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CORNALBA-HARRIS EQUALITY FOR SEMISTABLE HYPERELLIPTIC CURVES IN POSITIVE CHARACTERISTIC *

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Introduction. Let Y be a nonsingular projective curve over an algebraically closed field k and $f : X \to Y$ a generically smooth semistable curve of genus $g \geq 2$ with X nonsingular. Let $\omega_{X/Y}$ denote the relative dualizing sheaf of f. Relation between $\deg\left(f_*\omega_{X/Y}\right)$ and discriminant divisors has been studied by many people. Here we consider the case of f hyperelliptic, i.e., the case where there exists a Y-automorphism ι inducing the hyperelliptic involution on the geometric generic fiber. Then for each node x of type 0 in a fiber, we can assign a non-negative integer, called the subtype, to x or the pair $\{x, \iota(x)\}$ (c.f. [2] or §§ 1.2 for the definitions). Let $\delta_i(X/Y)$ denote the number of the nodes of type i, $\xi_0(X/Y)$ the number of nodes of subtype 0 and let $\xi_j(X/Y)$ denote the number of pairs of nodes $\{x, \iota(x)\}$ of subtype $j > 0$, in all the fibers. Cornalba and Harris proved in [2] an equality

$$\begin{aligned}(8g+4)\deg\left(f_*\omega_{X/Y}\right) =& g\xi_0(X/Y) + \sum_{j=1}^{\left[\frac{g-1}{2}\right]} 2(j+1)(g-j)\xi_j(X/Y)\\ &+ \sum_{i=1}^{\left[\frac{g}{2}\right]} 4i(g-i)\delta_i(X/Y)\end{aligned}$$

in case of $k = \mathbb{C}$, which we call *Cornalba-Harris equality.* It is the final result on the relation between the Hodge class and the discriminants for hyperelliptic curves in $\mathrm{char}(k) = 0$. Without the assumption of $\mathrm{char}(k) = 0$, the following results have been obtained.

(1) If $\mathrm{char}(k) \neq 2$, then Cornalba-Harris equality holds: Kausz in [7].

(2) In any characteristic, an inequality

$$\begin{aligned}(8g+4)\deg\left(f_*\omega_{X/Y}\right) \geq& g\xi_0(X/Y) + \sum_{j=1}^{\left[\frac{g-1}{2}\right]} 2(j+1)(g-j)\xi_j(X/Y)\\ &+ \sum_{i=1}^{\left[\frac{g}{2}\right]} 4i(g-i)\delta_i(X/Y)\end{aligned}$$

can be shown: the author in [17].

(3) A bound from the both sides

$$g\delta(X/Y) \leq (8g+4)\deg\left(f_*\omega_{X/Y}\right) \leq g^2\delta(X/Y)$$

can be shown in $\mathrm{char}(k) > 0$, where $\delta(X/Y) := \sum_{i=0}^{[g/2]} \delta_i(X/Y)$: Maugeais in [13].

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In this article, we shall show that Cornalba-Harris equality holds true in any characteristic—even in $\mathrm{char}(k) = 2$. That will be the last result on this problem for hyperelliptic curves.

Before dealing with the equality in positive characteristic, let us recall the proof over $\mathbb{C}$ in [2]. Let $\overline{\mathcal{I}}_{g,\mathbb{C}}$ be the moduli of stable hyperelliptic curves of genus g over $\mathbb{C}$ and $\mathcal{I}_{g,\mathbb{C}}$ the open dense subset consisting of smooth hyperelliptic curves. (They did not make clear what it is, but we shall give the precise definition later.) We denote by Δ_i the locus of curves with nodes of type i, and by Ξ_j that of curves with nodes of subtype j. They are Cartier divisors, and let δ_i and ξ_j denote the classes of Δ_i and Ξ_j respectively. If $f : X \to Y$ is a semistable hyperelliptic curve as at the beginning, then $\delta_i(X/Y)$ and $\xi_j(X/Y)$ are the degree of the pull-back of δ_i and ξ_j respectively by the Y-valued point $Y \to \overline{\mathcal{I}}_{g,\mathbb{C}}$ corresponding to f. Taking account that to give a stable hyperelliptic curve is the same as to give a tree of smooth rational curves and its $2g + 2$ smooth points modulo some group action, they compared $\overline{\mathcal{I}}_{g,\mathbb{C}}$ with the moduli of $(2g + 2)$-pointed stable curves of genus 0 via the moduli of admissible double coverings, and claimed that

(a) $\mathrm{Pic}(\mathcal{I}_{g,\mathbb{C}})$ is a torsion group, and
(b) the boundary components Δ_i for $1 \leq i \leq [g/2]$ and Ξ_j for $0 \leq i \leq [(g-1)/2]$ are irreducible.

The Hodge class λ is, accordingly, a linear combination of the classes δ_i's and ξ_j's up to torsion:

$$\lambda \equiv a_1\delta_1 + \cdots + a_{[g/2]}\delta_{[g/2]} + b_0\xi_0 + \cdots + b_{[(g-1)/2]}\xi_{[(g-1)/2]}$$

for some $a_1, \ldots, a_{[g/2]}, b_0, \ldots, b_{[(g-1)/2]} \in \mathbb{Q}$. Finally, they determined the coefficients using semistable hyperelliptic curves over a projective curve such that the configuration of their fibers is known and that their Hodge classes can be effectively calculated. (Such ones are constructed in the appendix of [14].)

How is it different in positive characteristic? It seems Cornalba-Harris equality can be shown by the same method if $\mathrm{char}(k) \neq 2$, and it can be actually done in all but finitely many characteristics. In the case of characteristic 2, however, the situation is different—wild ramifications prevent us from relating a hyperelliptic curve with a pointed stable curve of genus 0 easily.

Thus it does not seem that the argument in characteristic 0 works well, but we can use the result itself—we can specialize the result in characteristic 0 to obtain the result in positive characteristic. To explain what that indicates, let R be a discrete valuation ring and $f : \mathcal{X} \to \mathrm{Spec}(R)$ a flat morphism of finite type, where we assume $\mathcal{X}$ is a normal scheme for simplicity. Let L be an invertible sheaf on $\mathcal{X}$ trivial on the generic fiber. Then we can write $L = \mathcal{O}(D)$ where D is a Cartier divisor supported in the special fiber $\mathcal{X}_s$. Hence if $\mathcal{X}_s$ *is irreducible and reduced*, then $D = m\mathcal{X}_s$ for some $m \in \mathbb{Z}$, and thus we can conclude that L is trivial on $\mathcal{X}$. That is the idea that we would like to employ. We shall construct an algebraic stack $\overline{\mathcal{I}}_g$ over $\mathbb{Z}$ that is a compactification of the moduli of smooth hyperelliptic curves such that the specialization to any characteristic is irreducible (and automatically generically reduced), and define invertible sheaves on it corresponding to the classes δ_i's and ξ_j's. The result of Cornalba and Harris says that a certain non-trivial linear combination of the Hodge class, δ_i's and ξ_j's is trivial in characteristic 0. Thus we can conclude that it is trivial in any characteristic by the specialization argument as above.

The most important and essential part in our way is the irreducibility of the specialization of $\overline{\mathcal{I}}_g$ to characteristic 2. It is non-trivial at all, but Maugeais has recently proved in [13] that a stable hyperelliptic curve can be a special fiber of a generically smooth stable hyperelliptic curve over an equicharacteristic discrete valuation ring. What we have to do is quite clear now: to define $\overline{\mathcal{I}}_g$ and invertible sheaves δ_i's and ξ_j's precisely and apply the specialization argument to an algebraic stack carefully.

This article is organized as follows. In the first two sections, we shall carry out what we have just explained. In Appendix A, we shall give remark on the moduli of stable hyperelliptic curves and the relation of it with $\overline{\mathcal{I}}_g$ defined in Section 1. In the last section Appendix B, we shall talk on properties of algebraic substacks that will be used in this article.

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Notation and convention.

(1) We mean by "genus" the arithmetic genus. For a projective scheme X over a field, we denote by $p_a(X)$ its arithmetic genus.

(2) A *prestable* curve of genus g over S is a proper flat morphism $f : C \to S$ such that any geometric fiber is a reduced connected scheme of dimension 1 and with at most ordinary double points (called nodes) as singularities. A *stable (resp. semistable) curve* of genus g is a prestable curve of genus $g \geq 2$ such that a smooth rational component of its geometric fiber meets other irreducible components at no less that three (resp. two) points.

(3) The algebraic stack means the Artin or Deligne-Mumford algebraic stack over the category of schemes with a suitable Grothendieck topology. See [5] or [11] for algebraic stacks.

(4) We denote by $\overline{\mathcal{M}}_g$ the moduli stack of stable curves of genus g, and by $\overline{\mathcal{Z}}_g$ the universal curve over $\overline{\mathcal{M}}_g$. They are well-known to be Deligne-Mumford algebraic stacks.

1. Definitions and the statement.

1.1. Compactification of the moduli of hyperelliptic curves. Let us begin with basic definitions.

Definition 1.1. Let C be a (semi)stable curve over an algebraically closed field k and ι_C a k-automorphism of C. We call the pair (C, ι_C) a *(semi)stable hyperelliptic curve over* k if there exist a discrete valuation ring R with the residue field k, a (semi)stable curve $\mathcal{C} \to \operatorname{Spec} R$ and an R-automorphism $\iota_{\mathcal{C}}$ of $\mathcal{C}$ satisfying the following conditions.

(a) The geometric generic fiber is a smooth hyperelliptic curve and $\iota_{\mathcal{C}}$ is its hyperelliptic involution.

(b) The specialization of the pair $(\mathcal{C} \to \operatorname{Spec} R, \iota_{\mathcal{C}})$ coincides with (C, ι_C).

