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THE EINSTEIN-KÄHLER METRIC WITH EXPLICIT FORMULAS ON SOME NON-HOMOGENEOUS DOMAINS *

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Dedicated to Yum-Tong Siu on the occasion of his 60th birthday.

Abstract. In this paper we describe the Einstein-Kähler metric for the Cartan-Hartogs domains which are the special case of the Hua domains. First of all, we reduce the Monge-Ampère equation for the metric to an ordinary differential equation in the auxiliary function $X = X(z, w)$. This differential equation can be solved to give an implicit function in X. Secondly, for some cases, we obtained explicit forms of the complete Einstein-Kähler metrics on Cartan-Hartogs domains which are the non-homogeneous domains.

Let M be a complex manifold. Then a Hermitian metric $\sum_{i,j} g_{i,\overline{j}} dz^i \bigotimes d\overline{z}^j$ defined on M is said to be Kähler if the Kähler form $\Omega = \sqrt{-1}\sum_{i,j} g_{i,\overline{j}} dz^i \bigwedge d\overline{z}^j$ is closed. The Ricci form is given by $-\partial\overline{\partial}\log\det(g_{i,\overline{j}})$. If the Ricci form of the Kähler metric is proportional to the Kähler form, the metric is called Einstein-Kähler. If the manifold is not compact, we require the metric to be complete. Clearly for a noncompact complex manifold to admit such a metric, it is necessary that there exists a volume form, the negative of whose Ricci tensor defines a complete Kähler metric. The volume form of this Kähler metric must be equivalent to the original volume form. If we normalize the metric by requiring the scalar curvature to be minus one, then the Einstein-Kähler metric is unique. Cheng and Yau[CY] proved that any bounded domain D which is the intersection of domain with C^2 boundary admits a complete Einstein-Kähler. Without any regularity assumption on the domain D, Mok and Yau[MY] proved that the complete Einstein-Kähler metric always exists. This Einstein-Kähler metric is given by

$$E_D(z) := \sum \frac{\partial^2 g}{\partial z_i \partial \overline{z}_j} dz_i \overline{dz_j},$$

where g is an unique solution to the boundary problem of the Monge-Ampère equation:

$$\begin{cases} \det\left(\dfrac{\partial^2 g}{\partial z_i \partial \overline{z}_j}\right) = e^{(n+1)g} & z \in D, \\ g = \infty & z \in \partial D, \end{cases}$$

We call g the generating function of $E_D(z)$. It is obvious that if one determines g explicitly, then the Einstein-Kähler metric is also determined explicitly. Therefore if one would like to compute the Einstein-Kähler metric explicitly, it suffices to compute the generating function g in explicit formula.

The explicit formulas for the Einstein-Kähler metric, however, are only known on homogeneous domains. In his famous paper [Wu], H.Wu points out that among the four classical invariant metrics(i.e. the Bergman metric, Carathéodory metric,

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Kobayashi metric and Einstein-Kähler metric), the Einstein-Kähler metric is the hardest to compute because its existence is proved by complicated nonconstructive methods. The purpose of this paper is to compute the explicit formulas of complete Einstein-Kähler metrics on Cartan-Hartogs domains of the following four types:

$$
\begin{aligned}
Y_I(1,m,n;K) &:= \{w\in \mathbf{C}, Z\in R_I(m,n): |w|^{2K} < \det(I-Z\overline{Z}^T), K>0\} := Y_I,\\
Y_{II}(1,p;K) &:= \{w\in \mathbf{C}, Z\in R_{II}(p): |w|^{2K} < \det(I-Z\overline{Z}^T), K>0\} := Y_{II},\\
Y_{III}(1,q;K) &:= \{w\in \mathbf{C}, Z\in R_{III}(q): |w|^{2K} < \det(I-Z\overline{Z}^T), K>0\} := Y_{III},\\
Y_{IV}(1,n;K) &:= \{w\in \mathbf{C}, Z\in R_{IV}(n): |w|^{2K} < 1-2Z\overline{Z}^T+|ZZ^T|^2, K>0,\}\\
&:= Y_{IV}.
\end{aligned}
$$

