	5	255, 3855, 62211, 208947, 87381	0	9	22	0	0	18,	0,	6

$\sigma = 4$. $r(4) = 41$.

134	4	31, 8160, 25059, 238119, 1736, 1867799	7	7	0	11	9	0,	0,	0
135	4	31, 8160, 25059, 105991, 394851, 139649	7	7	0	7	21	0,	0,	0
136	4	31, 8160, 25059, 105991, 434281, 614571	7	7	0	3	9	12,	0,	0
137	4	31, 8160, 25059, 238119, 884843, 418183	6	12	0	3	15	24,	30,	6
138	4	31, 8160, 25059, 42605, 2724, 987586	6	12	0	2	6	18,	18,	0
139	4	31, 8160, 25059, 492069, 534498, 1812520	6	12	0	0	12	30,	40,	0
140	4	31, 8160, 25059, 238119, 372292, 29575	5	24	0	10	24	96,	192,	64
141	4	31, 8160, 25059, 105991, 26232, 43689	5	24	0	10	24	0,	0,	0
142	4	31, 8160, 25059, 238119, 884843, 1058259	5	16	0	2	16	44,	40,	24
143	4	31, 8160, 25059, 238119, 884843, 7297	5	16	0	2	16	20,	48,	0
144	4	31, 8160, 25059, 238119, 49277, 516264	5	16	0	1	11	53,	44,	44
145	4	31, 8160, 25059, 238119, 884843, 1409677	4	19	2	0	8	74,	64,	74
146	4	31, 8160, 25059, 238119, 884843, 52788	4	19	0	1	13	70,	71,	58
147	4	31, 8160, 25059, 238119, 884843, 1474759	4	19	0	1	9	66,	43,	36

148	4	31, 8160, 25059, 238119, 49277, 984106	4	19	0	0	12	78, 58, 86
149	4	31, 8160, 25059, 238119, 372292, 103644	3	29	0	1	23	152, 272, 152
150	4	31, 8160, 25059, 105991, 394851, 696425	3	29	0	1	15	184, 224, 272
151	4	31, 8160, 25059, 238119, 377379, 950861	3	29	0	1	15	160, 272, 192
152	4	31, 8160, 25059, 238119, 49277, 281774	3	21	4	0	6	111, 64, 174
153	4	31, 8160, 25059, 238119, 884843, 1475209	3	21	4	0	6	87, 96, 98
154	4	31, 8160, 25059, 238119, 884843, 1451537	3	21	2	0	10	95, 74, 104
155	4	31, 8160, 25059, 238119, 884843, 1352755	3	21	0	1	15	72, 0, 0
156	4	31, 8160, 25059, 105991, 42605, 141990	3	21	0	1	15	48, 128, 0
157	4	31, 8160, 25059, 238119, 372292, 699489	3	21	0	1	7	104, 64, 144
158	4	31, 8160, 25059, 238119, 1869504, 475241	2	30	0	0	12	186, 276, 244
159	4	31, 8160, 25059, 238119, 1869504, 1902665	2	30	0	0	12	162, 276, 180
160	4	31, 8160, 25059, 238119, 884843, 321232	2	22	8	0	8	110, 90, 150
161	4	31, 8160, 25059, 238119, 884843, 167565	2	22	8	0	4	122, 72, 192
162	4	31, 8160, 25059, 238119, 888353, 1355336	2	22	8	0	4	122, 64, 200
163	4	31, 8160, 25059, 124512, 494240, 700700	1	46	0	0	30	240, 1280, 0
164	4	31, 8160, 25059, 124512, 396941, 662065	1	38	0	0	14	240, 720, 192
165	4	31, 8160, 25059, 238119, 372292, 955584	1	30	16	0	14	176, 256, 192
166	4	31, 8160, 25059, 238119, 372292, 442537	1	30	8	0	6	192, 272, 256
167	4	31, 8160, 25059, 238119, 372292, 950861	1	30	8	0	6	192, 208, 240
168	4	31, 8160, 25059, 238119, 372292, 829089	1	22	24	0	6	120, 48, 176
169	4	31, 8160, 25059, 238119, 296165, 591468	1	22	20	0	2	128, 64, 220
170	4	255, 3855, 28951, 492145, 564498, 42406	0	45	18	0	0	270, 1440, 90
171	4	255, 3855, 28951, 492145, 564498, 722490	0	37	26	0	0	246, 640, 210
172	4	255, 3855, 28951, 492145, 564498, 1127602	0	29	34	0	0	190, 224, 266

173	4	255, 3855, 28951, 233577, 893570, 308270	0	21	42	0	0	126, 56, 238
174	4	255, 3855, 13107, 116021, 415508, 714818	0	21	42	0	0	126, 0, 210