A smooth hyperelliptic curve in the usual sense is of course a stable hyperelliptic curve in our sense.

Remark 1.2. In the case of $\operatorname{char}(k) \neq 2$, it is well-known that (C, ι_C) is hyperelliptic if and only if $\operatorname{ord}(\iota_C) = 2$ and $C/\langle \iota_C \rangle$ is a prestable curve of genus 0. We shall show that it holds even in $\operatorname{char}(k) = 2$ in Appendix A.

Definition 1.3. Let $f : C \to S$ be a (semi)stable curve and ι_C an S-automorphism of C. We call the pair (f, ι_C) a *(semi)stable hyperelliptic curve over* S if the restriction of (f, ι_C) to any geometric fiber is a (semi)stable hyperelliptic curve.

The moduli stack $\mathcal{I}_g$ of smooth hyperelliptic curves of genus g can be realized as a closed substack of the moduli stack of smooth curves (c.f [10]). We want a compactification of $\mathcal{I}_g$ of which boundary consists of stable hyperelliptic curves. Let $Aut_{\overline{\mathcal{M}}_g}\left(\overline{\mathcal{Z}}_g\right)$ be a category as follows: the objects are the pairs $(f : C \to S, \sigma)$ of stable curve f of genus g and an S-automorphism σ of C, and a morphism from $(f : C_1 \to S_1, \sigma_1)$ to $(f : C_2 \to S_2, \sigma_2)$ is a cartesian diagram

$$\begin{array}{ccc} C_1 & \xrightarrow{\phi} & C_2 \\ {\scriptstyle f_1}\big\downarrow & & \big\downarrow{\scriptstyle f_2} \\ S_1 & \longrightarrow & S_2 \end{array}$$

compatible with the automorphisms, namely $\sigma_2 \circ \phi = \phi \circ \sigma_1$. Then there exists a canonical morphism $Aut_{\overline{\mathcal{M}}_g}\left(\overline{\mathcal{Z}}_g\right) \to \overline{\mathcal{M}}_g$, which is well-known to be finite and unramified, and hence it is a Deligne-Mumford algebraic stack proper over $\mathbb{Z}$. Now let us embed $\mathcal{I}_g$ into $Aut_{\overline{\mathcal{M}}_g}\left(\overline{\mathcal{Z}}_g\right)$ via the hyperelliptic involution and let $\overline{\mathcal{I}}_g$ be the stack theoretic closure of $\mathcal{I}_g$ in $Aut_{\overline{\mathcal{M}}_g}\left(\overline{\mathcal{Z}}_g\right)$ (c.f. Definition B.2). Then, $\overline{\mathcal{I}}_g$ is a Deligne-Mumford algebraic stack proper over $\mathbb{Z}$ and each S-valued point of $\overline{\mathcal{I}}_g$ gives a stable hyperelliptic curve of genus g.

Remark 1.4. Over $\mathbb{Z}[1/2]$, let us consider the moduli stack $\overline{\mathcal{H}}_g$ of stable hyperelliptic curves of genus g, which is a substack of $Aut_{\overline{\mathcal{M}}_g}\left(\overline{\mathcal{Z}}_g\right)$. Let $(f : C \to \operatorname{Spec}(R), \iota)$ be an object of $Aut_{\overline{\mathcal{M}}_g}\left(\overline{\mathcal{Z}}_g\right)$ with R a discrete valuation ring. Taking the quotient by $\langle\iota\rangle$ is compatible with base-change, for 2 is a unit over $\mathbb{Z}[1/2]$. Therefore, taking account of Remark 1.2, we see that being hyperelliptic is a property stable under both specialization and generalization. That implies $\overline{\mathcal{H}}_g$ is an open and closed substack, containing $\mathcal{I}_g$ as an open dense substack. Accordingly, $\overline{\mathcal{I}}_g$ is a closed substack of $\overline{\mathcal{H}}_g$ containing the same open dense substack. But it is known that $\overline{\mathcal{H}}_g$ is smooth (c.f. [4]), hence $\overline{\mathcal{I}}_g = \overline{\mathcal{H}}_g$. Thus, $\overline{\mathcal{I}}_g$ is, at least over $\mathbb{Z}[1/2]$, the moduli stack of stable hyperelliptic curves. In Appendix A, we shall realize, over $\mathbb{Z}$, the moduli stack $\overline{\mathcal{H}}_g$ of hyperelliptic curves as a connected component of $Aut_{\overline{\mathcal{M}}_g}\left(\overline{\mathcal{Z}}_g\right)$ and see $(\overline{\mathcal{H}}_g)_{\mathrm{red}} = \overline{\mathcal{I}}_g$.

1.2. Boundary classes. Let C be a semistable curve over an algebraically closed field. Recall that for any node $x \in C$, we can assign a non-negative integer, called the type of x, in the following way: if the partial normalization C_x of C at x is connected, then the type of x is 0, and otherwise, the type is the minimum of the arithmetic genera of the two connected components of it. It is well-known that the locus of stable curves with a node of type i gives a divisor class, or an invertible sheaf δ_i, on $\overline{\mathcal{M}}_g$. We denote by the same symbol the pull-back of δ_i via the canonical morphism $\overline{\mathcal{I}}_g \to \overline{\mathcal{M}}_g$.

Let (C, ι) be a semistable hyperelliptic curve over an algebraically closed field. Then, each singular point $x \in C$ of type 0 has one of the following property.

(a) x is fixed by ι. In this case, we say x is of *subtype* 0.

(b) x is not fixed by ι. Then the partial normalization of C at $\{x, \iota(x)\}$ consists of two connected components of genus, say, j and $g-j-1$ $(0 \le j \le [(g-1)/2])$. In this case, we say $\{x, \iota(x)\}$ is of *subtype* j, or x is of *subtype* j by abuse of words. Note that if C is stable, then a pair of subtype 0 does not appear.

We would like to define invertible sheaves ξ_j on $\overline{\mathcal{I}}_g$, that is, roughly speaking, the sheaves of rational functions that may have a pole at the locus of stable hyperelliptic curves with pairs of nodes of subtype j. Over $\mathbb{C}$, the deformation theory of stable hyperelliptic curves of genus g is equivalent to that of $(2g+2)$-pointed stable curves of genus 0, and it is known that it is smooth and the locus of stable curve with nodes is a divisor. Therefore, we could define ξ_j's as a divisor class on $\overline{\mathcal{I}}_{g,\mathbb{C}}$ without being nervous (c.f. [2]). In our case, however, we do not have enough information on the geometry of $\overline{\mathcal{I}}_g$ in characteristic 2 and cannot easily defined them as the class of locuses. We shall define such boundary classes directly by giving for any stable hyperelliptic curve $(f : C \to S, \iota)$, an invertible sheaf $\xi_{j,S}$ on S which is functorial with respect to base-change.

Now let us begin with preliminary lemmas. For a stable hyperelliptic curve $f : C \to S$, we put $\mathrm{Sing}(f) := \{x \in C \mid f \text{ is not smooth at } x\}$.

Lemma 1.5. *Let $(f : C \to S, \iota)$ be a stable hyperelliptic curve. Then, the subset $\mathrm{Sing}(f) \cap (id_C, \iota)^{-1}(\Delta)$ is open and closed in $\mathrm{Sing}(f)$, where Δ is the diagonal of $C \times_S C$.*

Proof. The closedness is trivial. Now, we claim that the complement E is proper over S. We use the valuation criterion, so that assume that $S = \operatorname{Spec} R$, where R is a discrete valuation ring, and further we are to have a section $\sigma : S \to C$ such that $\sigma(\eta) \in E$, where η is the generic point of S. Then, the reduced closed subscheme $T := \sigma(S) \cup \iota(\sigma(S))$ is finite and flat over S of degree 2. Taking account that $\mathrm{Sing}(f) \to S$ is unramified in addition, we find that T is étale over S. Therefore $\sigma(s) \neq \iota(\sigma(s))$, where s is the closed point of S, and hence $\sigma(s) \in E$. ◻

Lemma 1.6. *Let $f : X \to S$ be a flat morphism and let Y be a closed subscheme of X flat over S. Then, the blowing-up of X along Y is flat over S.*

Proof. Since X and Y are flat over S, the ideal sheaf I_Y of Y is flat over $\mathcal{O}_S$, and hence an $\mathcal{O}_S$-algebra $\mathcal{A} := \mathcal{O}_X \oplus I_Y \oplus {I_Y}^2 \oplus \cdots$ is also flat. Therefore, $\mathbf{Proj}(\mathcal{A})$ is flat over S. ◻

Let N be the open and closed subset of $\mathrm{Sing}(f)$ defined by

$$N := \{x \in C \mid \text{the geometric point } \bar{x} \text{ is a node of type 0 in } C_{f(\bar{x})}\},$$

and put $N_0 := N \cap (id_C, \iota)^{-1}(\Delta)$, which is an open and closed subset of N by Lemma 1.5.

