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INTEGRALITY AND ARITHMETICITY OF CO-COMPACT LATTICE CORRESPONDING TO CERTAIN COMPLEX TWO-BALL QUOTIENTS OF PICARD NUMBER ONE *

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Dedicated to Professor Yum-Tong Siu on his sixtieth birthday

The main purpose of this paper is to provide some criteria for the integrality and arithmeticity of a cocompact lattice corresponding to a complex ball of dimension two.

Let G be a semi-simple Lie group of non-compact type and K be a maximal compact subgroup of G. G/K is then a symmetric space of non-compact type. Let Γ be a cocompact lattice of G. A lot of examples of such lattices are provided by arithmetic lattices (cf. [Bo1]). According to the Arithmeticity Theorem of Margulis [Mar], in the case that the symmetric space G/K has real rank at least 2, Γ has to be an arithmetic lattice. In the case that the symmetric space is either a quaternionic hyperbolic space or the Cayley hyperbolic plane, Corlette [C] establishes superrigidity of a lattice in those spaces, which together with the results of Gromov-Schoen [GS] established the arithmeticity of such lattices as well. For real hyperbolic spaces, there exist a lot of non-arithmetic lattices in each dimension as constructed by Gromov and Piatetski-Shapiro [GP]. According to the classification of semi-simple Lie groups, the only case remained is the class of complex hyperbolic manifolds of complex dimension at least 2. In this aspect, there are examples of non-arithmetic lattice in dimension 2 and 3 (cf. [Mos] [MD]). We do not know of examples in higher dimensions. In both real and complex hyperbolic cases, it is interesting to give criteria to characterize arithmetic lattices.

It is well-known since the work of Margulis (cf. [Zi]) that arithmeticity follows from superrigidity. In [MSY], a general Bochner type formula was formulated, the linear version of which yields vanishing theorems including Matsushima's Vanishing Theorem and Kazhdan's Property T for such lattices, while the semi-linear version yields a uniform proof of the superrigidity results of Margulis and Corlette for co-compact lattices. In this way, problems about vanishing theorems of cohomology groups and superrigidity are put in a equal footing. For the complex or real hyperbolic cases, the Bochner formula does not lead to vanishing or rigidity properties as in the other cases. In fact, there are examples with non-vanishing first Betti numbers (cf. [BW]). However, it naturally leads us to look for criteria for arithmeticity of complex and real rank one cases in terms of restrictions on cohomology groups. A question known in this direction is the following which was attributed to Rogawski (cf. [Re]).

CONJECTURE 1. *Let Γ be a torsion-free cocompact lattice of $PU(2,1)$ so that the corresponding ball quotient $M = \Gamma\backslash B^2_{\mathbf{C}}$ satisfies the conditions that the first Betti number $b_1(M) = 0$ and the Neron-Severi group $H^{1,1}(M) \cap H^2(M, \mathbf{Q}) = \mathbf{Q}$. Then Γ is arithmetic and comes from a division algebra with an involution of second kind.*

In the terminology of Definition 3 in Section 6, these are arithmetic lattices of second type. Arithmetic lattices of first type are the ones defined by a Hermitian form

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over a number field. It is verified by Rogawski [Ro] that cocompact lattices coming from congruence subgroups of a division algebra with an involution of second kind do share the two cohomological constraints that $H^1(M) = 0$ and $H^{1,1}(M) \cap H^2(M, \mathbf{Q}) = \mathbf{Q}$.

The main result of this paper is as follows. We refer the readers to §1 and the beginning of §5, §6 for the terminology used.

Main Theorem

A. *Let* Γ *be a torsion-free cocompact lattice of* $PU(2,1)$ *so that the quotient* $M = \Gamma\backslash PU(2,1)/P(U(2) \times U(1))$ *satisfies* $H^{1,1}(M) \cap H^2(M, \mathbf{Q}) = \mathbf{Q}$ *and* $b_1(M) = 0$. *Then* Γ *is integral.*

B. *Assume that* Γ *satisfies the hypothesis of Part (A) and furthermore that the canonical line bundle* K_M *is three times the generator of the Neron-Severi group* $H^{1,1}(M) \cap H^2(M, \mathbf{Z}) = \mathbf{Z}$ *modulo torsion. Then* Γ *is arithmetic.*

C. *Assume that* Γ *satisfies the hypothesis of Part (A) and furthermore that the corresponding ball quotient* M *contains no immersed totally geodesic curve. Then one of the followings holds.*
(i). Γ *is arithmetic of second type.*
(ii). Γ *is a non-arithmetic lattice of* $PU(2,1)$ *and is a subgroup of infinite index in an arithmetic lattice of second type of some* $PU(2,1)^p$ *with* $p > 1$.

We remark that the Main Theorem B remains valid for smooth ball quotients M_o which has a finite unramified covering M satisfying the hypothesis of Main Theorem B. Examples satisfying Main Theorem B include fake projective plane which are projective algebraic surfaces different from $P^2_{\mathbf{C}}$ but have the same rational cohomology ring as the projective plane and hence the canonical line bundle is three times the generator of the Neron-Severi group. The first such examples are constructed by Mumford [Mu2] (cf. [IK] for two more examples). Hence we have the following corollary.

Theorem 1. *Lattices associated with fake projective planes are arithmetic.*

We refer the readers to Kato [Ka] and Klingler [Kl] for other recent, independent works on the arithmeticity of Mumfords' fake projective planes. In particular, Theorem 1 is proved independently by Klingler in [Kl] which appears while the present paper is being refereed. Though more restrictive, the paper [Ka] gives a more precise description of the lattice in terms of Shimura varieties for the original example of fake projective plane constructed by Mumford.

From the results of Rogawski [Ro] and classification of algebraic groups, arithmetic lattices of $PU(2,1)$ of second type satisfy $b_1(M) = 0$, $H^{1,1}(M) \cap H^2(M, \mathbf{Q}) = \mathbf{Q}$ and contain no immersed totally geodesic hyperbolic curve (cf. Section 3). These lattices provide examples satisfying the conditions for Main Theorem C.

The proof of Main Theorem A is related to the results of [Y2], where we consider a conjecture of Borel on virtual positivity of the first Betti number of complex ball quotients formulated in a way similar to a conjecture of Thurston on real ball quotients.

Theorem 2 (Y2). *Let* Γ *be a co-compact torsion-free lattice of* $PU(2,1)$. *Let* M *be the associated compact complex two ball quotient* $\Gamma\backslash PU(2,1)/P(U(2) \times U(1)$.

Consider the realization of Γ *as a subgroup of* G_F *for an algebraic group* G *defined over a real number field* F *with* $G \otimes_F \mathbf{R} \cong PU(2,1)$. *Assume that* Γ *is non-integral in* G_F. *Then the first Betti number of* M *is virtually positive.*

We remark that in Main Theorem A, the hypothesis on b_1 is on M instead of some finite unramified covering of M as stated in Theorem 2. Neither result is stronger than the other, though the general approaches to the two problems are similar. In this paper, for completeness of arguments, we provide essentially all the necessary details for the proof of Main Theorem A .

The usual approach of geometric superrigidity does not work for smooth complex ball quotients. Approach such as [MSY] in general involves a Bochner type formula, the linear version gives rise to vanishing theorems while the non-linear version provides results in rigidity. We refer the readers to [Y1] for further discussions and examples. In view of the examples of complex ball quotients with non-trivial first Betti numbers as constructed by Kazhdan, Shimura and Borel-Wallach (cf. [BW]), we cannot expect superrigidity results for complex ball quotients to follow easily from an appropriate Bochner formula.

Our approach here is still geometric. The outline of the proof is as follows. From Weil's Local Rigidity Theorem, we know that a lattice in $G = PU(2,1)$ is locally rigid and can be defined over a real algebraic number field. If a lattice Γ satisfying the conditions of the Main Theorem is not integral, there is a prime p so that the representation in $G(F_p)$ is unbounded. From the results of Gromov-Schoen [GS], we know that there exists a non-trivial $\Gamma-$equivariant harmonic map from the two ball into the associated Bruhat-Tits Building. Bochner formula of Siu ([Siu1], [Sam]) still applies to show that the mapping is harmonic, leading to a multi-valued holomorphic one form η, which becomes single valued after passing to the spectral covering M_1 of M. The Bruhat-Tits Building can be of dimension one or dimension two.

For the case that the Bruhat-Tits Building is of dimension one, we use the Lefchetz type theorem and factorization theorem of Simpson ([Sim1]) to show that either the representation factors through a non-trivial homomorphism into the real line corresponding to an apartment of the building, or factors through a representation of the fundamental group of an orbicurve. The first case is ruled out by the fact that the stabilizer of an apartment in a building is a proper subgroup of $G(F_p)$. The second case implies that τ is indeed rational coming from the pull-back of Chern class of a line bundle.

In case that the building is of dimension 2, it is associated to $PL(3, F_p)$. There are two naturally occuring holomorphic one forms ω_1, ω_2 on the spectral covering of the manifold. We need to separate into three cases.
(a). ω_1 and ω_2 are linearly dependent.
(b). ω_1 and ω_2 are linearly independent but $\omega_1 \wedge \omega_2 \equiv 0$.
(c). ω_1 and ω_2 are linearly independent, $\omega_1 \wedge \omega_2 \not\equiv 0$.
Case (a) is handled in a way similar to the earlier situation that the building is a tree. Case (b) leads to a non-trivial morphism of the manifold into an orbicurve, contradictory to the assumption that the Picard number is one, similar to the classical Castelnuovo-de Franchis Theorem. Case (c) corresponds to a non-degenerate map into a two dimensional Euclidean building. First we show that the spectral mapping coming from the harmonic map is unramified, since existence of a ramification locus leads to a certain type of factorization, contradicting Zariski-denseness of the representation. As the action of the image of the fundamental group by the harmonic map is discrete on the building, we show that the Albanese map determined by ω_i's

projects real analytically but not holomorphically to a mapping into a real two torus. By considering the topology of a generic fiber of the projection map, we show that the first Betti number of M cannot be trivial. In this way, we conclude that the lattice has to be integral as stated in Main Theorem A.

Once we know that Γ is an integral lattice in a simple algebraic group $G \cong PU(2,1)$ defined over a real algebraic number field F, we consider the restriction of scalars $R_{F/\mathbf{Q}}(G) = \prod_{\sigma \in Gal(F/\mathbf{Q})} G(\mathcal{O}_F)^\sigma$. To show that such a lattice Γ is arithmetic, we need to show that all factors in $R_{F/\mathbf{Q}}(G)$ apart from the first one is compact. For Main Theorem B, suppose on the contrary that there is at least one more non-compact factor apart from the first one in the restriction of scalars, the algebraic group G^σ has to be isomorphic to $PU(2,1)$ from Lie algebra consideration. The machinery of harmonic maps and Bochner type formula can be applied to obtain an equivariant holomorphic map between the two complex two balls. From the assumption that the canonical line bundle is three times a generator of the Neron-Severi group modulo torsion, we show that such a holomorphic map has to be biholomorphic, which is a contradiction since the mapping is induced from the conjugation of the identity representation of Γ by a non-trivial element in the Galois group.