$\sigma = 3.$ $r(3) = 13.$

175	3	31, 8160, 25059, 238119, 884843, 1474759, 475241	9	18	0	20	18	0, 0, 0
176	3	31, 8160, 25059, 238119, 884843, 418183, 1451537	9	18	0	16	30	48, 96, 16
177	3	31, 8160, 25059, 238119, 884843, 418183, 57025	9	18	0	9	27	63, 102, 0
178	3	31, 8160, 25059, 238119, 884843, 418183, 699489	7	31	0	5	35	182, 374, 228
179	3	31, 8160, 25059, 238119, 884843, 1409677, 1058259	7	31	0	3	33	204, 368, 288
180	3	31, 8160, 25059, 238119, 372292, 29575, 955584	5	56	0	10	56	576, 2176, 1152
181	3	31, 8160, 25059, 238119, 884843, 1451537, 699489	5	40	0	2	32	324, 688, 608
182	3	31, 8160, 25059, 238119, 884843, 1451537, 1474759	5	40	0	1	27	357, 628, 804
183	3	31, 8160, 25059, 238119, 372292, 442537, 934222	3	61	0	1	39	744, 2640, 1800
184	3	31, 8160, 25059, 238119, 884843, 1451537, 167565	3	45	6	0	18	495, 774, 1476
185	3	31, 8160, 25059, 238119, 884843, 167565, 1352755	3	45	0	1	15	504, 672, 1520
186	3	31, 8160, 25059, 124512, 396941, 662065, 700700	1	78	0	0	30	1008, 6720, 1536
187	3	31, 8160, 25059, 238119, 372292, 442537, 955584	1	62	16	0	14	816, 2624, 2112

$\sigma = 2.$ $r(2) = 3.$

188	2	31, 8160, 25059, 238119, 884843, 418183, 1451537, 699489	13	28	0	46	60	96, 416, 0
189	2	31, 8160, 25059, 238119, 884843, 418183, 699489, 152785	9	66	0	12	90	864, 3672, 2448
190	2	31, 8160, 25059, 238119, 372292, 442537, 934222, 1844576	5	120	0	10	120	2880, 21120, 13440

$\sigma = 1.$ $r(1) = 1.$

191	1	31, 8160, 25059, 238119, 884843, 418183, 1451537, 699489, 929948	21	0	0	210	0	0, 0, 0

Remark 8.23. Using Proposition 5.19, we have also made the complete list of pairs $([\mathtt{C}], [\mathtt{C}'])$ of geometrically realizable classes of codes satisfying $[\mathtt{C}] < [\mathtt{C}']$.

8.4. Proof of Corollaries. In this subsection, we prove Corollaries 1.9, 1.10 and 1.11 that are stated in Introduction. We denote by $\mathbf{C}_\nu$ the geometrically realizable class of No. ν in the list.

Proof of Corollary 1.9. Note that

$$\mathcal{U}_\sigma = \bigsqcup_{11-\dim \mathtt{C}=\sigma} \mathcal{U}_{2,6,[\mathtt{C}]}.$$

Let $\widetilde{\mathcal{U}}_\sigma$ be the pull-back of $\mathcal{U}_\sigma$ by the étale covering $\widetilde{\mathcal{U}}_{2,6} \to \mathcal{U}_{2,6}$ constructed in the proof of Theorem 5.15. The code $\tau_G^{-1}(\mathcal{C}_G)$ in $\mathrm{Pow}(\mathtt{Z})$ does not vary when (G, τ_G) moves on an irreducible component of $\widetilde{\mathcal{U}}_\sigma$. Hence each irreducible component of $\mathcal{U}_\sigma$ is contained in a unique $\mathcal{U}_{2,6,[\mathtt{C}]}$ with $\dim \mathtt{C} = 11 - \sigma$. Therefore the number of the irreducible components of $\mathcal{U}_\sigma$ is greater than or equal to the number $r(\sigma)$ of geometrically realizable classes $[\mathtt{C}]$ of codes with $\dim \mathtt{C} = 11 - \sigma$. □

Proof of Corollary 1.10. Let G be a polynomial in $\mathcal{U}_{2,6}$. The Artin invariant of X_G is < 10 if and only if there exists a reduced irreducible curve of degree ≤ 2 splitting in X_G, or there exists a regular pencil of cubic curves splitting in X_G. If there is a line (resp. a smooth conic) splitting in X_G, then $G \in \mathcal{U}[51]$ (resp. $G \in \mathcal{U}[42]$) by Proposition 6.14. If there is a regular pencil of cubic curves splitting in X_G, then $G \in \mathcal{U}[33]$ by Corollary 8.17.