Where $R_I(m,n), R_{II}(p), R_{III}(q)$ and $R_{IV}(n)$ are the first, second, third and fourth Cartan domains respectively in the sense of Loo-Keng HUA[Hu], $\overline{Z}^T$ indicates the conjugate and transpose of Z, det indicates the determinant. The Cartan-Hartogs domains are introduced in 1998. The Bergman kernel functions on Cartan-Hartogs domains are obtained in explicit formulas in [Yin1,Yin2,Yin3,GY]. And these Bergman kernel functions are Bergman exhaustions, therefore all of the Cartan-Hartogs domains are bounded pseudoconvex domains. Some results on Hua domains can be found in [Yin1-Yin3,GY,YW,YWZ].

This paper is organized as follows. Section 1 presents some background material and known results from domain Y_I needed for us. In section 2, by using the noncompact automorphism group of Y_I and the biholomorphic invariance of the Einstein-Kähler metric, we reduce the Monge-Ampère equation for the metric to an ordinary differential equation in the auxiliary function $X = X(z,w) = |w|^2[\det(I-Z\overline{Z}^T)]^{-\frac{1}{K}}$, this differential equation can be solved to give an implicit function in X. In section 3, we give explicit forms of complete Einstein-Kähler metric on Y_I. And the explicit formulas of complete Einstein-Kähler metrics on Y_{II}, Y_{III} and Y_{IV} are given in section 4. At that time the Y_I, Y_{II}, Y_{III}, and Y_{IV} are the non-homogeneous domains.

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1. Preliminaries. $\Omega\times\mathbf{C}$ can first of all be interpreted as the total space of the anticanonical line bundle over Ω. The latter is endowed a Hermitian metric unique up to a global multiplicative constant. $\mathrm{Aut}(\Omega)$ acts canonically on $\Omega\times\mathbf{C}$ preserving the unit disk bundle $D\subset\Omega\times\mathbf{C}$, i.e., the set of all vectors of length <1. The action of $\mathrm{Aut}(\Omega)$ on Ω extends naturally to an action on $\Omega\times\mathbf{C}$ and hence on D. Write $D=\{(z,w): |w|<\varphi_o(z)\}$. The Cartan-Hartogs domains are of the form $D_\alpha=\{(z,w): |w|<\varphi_o^\alpha(z)\}$ for an arbitrary positive real constant α. We may now interpret D_α as the unit ball bundle of a positive power of the anticanonical line bundle represented by $(\Omega\times\mathbf{C}, h^\alpha)$, where the exponent is real. For an arbitrary $\alpha>0$ we do not have a canonical extension of the action of of $\mathrm{Aut}(\Omega)$ on Ω to an action on D_α. Instead, there is a semi-direct product $H=\mathrm{Aut}(\Omega)\triangleright S^1$ (an extension of $\mathrm{Aut}(\Omega)$ by S^1), acting on $\Omega\times\mathbf{C}$, such that S^1 is a normal subgroup acting on $\Omega\times\mathbf{C}$ by scalar mutliplication $(z,w)\to(z,e^{i\theta}w), \theta\in\mathbf{R}$, in the second factor, and such that H acts as holomorphic bundle isomorphisms of $(\Omega\times\mathbf{C},h^\alpha)$ and hence as automorphisms of D_α. When α is rational, $\alpha=p/q$, p, q coprime, H contains a proper subgroup $\mathrm{Aut}(\Omega)\triangleright\mu_q$, where $\mu_q\subset S^1$ denotes the group of q-th roots of unity, $\mu_q\cong\mathbf{Z}/q\mathbf{Z}$. D_α with α rational are thus particularly interesting geometrically. In fact, if $\Gamma\subset\mathrm{Aut}(\Omega)$ is a torsion-free cocompact discrete subgroup, then we have an action of $\Gamma\triangleright\mu_q$ on D_α, and the normal quotient space D_α/Γ carries the structure of a strongly pseudoconvex manifold whose exceptional set is the compact Hermitian locally symmetric manifold $\Omega/\Gamma = N$.