For more general situation considered in Main Theorem C without assumption on the canonical class, the lattice involved is either arithmetic or a subgroup of infinite index in some arithmetic subgroup of $PU(2,1)^p$ for some $p > 1$ following from restriction of scalars again. For the algebraic groups involved, $PU(2,1)$ or $PU(2,1)^p$ with $p > 1$, there are only two classes of such lattices, labelled as First and Second Type in Section 6. We show that for lattices of First type, there is always an immersed totally geodesic curve which is a complex one ball quotient, completing the proof of Main Theorem C.

The set up of the paper is as follows. We collect in Section 1 preliminary properties which are required for our later arguments. Then we formulate the proof of Main Theorem A in Section 2, for which the actual proof is carried out in Section 3 for rank one Bruhat-Tits buildings and Section 4 for rank two buildings.

The proof of Main Theorem B is given in Section 5, from which the applications to fake projective planes as stated in Theorem 1 follows. The proof for Main Theorem C is given in Section 6.

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1. Preliminaries. In this section, we recall some standard definitions and constructions concerning lattices of $PU(2,1)$ and their representations.

We begin with the notion of local rigidity. Let G be a complex algebraic group. Let Γ be a finitely generated group, which is taken to be a lattice in a semi-simple Lie group or the fundamental group of a locally symmetric space in our context. A representation $u \in \mathrm{Hom}(\Gamma, G)$ is said to be locally rigid if the orbit of u under G by conjugation $u \to u^g, u^g(x) = g \cdot u(x) \cdot g^{-1}$ is open in $\mathrm{Hom}(\Gamma, G)$ with respect to the topology defined by pointwise convergence. Following Weil [We], a representation u is locally rigid if $H^1(\Gamma, Ad \circ u) = 0$ (cf. [Rag], page 91). The same definition applies for a real algebraic group. A torsion-free lattice Γ of a connected real semisimple algebraic group H is said to be locally rigid if the inclusion $\Gamma \to H$ is a locally rigid representation. If Γ is a cocompact lattice in a simple Lie group of non-compact type which is not $SL(2,\mathbf{R})$, $H^1(\Gamma, Ad \circ u)$ vanishes from an appropriate Bochner formula

[We]. In particular, suppose that H is a connected real semisimple algebraic group which has no compact factors and is not locally isomorphic to $SL(2,\mathbf{R})$. Weil's Local Rigidity Theorem states that any cocompact irreducible lattice of H is locally rigid.

Consider now a lattice in $PU(2,1)$. The simple Lie group $G = PU(2,1)$ consists of the quotients of all 3×3 matrices of determinant one in $GL(3,\mathbf{C})$ preserving the diagonal Hermitian form $-|z_0|^2+|z_1|^2+|z_2|^2$ by scalar constants. Using the mapping

$$z = x+\sqrt{-1}y \longmapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix},$$

we may regard G as an algebraic $\mathbf{Q}$-group $G_{\mathbf{R}}$ consisting of the set of real points in an algebraic subgroup of $PGL(6,\mathbf{C})$. Let $\rho:\Gamma\to G_{\mathbf{R}}$ be the natural homomorphism given by embedding of the lattice. From Weil's Local Rigidity Theorem [We], we know that ρ is rigid and hence Γ can be defined over a real number field F after conjugating by an element in G (cf. [R] (6.6), (7.67)). That is, there is a faithful $\mathbf{R}-$rational representation $\pi : G\to PGL(m,\mathbf{C})$ for which $\pi(\Gamma)\subset PGL(m,F)$, and $\pi(G)$ as the Zariski closure of Γ is defined over F as well.

We make the following definitions.

Definition 1.
a. We say that a representation $\rho:\Gamma\to G = PU(2,1)$ is integral if there is a form G_F of G over a number field F such that a subgroup of finite index of Γ is contained in $G(\mathcal{O}_F)$. Hence it is conjugate to a representation whose coefficients in matrix form are rational numbers in a number field F with uniformly bounded denominators (cf. [Zi]). Let p be a finite place and F_p be the completion of F at p. From the finite generation of Γ, integrality of ρ is equivalent to the p-boundedness of ρ in $G(F_p)$ for all finite places p.
b. A lattice in G is arithmetic if there exists a semisimple algebraic $\mathbf{Q}$-group H and a surjective homomorphism $\varphi: H^o_{\mathbf{R}}\to G$ with compact kernel such that $\varphi(H_{\mathbf{Z}}\cap H^o_{\mathbf{R}})$ and Γ are commensurable.

It follows from definition that arithmetic lattices are integral. On the other hand, given an integral lattice Γ as above, we can define Weil's Restriction of Scalars $\mathrm{Res}_{F/\mathbf{Q}}(G)=\prod_{\sigma\in S^\infty}G^\sigma$, where S^∞ is the set of all Archimedean places. Suppose that G^σ is compact for all $\sigma\neq 1$, Γ is then an arithmetic lattice.

As explained earlier, to prove that a lattice is integral in G, it suffices for us to show that the induced representation of Γ in $G(F_p)$ is bounded for all finite place p. This is a special type of non-Archimedean superrigidity. To prove that Γ is arithmetic, we also need to show that the induced representation of Γ in G^σ is bounded for all $\sigma\neq 1$. This is a special case of Archimedean superrigidity. In the next few sections, we discuss first non-Archimedean superrigidity followed by the Archimedean superrigidity in more specialized settings.

2. Formulation of problem for integrality. Let $\rho:\Gamma\to G = PU(2,1)$ be the natural representation given by the embedding of the lattice. From the discussions in §1, we know that Γ can be regarded as a lattice of a semi-simple algebraic group G_F defined over a number field F and $G_{\mathbf{R}}\cong PU(2,1)$. Hence Γ is integral if ρ is $p-$bounded in $G(F_p)$ for all p. From our assumption that Γ is non-integral, we know that there exists p such that the homomorphism $\rho_p:\Gamma\to G(F_p)$ induced from ρ

is $p-$unbounded in $G(F_p)$. For simplicity of notations, we still denote ρ_p by ρ when there is no danger of confusion.

Denote by X the Bruhat-Tits building associated to $G(F_p)$. The Bruhat-Tits building associated to $G = PU(2,1)$ is a simplicial complex and is an Alexandrov space (cf. [GS]) with the naturally endowed Euclidean metric along simplices. Γ is now a cocompact lattice of G so that the natural homomorphism $\rho : \Gamma \to G$ has Zariski-dense image. As X is contractible, there exists a Lipschitz ρ-equivariant map $f_o : \Gamma\backslash G/K \to X$. It follows from the main result of Gromov-Schoen ([GS], Theorem 7.1) that there exists a ρ-equivariant harmonic map from G/K to X.

Moreover the regularity estimates of Gromov-Schoen implies that the singularity set $\widetilde{S}$ of f has Hausdorff codimension at least 2. At a regular value of f, the image lies locally in a subset of an apartment isometric to $\mathbf{R}^n$ corresponding to an apartment Σ of the Euclidean building X. Let x^1 be a linear function on Σ. Let $dz^1 = dx^1 \otimes \mathbf{C}$ be the complexification of dx^1. Consider the pull-back of dz^1 and let ∂f^1 be the $(1,0)-$part of f^*dz^1. As is shown in [GS], integration by parts still makes sense since $\widetilde{S}$ has high Hausdorff codimension. In this way, the $\partial\overline{\partial}-$Bochner formula of Siu [Siu] is still applicable to imply that

$$\int |\nabla\overline{\partial} f|^2 = \sum_{i,j} \int R^N(A_i, A_j, \overline{A_i}, \overline{A_j}),$$

where $A_i = \sum_\alpha f_i^\alpha \frac{\partial}{\partial y^\alpha} = f_*\big(\frac{\partial}{\partial z^i}\big)$ in terms of an orthonormal frame of tangent vectors $\{\frac{\partial}{\partial z^i}\}$ on M. We conclude that $\nabla\overline{\partial} f = 0$ or $\overline{\nabla}\partial f = 0$ since the image X has non-positive curvature operator. Hence ∂f^1 is a local holomorphic one form on a neighbourhood of $\widetilde{M} - \widetilde{S}$, where $\widetilde{S}$ is the set of singularities of the harmonic map f.

Let $x^i = \sigma^i(x^1), \sigma^r \in \overline{W}$, be the functions obtained from the action of the Weyl group $\overline{W}$. $\{x^i\}, i = 1, \ldots, l$ is canonically and globally defined on Σ up to the action of the Weyl group $\overline{W}$, where l is the order of the Weyl group of $\overline{W}$. Let ∂f^i be the $(1,0)-$part of $f^*(dx^i \otimes \mathbf{C})$. The same argument as above shows that ∂f^i are local holomorphic one forms on $\widetilde{M}$. Hence any symmetric polynomial α of $\partial f^i, i = 1, \ldots, l$, gives a multi-valued globally defined holomorphic one form ξ on $M - S$, where S is the image of $\widetilde{S}$ by the universal covering map. The multivalued form extends to S since df vanishes on such points (cf. [GS]).

Let $\alpha_1, \ldots, \alpha_l$ be the set of multivalued holomorphic forms thus obtained. They define on the total space of the cotangent bundle T^*M a subvariety M_1' with defining equation $t^l + \alpha_1(x)t^{l-1} \cdots + \alpha_n(x)$, where t is the canonical section of the tautological line bundle of T^*M. Let $p : M_1' \to M$ be the covering map. $p^*\partial f^i$ gives rise to a genuine holomorphic one form ω_i on M_1'. Let M_1 be a reduced, irreducible component of M_1'. We call M_1 the spectral covering of M_1' and $p|_{M_1} : M_1 \to M$ the spectral covering. Note that sometimes M_1' instead of M_1 is refered to be the spectral covering in literature. M_1 is a Galois covering of M from construction. Here we use the term Galois covering to describe a possibly ramified covering which is Galois outside of the ramifications locus. From definition, the singular values of the harmonic map f constructed by [GS] have to lie in walls of the apartments of the building X. Hence the singularity of $f \circ p : M_1 \to X$ has to lie in the ramification divisors corresponding $\omega_i = \omega_j$. The procedure described above is quite standard and can be found in earlier work of Simpson (cf. [Katz] and [Zu]). We summarized the discussions by the following lemma.

LEMMA 1. *The $\rho-$equivariant harmonic map of $\widetilde{M}$ to the building X gives rise to* *holomorphic one forms ω_i on some spectral covering M_1 on M. The spectral covering $M_1 \to M$ is a possibly ramified normal covering with order l dividing the order of the Weyl group of X with ramification loci given by $\omega_i = \omega_j$ for some $i \neq j$. The singularities of the induced harmonic map $f \circ p$ lie on the union of the divisors $\omega_i = \omega_j, i \neq j$ corresponding to the walls of the building X.*

Since $G \otimes_F \mathbf{R} \cong PU(2,1)$ is a Lie group of absolute rank 2, there are now two cases to consider.
Case (I). $rank_{F_p} G = 1$ so that the associated building X is a tree.
Case (II). $rank_{F_p} G = 2$ so that $G(F_p) \cong SL(3, \mathbf{C})$ and the associated building X is a two dimensional complex.

In the following two sections, we are going to consider the two cases separately for the proof of Theorem 1.

3. Boundedness of representations in rank one buildings. We consider non-Archimedean superrigidity in rank one p-adic algebraic groups, corresponding to rank one Bruhat-Tits buildings in Case (I) above.