Next we put $N_+ := N \setminus N_0$, and will decompose N_+ into open and closed subsets as follows. If we pull f back to N_+ by $\mathrm{res}(f) : N_+ \to S$, we obtain a nowhere smooth stable curve $g : C_{N_+} \to N_+$ and two sections arising from the inclusion $N_+ \to C$ and the composite morphism $N_+ \subset C \xrightarrow{\iota} C$, where we regard N_+ as a subscheme of C by endowing an arbitrary subscheme structure. Let $\tilde{N}_+$ be the union of that two sections, which is a disjoint union, and let $g^\dagger : C^\dagger_{N_+} \to N_+$ be the blowing-up of $g : C_{N_+} \to N_+$ along $\tilde{N}_+$. Then each fiber $\left(C^\dagger_{N_+}\right)_y$ of $g^\dagger$ at y is the blowing-up of $\left(C_{N_+}\right)_y$ at the two points $\left(\tilde{N}_+\right)_y$. It consists of two prestable curves, and by virtue of Lemma 1.6, the arithmetic genera of them are constant over each connected component of N_+. Therefore, the subset N_j defined by

$$N_j := \left\{ x \in N_+ \;\middle|\; \begin{array}{l} (C^\dagger_{N_+})_{\bar{x}} \text{ has exactly two connected component} \\ \text{which are of genus } j \text{ and of } g-j-1 \end{array} \right\}$$

is open and closed in N. Thus we have a decomposition

$$N = N_0 \amalg N_1 \amalg \cdots \amalg N_{[(g-1)/2]}$$

with $N_0, N_1, \ldots, N_{[(g-1)/2]}$ open and closed in N. Since this decomposition is made up to the property of geometric fibers only, it is of course independent of the choice of the subscheme-structure of N_+ above, and is compatible with base-change: more precisely, let $(f' : C' \to S', \iota')$ be the base-change by a morphism $S' \to S$ and let $N' = N'_0 \amalg N'_1 \amalg \cdots \amalg N'_{[(g-1)/2]}$ be the above decomposition for $(f' : C' \to S', \iota')$. Denote $q : C' \to C$ the canonical morphism. Then, for a geometric point $\bar{x} : \mathrm{Spec}(k) \to C'$, its image is in N'_j if and only if the image of $q \circ \bar{x}$ is in N_j.

Now for any stable hyperelliptic curve $f : C \to S$, we define subsheaves of the relative dualizing sheaf ω_f in the following inductive way. $(\Omega_f)_{-1} := \Omega_f$, where Ω_f is the sheaf of Kähler differentials on C over S. Suppose that $(\Omega_f)_{j-1}$ is defined, the sheaf $(\Omega_f)_j$ is defined by

$$(\Omega_f)_j = \begin{cases} (\Omega_f)_{j-1} & \text{on } C \setminus N_j, \\ \omega_f & \text{around } N_j. \end{cases}$$

According to the above argument, this $(\Omega_f)_j$ is functorial, i.e., for any cartesian diagram

$$\begin{array}{ccc} C_2 & \xrightarrow{\alpha} & C_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ S_2 & \longrightarrow & S_1 \end{array}$$

we have a canonical isomorphism $\alpha^*(\Omega_{f_1})_j \cong (\Omega_{f_2})_j$. Therefore, we can define invertible sheaves $\xi_j(f)$ on S by

$$\xi_j(f) := \det(\mathbf{R}f_*((\Omega_f)_j)) \otimes \det(\mathbf{R}f_*((\Omega_f)_{j-1}))^{-1}$$

that is compatible with base-change (c.f. [8] for "det"). Since S-valued points of $\overline{\mathcal{I}}_g$ are stable hyperelliptic curves, the above correspondences $f \mapsto \xi_j(f)$ define invertible sheaves ξ_j's on $\overline{\mathcal{I}}_g$. If $f : C \to S$ is smooth over all points of S of depth 0, then $\mathrm{Supp}(f_*((\Omega_f)_j/(\Omega_f)_{j-1}))$ has depth ≥ 1 for any j, and hence we have $\xi_j(f) \cong \mathcal{O}_S(\mathrm{Div} f_*((\Omega_f)_j/(\Omega_f)_{j-1}))$[1]. In particular, if $f : X \to Y$ is a semistable hyperelliptic curve of genus g as in the introduction, and if $h : Y \to \overline{\mathcal{I}}_g$ is the corresponding morphism[2], then $\xi_j(X/Y) = \deg(h^*\xi_j)$. Thus they are the boundary classes that we desire.

1.3. The statement and an application. For any stable curve $f : C \to S$ of genus g, we have a canonical invertible sheaf $\det(f_*\omega_f)$, called the Hodge class, and hence we have an invertible sheaf λ on $\overline{\mathcal{I}}_g$ corresponding to the Hodge class. Now we can propose our main result, where we employ additive notation instead of $\otimes$:

THEOREM 1.7. *The invertible sheaf*

$$(8g+4)\lambda - \left(g\xi_0 + \sum_{j=1}^{\left[\frac{g-1}{2}\right]} 2(j+1)(g-j)\xi_j + \sum_{i=1}^{\left[\frac{g}{2}\right]} 4i(g-i)\delta_i \right)$$

[1]The reader can consult [8] and [9] for details of the discussion here, but we give a comment here: if S is regular of dimension 1 and if G is a finitely supported $\mathcal{O}_S$-module, then $\mathrm{Div}(G)$ is an effective divisor such that for each prime divisor $s \in S$, the coefficient of s is equal to the length of G at s.

[2]There exists the corresponding morphism $Y \to Aut_{\overline{\mathcal{M}}_g}(\overline{\mathcal{Z}}_g)$. In general, let $\mathcal{X}$ be a noetherian algebraic stack, $\mathcal{Y}$ a closed substack of $\mathcal{X}$, and let $h : T \to \mathcal{X}$ be a morphism from a scheme T. Assume over an open subscheme T°, h factors through $\mathcal{Y}$. Then, $h^{-1}(\mathcal{Y})$ is a closed subscheme of T containing T°, hence $h|_{\overline{T^\circ}}$ factors through $\mathcal{Y}$. Accordingly, we have the morphism h into $\overline{\mathcal{I}}_g$.

on $\overline{\mathcal{I}}_g$ *is a torsion element in* $\operatorname{Pic}\left(\overline{\mathcal{I}}_g\right)$. *The set of classes* $\xi_0, \ldots, \xi_{[(g-1)/2]}, \delta_1, \ldots, \delta_{[g/2]}$ *is a basis of a* $\mathbb{Q}$*-vector space* $\operatorname{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{I}}_g\right) := \operatorname{Pic}\left(\overline{\mathcal{I}}_g\right) \otimes \mathbb{Q}$.

The proof of Theorem 1.7 will be given in the next section. As an immediate corollary, we have the result:

Corollary 1.8 (Cornalba-Harris equality in $\operatorname{char}(k) \geq 0$). *Let* Y *be a nonsingular projective curve over an algebraically closed field* k *and* $f : X \to Y$ *a generically smooth semistable hyperelliptic curve of genus* $g \geq 2$ *with* X *nonsingular. Then we have an equality*

$$\begin{aligned}(8g+4)\deg\left(f_*\omega_{X/Y}\right) = & g\xi_0(X/Y) + \sum_{j=1}^{\left[\frac{g-1}{2}\right]} 2(j+1)(g-j)\xi_j(X/Y) \\ & + \sum_{i=1}^{\left[\frac{g}{2}\right]} 4i(g-i)\delta_i(X/Y).\end{aligned}$$

Here we give an easy application. Szpiro's result [16, Proposition 3] says $\deg\left(f_*\omega_{X/Y}\right) > 0$ unless f is isotrivial in any characteristic. Hence we have the following result that is well-known in $\operatorname{char}(k) \neq 2$.

Corollary 1.9 ($\operatorname{char}(k) \geq 0$). *Let* Y *be a projective curve over an algebraically closed field* k. *Then any proper smooth curve* $f : X \to Y$ *of genus* $g \geq 2$ *with hyperelliptic geometric generic fiber is isotrivial.*

2. The proof. In this section, we give proof of Theorem 1.7 following the idea explained in the introduction. First we prepare some basic results concerning algebraic stacks and their invertible sheaves.

Lemma 2.1. *Let* $\mathcal{X}$ *be an algebraic stack,* S *a noetherian integral scheme and* $f : \mathcal{X} \to S$ *a flat morphism of finite type. Let* L *be an invertible sheaf on* $\mathcal{X}$ *of which restriction on the generic fiber of* f *is trivial. Then, there exists an open dense subscheme* U *of* S *such that* L *is trivial on* $f^{-1}(U)$.

Proof. Let $\mathcal{X}_\eta$ be the generic fiber of f and let $\phi : \mathcal{O}_{\mathcal{X}_\eta} \to L|_{\mathcal{X}_\eta}$ be an isomorphism. Let us take an atlas $\pi : Z \to \mathcal{X}$ of finite type and put $g := f \circ \pi$. Since g is flat and S is integral, $\mathcal{O}_Z$ and π^*L are subsheaves of $\mathcal{O}_{Z_\eta}$ and $\pi^*L|_{Z_\eta}$ respectively, and since g is also of finite type, we can extend $\pi^*(\phi)$ to be an isomorphism ψ over an open subscheme W with $Z_\eta \subset W$. Since g is of finite type and S is noetherian, $g(Z \setminus W)$ is a constructible set, and it does not contain the generic point. Therefore $U := S \setminus \overline{g(Z \setminus W)}$ is an open dense subset of S with $g^{-1}(U) \subset W$. Since the isomorphism ψ satisfies the cocycle condition over the generic fiber, it also does over $g^{-1}(U)$. Thus this isomorphism descends and we have a trivialization of L over $f^{-1}(U)$. □

We say that an algebraic stack $\mathcal{X}$ is *generically reduced* if there exists an reduced open dense substack of $\mathcal{X}$ (c.f. Definition B.5).

Lemma 2.2. *Let* $\mathcal{X}$ *be an algebraic stack,* S *a connected regular noetherian scheme of dimension* 1 *and* $f : \mathcal{X} \to S$ *a flat morphism of finite type. Let* L *be an invertible sheaf on* $\mathcal{X}$ *which is trivial on the generic fiber. Suppose that the fibers of* f *is irreducible and generically reduced. Then, there exists an invertible sheaf* M *on* S *with* $L = f^*M$.