It is obvious that the loci $\mathcal{U}[51]$, $\mathcal{U}[42]$ and $\mathcal{U}[33]$ are irreducible. Because the locus $k^\times G + \mathcal{V}_{2,6}$ is closed in $\mathcal{U}_{2,6}$ for any $G \in \mathcal{U}_{2,6}$, these loci are Zariski closed in $\mathcal{U}_{2,6}$. Because of the existence of the geometrically realizable class $\mathbf{C}_0$, Proposition 6.13 implies that $\mathcal{U}[51]$, $\mathcal{U}[42]$ and $\mathcal{U}[33]$ are proper subsets of $\mathcal{U}_{2,6}$. Therefore it remains to show that the codimension of these loci in $\mathcal{U}_{2,6}$ is ≤ 1.

Let $\widetilde{\mathcal{U}}_{2,6} \to \mathcal{U}_{2,6}$ be the étale covering that has appeared in the proof of Theorem 5.15. We choose six elements $\mathtt{P}_1, \ldots, \mathtt{P}_6$ of $\mathtt{Z}$, and consider the locus

$$\left\{ (G, \tau_G) \in \widetilde{\mathcal{U}}_{2,6} \;\middle|\; \begin{array}{l} \text{there exists a smooth conic passing through} \\ \tau_G(\mathtt{P}_1), \ldots, \tau_G(\mathtt{P}_6) \end{array} \right\} \tag{8.5}$$

of $\widetilde{\mathcal{U}}_{2,6}$. Because of the existence of the geometrically realizable class $\mathbf{C}_2$, for example, the locus (8.5) is non-empty. Since $\dim |\mathcal{O}_{\mathbb{P}^2}(2)| = 5$, the locus (8.5) is of codimension ≤ 1 in $\widetilde{\mathcal{U}}_{2,6}$. If (G, τ_G) is in the locus (8.5), then there exists a smooth conic splitting in X_G by Proposition 6.15, and hence G is contained in $\mathcal{U}[42]$ by Proposition 6.14. Therefore the codimension of $\mathcal{U}[42]$ in $\mathcal{U}_{2,6}$ is also ≤ 1. The fact that $\mathcal{U}[51] \subset \mathcal{U}_{2,6}$ is of codimension 1 is proved in a similar way.

Because of the existence of the geometrically realizable class $\mathbf{C}_3$, if G is a general point of $\mathcal{U}[33]$, then there exists only one regular pencil of cubic curves splitting in X_G. Consider the morphism

$$\varrho \,:\, H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)) \times H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)) \times k^\times \times H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)) \;\to\; H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6))$$

defined by

$$(G_3, G_3', c, H) \;\mapsto\; c\, G_3 G_3' + H^2.$$

Let G_3 and G_3' be general homogeneous polynomials of degree 3. Suppose that

$$\varrho(G_3, G_3', 1, 0) = \varrho(\Gamma_3, \Gamma_3', c, H).$$

Then the pencil of cubic curves spanned by the curves defined by $G_3 = 0$ and $G_3' = 0$ coincides with the pencil spanned by the curves defined by $\Gamma_3 = 0$ and $\Gamma_3' = 0$. Hence there exists an invertible matrix

$$\begin{pmatrix} s & t \\ u & v \end{pmatrix}$$

such that

$$G_3 = s\Gamma_3 + t\Gamma_3' \quad \text{and} \quad G_3' = u\Gamma_3 + v\Gamma_3'$$

hold. Then we have

$$c = sv + tu \quad \text{and} \quad H = \sqrt{su}\,\Gamma_3 + \sqrt{tv}\,\Gamma_3'.$$

Hence we have

$$\dim \mathcal{U}[33] = 3h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)) + 1 - \dim GL(2) = 27 = \dim \mathcal{U}_{2,6} - 1.$$

Therefore $\mathcal{U}[33]$ is a hypersurface of $\mathcal{U}_{2,6}$. □

Proof of Corollary 1.11. Let G_{DK} be the Dolgachev-Kondo polynomial (1.1). Note that $Z(dG_{\mathrm{DK}})$ coincides with the set $\mathbb{P}^2(\mathbb{F}_4)$ of $\mathbb{F}_4$-rational points of $\mathbb{P}^2$, and hence the set of lines splitting in $X_{G_{\mathrm{DK}}}$ is equal to the set $(\mathbb{P}^2)^\vee(\mathbb{F}_4)$ of $\mathbb{F}_4$-rational lines of $\mathbb{P}^2$.