PROPOSITION 1. *Let Γ be a non-integral lattice of $PU(2,1)$. Assume that $M = B^2_{\mathbf{C}}/\Gamma$ is smooth with Picard number 1. Then the representation of Γ in the isomorphism group of a rank one Bruhat-Tits building is always bounded.*

Proof. In this case, the Bruhat-Tits building X is a tree consisting of apartments Σ which are isomorphic to $\mathbf{R}$. The isotropy group of an apartment Σ in X is the affine Weyl group $W = \mathbf{Z} \times \overline{W}$, a semi-direct product where $\overline{W}$ is the usual Weyl group of G (cf. [Br]). $\mathbf{Z}$ acts by translation and $\overline{W} \cong \mathbf{Z}_2$ acts by reflections. Since $\overline{W}$ has order 2, the spectral covering $p : M_1 \to M$ has degree 2 or 1.

In general M_1 may have singularities. Let $p_2 : M_2 \to M_1$ be a desingularization of M_1. Let $\pi_2 : \widetilde{M_2} \to M_2$ be the univeral covering map of M_2. The mapping $q = p \circ p_2 \circ \pi_2 : \widetilde{M_2} \to M$ can be lifted to $\widetilde{q} : \widetilde{M_2} \to \widetilde{M}$. There exists a non-trivial holomorphic $1-$form α on M_2 obtained by pulling back the one from M_1.

Applying the result of Simpson [Sim3] to $\widetilde{Y} = \widetilde{M_2}$, we conclude that one of the following two conclusions holds.
Case Ia. The image of the Albanese map determined by the holomorphic one form α has dimension at least two.
Case Ib. The image of the Albanese map determined by the holomorphic one form α has dimension one.

LEMMA 2.
1. Case Ia leads to a contradiction.
2. In Case Ib, there exists a non-trivial holomorphic map from the smooth complex two ball quotient M to an orbicurve C.

Proof. The argument is actually a modification of an argument of Simpson in an unpublished manuscript concerning representation of Kähler groups in $SL(2)$. For our case, we consider representations in $PU(2,1)$.

For Case Ia, we define on the universal covering $\widetilde{M_2}$ of M_2 the function $g(y) = \int_{y_o}^{y} p^*(\alpha) \in \mathbf{C}$ by integrating the one form from a fixed point $y_o \in \widetilde{M_2}$. It is well defined since α is $d-$closed and $\widetilde{M_2}$ is simply connected. Recall that $g : \widetilde{M_2} \to \mathbf{C}$ has connected fibers. Hence we get a map $r : \mathbf{R} \to X$ so that $f_o \circ q = r \circ Re(g) : \widetilde{M_2} \to X$. The set of points $z \in \widetilde{M_2}$ for which $dg = \alpha$ or $dq = d(p \circ p_2 \circ \pi_2)$ is degenerate is a complex subvariety of $\widetilde{M_2}$. It follows that for each point $t \in \mathbf{R}$, there is z in the real hypersurface $(Re(g))^{-1}(t)$ such that both dg and dq are non-degenerate at $z \in \widetilde{M_2}$. Hence r is differentiable on $\mathbf{R}$. Furthermore, $|dr(t)| = 1$ for all $t \in \mathbf{R}$ by checking at such non-degenerate points $z \in (Re(g))^{-1}(t)$. Hence $Im(f)$ is an isometric copy of $\mathbf{R}$, that is, an apartment of Σ of the building X. It follows from our construction that Σ is the image of the $\pi_1(M)-$equivariant map $f_o : \widetilde{M} \to X$. It follows that $\rho(\pi_1(M))$ lies in the stabilizer of an apartment Σ in $G(F_p)$. Since the Zariski closure of the stabilizer is a proper subgroup of $G(F_p)$ (cf. [Br]), this contradicts the Zariski-denseness of ρ.

For Case Ib, the dimension of the the image $Im(a_2)$ of the Albanese map $a_2 : M_2 \to Alb(M_2)$ is of complex dimension one. Considering the normalization E of $Im(a_2)$, we get a holomorphic mapping $a_2' : M_2 \to E$.

As mentioned before, M can be considered as the quotient of spectral covering M_1 by an involution corresponding to the ambiguity of the two-valued form obtained from the Bruhat-Tits Building. From the result of Simpson [Sim3], we conclude that there is an biholomorphism $\tau : E \to E$ so that $\sigma^* a_2'^* \beta = a_2'^* \tau^* \beta$ corresponding to the involution σ, where β is given by $a_2'^* \beta = \alpha$.

The quotient of E by τ gives rise to a complex one dimensional orbicurve C. The mapping from M_2 to E coming from the Albanese mapping induces a holomorphic map $a : M \to C$. An orbicurve C in general is given by data (Y, n), where Y is the underlying topological manifold, and $n = n_{y_i}$ is the branching multiplicity of the orbifold singularity at each of the finite number of singular points $y_i, i = 1, \ldots, r$.

This concludes the proof of Lemma 2.

LEMMA 3. *Assume that the Picard number of an algebraic surface M is equal to 1. Then there is no non-trivial holomorphic map into an algebraic curve.*

Proof. Assume that $f : M \to R$ is a non-trivial holomorphic map into an algebraic curve R. Let a be a generator of the Neron-Severi group modulo torsion. Then the pull back of the fundamental class by f of R, c, is a non-torsion element in $H^{1,1}(M) \cap H^2(M, Z)$ and hence is a non-trivial multiple of a. Hence the push-forward of a as a cycle is non-trivial. A generic fibre of f is one dimensional. Let b be the cohomology class of a generic fibre. Then b is a non-trivial multiple of a as well. As the push-forward of b is trivial, the same is true for a. The contradiction establishes the Lemma.

Proposition 1 now follows from the previous two lemmas.

4. Boundedness of representations in rank two buildings. In this section, we treat Case (II) mentioned in §2, the more difficult case.

PROPOSITION 2. *Let Γ be a cocompact lattice of $G = PU(2,1)$. Assume that $M = \Gamma \backslash PU(2,1)/P(U(2) \times U(1))$ is smooth with Picard number 1 and $b_1(M) = 0$. Assume also that there is an unbounded representation of Γ in $PSL(3, F_p)$. Then the spectral covering map $p : M_1 \to M$ is an unramified covering and there exist at least two linearly independent holomorphic one forms on M_1.*

Proof. In this case, the Bruhat-Tits building X is a simplicial complex of rank two obtained as follows. Let V be the $F-$vector space of dimension 3 on which G acts. A lattice in V is a finitely generated $\mathcal{O}_F$ submodule of V which generates V. Two lattices on V are said to be equivalent if one is the multiple of another by an element in F_p^*. The vertices of X are given by the equivalence classes $[L]$ of lattices in V, such that an unordered edge is connected between $[L]$ and $[L']$ if $pL \subset L' \subset L$. An apartment of the building X is obtained by fixing a basis e_1, e_2, e_3 of V and considering the subcomplex Σ of X consisting of vertices corresponding to lattices of the form $L =< p^{r_1}e_1, p^{r_2}e_2, p^{r_3}e_3 >_{\mathcal{O}_F}$. Since the lattice $< p^{r_1}e_1, p^{r_2}e_2, p^{r_3}e_3 >_{\mathcal{O}_F}$ is equivalent to $< p^{x_1}e_1, p^{x_2}e_2, p^{x_3}e_3 >_{\mathcal{O}_F}$ with $x_i = r_i - \frac{\sum_{j=1}^3 r_j}{3}$, it follows that an apartment can be described as the two dimensional plane H defined on $\mathbf{R}^3$ by $x_1 + x_2 + x_3 = 0$, on which the vertices are the intersection of the integral lattices on $\mathbf{R}^3$ with H. In this way, the chamber system of X is of type $\widetilde{A}_2$. A chamber is thus given by an equilateral triangle. The isotropic group of an apartment is the affine Weyl group $W = L \times \overline{W}$ obtained as a semi-direct product, where $\overline{W} \cong S_3$ is the symmetric group of three elements, and $L \cong \mathbf{Z}^2$ is the translation group. On H, L acts by translation and $\overline{W}$ acts by permuting the coordinates x_1, x_2, x_3.

Since the representation $\rho : \Gamma \to G(F_p)$ is non-compact. Lemma 1 implies that there exist holomorphic one forms on a spectral covering M_1 of M. In terms of the notation described above, the holomorphic one forms on M_1 obtained from the harmonic map are given by $\omega^i = ((f \circ p)^*(dx_i \otimes \mathbf{C}))^{1,0}$. Obviously, $\omega_1 + \omega_2 + \omega_3 = 0$. Hence there are at most two linearly independent holomorphic one forms among ω_i.

Let $\mathrm{Alb}_{\overline{W},\{\omega_i\}}(M_1)$ be the Abelian variety defined as the quotient of the Albanese variety by the $\overline{W}$-invariant Abelian subvariety annihilated by $\omega_i, i = 1, 2, 3$. Let $\alpha : M_1 \to \mathrm{Alb}_{\overline{W},\{\omega_i\}}(M_1)$ the corresponding Albanese mapping. The standard construction of the usual Albanese maping (cf. [BPV]) is given on the universal covering by $\beta : \widetilde{M_1} \to \mathbf{C}^N$,

$$\beta(z) = (\int_o^z \eta_1, \ldots, \int_o^z \eta_N),$$

where z_o is a fixed point of $\widetilde{M_1}$ and $\eta_i, i = 1, \ldots, N$ forms a basis of the space of holomorphic one forms on M_1. Let $\theta_1, \ldots, \theta_k$ be an invariant set of holomorphic one forms on M_1 such that $\{\omega_1, \omega_2, \omega_3, \theta_1, \ldots, \theta_k\}$ is an $\overline{W}$-invariant spanning set of the space of the pull back by α of holomorphic one forms on $\mathrm{Alb}_{\overline{W},\{\omega_i\}}(M_1)$. Let $n = \dim_{\mathbf{C}} \mathrm{Alb}_{\overline{W},\{\omega_i\}}(M_1)$ and denote by $R(\theta_1, \ldots, \theta_k)$ the set of linear relations among θ_i on $\mathrm{Alb}_{\overline{W},\{\omega_i\}}(M_1)$. Then α is given on $\widetilde{M_1}$ by

$$\begin{aligned}\alpha(z) = (&\int_{z_o}^z \omega_1, \int_{z_o}^z \omega_2, \int_{z_o}^z \omega_3, \int_{z_o}^z \theta_1, \ldots, \int_{z_o}^z \theta_k)\\ &\in \{(y_1, y_2, \ldots, y_{k+3}) \in \mathbf{C}^{k+3} | y_1 + y_2 + y_3 = 0, R(y_4, \ldots, y_{k+3}) = 0\} \cong \mathbf{C}^n.\end{aligned}$$

For simplicity, we also denote $\mathrm{Alb}_{\overline{W},\{\omega_i\}}(M_1)$ by $\mathrm{Alb}(M_1)$ or A and just name it as Albanese variety.

There are now three cases to consider.

Case IIa. ω_1 and ω_2 are linearly dependent.
Case IIb. ω_1 and ω_2 are linearly independent but $\omega_1 \wedge \omega_2 \equiv 0$.
Case IIc. ω_1 and ω_2 are linearly independent and $\omega_1 \wedge \omega_2 \not\equiv 0$.