If it is furthermore possible to define a q-th root of the canonical line bundle on N, then $\Gamma \triangleright \mu_q$ is a direct product, so that Γ acts on D_α without any fixed point, and the Einstein-Kähler metric on D_α descends to an Einstein-Kähler metric on the strongly pseudoconvex manifold D_α/Γ. The exceptional set $N \subset D_\alpha/\Gamma$ is then a totally geodesic complex submanifold with respect to the Einstein-Kähler metric.

Lemma 1. The automorphism group $\mathrm{Aut}(Y_I)$ of Y_I consists of the following mappings $F(z, w; z_0, \theta_0)$:

$$\begin{cases} w^* &= e^{i\theta_0} w \det(I - Z_0\overline{Z}_0^T)^{\frac{1}{2K}} \det(I - Z\overline{Z}_0^T)^{-\frac{1}{K}}, \\ Z^* &= A(Z - Z_0)(I - \overline{Z}_0^T Z)^{-1} D^{-1}, \end{cases}$$

where $\overline{A}^T A = (I - Z_0\overline{Z}_0^T)^{-1}, \overline{D}^T D = (I - \overline{Z}_0^T Z_0)^{-1}, \quad Z_0, Z \in R_I(m, n), \theta_0 \in \mathbf{R}$.

Proof. See [Yi].

Obviously, the $F(z, w; z_0, \theta_0)$ maps points $(Z_0, w) \in Y_I$ on to points $(0, w^*)$ and $Z^* = A(Z - Z_0)(I - \overline{Z}_0^T Z)^{-1} D^{-1}$ is holomorphic automorphism of $R_I(m, n)$.

Lemma 2. Let $X = X(Z, w) = |w|^2[\det(I - Z\overline{Z}^T)]^{-\frac{1}{K}}$. Then X is invariant under the mapping of $\mathrm{Aut}(Y_I)$. That is $X(Z^*, w^*) = X(Z, w)$.

Proof. See [Yi].

Lemma 3. If $F = F(z, w; z_0, \theta_0) \in \mathrm{Aut}(Y_I)$. Let J_F be the Jacobi matrix of $F(z, w; z_0, \theta_0)$, i.e.

$$J_F = \begin{pmatrix} \dfrac{\partial z^*}{\partial z} & \dfrac{\partial w^*}{\partial z} \\ 0 & \dfrac{\partial w^*}{\partial w} \end{pmatrix},$$

where $z = (z_{11}, \cdots, z_{1n}, z_{21}, \cdots, z_{2n}, \cdots, z_{m1}, \cdots, z_{mn})$ is a vector, z_{jk} is the element of Z and $Z = (z_{jk})_{m\times n} \in R_I(m, n)$. Then one has

$$\frac{\partial z^*}{\partial z}\Big|_{z_0=z} = (A^T \cdot\times \overline{D}^T)\Big|_{z_0=z},$$

$$\frac{\partial w^*}{\partial z}\Big|_{z_0=z} = \frac{1}{K} e^{i\theta_0} \det(I - Z\overline{Z}^T)^{-\frac{1}{2K}} E(Z)^T w,$$

$$\frac{\partial w^*}{\partial w}\Big|_{z_0=z} = e^{i\theta_0} \det(I - Z\overline{Z}^T)^{-\frac{1}{2K}},$$

where $E(Z) = (\mathrm{tr}[(I - Z\overline{Z}^T)^{-1} I_{11}\overline{Z}^T], \mathrm{tr}[(I - \overline{Z}^T)^{-1} I_{12}\overline{Z}^T], \cdots, \mathrm{tr}[(I - Z\overline{Z}^T)^{-1} I_{mn}\overline{Z}^T])$ is a column vector with mn entries. I_{pq} is defined as a $m \times n$ matrix, the (p,q)-th entry of I_{pq}, i.e. the entry located at the junction of the p-th row and q-th column of I_{pq}, is 1, and others entries of I_{pq} are zero. The meaning of $\times$ is following(see [Lu]):

Let $A^* = (a_{ij})_{p\times q}$, $B^* = (b_{kl})_{r\times s}$, then $A^* \cdot\times B^* = (c_{\alpha\beta})_{pr\times qs}$ is defined as $c_{\alpha\beta} = a_{ik}b_{jl}$, where $\alpha = q(i-1)+j, \beta = s(k-1)+l$.

Proof. It can be obtained by direct computation.