Let G be a polynomial in $\mathcal{U}_{2,6}$ such that the Artin invariant of X_G is 1. It is enough to show that, if we choose homogeneous coordinates of $\mathbb{P}^2$ appropriately, then G is contained in $k^\times G_{\mathrm{DK}} + \mathcal{V}_{2,6}$. Let $\mathcal{L}_G$ be the set of lines splitting in X_G. Since there exists only one geometrically realizable class $\mathbf{C}_{191}$ with Artin invariant 1, the configuration $(\mathcal{L}_G, Z(dG))$ of lines and points is isomorphic as abstract configurations (see [5]) to $((\mathbb{P}^2)^\vee(\mathbb{F}_4), \mathbb{P}^2(\mathbb{F}_4))$. In particular, for any two points $P, Q \in Z(dG)$, the line $\overline{PQ}$ passing through P and Q is in $\mathcal{L}_G$. By choosing suitable homogeneous coordinates $[X_0, X_1, X_2]$ and numbering the lines $\mathcal{L}_G = \{L_0, \dots, L_{20}\}$ appropriately, we can assume that

$$L_0 = \{X_2 = 0\}, \quad L_1 = \{X_1 = 0\}, \quad L_2 = \{X_1 = X_2\}, \quad L_3 = \{X_0 = 0\},$$
$$L_4 = \{X_0 = X_2\}, \quad L_5 = \{X_0 = X_1\}, \quad L_6 = \{X_0 + X_1 + X_2 = 0\}.$$

The following points are in $Z(dG)$:

$$P_0 := L_0 \wedge L_1 = [1,0,0], \quad P_1 := L_0 \wedge L_3 = [0,1,0], \quad P_2 := L_3 \wedge L_1 = [0,0,1].$$

There exists a point $Q_0 := [\alpha, 0, 1]$ in $L_1 \cap Z(dG)$ with $\alpha \neq 0, 1$. Then we have

$$\begin{aligned}
L_7 &:= \overline{P_1 Q_0} = \{X_0 = \alpha X_2\} \in \mathcal{L}_G, \\
Q_1 &:= L_5 \wedge L_7 = [\alpha, \alpha, 1] \in Z(dG), \\
L_8 &:= \overline{P_0 Q_1} = \{X_1 = \alpha X_2\} \in \mathcal{L}_G, \\
Q_2 &:= L_6 \wedge L_8 = [1+\alpha, \alpha, 1] \in Z(dG), \\
L_9 &:= \overline{P_1 Q_2} = \{X_0 = (1+\alpha) X_2\} \in \mathcal{L}_G.
\end{aligned}$$

The five points consisting $L_9 \cap Z(dG)$ are therefore

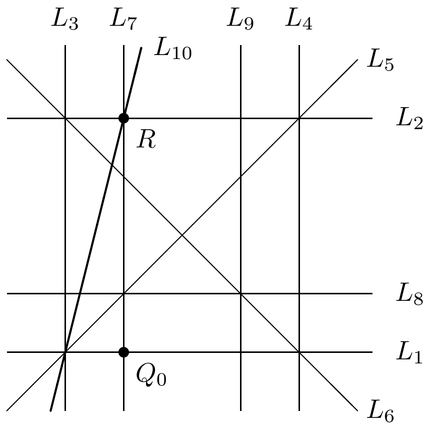

FIG. 8.4. *Lines in* $\mathcal{L}_G$

$$P_1 = [0,1,0], \quad Q_2 = [1+\alpha, \alpha, 1], \quad L_2 \wedge L_9 = [1+\alpha, 1, 1],$$
$$L_5 \wedge L_9 = [1+\alpha, 1+\alpha, 1], \quad \text{and} \quad L_1 \wedge L_9 = [1+\alpha, 0, 1].$$

On the other hand, the point $R := L_7 \wedge L_2 = [\alpha, 1, 1]$ is contained in $Z(dG)$, and hence a line

$$L_{10} := \overline{P_2 R} = \{X_0 + \alpha X_1 = 0\}$$

is an element of $\mathcal{L}_G$. The point

$$L_{10} \wedge L_9 = [\alpha^2 + \alpha, \alpha + 1, \alpha]$$

is therefore among the five points above. Because $\alpha \neq 0, 1$, this point must be Q_2, and α is a root of $t^2+t+1=0$. Then we can show that all points of $Z(dG)$ are $\mathbb{F}_4$-rational, and hence $Z(dG) = Z(dG_{\mathrm{DK}})$ holds. By the uniqueness assertion of Theorem 2.1, we have $dG = c \cdot dG_{DK}$, where c is a non-zero constant. Since $\mathcal{V}_{2,6}$ is the kernel of the linear homomorphism $G \mapsto dG$, we have $G \in k^{\times} G_{\mathrm{DK}} + \mathcal{V}_{2,6}$. ∎