Lemma 4.
1. In Cases IIa and IIb, there exists a non-constant holomorphic map from M to an orbicurve.
2. In Case IIc, the spectral covering $p : M_1 \to M$ is an unramified covering of index at most 6 and the Albanese mapping determined by $\{\omega_1, \omega_2, \omega_3\}$ has complex rank at least two.

Proof. This is given by case by case consideration.

Case IIa. Since ω_1 and ω_2 are linearly dependent, there exist constants k_1, k_2 satisfying $\omega_2 = k_1\omega_1$ and $\omega_3 = k_2\omega_1$. Since the holomorphic one forms are obtained from the harmonic map into the building, the image of the harmonic map actually lies in a one dimensional affine space along each apartment. The image is actually a one dimensional subcomplex of the building X. The argument of Proposition 1 shows that either the image of the harmonic map Φ lies in a line in an apartment isometric to R, which contradicts the Zariski denseness of the representation ρ; or there exists a surjective holomorphic map from M to an orbicurve, which is the conclusion we sought.

Case IIb. In this case, ω_1 and ω_2 are linearly independent and $\omega_1 \wedge \omega_2 \equiv 0$ on M_1. According to a classical theorem of Castelnuovo-de Franchis (cf. [BPV], Proposition IV 4.1), there is a holomorphic mapping $f_1 : M_1 \to R$, a Riemann surface, such that ω_1 and ω_2 are the pull-back of some holomorphic one forms from R. The algebraic curve R can be canonically obtained from the span of $\omega_1, \omega_2, \omega_3$ in the following sense (cf. [Ran]). The Albanese variety A of M_1 has dimension at least two from the existence of two linearly independent holomorphic one forms. Consider the Albanese map $\alpha : M_1 \to A$. If the Albanese variety A has dimension 2 or if the dimension of $\alpha(M_1)$ is one, it suffices to let R to be the image of the Albanese map. Hence we may assume that $dim(A) = n \geq 3$ and $dim(\alpha(M_1)) = 2$. The following is the summary of the construction in [Ran] page 464-466, which shows that R can be canonically constructed. We refer the readers to [Mu1] for general discussions on Abelian varieties. From our degeneracy condition, the image of α lies in $dz_1 \wedge dz_2 = 0$. There exists a linear subspace $L = \oplus_{i=3}^n \frac{\partial}{\partial z^i} \otimes \mathbf{C} \subset T_oA$ of dimension $n-2$ containing all $T_y(\alpha(M_1))$, where $y \in \alpha(M_1)$. Take a generic hyperplane H containing L in $\mathbf{C}^n$ and an irreducible component F_H of $\{y \in \alpha(M_1) : T_y(\alpha(M_1)) \subset H\}$ containing a generic point of $\alpha(M_1)$. F_H has dimension greater than 0 and generates an Abelian subvariety B of A. B is proper since T_oB is generated by the set $\{T_y(\alpha(M_1)), y \in \alpha(M_1)\}$ and hence is a subspace of $H \subset \mathbf{C}^n$. Consider the quotient map $\pi : A \to A/B$ into the quotient Abelian variety, which exists from Poincare's Complete Irreducibility Theorem (cf. [Mu1]). $R = \pi(\alpha(M_1))$ has dimension less than $\alpha(M_1)$ and is non-trivial since ω_i, the one forms that we start with, are the pull-back of one-forms from R.

Consider now the finite group $\overline{W}$ acting on M_1. The action leaves the space spanned by $\omega_i, i = 1, 2, 3$ invariant. Since the above construction of R is canonical, $\overline{W}$ induces a finite group action of R. The quotient is an orbicurve C. We conclude that there exists a holomorphic map $a : M \to C$.

Case IIc. In this case, the conditions that ω_1 and ω_2 are linearly independent and $\omega_1 \wedge \omega_2 \not\equiv 0$ imply that the Albanese map $\alpha : M_1 \to A$ is a holomorphic mapping of complex rank two into an Abelian variety A of complex dimension at least two. Recall that M_1 is a connected Galois covering of M with the covering group corresponding to a subgroup of S_3. The index of the covering is then an integer dividing 6. We are going to prove that the spectral covering $p : M_1 \to M$ is unramified. For this purpose, we need to rule out Subcases IIci, IIcii in the following corresponding to existence of a non-trivial ramification divisor.

Subcase IIci, that there is a ramification divisor of p which is not contracted by the Albanese map α on M_1

According to Lemma 1, the divisor corresponds to $\omega_i = \omega_j$ for some distinct pair of indices $i \neq j$ in $\{1, 2, 3\}$. For simplicity, let us just denote the divisor by $\omega_i = \omega_j$

with the understanding that i, j take distinct values in $\{1,2,3\}$. Let $\omega_o = \omega_i - \omega_j$. Then ω_o vanishes along a divisor D_o but is not identically zero on M_1. From definition, $\tilde{\alpha}(\widetilde{D}_o)$ lies as a complex subvariety of complex dimension one in a linear space L on $\mathbf{C}^n$ defined by $z_i = z_j + c$. The subvariety D_o, or more precisely some translates of $\alpha(D_o)$, generates an Abelian subvariety B with $T_oB = \{T_yD, y \in D_o\}$ as in Case IIb earlier. Let $\pi : A \to A/B$ be the quotient map into the quotient Abelian variety A/B. On one hand, $\pi(\alpha(M_1))$ has dimension less than 2 since fibres in the direction of $ker(\omega_o)$ in M_1 are contracted. On the other hand, $\pi(\alpha(M_1))$ has dimension greater than 0 since π is non-trivial in the non-vanishing direction of $\omega_o = \alpha^*dz_i - \alpha^*dz_j$. Hence $E = \dim(\pi \circ \alpha(M_1))$ is of dimension 1.

E is however not canonical and depends on D and ω_o. Let $h = \pi \circ \alpha : M_1 \to E$. Let $\widetilde{h} : \widetilde{M_1} \to \widetilde{E}$ be the mapping given by $z \mapsto \int_{z_o}^{z} \omega_o$, where z_o is some arbitrary fixed point on $\widetilde{M_1}$. We may assume that a fibre of h is connected after replacing E by a finite covering through Stein factorization. It follows from definition of h and $\widetilde{h}$ that a lift of a fiber $h^{-1}(a)$ of h at a regular value $a \in E$ to $\widetilde{M}$ is a component of a fibre of $\widetilde{h}$. The set of regular values of h is a Zariski open set E_o of E. E_o is an affine curve. Consider the homotopy sequence of the fibration,

$$1 = \pi_2(E_o) \to \pi_1(h^{-1}(a)) \stackrel{i_{1*}}{\to} \pi_1(h^{-1}(E_o))) \to \pi_1(E_o) \to \cdots,$$

where i_{1*} is the homomorphism induced by the imbedding. We conclude that the fundamental group of a fiber $\pi_1(h^{-1}(a)) = \ker(i_{1*})$ is a normal subgroup of $\pi_1(h^{-1}(E_o))$. $h^{-1}(E_o)$ is a Zariski-open subset of M_1 and the homomorphism $i_{2*} : \pi_1(h^{-1}(E_o)) \to \pi_1(M_1)$ is surjective. Pulling back the homomorphism $\rho : \pi_1(M) \to G(F_p)$ by the projection $p : M_1 \to M$ and restricting to $h^{-1}(a)$, we get a homomorphism $\sigma = \rho \circ p_* \circ i_{2*} \circ i_{1*} : \pi_1(h^{-1}(a)) \to G(F_p)$. The Zariski closure $\overline{\sigma(\pi_1(h^{-1}(a)))}$ of the image of σ is a normal algebraic subgroup of $G(F_p)$. Since $G(F_p)$ with $G \cong PSL(3)$ is simple, $\overline{\sigma(\pi_1(h^{-1}(a)))}$ is either trivial or $G(F_p)$. Let $\widetilde{h^{-1}(a)}$ be the universal covering of $h^{-1}(a)$. The $\partial\overline{\partial}$-Bochner formula of Siu implies that the harmonic map f is pluriharmonic. It follows that $f \circ \widetilde{p} : \widetilde{h^{-1}(a)} \to X$ is a $\sigma-$equivariant harmonic mapping into the building. Here $\widetilde{p} : \widetilde{h^{-1}(a)} \to \widetilde{M}$ is the lift of $p|_{h^{-1}(a)} : h^{-1}(a) \to M$ to the corresponding universal covering spaces. Note that in the definition of $\alpha : \widetilde{M_1} \to \mathbf{C}^n$, the real part of the first three components $(\int_{z_o}^{z} \omega_1, \int_{z_o}^{z} \omega_1, \int_{z_o}^{z} \omega_1)$, where $\omega_1 + \omega_2 + \omega_3 = 0$, are constructed from the pluriharmonic map into the Bruhat-Tits building that we start with. It follows that the image of $Re(f \circ \widetilde{p}|_{\widetilde{h^{-1}(a)}})$ is unbounded corresponding to the real part of $\alpha(z)$ transversal to $\int_{z_o}^{z} \omega_o$ in the Euclidean space. As a fundamental domain of $h^{-1}(a)$ is compact, we conclude that σ and hence $\overline{\sigma(\pi_1(h^{-1}(a)))}$ is non-trivial. Since $\overline{\sigma(\pi_1(h^{-1}(a)))}$ is a normal subgroup of $G(F_p)$, which is simple, this implies that $\overline{\sigma(\pi_1(h^{-1}(a)))} = G(F_p)$. We claim that this cannot be the case for all generic $a \in E$.