LEMMA 4. If $F = F(z,w;z_0,\theta_0) \in \mathrm{Aut}(Y_I)$ and $T = T[(z,w),\overline{(z,w)}]$ is the metric matrix of the Einstein-Kähler metric of Y_I, one has

$$T[(z,w),\overline{(z,w)}] = J_F|_{z_0=z}T[(0,w^*),\overline{(0,w^*)}]\overline{J_F}^T|_{z_0=z},$$

and $|J_F|^2_{z_0=z} = \det(I - Z\overline{Z}^T)^{-(m+n+\frac{1}{K})}$, where $|J_F| = \det J_F$.

Proof. It can be proved by using the invariance of the Einstein-Kähler metric under the holomorphic automorphism of Y_I.

2. Reduction of the Monge-Ampère equation to an ordinary differential equation. Let $Z = (z_{jk})_{m\times n} \in R_I(m,n)$. We denote

$$(z,w) = (z_{11},\cdots,z_{1n},z_{21},\cdots,z_{2n},\cdots,z_{m1},\cdots,z_{mn},w) = (z_1,z_2,\cdots,z_N),$$

where $N = mn+1$, and

$$g_{\alpha\overline{\beta}}(z,w) = \frac{\partial^2 g}{\partial z_\alpha \partial \overline{z}_\beta} \qquad \alpha,\beta = 1,2,\cdots,N.$$

Note that $z_N = w$.

Suppose $g(z,w)$ generates the Einstein-Kähler metric of Y_I. Then $g(z,w)$ is a solution to the boundary problem of the Monge-Ampère equation:

$$\begin{cases} \det(g_{\alpha\overline{\beta}}(z,w)) = e^{(N+1)g(z,w)} & (z,w)\in Y_I, \\ g = \infty & (z,w)\in \partial Y_I. \end{cases} \tag{1}$$

Let $F : (z,w) \longrightarrow (z^*,w^*)$, $F = F(z,w;z_0,\theta_0) \in \mathrm{Aut}(Y_I)$. Because of the invariance of the metric, it is easy to show that $\det(g_{\alpha\overline{\beta}}(z,w)) = |J_F|^2\det(g_{\alpha\overline{\beta}}(z^*,w^*))$. So

$$e^{(N+1)g(z,w)} = |J_F|^2 e^{(N+1)g(z^*,w^*)}.$$

Thus

$$e^{-g(z^*,w^*)} = |J_F|^{\frac{2}{N+1}} e^{-g(z,w)}.$$

For arbitrary $(z,\ w) \in Y_I$, especially take $z_0 = z,\quad \theta_0 = -\arg w,\quad$ that is $F_0 = F(z,w;z,-\arg w)$. We have

$$e^{-g(0,w^*)} = |J_{F_0}|^{\frac{2}{N+1}} e^{-g(z,w)} = \det(I - Z\overline{Z}^T)^{-\frac{(m+n+1/K)}{N+1}} e^{-g(z,w)},$$

where $w^* = |w^*| = X^{\frac{1}{2}}$. If $\lambda = K(m+n)+1$, then $|J_{F_0}|^2 = X^\lambda |w|^{-2\lambda}$.
Let $h(X) = e^{-g(0,X^{\frac{1}{2}})} = e^{-g(0,w^*)}$. We obtain

$$h(X) = |J_{F_0}|^{\frac{2}{N+1}} e^{-g(z,w)} = X^{\frac{\lambda}{N+1}} |w|^{-2\frac{\lambda}{N+1}} e^{-g(z,w)}.$$

Hence

$$\frac{\partial h}{\partial w} = h'(X)\frac{\partial X}{\partial w} = |J_F|^{\frac{2}{N+1}} e^{-g(z,w)} \frac{\partial(-g)}{\partial w}.$$

It is obvious that

$$\begin{aligned} \frac{\partial X}{\partial w} &= \frac{X}{w}, \\ \frac{\partial X}{\partial \overline{w}} &= \frac{X}{\overline{w}}. \end{aligned} \tag{2}$$

Where

$$\frac{X}{w} = \begin{cases} \overline{w}\det(I - Z\overline{Z}^T)^{-1/K} & w \neq 0, \\ 0 & w = 0. \end{cases}$$