9. The algorithm.

9.1. The description of the algorithm. We present an algorithm that calculates the code $\mathcal{C}_G$ from a given homogeneous polynomial $G \in \mathcal{U}_{2,6}$. From the results in the previous sections, we obtain the following:

COROLLARY 9.1. *Let G be a polynomial in $\mathcal{U}_{2,6}$.*

(1) *A subset $B \subset Z(dG)$ of weight* 5 *is contained in $\mathcal{C}_G$ if and only if the points of B are collinear.*

(2) *Let $B \subset Z(dG)$ be a subset of weight* 8 *such that no three points of B are collinear. Then B is contained in $\mathcal{C}_G$ if and only if there exists a conic containing B.* (*Note that, if such a conic exists, then it must be smooth because no three points of B are collinear.*)

COROLLARY 9.2. *Let G be a polynomial in $\mathcal{U}_{2,6}$, and let $B \subset Z(dG)$ be a subset of weight* 9 *such that no three points of B are collinear. Then B is contained in $\mathcal{C}_G$ if* *and only if the following hold;* (i) *the linear system* $|\mathcal{I}_B(3)|$ *of cubic curves containing* B *is of dimension* 1, *and* (ii) *if* $H^0(\mathbb{P}^2, \mathcal{I}_B(3))$ *is generated by* G_E *and* $G_{E'}$, *then* $G_E G_{E'}$ *is contained in* $k^\times G + \mathcal{V}_{2,6}$.

Proof. If $B \in \mathcal{C}_G$, then (i) and (ii) hold by Proposition 8.15 and Corollaries 8.11 and 8.17. Suppose that (i) and (ii) hold, and let E and E' be the cubic curves defined by $G_E = 0$ and $G_{E'} = 0$. Then E and E' are splitting in X_G, and

$$B = E \cap E' = w_G(E) = w_G(E')$$

holds by Proposition 6.13. Hence B is contained in $\mathcal{C}_G$. □

Remark 9.3. In Corollary 9.2, the condition (i) alone is not enough for B to be contained in $\mathcal{C}_G$. See Example 9.7.

ALGORITHM 9.4. Suppose that we are given a homogeneous polynomial $G \in \mathcal{U}_{2,6}$. This algorithm outputs a set $\texttt{Gen} = \{A_0, \dots, A_{k-1}\} \subset \mathrm{Pow}(Z(dG))$ that generates $\mathcal{C}_G$, and the Artin invariant of X_G.

Step 1. Set `Gen` to be an empty set $\emptyset$.

Step 2. Calculate the coordinates of the points $P_0, \dots, P_{20}$ of $Z(dG)$ by solving

$$\frac{\partial G}{\partial X_0} = \frac{\partial G}{\partial X_1} = \frac{\partial G}{\partial X_2} = 0.$$

Step 3. Put the word $Z(dG) = \{P_0, \dots, P_{20}\}$ in `Gen`.

Step 4. Make the list `Col` of all triples $\{P_i, P_j, P_k\}$ of points of $Z(dG)$ that are collinear.

Step 5. Using `Col`, list up all 5-tuples $\{P_{i_1}, \dots, P_{i_5}\}$ that are collinear, and put them in `Gen`. By Proposition 6.15, every triple in `Col` must extend to a collinear 5-tuple.

Step 6. For each 8-tuple $B = \{P_{i_1}, \dots, P_{i_8}\}$ of points of $Z(dG)$, check whether there exist collinear three points of B by using `Col`. If there are no such three points, then check whether there exists a conic that passes through the points of B. If such a conic exists, then put B in `Gen`.

Step 7. For each 9-tuple $B = \{P_{i_1}, \dots, P_{i_9}\}$, check whether there exist collinear three points of B by using `Col`. If there are no such three points, then calculate $\dim |\mathcal{I}_B(3)|$. If $\dim |\mathcal{I}_B(3)| = 1$, choose polynomials G_E and $G_{E'}$ that span $H^0(\mathbb{P}^2, \mathcal{I}_B(3))$, and check whether $G_E G_{E'}$ is contained in $k^\times G + \mathcal{V}_{2,6}$ or not by using the method described in Remark 3.2. If $G_E G_{E'} \in k^\times G + \mathcal{V}_{2,6}$, then put B in `Gen`.

Step 8. Calculate the code $\mathcal{C}_G$ generated by the words in `Gen`. The Artin invariant of X_G is $11 - \dim \mathcal{C}_G$.