From construction, we know that the restriction of $f \circ \widetilde{p}(h^{-1}(a))$ to each apartment Σ of the building lies in the kernel of some differential form $dx_i - dx_j$ for some linear function on the apartment $\Sigma \cong \mathbf{R}^2$, where x_i, x_j are some linear functions on Σ as explained earlier. In fact, $(f \circ \widetilde{p})^*(dx_i - dx_j)$ is the real part of ω_o. Hence the image $f \circ p(h^{-1}(a)) \cap \Sigma$ is contained in a line l_c defined on Σ by $x_i - x_j = c$. As a can take generic values, and all such l_c are isometric straight lines with the same slope on $\Sigma \cong \mathbf{R}^2$, we may assume that $c = 0$ after moving by some isometry if necessary, so that we may regard l_c as a one dimensional subcomplex of the apartment $\Sigma \subset X$. Since

$f\circ\widetilde{p}(h^{-1}(a))$ is connected in X, an isometry carrying l_c to a wall, a one-dimensional subcomplex, in an apartment Σ in X actually extends to an isometry of $f\circ\widetilde{p}(h^{-1}(a))$ to a wall in X. Hence $f\circ\widetilde{p}(h^{-1}(a))$ can be taken as a tree, which is a totally geodesic subcomplex in X. As explained earlier, vertices of an apartment Σ of X correspond to equivalence classes $[L]$ of lattices $L=<p^{r_1}e_1,p^{r_2}e_2,p^{r_3}e_3>_{\mathcal{O}_F}$ with integer values r_i in a three dimensional vector space $V=<e_1,e_2,e_3>$. Without loss of generality, we assume that $\omega_o=\omega_1-\omega_3$ and the image $f\circ p(h^{-1}(a))\cap\Sigma$ is described by $x_1-x_3=0$. Since $x_i=r_i-\frac{\sum_{j=1}^3 r_j}{n}$, this corresponds to $r_1=r_3$ in all the lattices corresponding to the image $f\circ p(h^{-1}(a))\cap\Sigma$. Consider a two dimensional vector subspace $V'_\Sigma=<e_1+e_3,e_2>$ of V. The set of lattices in V'_Σ with $e'_1=e_1+e_3$ and $e'_2=e_2$ as a basis contributes to an apartment Σ' of a rank one building X'_Σ associated to V'_Σ. For two different apartments Σ_1 and Σ_2 in X sharing a common chamber C, the corresponding subspace V'_{Σ_1} and V'_{Σ_2} are the same since the geodesic subcomplex $f\circ\widetilde{p}(h^{-1}(a))\cap\Sigma_i$ is determined by the subset $f\circ\widetilde{p}(h^{-1}(a))\cap C$ containing an open subset of Σ_i for $i=1,2$. As any two apartments of X share a common chamber from the definition of a building, V' is a well-defined two dimensional subspace of V for $f\circ\widetilde{p}(h^{-1}(a))$. The associated building X' is a tree on which $G'(F_p)=PSL(V',F_p)\cong PSL(2,F_p)$ as a subgroup of $G(F_p)=PSL(V,F_p)\cong PSL(3,F_p)$ acts. Since $\overline{\sigma(h^{-1}(a))}$ lies in $G'(F_p)$, a proper subgroup of $G(F_p)$, the claim is proved. It follows from the claim and the discussions before its statement that Subcase IIc(i) does not occur.

Subcase IIcii, that all divisors in the ramification locus, which is not empty, are contracted by the Albanese map α.

Let D be a connected component in the ramification locus of p. Let E be the image in M. From Lemma 1, D has to lie on $\omega^i=\omega^j$ for some $i\neq j$, where $\omega^i=((f\circ p)^*(dx^i\otimes\mathbf{C}))^{1,0}$. As f is holomorphic, $\omega^i=(f\circ p)^*(dz^i)$, where dz^i is the $(1,0)-$part of $dx^i\otimes\mathbf{C}$. Since $\alpha(D)$ is a point, it follows from the construction of α that $f(E)=f(p(D))$ is a point $x_o\in X$. Hence the restriction of ρ to $\pi_1(E)$ or $\pi_1(D)$ is bounded and both D and E are contractible by some holomorphic maps. We would only use the second fact, where D is contracted by the Albanese map $\alpha: M_1\to \mathrm{Alb}(M_1)$ and E is constracted by the map $\alpha_o: M\to\mathrm{Alb}(M_1)/\overline{W}$ induced by α. Let $N_1=\alpha(M_1)$ and $N_2=\alpha_o(M)$. Then $\alpha: M_1\to N_1$ contracts all components D of the ramification divisor of p to points $\{A_i\}$ on N_1, and $\alpha_o: M\to N$ contract all components $E=p(D)$ of the branching divisor of p to points $\{B_j\}$ on N. From construction, there is a holomorphic map $q: N_1\to N$ whose restriction $q: N_1-\cup_i A_i\to N-\cup_j B_j$ is a finite unramified covering. Let $n_1: P_1\to N_1$ and $n: P\to N$ be the normalizations of N_1 and N respectively. q induces a holomorphic map $s: P_1\to P$ (cf. [BPV]). Since a holomorphic map into an algebraic variety factors through its normalization, $\alpha_o: M\to N$ can be factored as $n\circ r$ with a holomorphic mapping $r: M_1\to P_1$, and $\alpha_1: M_1\to N_1$ can be written as $n_1\circ r_1$ with a holomorphic mapping $r_1: M_1\to P_1$.

Chasing the commutative diagram from our construction, we conclude that $s\circ r_1=r\circ p: M_1\to P$. The sets $n_1^{-1}(\cup_i P_i)\subset P_1$ and $n^{-1}(\cup_j Q_j)\subset P$ are both finite set of points as only normalizations are involved. We see that $s: P_1\to P$ is an unramified covering when restricted to $s: P_1-n_1^{-1}(\cup_i A_i)\to P-n^{-1}(\cup_j B_j)$. However, the ramification locus of s, a holomorphic map between two normal varieties, has to contain a ramification divisor of codimension one if the ramification locus is non-empty. Since $n_1^{-1}(\cup_i P_i)\subset P_1$ and $n^{-1}(\cup_j Q_j)\subset P$ have codimension two in N'_1 and N' respectively, we conclude that s is actually an unramified covering. It is clear then that $p : M_1 \to M$ is unramified. Hence Subcase IIcii actually does not occur.

As both Subcases IIci and IIcii are ruled out, we conclude that $p : M_1 \to M$ is an unramified covering. Again, the assumption in this case implies that there are at least two linear independent holomorphic one forms on M_1 coming from pulling back of the one forms from the Albanese of M_1. This concludes the proof of the proposition.

PROPOSITION 3. *Let Γ be a torsion-free cocompact lattice of $PU(2,1)$ so that $M = \Gamma\backslash PU(2,1)/P(U(2)\times U(1))$ is a smooth complex ball quotient with Picard number 1 and the first Betti number $b_1(M) = 0$. Then any representation of Γ into $G(F_p) \cong PSL(3, F_p)$ is bounded.*

Proof. Assume for the purpose of proof by contradiction that there exists an unbounded representation $\rho : \Gamma \to PSL(3, F_p)$. We only need to consider Case IIc, so that the Albanese variety A arising from the Albanese mapping $\alpha : M_1 \to A$ determined by $\omega_1, \omega_2, \omega_2$ has complex dimension at least two.

LEMMA 5. *There exists a harmonic map $h_R : M_1 \to T_2$ into a real two torus.*

Proof. The real part of the map is $\widetilde{h}_R : \widetilde{M_1} \to \mathbf{R}^2$ defined by

$$\begin{aligned}\widetilde{h}_R(z) = (&\int_{z_o}^{z}(f\circ p)^* dx_1, \int_{z_o}^{z}(f\circ p)^* dx_2, \int_{z_o}^{z}(f\circ p)^* dx_3) \\ &\in \{(y_1,y_2,y_3)\in \mathbf{R}^3 | y_1+y_2+y_3=0\} \cong \mathbf{R}^2,\end{aligned}$$

where f is the harmonic map into the building and dx_i's are the local coordinates on the two dimensional building as discussed before.

We claim that the set $\widetilde{h}_R(\pi_1(M_1)z)$ is discrete in $\mathbf{R}^2$. We need to show the discreteness around any point $z \in \widetilde{M_1}$. However, around any point $z \in \widetilde{M_1}$, the differential of $\widetilde{h}_R(\widetilde{M_1})$ is the same as the differential of $\widetilde{f_1} : \widetilde{M_1} \to X$ induced from the harmonic map originally obtained from $\widetilde{M}$ to X. This is particularly clear in the case that $p(z) \in \widetilde{M}$ does not lie in the singularity of the harmonic map $f : \widetilde{M} \to X$. In such case, both $\mathbf{R}^2$ and locally around $f \circ p(z)$ in the building X, the metrics involved are the Euclidean metric, the actions of $\pi_1(M_1)$ on $h_R(\widetilde{M_1}) = \mathbf{R}^2$ and $f(\widetilde{M_1}) = X$ are locally isometric. Hence non-discreteness of the action of $(\widetilde{h}_R)_*(\pi_1(M_1))$ around any point on $\mathbf{R}^2$ would be reflected by the non-discreteness of the action of $(\rho\circ p)_*(\pi_1(M_1)$ on X. The action of the latter group is however known to be discrete, since it lies in the affine Weyl group of the building. The same argument works for the case that $p(z)$ lies in the singular set of the harmonic map f as well. In fact, around such a point, the differentials dh_R and df are still the same. $f(\pi(z))$ has to lie in a wall on the building X. The action of $(\widetilde{h}_R)_*(\pi_1(M_1))$ around $\widetilde{h}_R(z)$ is the same as the action of $f_* \circ p_*(\pi_1(M_1))$ around $f \circ p(z)$. The latter is discrete from the $\pi_1(M)-$equivariance of f and the fact that the affine Weyl group still acts discretely on and around the walls of the building X. The claim is thus proved.

It follows that $\widetilde{h}_R$ induces a pluriharmonic map of real rank two $h_R : M_1 \to T = R_2/\pi_1(M_1)$, a real 2−torus.

Recall from our earlier discussions in this section that the lift of the Albanese map $\alpha : M_1 \to A$ determined by ω_i's is defined on $\widetilde{M_1}$ by

$$\alpha(z) = (\int_{z_o}^{z}(f\circ p)^*(dx^1\otimes \mathbf{C})^{1,0}, \int_{z_o}^{z}(f\circ p)^*(dx^2\otimes \mathbf{C})^{1,0},$$

$$\int_{z_o}^{z}(f\circ p)^*(dx^3\otimes \mathbf{C})^{1,0},\int_{z_o}^{z}\theta_1,\dots,\int_{z_o}^{z}\theta_k)$$
$$\in\{(y_1,\dots,y_{k+3})\in \mathbf{C}^{k+3}|y_1+y_2+y_3=0,R(y_4,\dots,y_{k+3})=0\}\cong \mathbf{C}^n.$$

It follows that on $\widetilde{M}_1$, the pull back of h_R is just the real part of the projections to the first three factors of the pull back of α. Regarding both maps as Γ-equivarient mappings on the universal covering $\widetilde{M}_1$, it follows that there exists a projection $q: A\to T$ so that $h_R=q\circ\alpha$.

Our goal is to prove that such a picture contradicts our hypothesis on M. For the clarity of presentation, let us consider first the simpler case of $\dim_{\mathbf{C}} A=2$ which is geometrically clearer. Necessary modifications will be added to prove the general case later on. The two dimensional case is of particular interest since it is satisfied by the fake projective planes, the focus of Theorem 1 (cf. §5 for definition). This can be seen from the the following simple observations. The Galois group of the spectral covering is a normal subgroup of the symmetry group S_3. However, it cannot be the alternating group A_3, otherwise $\Omega=\sum_i\omega_i\wedge\omega_{i+1}$ is A_3-invariant and descends to give a non-trivial element in $H^{2,0}(M)$, where we let $\omega_4=\omega_1$. Hence Galois group of the spectral covering is S_3. Let M' be the covering of M corresponding to A_3. It follows from Riemann-Roch Theorem that $h^{2,0}(M')=2$. However, if there exists a holomorphic one form η on M_1 not contained in the linear span of $\omega_i, i=1,2,3$, η cannot be A_3 invariant for otherwise $\sqrt{-1}\eta\wedge\overline{\eta}$ would lead to an element in $H^{1,1}(M)$ linearly independent of the Kähler class. Hence $\sum_{\sigma\in A_3}\sqrt{-1}\eta\wedge\sigma(\eta)$ gives rise to a holomorphic two form on M' independent of Ω following the same construction as for ω, contradicting the conclusion from Riemann Roch. We conclude that $\omega_i, i=1,2$ are the only linearly independent holomorphic one forms on M_1 if M is a fake projective plane.