Proof. By Lemma 2.1, there exists a finite subset B of closed points of S such that L is trivial over $\mathcal{V} := f^{-1}(S \setminus B)$, hence let $\phi : \mathcal{O}_{\mathcal{V}} \to L|_{\mathcal{V}}$ be an isomorphism. Let us take an atlas $\pi : Z \to \mathcal{X}$ with $g := f \circ \pi$ of finite type. Then we have an isomorphism of invertible sheaves $\tilde{\phi} := \pi^*(\phi)$ over $W := \pi^{-1}(\mathcal{V})$. Since g is flat and of finite type, Z_s is an effective Cartier divisor with finite irreducible components for any $s \in B$. Therefore, there exists an integer n_s such that $\tilde{\phi}$ extends to a homomorphism $\tilde{\psi} : \mathcal{O}_Z \to (\pi^* L)(n_s Z_s)$. Note that it descends to a homomorphism $\psi : \mathcal{O}_{\mathcal{X}} \to L(n_s \mathcal{X}_s)$. Now we take such n_s's to be minimal. It is enough to show that $\tilde{\psi}$ is an isomorphism, but since it is between invertible sheaves, enough just to show it surjective, so that, we may assume not only that S is the spectrum of a complete discrete valuation ring A with algebraically closed residue field but that Z is normal.

Let $\mathcal{X}'_s$ be a smooth open dense substack of ${\mathcal{X}_s}$[3], which is irreducible, and let $Z_{s,1}, \dots, Z_{s,l}$ be the irreducible components of the special fiber Z_s. Since $\mathcal{X}'_s$ is smooth, $\pi^{-1}(\mathcal{X}'_s)$ is smooth and we have $\pi^{-1}(\mathcal{X}'_s) = \coprod_i (Z_{s,i} \cap \pi^{-1}(\mathcal{X}'_s))$. Next we put $\mathcal{X}_s^\circ := \bigcap_i \pi(Z_{s,i} \cap \pi^{-1}(\mathcal{X}'_s))$, which is an open dense substack of $\mathcal{X}_s$ (c.f. Lemma B.8 for the image of a morphism of algebraic stacks), and put $Z_{s,i}^\circ := Z_{s,i} \cap \pi^{-1}(\mathcal{X}_s^\circ)$. Then, from the construction, we see that

(a) $Z_{s,i}^\circ$ is an open dense subscheme of $Z_{s,i}$,
(b) $Z_{s,1}^\circ, \dots, Z_{s,l}^\circ$ is contained in the smooth locus of g, and
(c) $\pi\left(Z_{s,1}^\circ\right) = \cdots = \pi\left(Z_{s,l}^\circ\right) = {\mathcal{X}_s}^\circ$.

Since the special fiber is reduced and n_s is taken to be minimal, $\tilde{\psi}$ is an isomorphism over one of $Z_{s,1}^\circ, \dots, Z_{s,l}^\circ$, say $Z_{s,1}^\circ$. On the other hand, we can take, for any $x : \{s\} \to {\mathcal{X}_s}^\circ$, a section $\sigma : S \to \mathcal{X}$ with $\sigma(s) = x$, and moreover, can take a section $\sigma_i : S \to Z$ for any $1 \leq i \leq l$ such that $\sigma_i(s) \in Z_{s,i}^\circ$ and $\pi \circ \sigma_i = \sigma$. Then, ${\sigma_1}^*(\tilde{\psi})$ is an isomorphism, so is $\sigma^*(\psi)$, and hence ${\sigma_i}^*(\tilde{\psi})$ is an isomorphism for any i. That implies that the Weil divisor determined by $\tilde{\psi}$ is trivial and hence $\tilde{\psi}$ is an isomorphism. □

Now we are ready for proving it. The morphism $\mathcal{I}_g \to \operatorname{Spec}\mathbb{Z}$ is smooth, and flat in particular, hence $\overline{\mathcal{I}}_g \to \operatorname{Spec}\mathbb{Z}$ is flat by virtue of Corollary B.4. Let us look at the fibers of $\overline{\mathcal{I}}_g \to \operatorname{Spec}\mathbb{Z}$. It is well-known that it is smooth and has geometrically connected fibers over $\mathbb{Z}[1/2]$ (c.f. Remark 1.4). How about over the prime (2)? By [12], $\mathcal{I}_g \to \operatorname{Spec}\mathbb{Z}$ has irreducible geometric fibers, and by [10], it is smooth, even over (2). On the other hand, Maugeais proved in [13] the following important result. (If $\operatorname{char}(k) \neq 2$, it had been well-known. Maugeais' contribution is the case of $\operatorname{char}(k) = 2$.)

THEOREM 2.3 (Corollary 53 in [13]). *Let k be an algebraically closed field and C a stable hyperelliptic curve over k. Then there exist an equicharacteristic discrete valuation ring R of which residue field is k and a curve $\mathcal{C} \to \operatorname{Spec}(R)$ of which generic fiber is a smooth hyperelliptic curve and of which special fiber coincides with C.*

That tells us for any algebraically closed field k with $\operatorname{char}(k) = 2$, the open substack $\mathcal{I}_g \otimes k$ is dense in $\overline{\mathcal{I}}_g \otimes k$ (c.f. Proposition B.7). Thus, in summary, we find that the morphism $\overline{\mathcal{I}}_g \to \operatorname{Spec}\mathbb{Z}$ is flat and has irreducible and generically reduced

[3]If Z'_s is the smooth locus, then we can check it has the property required in Proposition B.1, hence we can take $\pi_s(Z'_s)$ as $\mathcal{X}'_s$. In our applying this lemma below, however, such an open substack $\mathcal{X}'_s$ will has been already given.

fibers. By [2], we know that

$$(8g+4)\lambda - \left(g\xi_0 + \sum_{j=1}^{\left[\frac{g-1}{2}\right]} 2(j+1)(g-j)\xi_j + \sum_{i=1}^{\left[\frac{g}{2}\right]} 4i(g-i)\delta_i \right),$$

where we employ the additive notation instead of $\otimes$, is a torsion element on the generic fiber of $\overline{\mathcal{I}}_g$, hence by Lemma 2.2, it is a torsion element whole on $\overline{\mathcal{I}}_g$. Thus we obtain the first part of Theorem 1.7. Since the same statement in characteristic 0 holds, the latter part also follows from the same argument as above.

Appendix A. The moduli of stable hyperelliptic curves. In this appendix, we shall construct the moduli stack $\overline{\mathcal{H}}_g$ of stable hyperelliptic curves genus g over $\mathbb{Z}$, as an open and closed substack of $Aut_{\overline{\mathcal{M}}_g}\left(\overline{\mathcal{Z}}_g\right)$. Further, we consider Theorem 1.7 on $\overline{\mathcal{H}}_g$.

A.1. Quotient of prestable curves over a discrete valuation ring by a finite group. The purpose of this subsection is to give technical remarks on the quotient of prestable curves over a discrete valuation ring by a finite group. They may be well-known facts, though the author does not know complete references.

For a ring A with a group G-action, we denote by A^G the ring of G-invariants of A. For a while, let R be a complete discrete valuation ring.

Lemma A.1 (c.f. Claim 3.1 of [6]). *Let G be a finite subset of $\mathrm{Aut}_R(R[[x]])$. Then, we have $R[[x]]^G = R[[z]]$, where $z = \prod_{g\in G} g(x)$.*

Proof. (Same proof as that of [6, Claim 3.1].) By virtue of [1], we can see that $R[[x]]$ is a free $R[[z]]$-module with a basis $\{1, x, \ldots, x^{s-1}\}$, where $s := |G|$. On the other hand, taking account of $[Q(R[[x]]) : Q(R[[x]]^G)] = s$, we have $Q(R[[x]]^G) = Q(R[[z]])$, where $Q(*)$ denotes the quotient field. Since $R[[x]]^G$ is integral over $R[[z]]$ that is integrally closed, they coincide with each other. ∎

Let us consider the case where G is a finite subgroup of $\mathrm{Aut}_R(A)$ with $A = R[[x,y]]/(xy)$. We can naturally regard A as a subring of $B := R[[x]] \times R[[y]]$ by $A = \{(f,g) \in B \mid f(0) = g(0)\}$, and G as a subgroup of $\mathrm{Aut}_R(B)$. Put

$$H := \{g \in G \mid g(R[[x]] \times \{0\}) = R[[x]] \times \{0\}\},$$

which is a normal subgroup of index 1 or 2.

Lemma A.2. *Let A, G and H be as above.*
(1) *If $G \neq H$, then $A^G = R[[z]]$ for some $z \in xA + yA$.*
(2) *If $G = H$, then $A^G = R[[z,w]]/(zw)$ for some $z \in xA$ and $w \in yA$.*

Proof. The subgroup H acts on the subrings $R[[x]]$ and $R[[y]]$ of B and we have

$$B^G = \left((R[[x]]^H \times R[[y]]^H\right)^{G/H}.$$

Lemma A.1 tells us $R[[x]]^H = R[[z]]$ and $R[[y]]^H = R[[w]]$ for some $z \in xR[[x]]$ and $w \in yR[[y]]$.

If G/H has a non-trivial element ι, then it gives an isomorphism between $R[[x]]^H$ and $R[[y]]^H$, and $R[[y]]^H = R[[\iota(z)]]$. Therefore, we have

$$\left(R[[x]]^H \times R[[y]]^H\right)^{G/H} = \{(f, \iota(f)) \in R[[z]] \times R[[\iota(z)]]\} \cong R[[z]].$$

Taking account that G is acting on A, we can see that the constant term of $\iota(f)$ coincides with that of f, and hence the above ring B^G is contained in A. Accordingly, we have $A^G \cong R[[z]]$.

If $G = H$, then $B^G = R[[z]] \times R[[w]]$. Since $(f, g) \in R[[z]] \times R[[w]]$ living in A is equivalent to $f(0) = g(0)$, we have $A^G = R[[z, w]]/(zw)$. □

Now we can obtain the following proposition. It is stated in [15] in the case where f is generically smooth.