9.2. Examples.

EXAMPLE 9.5. The code $\mathcal{C}_G$ of the polynomial G in Example 1.4 is in the class $\mathbf{C}_{135}$. Let us consider the polynomial

$$G' := {X_0}^5 X_2 + {X_0}^4 X_1 X_2 + {X_0}^3 {X_1}^2 X_2 + {X_0}^2 {X_1}^3 X_2 + \\ + X_0 {X_1}^4 X_2 + X_0 {X_1}^3 {X_2}^2 + X_0 X_1 {X_2}^4.$$

The points of $Z(dG')$ are defined over $\mathbb{F}_{2^{24}}$. Under the Frobenius morphism over $\mathbb{F}_2$, they are decomposed into six orbits, the cardinalities of which are 1, 1, 3, 4, 4, 8. The set of curves of degree ≤ 3 splitting in $X_{G'}$ consists of seven lines, which are

$$
\begin{aligned}
&P_0 = [\alpha^5 + \alpha^3 + \alpha + 1, \alpha^3 + \alpha^2 + \alpha + 1, 1],\\
&P_\nu = \mathrm{Frob}^\nu(P_0) \quad (\nu = 1, \dots, 5),\\
&P_6 = [1, 1, 1], \quad P_7 = [1, 0, 1],\\
&P_8 = [\alpha^4 + \alpha^3 + \alpha^2 + \alpha, \alpha + 1, 1],\\
&P_{8+\nu} = \mathrm{Frob}^\nu(P_8) \quad (\nu = 1, \dots, 5),\\
&P_{14} = [0, 0, 1],\\
&P_{15} = [\alpha^5 + \alpha^4 + \alpha^3 + \alpha^2 + 1, \alpha^5 + \alpha^4 + \alpha^3 + \alpha^2 + \alpha, 1],\\
&P_{15+\nu} = \mathrm{Frob}^\nu(P_{15}) \quad (\nu = 1, \dots, 5).
\end{aligned}
$$

TABLE 9.1. *Points of $Z(dG)$ in Example 9.7*

decomposed into four Frobenius orbits of cardinalities 1, 1, 1, 4, and seven smooth conics, which are decomposed into three Frobenius orbits of cardinalities 1, 2, 4. The class $[\mathcal{C}_{G'}]$ is $\mathbf{C}_{134}$.

EXAMPLE 9.6. Consider the polynomial

$$G := {X_0}^4 X_1 X_2 + {X_0}^3 {X_1}^3 + X_0 {X_1}^4 X_2 + X_0 X_1 {X_2}^4.$$

The subscheme $Z(dG)$ is reduced of dimension 0, and each point is defined over $\mathbb{F}_{2^4}$. The class of the code $\mathcal{C}_G$ is $\mathbf{C}_{190}$. In particular, the Artin invariant of X_G is 2.

EXAMPLE 9.7. We will give an example of X_G with Artin invariant 3. Consider the polynomial

$$
\begin{aligned}
G := {X_0}^5 X_2 + {X_0}^4 X_1 X_2 + {X_0}^3 {X_1}^3 + {X_0}^3 {X_1}^2 X_2 + {X_0}^3 {X_2}^3 + {X_0}^2 {X_1}^3 X_2 + \\
+ X_0 {X_1}^3 {X_2}^2 + X_0 X_1 {X_2}^4 + {X_1}^5 X_2.
\end{aligned}
$$

Let α be a root of the irreducible polynomial

$$t^6 + t^5 + t^3 + t^2 + 1 \in \mathbb{F}_2[t].$$

Then $Z(dG)$ consists of the points in Table 9.1. The words of weight 5 in $\mathcal{C}_G$ are

$$\{0, 3, 6, 16, 19\}, \quad \{1, 4, 6, 17, 20\}, \quad \{2, 5, 6, 15, 18\},$$

which form one Frobenius orbit, where the set $\{P_{i_1}, \dots, P_{i_k}\}$ is simply denoted by $\{i_1, \dots, i_k\}$. There are 45 irreducible words of weight 8 in $\mathcal{C}_G$. The cardinalities of Frobenius orbits are

$$1, 6, 6, 2, 2, 6, 6, 3, 6, 6, 1.$$

There are no irreducible words of weight 9 in $\mathcal{C}_G$. The class $[\mathcal{C}_G]$ is $\mathbf{C}_{185}$. In particular, the Artin invariant of X_G is 3.