Then

$$\frac{\partial g}{\partial w} = -\frac{X}{w} \cdot \frac{h'(X)}{h(X)}.$$

Since $h(X) = X^{\frac{\lambda}{N+1}} |w|^{-2\frac{\lambda}{N+1}} e^{-g(z,w)}$, we have

$$\frac{\partial h}{\partial z_\alpha} = h'(X)\frac{\partial X}{\partial z_\alpha} = -h(X)\frac{\partial g}{\partial z_\alpha} + \frac{\lambda}{N+1}X^{-1}\frac{\partial X}{\partial z_\alpha}h(X),$$

where $\alpha = 1, 2, \cdots, N-1$.
Then

$$\frac{\partial g}{\partial z_\alpha} = \Big(\frac{\lambda}{N+1}X^{-1} - \frac{h'(X)}{h(X)}\Big)\frac{\partial X}{\partial z_\alpha}.$$

Let

$$Y(X) = \frac{\lambda}{N+1} - X\frac{h'(X)}{h(X)}, \tag{3}$$

then

$$\begin{aligned} \frac{\partial g}{\partial z_\alpha} &= YX^{-1}\frac{\partial X}{\partial z_\alpha}, \qquad \alpha = 1, 2, \cdots, N-1. \\ \frac{\partial g}{\partial w} &= \left(Y - \frac{\lambda}{N+1}\right)\frac{1}{w}. \end{aligned}$$

So, we have

$$\begin{aligned} \frac{\partial^2 g}{\partial w \partial \overline{w}} &= Y'\frac{X}{|w|^2}, \\ \frac{\partial^2 g}{\partial w \partial \overline{z}_\beta} &= \frac{1}{w}Y'\frac{\partial X}{\partial \overline{z}_\beta}, \\ \frac{\partial^2 g}{\partial z_\alpha \partial \overline{w}} &= Y'X^{-1}\frac{\partial X}{\partial \overline{w}} \cdot \frac{\partial X}{\partial z_\alpha} + YX^{-1}\frac{\partial^2 X}{\partial z_\alpha \partial \overline{w}} - YX^{-2}\frac{\partial X}{\partial \overline{w}} \cdot \frac{\partial X}{\partial z_\alpha}, \\ \frac{\partial^2 g}{\partial z_\alpha \partial \overline{z}_\beta} &= Y'X^{-1}\frac{\partial X}{\partial \overline{z}_\beta} \cdot \frac{\partial X}{\partial z_\alpha} + YX^{-1}\frac{\partial^2 X}{\partial z_\alpha \partial \overline{z}_\beta} - YX^{-2}\frac{\partial X}{\partial \overline{z}_\beta} \cdot \frac{\partial X}{\partial z_\alpha}, \end{aligned} \tag{4}$$

where $\alpha, \beta = 1, 2, \cdots, N-1$.
Because

$$\begin{aligned} \frac{\partial X}{\partial z_{pq}} &= \frac{1}{K} X \mathrm{tr}[(I - Z\overline{Z}^T)^{-1} I_{pq} \overline{Z}^T], \\ \frac{\partial X}{\partial \overline{z}_{st}} &= \frac{1}{K} X \mathrm{tr}[(I - Z\overline{Z}^T)^{-1} Z I_{st}^T]. \end{aligned} \tag{5}$$

From (2) and (5), we have

$$\begin{aligned} &\frac{\partial X}{\partial w}\Big|_{z=0} = \overline{w}, && \frac{\partial X}{\partial \overline{w}}\Big|_{z=0} = w, \\ &\frac{\partial X}{\partial z_\alpha}\Big|_{z=0} = \frac{\partial X}{\partial z_{pq}}\Big|_{z=0} = 0, && \frac{\partial X}{\partial \overline{z}_\beta}\Big|_{z=0} = \frac{\partial X}{\partial \overline{z}_{st}}\Big|_{z=0} = 0, \end{aligned} \tag{6}$$

where $\alpha = n(p-1) + q, \beta = n(s-1) + t$.
And from (5), we have

$$\begin{aligned} \frac{\partial^2 X}{\partial z_{pq} \partial \overline{z}_{st}} = & \frac{1}{K} \frac{\partial X}{\partial \overline{z}_{st}} \mathrm{tr}[(I - Z\overline{Z}^T)^{-1} I_{pq} \overline{Z}^T] + \frac{X}{K} \mathrm{tr}[(I - Z\overline{Z}^T)^{-1} I_{pq} I_{st}^T] \\ & + \frac{X}{K} \mathrm{tr}[(I - Z\overline{Z}^T)^{-1} Z I_{st}^T (I - Z\overline{Z}^T)^{-1} I_{pq} \overline{Z}^T]. \end{aligned} \tag{7}$$