Proposition A.3. *Let R be a discrete valuation ring and $f : C \to S := \mathrm{Spec}(R)$ a flat morphism of finite type. Suppose that each geometric fiber is reduced curve and has at most ordinary double points as singularities. Let G be a finite subgroup of $\mathrm{Aut}_S(C)$. Then $C/G \to S$ is also a flat morphism of finite type such that any geometric fiber is a reduced curve and has at most ordinary double points as singularities.*

Proof. We may assume that R is complete and that its residue field is algebraically closed. If C° is the open subscheme of normal points of C, then $C^\circ/G \to S$ is a curve with the required property by virtue of [15]. Hence we only have to look at $C/G \to S$ around the image of each non-normal point.

Let x be a non-normal closed point of C. Then the completion $\hat{\mathcal{O}}_{C,x}$ of the local ring at x is R-isomorphic to $R[[u, v]]/(uv)$. Since $\hat{\mathcal{O}}_{C/G,\pi(x)} \cong (\hat{\mathcal{O}}_{C,x})^{G_x}$, where $\pi(x)$ is the image of x by the quotient and G_x is the stabilizer of x, it follows from Lemma A.2. □

The following corollary is an immediate consequence.

Corollary A.4. *With the same notation as above, suppose f prestable. Then $g : C/G \to S$ is a prestable curve.*

A.2. Automorphisms of order 2 and 2-admissible coverings. Let X and Y be pre-stable curves over an algebraically closed field k of characteristic $p > 0$. We call a finite k-morphism $\pi : X \to Y$ of degree p over any irreducible component of Y, a *p-covering.* Let ι be a k-automorphism of order p of a semistable curve X and suppose that $Y := X/\langle\iota\rangle$ is a prestable curve of genus 0. If ι acts on an irreducible component Z of X trivially, then $Z \cong \mathbb{P}^1_k$. Taking account that, let us define a finite surjective morphism $\phi : Y \to Y$, which is the identity set-theoretically, characterized by the following condition: if $\langle\iota\rangle$ acts trivially on an irreducible component Z, then $\mathrm{res}(\phi) : Z \to Z$ is the relative Frobenius morphism, i.e., the morphism given by $t \mapsto t^p$ for a coordinate t, and otherwise, it is the identity. Then $\pi := \phi \circ q$ is a p-covering, which we call the *standard p-covering arising from ι.*

Here we recall the notion of the conductor (c.f. [13]). Let $\pi : X \to Y$ be a p-covering of irreducible curves, and let $y \in Y$ be a regular closed point. Suppose that y is the image of a regular point $x \in X$, π is ramified over y, and there exists an open subset $U \subset Y$ such that $\pi^{-1}(U) \to U$ is a G-torsor, where G is $\mathbb{Z}/p\mathbb{Z}$ or a local additive group scheme α_p over k of length p. We define an integer $m(y)$ as follows. If $G = \mathbb{Z}/p\mathbb{Z}$, then $m(y)$ is the Hasse conductor of the extension $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. If $G = \alpha_p$, then $m(y) = -(1+\mathrm{ord}_y(du))$ where u is a regular function on U corresponding to this torsor.

Maugeais introduced in [13] the notion of p-admissible covering. It consists of suitable data $\left(\pi : X \to Y, \{(G_Z, u_Z)\}_{Z\in\mathrm{Irr}(Y)}\right)$, where $\pi : X \to Y$ is a p-covering, G_Z is a certain group scheme and u_Z is a rational function on Z. We do not recall the precise definition here, but remark that $\left(\pi : X \to Y, \{(G_Z, u_Z)\}_{Z\in\mathrm{Irr}(Y)}\right)$ as follows is a p-admissible covering: the data consisting of

(a) a p-covering $\pi : X \to Y$,

(b) for each irreducible component Z of Y, a pair (G_Z, u_Z), where G_Z is a group scheme $\mathbb{Z}/p\mathbb{Z}$ or α_p and u_Z is a rational function on Z,

with the following properties.

(1) $\pi^{-1}(Y_{\mathrm{reg}}) = X_{\mathrm{reg}}$, where $*_{\mathrm{reg}}$ indicates the regular locus.

(2) For each irreducible component Z of Y, there exists an open subset U_Z of Z such that $\pi^{-1}(U_Z) \to U_Z$ is a G_Z-torsor defined by u_Z, i.e., if $U_Z = \mathrm{Spec}(B)$, it is a G_Z-torsor given by

$$\begin{cases} B[z]/(z^p - z - u) & \text{if } G_Z = \mathbb{Z}/p\mathbb{Z} \\ B[z]/(z^p - u) & \text{if } G_Z = \alpha_p. \end{cases}$$

(3) Let y be a node that is an intersection point of two irreducible components Z_0 and Z_1 of Y, and suppose $\#\pi^{-1}(y) = 1$. Then $m_{Z_0}(y) + m_{Z_1}(y) = 0$, where $m_{Z_i}(y)$ is the conductor m of $\pi^{-1}(Z_i) \to Z_i$ at y defined above.

(Note that our notation is a little different from that in [13], where the conductor is defined for not a critical value but a critical point.)

Before proposing a result that we would like to show in this subsection, let us fix our terminology. We call an irreducible component E of a prestable curve $\tilde{X}$ a $(-i)$-*curve* for $i = 1, 2$ if $E \cong \mathbb{P}^1_k$ and exactly i nodes of $\tilde{X}$ lie on E. We call a morphism of prestable curves $\rho : \tilde{X} \to X$ a *contraction* if any irreducible component E of $\tilde{X}$ such that $\mathrm{res}(\rho) : E \to \rho(E)$ is not an isomorphism, is an $(-i)$-curve ($i = 1, 2$) and $\rho(E)$ is a point. It is well-known that if E is a (-2)-curve of a prestable $\tilde{X}$, then we can contract E to a node x and obtain another prestable curve X. From the viewpoint of X, $\tilde{X}$ is a prestable curve obtained by replacing a node x with $\mathbb{P}^1_k$ in a suitable way. We call that modification to obtain $\tilde{X}$ from X the *inverse contraction at* x, and call that E an exceptional curve over x.

In the rest of this subsection, we shall give proof of the following result. (It makes sense in any characteristic, but we deal with the case of $\mathrm{char}(k) = 2$ only.)

PROPOSITION A.5. *Let X be a semistable curve over k and ι a k-automorphism of X of order* 2. *Then the following statements are equivalent:*

(a) *(X, ι) is a semistable hyperelliptic curve.*

(b) *There exist a prestable curve Y of genus* 0 *and a* 2*-covering $\pi : X \to Y$ with the following property: there exists a morphism $g : X/\langle\iota\rangle \to Y$ such that g is a factorization of π through $X \to X/\langle\iota\rangle$ and is a homeomorphism.*

(c) *$X/\langle\iota\rangle$ is a prestable curve of genus* 0.

(d) *There exists a* 2*-admissible covering $\tilde{\pi} : \tilde{X} \to \tilde{Y}$ with $p_a(\tilde{Y}) = 0$ such that there exist a* 2*-covering $\pi : X \to Y$ that factors through the quotient $X \to X/\langle\iota\rangle$, and contractions $\rho : \tilde{X} \to X$ and $\rho' : \tilde{Y} \to Y$ with $\pi \circ \rho = \rho' \circ \tilde{\pi}$.*

Proof. It is shown in [13] that (d) implies (a) (c.f. [13, Corollary 43, Theorem 49 and Proposition 50]), and it is immediate that (a) implies (b) from the definition. Assume (b). Since $X/\langle\iota\rangle$ and Y are prestable curve and g is a homeomorphism, we have $p_a(X/\langle\iota\rangle) = p_a(Y) = 0$. It only remains to show (c) implies (d), which we do in several steps.

Step 1. Let X_0 be the inverse contraction of X at those ι-fixed nodes around which ι acts as an exchange of the branches. Then we can naturally make ι act on X_0 in order 2, so that X_0 does not have an ι-fixed node around which ι acts as the exchange of the branches, and $Y_0 := X_0/\langle\iota\rangle$ is a prestable curve of genus 0. Now let $\pi_0 : X_0 \to Y_0$ be a standard 2-covering. Note that $(\pi_0)^{-1}(Y_{0,\mathrm{reg}}) = X_{0,\mathrm{reg}}$.

Step 2. Let us look at $\pi_0 : X_0 \to Y_0$. For each irreducible component Z of Y_0, we have one and only one case:

(a) π_0 is separable over Z, and only one irreducible component lies over it. We denote the set of such irreducible components by I_{sep}.

(b) π_0 is an inseparable over Z. Then we have $Z \cong \mathbb{P}^1_k$. We denote the set of such irreducible components by I_{ins}.

(c) $(\pi_0)^{-1}(Z)$ consists of two irreducible components. In this case, it is a disjoint union of two $\mathbb{P}^1_k$'s. We denote the set of such irreducible components by I_{et}.

For each $Z \in I_{ins}$, fix a closed point $\infty \in Z$. Then, over $Z \setminus \{\infty\}$, π_0 can be regarded as an α_2-torsor given by

$$k[t] \to k[s,t]/(s^2 - t).$$

Thus, we are in the following situation:

(a) Over $Z \in I_{sep}$, π_0 is a $\mathbb{Z}/2\mathbb{Z}$-torsor except at the critical values. The conductor at each critical value is a positive odd number.

(b) Over $Z \in I_{ins}$, π_0 is an α_2-torsor except at ∞, corresponding to the form dt, where t is an affine coordinate of $Z \setminus \{\infty\}$. In particular, the conductor at each point of $Z \setminus \{\infty\}$ is -1.

(c) Over $Z \in I_{et}$, π_0 is a trivial $\mathbb{Z}/2\mathbb{Z}$-torsor.