Now we begin with proof for the case of $\dim_{\mathbf{C}} A=2$.

Simplified case: $\dim_{\mathbf{C}} A=2$

Let p be a generic point on the real two torus T. The Galois group of the spectral covering $H<S_3$ acts on T by permuting the coordinates x_1,x_2,x_3 as before. As p is generic, $h(p)\neq p$ for $h\in H$. As A is topologically a four torus, $q^{-1}(p)$ is a real two torus on A. Since $\dim_{\mathbf{C}}(A)=2$, $\alpha: M_1\to A$ is surjective. Let $C=\alpha^{-1}(q^{-1}(p))$ as a set. Then C is a two dimensional compact manifold on M_1. Under the action of $h\in H$, $h\neq 1$, $h(C)$ is an isometric copy of C. $h(C)$ is disjoint from C since $h(p)$ is disjoined from p. $p_1(C)$ is a two dimensional compact manifold on M. Similar to C, $p_1(C)$ is not a complex subvariety of M.

Let R be the ramification locus and $B=\alpha(R)$ the branching locus of α. It is clear that there is no contracted divisors for α, otherwise the divisor descends to M and contradicts $\mathrm{Pic}(M)=1$. We claim that for a generic p, $q^{-1}(p)\cap B=\emptyset$. Note that h_R is analytic apart from some real one dimensional subvarieties R_1 on T corresponding to the fixed point sets of elements of the Weyl group. The albanese map α induced by ω_i's is obviously complex analytic. Hence q is analytic as a real differentiable map apart from the preimage of R_1 and can be considered as projection into the purely real part of the torus A. However R is a complex analytic subvariety of A. Its projection to T cannot be the whole T unless $R=A$. The Claim is proved.

Recall that $C=\alpha^{-1}(q^{-1}(p))$. It follows from the Claim that $\alpha: C\to q^{-1}(p)$ is an unramified covering.

Let $\gamma \cong S^1$ be a generating cycle of $q^{-1}(p)$. Clearly, γ is a generator of $H_1(A, \mathbf{R})$ and is not homologous to a trivial cycle on A. Consider $\gamma_1 = \alpha^{-1}(\gamma)$ as a set. γ_1 defines a cycle on $C \subset M_1$. $\gamma_o = p_1(\gamma_1)$ is then a cycle on M.

We claim that γ_o is a non-trivial one cycle on M. Suppose on the contrary that γ_o is the boundary of a two cycle σ on M. As $p_1 : M_1 \to M$ is an unramified covering, we know from our construction that $p_1^{-1}(C)$ is a disjoint union of $[M_1 : M]$ copies of cycles each isometric to C. We may lift σ to a surface $\sigma_1 \subset M_1$ so that the boundary of σ_1 is simply γ_1. In fact, it suffices to take the component of $p_1^{-1}(\sigma)$ which has γ_1 as the boundary. The push forward by the Albanese map $\alpha(\sigma_1)$ would then be a two dimensional manifold with γ as the boundary. This however contradicts our choice that γ is a non-trivial 1-cycle on A, a real four torus as a differentiable manifold. The claim is proved.

The same argument implies that γ_o is not a torsion element in $H^1(M, \mathbf{R})$, otherwise it would lead to a torsion element in $H^1(A, \mathbf{R})$, again a contradiction. It follows that the first Betti number of M is non-trivial. This contradicts our hypothesis on M.

General case: $\dim_{\mathbf{C}} A = n \geq 2$

We use the notation as in the case of $\dim_{\mathbf{C}} A = 2$. In this case the fiber of q, $T_1 = q^{-1}(p)$, for a generic p is a real $2n-2$ torus and $\alpha(M_1) \cap q^{-1}$ is a real two dimensional variety on $T_1 = q^{-1}(p)$. The same argument as in $\dim_{\mathbf{C}} A = 2$ case implies that the branching locus B should be projected a proper real subvariety of T. Hence for a generic $p \in T$, $\alpha : C = \alpha^{-1}(q^{-1}(p)) \to q^{-1}(p)$ is an unramified finite covering.

Topologically, T_1 is diffeomorphic to a cartesian product $\prod_{i=1}^{n-2}(S_i)$, where each S_i is diffeomorphic to a circle S^1. Since $\alpha(M_1) \cap q^{-1}$ is a real two dimsional subvariety of T_1, there exists a projection into direct factors $\tau_{ij} : T_1 = (S^1)^{n-2} \to S_i \times S_j$ such that the restriction of the projection to $\alpha(M_1) \cap q^{-1}$ is surjective. Renaming the index if necessary, we may assume that $p_{12} : \alpha(M_1) \cap q^{-1} \to S_1 \times S_2 \cong (S^1)^2$ is surjective. Again, let γ_{-1} be a non-trivial one cycle on $S_1 \times S_2$. Let γ be a connected component of $p_{12}^{-1}(\gamma_{-1})$. Let γ_1 be a connected component of $\alpha^{-1}(\gamma)$ on M_1. Let $\gamma_o = \pi(\gamma_1)$. Suppose now that γ_o is homologically trivial on M. Then the same argument as in the case of $\dim_{\mathbf{C}} A = 2$ shows that γ_{-1} has to be homologically trivial on $S_1 \times S_2 \cong (S^1)^2$, a two torus. This certainly contradicts our choice of γ_{-1} on $S_1 \times S_2$. Similarly, γ_o is a non-torsion element in $H^1(M, \mathbf{R})$, otherwise γ_{-1} would be a torsion element in $S_1 \times S_2$. This completes the proof of the proposition.

Main Theorem A. *Let Γ be a torsion-free cocompact lattice of $PU(2,1)$ so that the quotient $M = \Gamma \backslash PU(2,1)/P(U(2) \times U(1))$ satisfies $H^{1,1}(M) \cap H^2(M, \mathbf{Q}) = \mathbf{Q}$ and $b_1(M) = 0$. Then Γ is integral.*

Proof. This follows immediately from the formulation in §2 and Propositions 1, 2 and 3.

5. Results in arithmeticity. For a complex two ball quotient M of Picard number 1, a rational multiple of the canonical line bundle K_M is a generator of the Neron-Severi group modulo torsion. The dual of a complex two ball quotient is the complex projective plane $P^2_{\mathbf{C}}$, for which the canonical line bundle is $K_{P^2_{\mathbf{C}}} = -3H_{P^2_{\mathbf{C}}}$, where $H_{P^2_{\mathbf{C}}}$ is the hyperplane line bundle. In other words, a cubic root of $K_{P^2_{\mathbf{C}}}$ exists as a line bundle. Realizing a complex two ball as a domain of $P^2_{\mathbf{C}}$ in the standard way

$$B^2_{\mathbf{C}} = \{[z_0, z_1, z_2] : |z_0|^2 > |z_1|^2 + |z_2|^2\} \hookrightarrow P^2_C,$$

The canonical line bundle $K_{B^2_{\mathbf{C}}}$ can be written as $3H = 3H_{P^2_{\mathbf{C}}}|_{B^2_{\mathbf{C}}}$ as a $SU(1,2)$ invariant line bundle.

In general, a cocompact lattice Γ of the complex two ball $B^2_{\mathbf{C}}$ is a lattice in the automorphism group of $B^2_{\mathbf{C}}$, $PU(2,1) = SU(2,1)/\{\epsilon I\}$, where $\{\epsilon I\}$ consists of all cubic roots of unity ϵ multiplied to the identity I. The canonical line bundle $K_{B^2_{\mathbf{C}}}$ descends to $M = B^2_{\mathbf{C}}/\Gamma$, but the cubic root H may not descend since there may not be a consistent way of taking the cubic root of the transition functions of $K_{B^2_{\mathbf{C}}}$ with respect to Γ which takes values in $SU(2,1)/\{\epsilon I\}$. However, if Γ is defined in $SU(2,1)$, we can take cubic root of the transition function of the canonical line bundle so that the bundle descends to M. In such a case $K_M = 3H_M$ for a positive line bundle H_M.

Conversely, consider an arbitrary smooth complex two ball quotient M on which $K_M = 3H_M$. As is observed in [Ko], page 96, there exists a finite unramified covering $p : M_1 \to M$ for which the corresponding normal sublattice Γ_1 of finite index in Γ can be embedded in $SU(2,1)$ so that $K_{M_1} = 3H_{M_1}$ for an ample line bundle H_{M_1} descending from H on $B^2_{\mathbf{C}}$. It follows that p^*H_M is linear equivalent to H_{M_1} which is the restriction of the cubic root H of $K_{B^2_{\mathbf{C}}}$ when pulled back to the universal covering $B^2_{\mathbf{C}}$. Hence the pull-back of H_M to $B^2_{\mathbf{C}}$ by the universal covering map can be considered as a cubic root of $K_{B^2_{\mathbf{C}}}$. It other words, a cubic root of the transition functions of the canonical line bundle $K_{B^2_{\mathbf{C}}}$ can be taken on $B^2_{\mathbf{C}}$ consistent with the action of Γ. Hence there is a lifting of Γ to $SU(2,1)$ and Γ can be considered as a lattice in $SU(1,2)$.

Note that even if the condition $K_M = 3H_M$ is satisfied and M has Picard number one, H_M may not be the generator of the torsion free part of the Neron-Severi group. Here we give a characterization for the case that the canonical line bundle of the ball quotient is three times the generator of the Neron-Severi group modulo torsion. This is also the case discussed in [Re] and [Wa].

Main Theorem B *Let Γ be a cocompact lattice of $G = PU(2,1)$ so that the quotient $M = \Gamma\backslash PU(2,1)/P(U(2)\times U(1))$ has Picard number 1 and that the canonical line bundle K_M is three times the generator of the Neron-Severi group modulo torsion. Then Γ is arithmetic.*

Proof. From Main Theorem A, we know that Γ can be considered as an integral lattice in an algebraic group G which is $F-$isomorphic to $PU(2,1)$, where F is a real algebraic number field. To prove arithmeticity, it suffices for us to prove that $G^\sigma(R)$ is compact for each $\sigma \in Gal(F/\mathbf{Q}) - \{1\}$.

Lemma 6. *Let Γ be a co-compact lattice of $G_{\mathbf{R}} \cong PU(2,1)$, where Γ and G are defined over a real algebraic number field F. Then the Restriction of Scalars for G is*

$$\mathcal{R} = \prod_{\sigma\in S^\infty} G^\sigma = PU(2,1)^p \times PU(3)^{d-p}$$

for some $1 \le p \le d$, where $G_{\mathbf{R}} \cong PU(2,1)$ is the first factor of the Restriction of Scalars.

Proof. Two field embeddings $\sigma_1, \sigma_2 : F \to \mathbf{C}$ are said to be equivalent if they are complex conjugate of one another. An Archimedean place σ of F is an equivalence class of embeddings of F into $\mathbf{C}$. Denote by S^∞ the set of all such Archimedean places and σ_1 the identity.

For an Archimedean place σ of F, we let $F^\sigma = \sigma(F)$ and F_σ the completion of F at σ, which is $\mathbf{R}$ if F^σ is real and $\mathbf{C}$ if F^σ is complex. Suppose G as an algebraic variety is defined by a set $\mathcal{P}$ of polynomials over F on a vector space isomorphic to

$F^l_{\sigma_1} \cong \mathbf{R}^l$ for some integer l. We denote by G^σ the algebraic group defined by the set of polynomials $P^\sigma, P \in \mathcal{P}$, over F^σ on a vector space isomorphic to F^l_σ.