Consider the following word of weight 9;

$$A := \{0, 1, 2, 3, 7, 8, 9, 15, 20\}.$$

Note that no three points of A are collinear. There exists a pencil of cubic curves whose base locus is A, which is spanned by

$$\begin{aligned} & {X_0}^2X_1 + \left(\alpha^4+\alpha^3+\alpha^2\right){X_0}^2X_2 + \left(\alpha^5+\alpha^4+\alpha^2\right)X_0{X_1}^2 + \\ & \quad + \left(\alpha^5+\alpha^4+\alpha+1\right)X_0{X_2}^2 + \left(\alpha^4+\alpha^3+1\right){X_1}^3 + \left(\alpha^4+\alpha^3+\alpha\right){X_1}^2X_2 + \\ & \qquad + \left(\alpha^4+\alpha^3+1\right)X_1{X_2}^2 + \left(\alpha^5+\alpha^3+\alpha^2+\alpha+1\right){X_2}^3 = 0, \end{aligned}$$

and

$$\begin{aligned} & {X_0}^3 + \left(\alpha^4+\alpha\right){X_0}^2X_2 + \left(\alpha^5+\alpha^3\right)X_0{X_1}^2 + \left(\alpha^3+\alpha^2+1\right)X_0{X_2}^2 + \\ & \quad + \alpha^3{X_1}^3 + \left(\alpha^5+\alpha^4+\alpha^2+\alpha+1\right){X_1}^2X_2 + \left(\alpha^3+1\right)X_1{X_2}^2 + \\ & \qquad + \left(\alpha^4+\alpha^3+\alpha^2+\alpha\right){X_2}^3 = 0. \end{aligned}$$

However this pencil is not splitting in X_G.

9.3. Irreducibility of $\mathcal{U}_{2,6,\mathbf{C}}$ for some $\mathbf{C}$. For some geometrically realizable classes $\mathbf{C}$, we can prove the irreducibility of the locus $\mathcal{U}_{2,6,\mathbf{C}}$, and give a homogeneous polynomial G that corresponds to the generic point of $\mathcal{U}_{2,6,\mathbf{C}}$.

Definition 9.8. For a non-increasing sequence $[a_1 \dots a_k]$ of positive integers with $a_1 + \cdots + a_k = 6$, we denote by $\mathcal{U}[a_1 \dots a_k]$ the locus of $G \in \mathcal{U}_{2,6}$ such that there exist homogeneous polynomials $G_{a_1}, \dots, G_{a_k}$ of degrees $a_1, \dots, a_k$ satisfying

$$G_{a_1} \cdots G_{a_k} \in k^\times G + \mathcal{V}_{2,6}.$$

It is obvious that $\mathcal{U}[a_1 \dots a_k]$ is an irreducible Zariski closed subset of $\mathcal{U}_{2,6}$.

Example 9.9. Let G be a point of $\mathcal{U}[2211]$. By Proposition 6.13, there exist splitting lines L_1, L_2 and splitting smooth conics Q_1, Q_2 such that the union $L_1 \cup L_2 \cup Q_1 \cup Q_2$ has only ordinary nodes as its singularities. Hence $\mathcal{C}_G$ contains words A_1, A_2, B_1, B_2 satisfying the following:

- $|A_1| = |A_2| = 5$, $|B_1| = |B_2| = 8$,
- B_1 and B_2 are irreducible in $\mathcal{C}_G$,
- $|A_i \cap B_j| = 2$ for $i, j = 1, 2$, and $|B_1 \cap B_2| = 4$,
- $|A_1 \cap A_2 \cap B_j| = |A_i \cap B_1 \cap B_2| = 0$ for $i, j = 1, 2$.

Conversely, suppose that the code $\mathcal{C}_G$ of a polynomial $G \in \mathcal{U}_{2,6}$ contains words A_1, A_2, B_1, B_2 satisfying the conditions above. By Propositions 8.5 and 8.8, there exist lines L_1, L_2 and smooth conics Q_1, Q_2 splitting X_G such that $L_i \cap Z(dG) = A_i$ and $Q_j \cap Z(dG) = B_j$ hold. By Remarks 8.6, 8.9 and 8.10, the union $L_1 \cup L_2 \cup Q_1 \cup Q_2$ has only ordinary nodes as its singularities. Hence, by Proposition 6.14, G is a point of $\mathcal{U}[2211]$.

If $[\mathcal{C}_G] = \mathbf{C}_{15}$, then $\mathcal{C}_G$ contains words A_1, A_2, B_1, B_2 satisfying the conditions above. Conversely, from the complete list of geometrically realizable classes of codes, we see that if $\mathcal{C}_G$ contains words A_1, A_2, B_1, B_2 satisfying the conditions above, then $\mathbf{C}_{15} \leqq [\mathcal{C}_G]$ holds. Hence we have

$$\mathcal{U}_{2,6,\mathbf{C}_{15}} \quad \subset \quad \mathcal{U}[2211] \quad \subset \quad \mathcal{U}_{2,6,\geq \mathbf{C}_{15}}.$$

Therefore $\mathcal{U}_{2,6,\mathbf{C}_{15}}$ is irreducible and its generic point coincides with the generic point of $\mathcal{U}[2211]$.