Step 3. We shall modify π_0 so that it satisfies the conditions on the conductors at nodes. Let $y \in Y_0$ be a node and let Z_0 and Z_1 be the irreducible components with $Z_0 \cap Z_1 = \{y\}$.

If π_0 is étale over y or $m_{Z_0}(y) + m_{Z_1}(y) = 0$, we do not perform any modification there.

Otherwise, let $x \in X_0$ the node over y. Let $Y_{0,x}$ (resp. $X_{0,y}$) be the inverse contraction of Y_0 (resp. X_0) at y (resp. x) and let E_y (resp. E_x) be the exceptional curve. Let y_i (resp. x_i) be the intersection of Z_i and E_y (resp. $(\pi_0)^{-1}(Z_i)$ and E_y). We fix an inhomogeneous coordinate on E_y (resp. E_x) in which y_i (resp. x_i) as a point on E_y (resp. E_x) corresponds to i $(= 0, 1)$. Here we use the following lemma.

Lemma A.6 (char$(k) = 2$). *For any odd integers m_0 and m_1, there exists a 2-covering $\pi : \mathbb{P}^1_k \to \mathbb{P}^1_k$ such that $\pi(x) = x$ for $x = 0, 1, \infty$, and that it has an α_2-torsor structure generically corresponding to the form du, where u is a rational function on $\mathbb{P}^1_k$ with*

$$\operatorname{div}(u) = m_0[0] + m_1[1] - (m_0 + m_1)[\infty].$$

Proof. Let t be an inhomogeneous coordinate on $\mathbb{P}^1_k$ and u a rational function as above. Let $\pi' : E \to \mathbb{P}^1_k$ be the finite morphism from a *smooth* projective curve E *generically* defined by $k[t] \to k[t,s]/(s^2 - u)$. It is an α_2-torsor generically corresponding to du. Since π is inseparable of degree 2, E is isomorphic to $\mathbb{P}^1_k$. Moreover, we can chose an isomorphism $\phi : \mathbb{P}^1_k \to E$ such that $\pi'(\phi(x)) = x$ for $x = 0, 1, \infty$. Thus $\pi := \pi' \circ \phi$ is a required one. □

Let $\pi_{x,y} : E_x \to E_y$ be the covering in Lemma A.6 for $m_i = m_{Z_i}(y)$. Using that, we construct a 2-covering $X_{0,x} \to Y_{0,y}$ from π_0, that is, it coincides with π_0 except over E_y and coincides with above $\pi_{x,y}$ over E_y. By the construction, if y_i is a node of $Y_{0,y}$ sitting on Z_i and E_y, then $m_{Z_i}(y_i) + m_{E_y}(y_i) = 0$.

Now let $\tilde{\pi} : \tilde{X} \to \tilde{Y}$ be the 2-covering obtained by the above modification at all such nodes. Then, it has a structure of 2-admissible covering. □

We would like to remark one thing. Proposition 38 in [13] for $p = 2$ says that (a) implies (d), and in that proof, it is essential that X is the specialization of a smooth projective curve of characteristic 0 with an action of $\mathbb{Z}/2\mathbb{Z}$. In our proof, however, we can reach the conclusion via combinatorial way from (c), which is a condition in terms of geometry over k.

A.3. The moduli stack of hyperelliptic curves. Let $Aut_{\overline{\mathcal{M}}_g}(\overline{\mathcal{Z}}_g)_{(2)}$ be the full subcategory of $Aut_{\overline{\mathcal{M}}_g}(\overline{\mathcal{Z}}_g)$ of which objects are the pairs of a stable curve and an automorphism of order 2. It is a not only close but also open substack.

We shall show that the open substack $Aut_{\overline{\mathcal{M}}_g}(\overline{\mathcal{Z}}_g)_{(2)} \setminus \overline{\mathcal{I}}_g$ is closed. Let R be a discrete valuation ring, and s (resp. η) the special (resp. generic) point of $\mathrm{Spec}(R)$. Let $f : C \to \mathrm{Spec}(R)$ be a stable curve of genus g and ι an R-automorphism such that the special fiber of (f, ι) is a stable hyperelliptic curve. By Proposition A.5, we have $p_a(C_s/\langle \iota_s \rangle) = 0$. On the other hand, $(C/\langle \iota \rangle)_s$ is a prestable curve by Corollary A.4. Since the canonical morphism $C_s/\langle \iota_s \rangle \to (C/\langle \iota \rangle)_s$ is a homeomorphism, we have $p_a((C/\langle \iota \rangle)_s) = p_a(C_s/\langle \iota_s \rangle) = 0$. Therefore the generic fiber $(C/\langle \iota \rangle)_\eta$ is also of genus 0 by Corollary A.4. Hence, taking account of $(C/\langle \iota \rangle)_\eta = C_\eta/\langle \iota_\eta \rangle$, we find that (C_η, ι_η) is a stable hyperelliptic curve by virtue of Proposition A.5. That implies $Aut_{\overline{\mathcal{M}}_g}(\overline{\mathcal{Z}}_g)_{(2)} \setminus \overline{\mathcal{I}}_g$ is stable under specialization, hence it is closed.

Now let $\overline{\mathcal{H}}_g$ be the connected component of $Aut_{\overline{\mathcal{M}}_g}(\overline{\mathcal{Z}}_g)_{(2)}$ containing $\overline{\mathcal{I}}_g$. Then we have $(\overline{\mathcal{H}}_g)_{\mathrm{red}} = \overline{\mathcal{I}}_g$ by the definition. Further, for any hyperelliptic curve $(f : C \to S, \iota)$, which is an object of $Aut_{\overline{\mathcal{M}}_g}(\overline{\mathcal{Z}}_g)_{(2)}$, the hyperelliptic curve $(f_0 : C_0 \to S_0, \iota)$, where f_0 is the restriction of f over the reduced structure S_0 of S, is an object of $(\overline{\mathcal{H}}_g)_{\mathrm{red}}$ and hence $(f : C \to S, \iota)$ is an object of $\overline{\mathcal{H}}_g$. In summary we have the following theorem.

Theorem A.7. *The moduli stack $\overline{\mathcal{H}}_g$ of stable hyperelliptic curves of genus g exists, which is a Deligne-Mumford algebraic stack proper over $\mathbb{Z}$. It is an open and closed substack of $Aut_{\overline{\mathcal{M}}_g}(\overline{\mathcal{Z}}_g)$, and $(\overline{\mathcal{H}}_g)_{\mathrm{red}} = \overline{\mathcal{I}}_g$.*

The sheaves λ, δ_i's and ξ_j's in § 1 are also defined over $\overline{\mathcal{H}}_g$ and so is an invertible sheaf

$$L := (8g+4)\lambda - \left(g\xi_0 + \sum_{j=1}^{\left[\frac{g-1}{2}\right]} 2(j+1)(g-j)\xi_j + \sum_{i=1}^{\left[\frac{g}{2}\right]} 4i(g-i)\delta_i \right)$$

(in the additive notation). By Theorem 1.7, it is a torsion element in $\mathrm{Pic}\big((\overline{\mathcal{H}}_g)_{\mathrm{red}}\big)$. We would finally like to conclude that it is a torsion element in $\mathrm{Pic}\big(\overline{\mathcal{H}}_g\big)$, so that we claim the following.

Lemma A.8. *Let $\mathcal{X}$ be an algebraic stack, L an invertible sheaf on $\mathcal{X}$ and N a quasi-coherent ideal sheaf such that $N^l = 0$ for some $l \in \mathbb{Z}$ and that N is annihilated by an integer a. If L is trivial on the closed substack $\mathcal{X}_0$ defined by N, then $L^{\otimes a^e}$ is trivial for some $e \in \mathbb{Z}$. Therefore, if L is a torsion element in* $\mathrm{Pic}(\mathcal{X}_0)$*, then so it is in* $\mathrm{Pic}(\mathcal{X})$.

Proof. Let $\pi : X \to \mathcal{X}$ be an atlas such that any connected component of X is affine, and put $Y := X \times_{\mathcal{X}} X$. Let $q_i : Y \to X$ denote the i-th projection $(i = 1, 2)$, and $q : Y \to \mathcal{X}$ the natural morphism. Put also $X_0 := X \times_{\mathcal{X}} \mathcal{X}_0$, and $Y_0 := X_0 \times_{\mathcal{X}_0} X_0$. Since L is trivial on $\mathcal{X}_0$, we have an isomorphism $\phi : \mathcal{O}_{\mathcal{X}_0} \to L|_{\mathcal{X}_0}$

and its pull-back $\pi^*(\phi)$. Since any connected component is affine, we can extend $\pi^*(\phi)$ to a homomorphism $\psi : \mathcal{O}_X \to \pi^* L$, which is an isomorphism. Then the ratio ${q_1}^*(\psi)/{q_2}^*(\psi)$ gives a unit function on Y, and since ${q_1}^*(\psi)$ coincides with ${q_2}^*(\psi)$ on Y_0, we can write ${q_1}^*(\psi)/{q_2}^*(\psi) = 1+\epsilon$ over any affine open subscheme V of Y, where $\epsilon \in (q|_V)^* N$. Therefore, we have, for a large integer e depending only on l and a,

$$ {q_1}^*(\psi^{\otimes a^e})/{q_2}^*(\psi^{\otimes a^e}) = (1+\epsilon)^{a^e} = 1. $$

That implies the isomorphism $\psi^{\otimes a^e} : \mathcal{O}_X \to \pi^* L^{\otimes a^e}$ descends to an isomorphism $\mathcal{O}_{\mathcal{X}} \cong L^{\otimes a^e}$, thus we have our assertion. □

Since $\overline{\mathcal{H}}_g$ is reduced over $\mathbb{Z}[1/2]$ (c.f. Remark 1.4), we can apply the above lemma to L and obtain the following theorem.