We now apply the well-known fact that $G \otimes_F \mathbf{C}$ is isogenous to $G^\sigma \otimes_{F^\sigma} \mathbf{C}$, which follows from the observation that the Lie algebra $\mathbf{g}$ satisfies $\mathbf{g} \otimes_F \mathbf{C} \cong \mathbf{g}^\sigma \otimes_{F^\sigma} \mathbf{C}$ by comparing the structure cosntants of the Lie algebra (cf.Lemma 6.57 of [Wi]). Since $G \otimes \mathbf{C} \cong SL(3, \mathbf{C})$, we conclude that $G^\sigma \otimes_{F^\sigma} C \cong SL(3, \mathbf{C})$ as well. Direct checking going through the list of classical groups (cf. [Wi], §3E) implies that G^σ is isogenous to either $SL(3, F_\sigma)$ or $PU(p,q)$ with $p+q=3$. We are going to rule out the first case.

The inclusion $\rho : \Gamma \to G$ is a $k-$homomorphism. An Archimedean place $\sigma : F \to C$ induces $\rho^\sigma \in R(\Gamma, G^\sigma)$ so that $\rho^\sigma(\Gamma) = \Gamma^\sigma$. We claim that $H^1(\Gamma, Ad \circ \rho^\sigma) = 0$. It suffices to show that every cocycle is actually a coboundary. A cocycle ϕ satisfies $\phi(xy) = \phi(x) + Ad \circ \rho^\sigma(x)\phi(y)$ for all $x, y \in G(F)$. ϕ is a coboundary if there exist v in the Lie algebra of $G(F)$ such that $\phi(x) = Ad \circ \rho^\sigma(x)v - v$. We know from [We] that $H^1(\Gamma, Ad \circ \rho) = 0$. It follows from conjugation by σ that $H^1(\Gamma, Ad \circ \rho^\sigma) = 0$ as well. The claim is proved.

It follows from the claim and Weil's work (cf. [Rag] page 91) that ρ^σ is a rigid representation. Since Γ is Zariski dense in G, Γ^σ is Zariski dense in G^σ considered as a real algebraic group. Hence we may apply a theorem of Simpson [Sim1] to conclude that Γ^σ is an algebraic group of Hodge type in the sense of Simpson. However, neither $SL(3, \mathbf{R})$ nor $SL(3, \mathbf{C})$ is an algebraic group of Hodge type (cf. [Sim1], page 50). We conclude that G^σ is isogenous to $PU(p,q)$ for some $p+q=3$. $\mathcal{R}$ is now obtained from Weil's Restriction of Scalars $\mathcal{R} = \mathrm{Res}_{F/\mathbf{Q}}(G) = \prod_{\sigma \in S^\infty} G^\sigma$, which without loss of generality is written as $PU(2,1)^p \times PU(3)^{d-p}$.

Hence to prove Main Theorem B, it suffices for us to show that the form G^σ_F obtained from the conjugate of G_F by σ has signature $(3,0)$ or $(0,3)$. Assume for the purpose of proof by contradiction that the form P^σ is not definite so that it is a non-degenerate Hermitian form with signature $(2,1)$ or $(1,2)$. In either case, the corresponding group $\Gamma^\sigma = \sigma(\Gamma)$ is a subgroup of another complex two ball M_σ. This is not possible from the following argument.

The lattice Γ^σ is Zariski dense in $B^2_{\mathbf{C}}$ from the Zariski denseness of Γ in G. Hence the arguments of Eells-Sampson implies the existence of $\Gamma-$equivariant map $\Phi : B^2_{\mathbf{C}} = \widetilde{M} \to \widetilde{M}_\sigma = B^2_{\mathbf{C}}$ (cf. [La]).

We claim that Φ is holomorphic. In case that the real rank r of the mapping Φ is at least 3, this follows from Siu's $\partial\overline{\partial}-$Bochner formula. The cases $r \leq 2$ can be treated similar to [CT], page 173-201 or [Siu2]. Following an earlier observation of Sampson, $r = 1$ implies that the image is a geodesic line whose stabilizer contains the image of Γ, contradicting Zariski denseness of image of Γ. The case of $r = 2$ leads to factorization of the map Φ through a holomorphic map into a Riemann surface. Pulling back the canonical Kähler form from the Riemann surface would contradict the hypothesis that the Picard number of M is 1 as argued earlier.

Alternately, from the results of Simpson [Sim1], we know that Φ has to come from a variation of Hodge structure. Φ can then be lifted to a holomorphic map Φ' from $\widetilde{M}$ to a Griffiths' Period Domain N^σ_1 above N^σ. Since $N^\sigma = B^2_{\mathbf{C}} \cong PU(2,1)/P(U(2) \times U(1))$, the only Griffiths' Period Domain above N^σ is N^σ itself. We conclude that Φ is holomorphic again. This provides another proof for the claim.

As explained in the paragraph above the statement of Main Theorem B in this section, Γ is actually a subgroup in $SU(2,1)$ and hence so is Γ^σ from definition. Therefore $\Phi^* K_{\widetilde{M}_\sigma} = 3\Phi^* H_{\widetilde{M}_\sigma}$, the line bundle on $\widetilde{M}^\sigma \subset \mathbf{P}^2_{\mathbf{C}}$ obtained from the restriction of hyperplane line bundle from $\mathbf{P}^2_{\mathbf{C}}$, Let $K_{\widetilde{M}_\sigma}$ be the $SU(2,1)$-invariant line bundle on $\widetilde{M}_\sigma$. $K_{\widetilde{M}_\sigma}$ is then Γ^σ-invariant. Hence $\Phi^* K_{\widetilde{M}_\sigma}$ is then a Γ-invariant holomorphic line bundle on $\widetilde{M}$, since Φ is Γ-equivariant.

Pulling back local canonical section $dz^1 \wedge dz^2$ on $\widetilde{M}^\sigma$ by Φ gives rise to a local holomorphic section of $K_{\widetilde{M}}$ vanishing along the ramification divisor R of Φ. It follows that

$$K_{\widetilde{M}} = \Phi^* K_{\widetilde{M}^\sigma} + R$$

as a Γ-invariant line bundle on $\widetilde{M}$.

Recall that $K_{\widetilde{M}_\sigma} = 3H_{M_\sigma}$, and R is a non-negative multiple of the generator of the Neron-Severi group of M. It follows from the Γ-invariance of line bundles involved and the fact that $K_M = K_{\widetilde{M}}$ is three times the generator of the Neron-Severi group modulo torsion that $R = 0$. We conclude that Φ is an unramified holomorphic mapping. Hence the Bergman metric on $\widetilde{M}_\sigma$ is the same as the Bergman metric on $\widetilde{M}$. It follows that Φ is a totally geodesic immersion. As is well-known, this induces a rational isomorphism $\Xi : G_F(R) \to G_{F^\sigma}(R)$, extending the homomorphism $\Phi : \Gamma \to \Gamma^\sigma$. As Γ is Zariski dense in G and $\Xi|_\Gamma = \Phi|_\Gamma$, the rational homomorphism has to be the same as the conjugation map $\sigma : G \to G^\sigma$ everywhere on G. However, the conjugation σ is not even continuous. This contradiction implies that $p = 1$ and hence Γ is arithmetic. This concludes the proof of Main Theorem B.

Concrete examples satisfying conditions of Main Theorem B includes fake projective planes.

DEFINITION 2. *A fake projective plane is a smooth complex surface satisfying one of the following equivalent conditions.*
a. $b_1(X) = b_2(X) = 0$ *and* $\pi_1(X)$ *is infinite.*
b. $c_1^2(X) = 3c_2(X)$, *Euler number* $\chi(X) = 3$ *and the curvature of* X *is negative.*
c. X *is a smooth complex ball quotient with rational (co)homology ring the same as the corresponding one of* $\mathbf{P}^2_{\mathbf{C}}$.

The equivalence of the above conditions follows easily from standard facts in the study of complex surfaces, such as Riemann-Roch Theorem, Hodge Index Theorem, Noether Formula and the solution of Calabi Conjecture by Aubin and Yau (cf. [BPV], page 136). An example of fake projective plane is first given by Mumford [M], using p-adic uniformization. Later on modification of the techniques is given by Ishida-Kato[IK] to give two more similar examples.

Proof of Theorem 1. It follows from definition above that a fake projective plane has Hodge number $h^{1,1} = 1$ and hence Picard number equal to 1. Furthmore, $b_1 = 0$ and it is a ball quotient with K being three times the generator of the Picard group, since the same is true for $\mathbf{P}^2_{\mathbf{C}}$. The arithmeticity of the lattice involved now follows from Main Theorem B.

6. More general cases. In this section, we consider smooth complex two ball quotients with $b_1(M) = 0$ and $\mathrm{Pic}(M) = 1$ but no longer require that K_M is three times a generator of the Neron-Severi group. To facilitate our later discussion, we recall some basic results from the classification of algebraic groups.

Algebraic groups defined over a number field F has been classified. A standard reference is the article of Tits [T], page 54-61. Regarding as lattices in an algebraic group, arithmetic lattices are classified as well. The procedure for the construction of any algebraic group G defined over an algebraic number field F such that $G_{\mathbf{R}} = PU(p,q)$ is as follows. Let D be a central division algebra of degree d defined over a quadratic extension L of F with an involution of second kind τ so that $F = \{a \in L | a^\tau = a\}$. Let V be a vector space over D and B be a non-degenerate τ-hermitian form over V so that apart from additivity,

$$B(ax,y) = aB(x,y), B(x,y) = (B(y,x))^\tau$$

for vectors $x, y \in V$ and $a \in D$. B may be regarded as a $n \times n$ matrix defined over D for some integer n. Let

$$U(B,D,\tau) = \{X \in GL_n(V) | (X^t)^\tau BX = F\},$$
$$PU(B,D,\tau) = U(B,D,\tau) \cap PSL_n(D).$$

Then $G = PU(B,D,\tau) \otimes_F \mathbf{R}$, with $p+q = nd$.

We are interested in the cases of $p+q = 3$. Hence either

Case (a). $d = 1, n = 3$ so that $G(\mathcal{O}_F)$ is defined by a hermitian form B over a number field, or

Case (b). $d = 3, n = 1$ so that $G(\mathcal{O}_F)$ is defined by a hermitian form B over a division algebra.

In Case (b), the hermitian form can be understood by noting that $D \otimes_F \mathbf{R} \cong M_3(\mathbf{R})$ so that $B \in D$ and the extension of B over the splitting field of F is a non-degenerate Hermitian 3×3 matrix with one negative and two positive eigenvalues.

Let $\sigma \neq 1$ be an embedding of F into $\mathbf{C}$ such that G^σ is non-compact. We claim that $F^\sigma \subset \mathbf{R}$. This is equivalent to proving that F_σ, the completion of F at an Archimedean place σ, is $\mathbf{R}$ for all σ. From the above discussions, G is defined by a hermitian form B over a number field $F \subset \mathbf{R}$ or a division algebra over $F \subset \mathbf{R}$. The conjugate of the defining hermitian form B^σ is defined over F_σ, the completion of F^σ at σ. Since G^σ is either the real algebraic group $PU(1,2)$ or a Lie group of compact type according to the arguments in Lemma 6, this is possible only if the quadratic form B^σ is defined over a real number field. This is our claim.