ν	4	6	8	13	15	35	77
σ	8	8	8	7	7	6	5
$[a_1 \dots a_k]$	[411]	[321]	[222]	[3111]	[2211]	[21111]	[111111]

TABLE 9.2. *The pairs of* $\mathbf{C}_\nu$ *and* $[a_1 \dots a_k]$

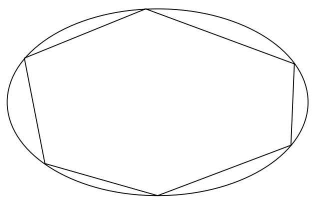

FIG. 9.1. *The Pascal configuration*

By the same argument, we obtain Table 9.2 of the pairs of $\mathbf{C}_\nu$ and $[a_1 \dots a_k]$ such that $\mathcal{U}_{2,6,\mathbf{C}_\nu}$ is irreducible, and that the generic point of $\mathcal{U}_{2,6,\mathbf{C}_\nu}$ coincides with the generic point of $\mathcal{U}[a_1 \dots a_k]$.

EXAMPLE 9.10. Let G be a polynomial of $\mathcal{U}_{2,6}$, and let $A_1, \dots, A_6$ and B be distinct words of $\mathcal{C}_G$. We say that $(A_1, \dots, A_6, B)$ is a *Pascal configuration* if the following hold:

- The words $A_1, \dots, A_6$ are of weight 5.
- The word B is of weight 8 and irreducible in $\mathcal{C}_G$.
- Let P_{ij} be the point of $A_i \cap A_j$ for $i \neq j$. Then the six points P_{12}, P_{23}, P_{34}, P_{45}, P_{56} and P_{61} are distinct and contained in B.

The code $\mathcal{C}_G$ contains a Pascal configuration if and only if there exists a hexagon $L_1L_2L_3L_4L_5L_6$ formed by lines splitting in X_G that is inscribed in a smooth conic Q. (See Figure 9.1.) Note that the conic Q is also splitting in X_G by Proposition 6.15. If $\mathcal{C}_G$ is in the class $\mathbf{C}_{136}$, then $\mathcal{C}_G$ contains a Pascal configuration. If $\mathcal{C}_G$ contains a Pascal configuration, then $\mathbf{C}_{136} \leqq [\mathcal{C}_G]$ holds. Because the moduli of pairs of a smooth conic Q and a hexagon inscribed in Q is irreducible, we conclude that the locus $\mathcal{U}_{2,6,\mathbf{C}_{136}}$ is irreducible.

We fix a smooth conic $Q_1 \subset \mathbb{P}^2$, and let $P_1, \dots, P_6$ be general points on Q_1. We put

$$L_i := \overline{P_iP_{i+1}} \quad (i = 1, \dots, 5), \qquad L_6 := \overline{P_6P_1}.$$

Let $G_{L_i} = 0$ be a defining equation of the line L_i. Then

$$G := G_{L_1}G_{L_2}G_{L_3}G_{L_4}G_{L_5}G_{L_6}$$

is a point of $\mathcal{U}_{2,6,\mathbf{C}_{136}}$. The points $L_1 \wedge L_4$, $L_2 \wedge L_5$, and $L_3 \wedge L_6$ are distinct, because $P_1, \dots, P_6$ are general on Q_1. By Pascal's theorem, these three points are on a line M. By the converse to Pascal's theorem, the hexagons

$$L_1L_5L_3L_4L_2L_6, \quad L_1L_2L_6L_4L_5L_3, \quad \text{and} \quad L_1L_5L_6L_4L_2L_3,$$

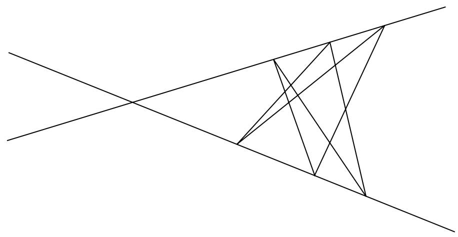

FIG. 9.2. *The Pappos configuration*

are also inscribed in smooth conics. Let Q_2, Q_3 and Q_4 be those conics. Then the lines $L_1, \ldots, L_6, M$ and the smooth conics $Q_1, \ldots, Q_4$ are splitting in X_G.

EXAMPLE 9.11. The class $\mathbf{C}_{177}$ corresponds to the *Pappos configuration* (Figure 9.2) in the same way as $\mathbf{C}_{136}$ corresponds to the Pascal configuration. Hence $\mathcal{U}_{2,6,\mathbf{C}_{177}}$ is irreducible.

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