THEOREM A.9. *We have* $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{H}}_g) \cong \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{I}}_g)$. *In particular, the invertible sheaf*

$$ (8g+4)\lambda - \left(g\xi_0 + \sum_{j=1}^{\left[\frac{g-1}{2}\right]} 2(j+1)(g-j)\xi_j + \sum_{i=1}^{\left[\frac{g}{2}\right]} 4i(g-i)\delta_i \right) $$

on $\overline{\mathcal{H}}_g$ *is a torsion element in* $\mathrm{Pic}(\overline{\mathcal{H}}_g)$.

Appendix B. Remarks on algebraic substacks. In this appendix, we deal with some algebraic substacks of algebraic stacks. Most of them may be well-known although the author can not find precise references.

We should make sure our terminology and notation here. A stack is a fibered in groupoid over a site, with some properties, and we deal with here stacks over the category (Sch) of schemes with a suitable Grothendieck topology (e.g. étale or fppf) only. (Everything can be done relatively of course, but we do not mention it for simplicity.) We denote this site by $(Sch)_{et}$ (étale site), $(Sch)_{fppf}$ (fppf site), or (Sch) when we do not have to emphasize its topology. When we say $\{T_i \to T\}_i$ is a covering, it is a covering of the site (Sch). Any scheme is regarded as a stack in the usual manner: if X is a scheme, its sections over a scheme T are the morphisms $T \to X$.

Let us recall the definition of substacks ([11, Definition 2.5], [5, Definition 2.35]). A *substack* of a stack $\mathcal{X}$ is a full subcategory $\mathcal{Y}$ of $\mathcal{X}$ such that

(a) all objects of $\mathcal{X}$ isomorphic to an object of $\mathcal{Y}$ are in $\mathcal{Y}$,
(b) for any morphism $\phi : T' \to T$ of schemes and any object x of $\mathcal{Y}_T$, the pull-back $\phi^* x$ is an object of $\mathcal{Y}_{T'}$, where $\mathcal{Y}_T$ is the groupoid of the sections of $\mathcal{Y}$ over T,
(c) for any object x of $\mathcal{X}$ over T and for any covering $\{T_i \to T\}_i$, if $x|_{T_i}$ is an object of $\mathcal{Y}$ for all i, then x is an object of $\mathcal{Y}$.

Let $F : \mathcal{X} \to \mathcal{Y}$ be a morphism of stacks. If $\mathcal{V}$ is a substack, we denote by $F^{-1}(\mathcal{V})$ the full subcategory of $\mathcal{X}$ of which objects are those mapped into $\mathcal{V}$ by F. It is a substack of $\mathcal{X}$, and the canonical morphism $\mathcal{V} \times_{\mathcal{Y}} \mathcal{X} \to F^{-1}(\mathcal{V})$ is an isomorphism of stacks. (Its inverse is given by $x \mapsto (F(x), x, id_{F(x)})$. Hence we often identify one with the other.

For a substack $\mathcal{U}$ of $\mathcal{X}$, let $F(\mathcal{U})^-$ denote the full subcategory of $\mathcal{Y}$ whose objects are those isomorphic to an object coming from $\mathcal{U}$. It is clearly a prestack, but not a stack in general. Its associated stack is however constructed as a substack of $\mathcal{Y}$ in the following way. We define a full subcategory $F(\mathcal{U})$ of $\mathcal{Y}$ so that an object y of $\mathcal{Y}$ over

T is an object of $F(\mathcal{U})$ if and only if, for some covering $\{T_i \to T\}_i$, $y|_{T_i}$ is an object of $F(\mathcal{U})^-$ over T_i for all i. We call this substack $F(\mathcal{U})$ the *stack-theoretic image* or simply the *image* of $F|_{\mathcal{U}}$. Throughout this section, we always use $F(\mathcal{U})$ in this sense.

B.1. The fundamental correspondence. A substack $\mathcal{Y}$ of an algebraic stack $\mathcal{X}$ is said to be *locally closed (resp. open, resp. closed)* if the canonical morphism $\mathcal{Y} \to \mathcal{X}$ is represented by a locally closed (resp. open, resp. closed) immersion. Let $\pi : X \to \mathcal{X}$ be an atlas. If $\mathcal{Y}$ is a locally closed substack of $\mathcal{X}$, then $Y := \pi^{-1}(\mathcal{Y}) \cong \mathcal{Y} \times_{\mathcal{X}} X$ is naturally a subscheme[4] of X. Moreover, since $Y \times_{\mathcal{X}} Y = Y \times_{\mathcal{Y}} Y$ and $Y \times_{\mathcal{Y}} Y \cong Y \times_{\mathcal{Y}} \mathcal{Y} \times_{\mathcal{X}} X \cong Y \times_{\mathcal{X}} X$, we have (in $Y \times_{\mathcal{X}} X$)

$$Y \times_{\mathcal{X}} Y = Y \times_{\mathcal{X}} X. \tag{$\spadesuit$}$$

We denote by $\mathcal{S}$ the set of subschemes of X with this property $\spadesuit$. The following proposition is a very fundamental assertion on locally closed substacks.

PROPOSITION B.1. *Once an atlas of $\mathcal{X}$ fixed, the map $\mathcal{Y} \mapsto \pi^{-1}(\mathcal{Y})$ given above from the set of locally closed substack of $\mathcal{X}$ to $\mathcal{S}$ is a bijection. Moreover, open (resp. closed) substacks correspond to the open (resp. closed) ones in $\mathcal{S}$.*

Proof. Let us construct the inverse. For a subscheme $Y \subset X$ with $Y \times_{\mathcal{X}} Y = Y \times_{\mathcal{X}} X$, put $\mathcal{Y} := \pi(Y)$. From now on, we will show that the correspondence $Y \mapsto \mathcal{Y}$ gives the inverse.

First we claim that the canonical monomorphism $\pi(\pi^{-1}(\mathcal{Y})) \to \mathcal{Y}$ is an isomorphism. Let y be an object of $\mathcal{Y}$ over T. Then there is a covering $\{T_i \to T\}_i$ and objects y_i of X over T_i with $\pi(y_i) \cong y_{T_i}$. That implies that y is in $\pi(\pi^{-1}(\mathcal{Y}))$, and hence $\pi(\pi^{-1}(\mathcal{Y})) \to \mathcal{Y}$ is an epimorphism. Thus it is an isomorphism.

Next we prove the natural monomorphism $Y \to \pi(Y) \times_{\mathcal{X}} X$, described as $y \mapsto (\pi(y), y, id_{\pi(y)})$, is an isomorphism. Let $(y, x, \phi : y \to \pi(x))$ be an object of $\pi(Y) \times_{\mathcal{X}} X$ over T. From the definition of $\mathcal{Y}$, there exist a covering $\{T_i \to T\}_i$, an object y_i of Y over T_i and an isomorphism $\psi_i : \pi(y_i) \to y|_{T_i}$ for each i, such that $(y_i, x|_{T_i}, \phi|_{T_i} \circ \psi_i)$ is an object of $Y \times_{\mathcal{X}} X$ over T_i. Taking account of the condition $Y \times_{\mathcal{X}} X = Y \times_{\mathcal{X}} Y$, we see that the object $x|_{T_i}$ is an object of Y for all i, and hence x is an object of Y over T. An object $(\pi(x), x, id_{\pi(x)})$, coming from Y, is isomorphic to (y, x, ϕ), which implies the natural morphism is epimorphic. Thus we see that our correspondence $Y \mapsto \pi(Y)$ gives the inverse. The last part of the proposition is obvious. □

B.2. Some algebraic substacks and their properties. Here we will talk on the stack-theoretic closure, irreducibility of stacks and the image of a morphism.

Let $\{\mathcal{Y}_\lambda\}_\lambda$ be a family of closed substacks of an algebraic stack $\mathcal{X}$. We define a substack $\bigcap_\lambda \mathcal{Y}_\lambda$ of $\mathcal{X}$ to be the full subcategory of which objects are those of all $\mathcal{Y}_\lambda$. Let us fix an atlas $\pi : X \to \mathcal{X}$ and put $Y_\lambda := \pi^{-1}(\mathcal{Y}_\lambda)$. Then we can see that a diagram

$$\begin{array}{ccc} \bigcap_\lambda Y_\lambda & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ \bigcap_\lambda \mathcal{Y}_\lambda & \longrightarrow & \mathcal{X} \end{array}$$

[4]For simplicity, all representable morphisms here are supposed to be schemetic. That is in fact enough for our application.

is cartesian, hence $\bigcap_\lambda \mathcal{Y}_\lambda$ is a closed substack. Accordingly, the following definition makes sense.

Definition B.2. Let $\mathcal{Y}$ be a locally closed substack of an algebraic stack. We call the smallest closed substack containing $\mathcal{Y}$ the *closure* of $\mathcal{Y}$. We denote it by $\overline{\mathcal{Y}}$.

The following proposition shows us the stack-theoretic closure is compatible with an atlas.

Proposition B.3. *Let $\mathcal{Y}$ be a locally closed substack of a locally neotherian algebraic stack $\mathcal{X}$, $\pi : X \to \mathcal{X}$ an atlas and put $Y := \pi^{-1}(\mathcal{Y})$. Then we have $\pi^{-1}(\overline{\mathcal{Y}}) = \overline{Y}$.*

Proof. Clearly $\overline{Y} \subset \pi^{-1}(\overline{\mathcal{Y}})$, hence it is enough to show $\overline{Y} \times_{\mathcal{X}} X = \overline{Y} \times_{\mathcal{X}} \overline{Y}$, by virtue of Proposition B.1. Here we need a claim.