It follows from our claim that the defining number field F is totally real. We summarize our discussions into the following definition and Lemma.

Definition 3.

a. Let k be a totally real number field of degree a over the field of rational numbers $\mathbf{Q}$ and F be a quadratic imaginary extension of k. Let D be a central division algebra defined over F with an involution τ of second kind so that k is the set of elements of F fixed by τ. Let B be a non-degenerate $\tau-$hermitian form defined over a $n-$dimensional $D-$vector space. Let $G = PU(B,D,\tau)$. Assume that $nd = 3$. Assume furthermore that there are exactly p non-equivalent embeddings $\sigma_i, i = 1, \ldots, p$, of k into $\mathbf{C}$ such that $G^{\sigma_i} \otimes \mathbf{R}$ are isomorphic to $PU(1,2)$ and the rest to the semi-simple Lie group of compact type. Let Λ be a lattice of $PU(1,2)^p$ isomorphic to $\prod_{i=1}^p G^{\sigma_i}(\mathcal{O}_{F^{\sigma_i}})$. Λ is said to be an arithmetic lattice of First Type if $n = 3$ and $d = 1$. In such a case, Λ are defined by hermitian forms defined over a number field. Λ is said to be an arithmetic lattice of Second Type if $n = 1$ and $d = 3$. In such a case, Λ are defined by hermitian forms defined over a division algebra.

b. Suppose Γ is an integral lattice of G_F over an algebraic number field F. We say that Γ is of First Type (respectively Second Type) if and only $G(\mathcal{O}_F)$ is of First Type (respectively Second Type).

Our earlier discussions can be summarized by the following lemma.

Lemma 7. *Let Γ be a co-compact torsion-free lattice of $PU(2,1)$. Consider the realization of Γ as a subgroup of G_F for an algebraic group G defined over a real number field F with $G \otimes_F \mathbf{R} \cong PU(2,1)$ and assume that Γ is integral in G_F. Let $\mathcal{R} = \prod_{\sigma \in S^\infty} G^\sigma = \mathcal{R}_{nc} \times \mathcal{R}_c$, where $\mathcal{R}_{nc} \cong PU(2,1)^p$ is the non-compact and $\mathcal{R}_c \cong PU(3)^p$ the compact part of $\mathcal{R}$. Let $p_1 : \mathcal{R} \to \mathcal{R}_{nc}$ be the projection into the non-compact part. Then $p_1(\mathcal{R}(\mathcal{O}_{\mathbf{Q}}))$ is an arithmetic lattice of $\mathcal{R}_{nc} \cong PU(2,1)^p$ of First or Second Type, $G(\mathcal{O}_F) \cong \mathcal{R}(\mathcal{O}_{\mathbf{Q}})$ and $p_1(\Gamma)$ is a subgroup of $p_1(\mathcal{R}(\mathcal{O}_{\mathbf{Q}}))$.*

Here is the main result of this section.

Main Theorem C. *Let Γ be a cocompact lattice of $G = PU(2,1)$ so that the quotient $\Gamma \backslash PU(2,1)/P(U(2) \times U(1))$ has Picard number 1, $b_1(M) = 0$ and contains no immersed totally geodesic hyperbolic algebraic curves. Then one of the followings holds.*
(i). Γ is arithmetic of second type, that is, comes from a division algebra of E of rank 3 with an involution of the second type.
(ii). Γ is a non-arithmetic lattice of $PU(2,1)$ but is a subgroup of an arithmetic lattice of second type of some $PU(2,1)^p$ with $p > 1$.

Remarks.
1. It is proved by Rogawski [Ro] that cocompact lattices of $PU(2,1)$ of case (i) above coming from all congruence subgroups of a division algebra of $E|_Q$ of rank 3 with an involution of the second type really satisfies the property that the Picard number is 1 and there is no holomorphic $1-$forms.
2. It is not known to the author whether non-arithmetic lattice of type (ii) exists. Examples of non-arithmetic lattice of $PU(2,1)$ which is a subgroup of an arithmetic lattice of first type of some $PU(2,1)^p$ with $p > 1$ can be found in [DM].
3. If an unramified covering of M satisfies the condition that the canonical line bundle is three times the generator of the Neron-Severi group modulo torsion, then the arguments of Main Theorem B exclude case (ii) from the above conclusion.

We begin with the following simple observations for arithmetic lattice of first type in $PU(2,1)$. It is commensurable to a lattice $G(P,\mathcal{O})$, the first factor of $H_R^o \cong PU(2,1) \times PU(3)^{d-1}$, where $\mathcal{O}$ is the set of integers in F. We can diagonalize the form P by an element $g \in GL(3,k)$ such that the first factor in H_Z^o, $G(P,\mathcal{O})$ is commensurable with $G(P^g,\mathcal{O})$ for some diagonal P^g. Hence our lattice is commensurable with $G(P^g,\mathcal{O})$ with P^g of form

$$l_o|x_0|^2 - l_1|x_1|^2 - l_2|x_2|^2,$$

where $l_i \in k$ and $l_i > 0$. We may regard Γ as a co-compact lattice consisting of elements γ satisfying $\gamma P_o \gamma^t = P_o$, where P_o is the diagonal matrix with diagonal values given by $[l_o, l_1, l_2]$. The complex ball of dimension two $B^2_{\mathbf{C}}$ is realized as $\{[z_o, z_1, z_2] \in P^2_{\mathbf{C}} : l_o|z_0|^2 - l_1|z_1|^2 - l_2|z_2|^2 < 0\}$. Consider the geodesic subspace $\widetilde{N} \cong B^1_{\mathbf{C}}$ of $\widetilde{M} \cong B^2_{\mathbf{C}}$ defined by $\{[z_o, z_1, 0] \in P^2_{\mathbf{C}} : l_o|z_0|^2 - l_1|z_1|^2 < 0\}$. Let ι be the reflection on $\widetilde{M} = B^2_{\mathbf{C}}$ about the hyperplane $z_2 = 0$. Let $\Lambda = \Gamma \cap \widetilde{N}$.

Lemma 8. *Assume that $M = B^2_C/\Gamma$, where Γ is a torsion free lattice of $PU(2,1)$. Then $\widetilde{N}/\Lambda$ is an immersed totally geodesic complex one ball quotient in M. Furthermore, the curve is a smoothly embedded one provided that ι normalizes Γ.*

Proof. Since $\widetilde{N}$ is the fixed point set of the involution ι obtained by reflection about the hyperplane $z_2 = 0$, Λ is a uniform lattice $\widetilde{N}$ (cf. [Rag], page 24). Hence $R = \widetilde{N}/\Lambda$ is a totally geodesic complex curve on M. It remains to prove that R is smooth. Since M is smooth, the only singularity that can occur is the self intersection of the image of $\widetilde{N}$ by the covering map, which occurs when there exists $\gamma \in \Gamma - \Lambda$ and $x \in \widetilde{N}$ such that $\gamma x \in \widetilde{N}$. Assume for the purpose of proof by contradiction that such γ and x exist. Then $\gamma^{-1}\iota\gamma\iota x = x$ as $\widetilde{N}$ is fixed by ι. Since ι normalizes Γ, $\gamma^{-1}\iota\gamma\iota \in \Gamma$. As M is smooth, Γ is torsion free. It follows that $\gamma^{-1}\iota\gamma\iota = e$, the identity. Hence γ commutes with ι, which implies that $\gamma \in \Lambda$, contradictory to our assumption.

Proof of Main Theorem C. It follows from Main Theorem A that $\Gamma \subset G(\mathcal{O}_F)$, where $\mathcal{O}_F$ is the set of integers of a real algebraic number field F as discussed earlier. Consider the restriction of scalars $R_{F/\mathbf{Q}}(G) = \prod_{\sigma \in Gal(F/\mathbf{Q})} G^\sigma \cong PU(2,1)^p \times PU(3)^{d-p}$, where p is the number of embeddings of F into C for which the Hermitian forms P^σ associated to G^σ are indefinite.

Suppose $p = 1$ and lattice is of first type. It follows from definition that Γ is an arithmetic lattice. From Lemma 8, we know that the image of N is an immersed totally geodesic one ball quotient. This rules out arithmetic lattices of first type and leads to conclusion (i).

Consider now the case $p > 1$. Let

$$\alpha : g \longmapsto (\sigma_1(g), \cdots, \sigma_d(g))$$

be the diagonal embedding from $G(F)$ to $R_{F/\mathbf{Q}}(G)$. From the restriction of scalars, $(R_{F/\mathbf{Q}}(G))_{\mathbf{Q}} = \alpha(G_F)$ and $(R_{F/\mathbf{Q}}(G))_{\mathbf{Z}} = \alpha(G(\mathcal{O}_F))$. The projection $p : R_{F/\mathbf{Q}}(G) \to G_F$ is a bijection with inverse α. By a theorem of Borel-Harish-Chandra (cf. [Z] p. 36), $(R_{F/\mathbf{Q}}(G))_{\mathbf{Z}}$ is a lattice of $(R_{F/\mathbf{Q}}(G))_R$. Since $(R_{F/\mathbf{Q}}(G))_{\mathbf{R}}$ as a semi-simple Lie group has real rank $p > 1$, Margulis' Arithmeticity theorem implies that $(R_{F/\mathbf{Q}}(G))_{\mathbf{Z}}$ is an arithmetic lattice in $(R_{F/\mathbf{Q}}(G))_{\mathbf{R}}$ and hence is either of first or second type classified before. We claim that $(R_{F/\mathbf{Q}}(G))_{\mathbf{Z}}$ cannot be of first type. Suppose on the contrary that $(R_{F/\mathbf{Q}}(G))_{\mathbf{Z}}$ is of first type. It follows that $G(\mathcal{O}_F) = G(P, \mathcal{O})$, the group of units of a Hermitian form P over k, except that now P^σ is not definite as a quadratic form for some $\sigma \neq 1$. The arguments of Lemma 8 do not make any reference concerning the properties of P^σ and hence the same proof implies that there exists an immersed totally geodesic one ball quotient in M. As $G(\mathcal{O}_F)$ is bijective to $(R_{F/\mathbf{Q}}(G))_{\mathbf{Z}}$ by the natural homomorphisms given by α and p, we conclude that $G(\mathcal{O}_F)$ is a subgroup of the lattice $(R_{F/\mathbf{Q}}(G))_{\mathbf{Z}}$. Since $R_{F/\mathbf{Q}}(G)$ has more than one non-compact factors in $PU(2,1)$, it follows from Weak Approximation Theorem of algebraic groups over a number field that $G(\mathcal{O}_F)$, the projection of the lattice $(R_{F/\mathbf{Q}}(G))_{\mathbf{Z}}$ to the first factor of the product, is dense in the point-set topology (cf. [Kn]). Hence Γ cannot be finite indexed in $(R_{F/\mathbf{Q}}(G))_{\mathbf{Z}} = G(\mathcal{O}_F)$, for otherwise the latter group would be a discrete subgroup in G. This concludes the proof of Main Theorem C.

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