Claim. Let S and W be locally noetherian scheme and let $f : W \to S$ be a flat morphism locally of finite type. Let T be a locally closed subscheme of S. Then $f^{-1}(\overline{T}) = \overline{f^{-1}(T)}$.

Proof. $\overline{f^{-1}(T)}$ is a closed subscheme of $f^{-1}(\overline{T})$, and hence let us consider the exact sequence

$$0 \to \mathcal{N} \to \mathcal{O}_{f^{-1}(\overline{T})} \to \mathcal{O}_{\overline{f^{-1}(T)}} \to 0.$$

Since $\mathcal{N}$ is supported in the boundary $f^{-1}(\overline{T}) \setminus f^{-1}(T)$, we have $f\left(\mathrm{Ass}_{\mathcal{O}_{f^{-1}(\overline{T})}}(\mathcal{N})\right) \subset \overline{T} \setminus T$. On the other hand, we have

$$f\left(\mathrm{Ass}_{\mathcal{O}_{f^{-1}(\overline{T})}}(\mathcal{N})\right) \subset f\left(\mathrm{Ass}_{\mathcal{O}_{f^{-1}(\overline{T})}}(\mathcal{O}_{f^{-1}(\overline{T})})\right) \subset \mathrm{Ass}_{\mathcal{O}_{\overline{T}}}(\mathcal{O}_{\overline{T}}),$$

where the last inclusion holds because $f^{-1}(\overline{T}) \to \overline{T}$ is flat. Since no associated point appears in the boundary of the closure, we have $\mathrm{Ass}_{\mathcal{O}_{\overline{T}}}(\mathcal{O}_{\overline{T}}) = \mathrm{Ass}_{\mathcal{O}_T}(\mathcal{O}_T)$. Therefore

$$f\left(\mathrm{Ass}_{\mathcal{O}_{f^{-1}(\overline{T})}}(\mathcal{N})\right) \subset (\overline{T} \setminus T) \cap \mathrm{Ass}_{\mathcal{O}_{\overline{T}}}(\mathcal{O}_{\overline{T}}) = (\overline{T} \setminus T) \cap \mathrm{Ass}_{\mathcal{O}_T}(\mathcal{O}_T) = \emptyset,$$

which implies $\mathcal{N} = 0$. □

By virtue of this claim, we have $\overline{Y} \times_{\mathcal{X}} X = \overline{Y \times_{\mathcal{X}} X}$ in $X \times_{\mathcal{X}} X$. On the other hand, we have

$$Y \times_{\mathcal{X}} X = Y \times_{\mathcal{X}} Y \subset \overline{Y} \times_{\mathcal{X}} \overline{Y}.$$

Accordingly we conclude $\overline{Y} \times_{\mathcal{X}} X \subset \overline{Y} \times_{\mathcal{X}} \overline{Y}$. The other inclusion is trivial. □

Corollary B.4. *Let $\mathcal{Y}$ be a locally closed substack of a locally noetherian algebraic stack $\mathcal{X}$ over a regular scheme S of dimension* 1*. If $\mathcal{Y}$ is flat over S, then so is $\overline{\mathcal{Y}}$.*

Proof. (We use the same notation as that in Proposition B.3.) By virtue of Proposition B.3, it is enough to show that $\overline{Y}$ is flat over S. Since Y is flat over S, there is no associated point of $\mathcal{O}_Y$ mapping to a closed point of S. Since $\mathrm{Ass}_{\mathcal{O}_{\overline{Y}}}(\mathcal{O}_{\overline{Y}}) = \mathrm{Ass}_{\mathcal{O}_Y}(\mathcal{O}_Y)$, there is no associated point of $\mathcal{O}_{\overline{Y}}$ mapping to a closed point of S either. Taking account that S is regular of dimension 1, we conclude therefore $\overline{Y}$ is flat over S. □

An algebraic stack $\mathcal{X}$ is said to be *irreducible* if for any two non-empty open substacks $\mathcal{Y}_1$ and $\mathcal{Y}_2$, we have $\mathcal{Y}_1 \cap \mathcal{Y}_2 \neq \emptyset$ ([11, Definition 4.11]). A non-empty open substack of an irreducible substack is irreducible of course. In case of schemes, the irreducibility of a scheme is led from that of a *dense* open subscheme. We would like to say the same thing for algebraic stacks.

Definition B.5. Let $\mathcal{Y}$ be a locally closed substack of an algebraic stack $\mathcal{X}$. We say $\mathcal{Y}$ is dense in $\mathcal{X}$ if $\overline{\mathcal{Y}}_{\mathrm{red}} = \mathcal{X}_{\mathrm{red}}$.

It is obvious that if $\mathcal{X}$ is a scheme, the meaning of denseness is same as the usual topological one.

Lemma B.6. *Let $\mathcal{Y}$ be an open substack of an algebraic stack $\mathcal{X}$. If $\mathcal{Y}$ is irreducible and dense in $\mathcal{X}$, then $\mathcal{X}$ is irreducible.*

Proof. We may assume $\mathcal{X}$ reduced. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be non-empty open substacks of $\mathcal{X}$. If $\mathcal{Y} \cap \mathcal{X}_i = \emptyset$ $(i = 1, 2)$, then $\mathcal{Y}$ is an open substack contained in a proper closed substack of $\mathcal{X}$. That contradicts to the assumption of the denseness, hence $\mathcal{Y} \cap \mathcal{X}_i \neq \emptyset$ $(i = 1, 2)$. Since $\mathcal{Y}$ is irreducible, we have $(\mathcal{Y} \cap \mathcal{X}_1) \cap (\mathcal{Y} \cap \mathcal{X}_2) \neq \emptyset$, which implies $\mathcal{X}_1 \cap \mathcal{X}_2 \neq \emptyset$. □

Thus, in examining the irreducibility, we are sometimes interested in the denseness of an irreducible open substack. The following proposition gives a sufficient condition of denseness.

Proposition B.7. *Let $\mathcal{X}$ be an algebraic stack and let $\mathcal{Y}$ be a locally closed substack of $\mathcal{X}$. Then $\mathcal{Y}$ is dense in $\mathcal{X}$ if the following holds: Let k be an algebraically closed field and let $x : \mathrm{Spec}(k) \to \mathcal{X}$ be a morphism. Then there exist a discrete valuation ring R with the residue field k and the quotient field K, and a morphism $\mathrm{Spec}(R) \to \mathcal{X}$ such that the composition $\mathrm{Spec}(k) \to \mathrm{Spec}(R) \to \mathcal{X}$ coincides with x and that the composition $\mathrm{Spec}(K) \to \mathrm{Spec}(R) \to \mathcal{X}$ factors through $\mathcal{Y}$.*

Proof. Let us take an atlas $\pi : X \to \mathcal{X}$ and put $Y := \mathcal{Y} \times_{\mathcal{X}} X$. By the assumption, we can take a discrete valuation ring R and a morphism $x_R : \mathrm{Spec}(R) \to \mathcal{X}$ as in the proposition. Here we may assume that R is strictly henselian. Then we can lift x_R to a morphism $\tilde{x}_R : \mathrm{Spec}(R) \to X$. Since the generic point goes into $\overline{Y}$, so does the special point, hence $x = \pi \circ x_R|_{\mathrm{Spec}(k)}$ factors through $\overline{\mathcal{Y}}$. That implies $\overline{\mathcal{Y}}_{\mathrm{red}} = \mathcal{X}_{\mathrm{red}}$ (c.f. [11, Proposition 5.4.(i)]). □

Finally, let us talk on the stack-theoretic image of morphisms of algebraic stacks. It should be remarked that the image in the sense of stack is not necessarily algebraic even if F is a morphism of schemes. However, the image in the stack-theoretic sense and that in the scheme-theoretic sense are sometimes the same. Let $F : X \to Y$ be a morphism of schemes. Suppose that F is universally open and for any open subscheme U of X, the morphism $F|_U : U \to V$ has a local section for some open subscheme V, where a local section means a pair of a covering $\{V_i \to V\}_i$ and a family of morphisms $\{V_i \to U\}_i$ over V. It is clear $F(U)$ is, in the sense of stack, a substack of V. To show $V \subset F(U)$, let $v : T \to V$ be a morphism of schemes. By the assumption, it has a lift to U after taking a covering, which implies that it is an object of $F(U)$. Thus we have $F(U) = V$ in this case. Note that the assumption above holds if F is smooth. That is the only case we need, and we can show the following assertion.

Lemma B.8. *Let $F : \mathcal{X} \to \mathcal{Y}$ be a representable smooth morphism of algebraic stacks and let $\mathcal{U}$ be an open substack of $\mathcal{X}$. Then $F(\mathcal{U})$ is an open substack of $\mathcal{Y}$.*

Proof. We may assume $\mathcal{U} = \mathcal{X}$. Let $\pi : Y \to \mathcal{Y}$ be an atlas and consider the following cartesian diagram

$$\begin{tikzcd} Y \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d, "F_Y"'] & \mathcal{X} \arrow[d] \\ Y \arrow[r] & \mathcal{Y}. \end{tikzcd}$$

Then, we can verify without difficulty that

$$F_Y\left(Y \times_{\mathcal{Y}} \mathcal{X}\right) \times_{\mathcal{Y}} Y = F_Y\left(Y \times_{\mathcal{Y}} \mathcal{X}\right) \times_{\mathcal{Y}} F_Y\left(Y \times_{\mathcal{Y}} \mathcal{X}\right),$$

and we see moreover from the above discussion that $F_Y\left(Y \times_{\mathcal{Y}} \mathcal{X}\right)$ is an open subscheme of Y. Therefore, by Proposition B.1, $\pi\left(F_Y\left(Y \times_{\mathcal{Y}} \mathcal{X}\right)\right) = F(\mathcal{X})$ is an open substack of $\mathcal{Y}$. $\square$

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