From (2), (5), (6), (7), by computation, we obtain

$$\begin{aligned} &\frac{\partial^2 X}{\partial z_{pq} \partial \overline{z}_{st}}\Big|_{z=0} = \frac{X}{K} \delta_{ps} \delta_{qt}, \\ &\frac{\partial^2 X}{\partial z_{pq} \partial \overline{w}}\Big|_{z=0} = 0, \\ &\frac{\partial^2 X}{\partial w \partial \overline{z}_{st}}\Big|_{z=0} = 0, \end{aligned} \tag{8}$$

where

$$\delta_{ps} = \begin{cases} 1 & p = s, \\ 0 & p \neq s. \end{cases} \qquad \delta_{qt} = \begin{cases} 1 & q = t, \\ 0 & q \neq t. \end{cases}$$

From (6), (8) and (4), we obtain

$$\begin{aligned} &\frac{\partial^2 g}{\partial w \partial \overline{w}}\Big|_{z=0} = Y', \\ &\frac{\partial^2 g}{\partial w \partial \overline{z}_\beta}\Big|_{z=0} = 0, \\ &\frac{\partial^2 g}{\partial z_\alpha \partial \overline{w}}\Big|_{z=0} = 0, \\ &\frac{\partial^2 g}{\partial z_\alpha \partial \overline{z}_\beta}\Big|_{z=0} = Y K^{-1} \delta_{\alpha\beta}, \end{aligned}$$

where $\alpha, \beta = 1, 2, \cdots, N-1$.
Therefore

$$\det(g_{\alpha\overline{\beta}}(0, w^*)) = \det \begin{pmatrix} \frac{Y}{K} & & & 0 \\ & \ddots & & \\ & & \frac{Y}{K} & \\ 0 & & & Y' \end{pmatrix} = \left(\frac{Y}{K}\right)^{N-1} Y'.$$

Since

$$\det(g_{\alpha\overline{\beta}}(z, w)) = \det(g_{\alpha\overline{\beta}}(0, w^*))|J_{F_0}|^2 = \left(\frac{Y}{K}\right)^{N-1} Y'|J_{F_0}|^2,$$

and

$$e^{(N+1)g(z,w)} = |J_{F_0}|^2 e^{(N+1)g(0,w^*)} = |J_{F_0}|^2 h^{-(N+1)},$$

we reduce the Monge-Ampère equation to an ordinary differential equation:

$$\left(\frac{Y}{K}\right)^{N-1} Y' = h^{-(N+1)}.$$

It is equivalent to

$$\log(Y^{N-1}Y') + (N+1)\log h - \log K^{N-1} = 0.$$

Differentiating with respect to X

$$\frac{(Y^{N-1}Y')'}{(Y^{N-1}Y')} + (N+1)\frac{h'}{h} = 0.$$

By computation and (3), we obtain

$$\frac{(Y^{N-1}Y')'}{(Y^{N-1}Y')} + \frac{\lambda}{X} - (N+1)\frac{Y}{X} = 0.$$

So we have

$$[X(Y^{N-1}Y')]' = (Y^{N+1})' - \frac{\lambda - 1}{N}(Y^N)'.$$

It is equivalent to

$$XY^{N-1}Y' = Y^{N+1} - \frac{\lambda - 1}{N} Y^N + C, \tag{9}$$

where C is a constant. Because

$$\frac{\partial g}{\partial w} = (Y - \frac{\lambda}{N+1})\frac{1}{w}.$$

So

$$Y = w\frac{\partial g}{\partial w} + \frac{\lambda}{N+1},$$

it holds for $\forall(z,w)\in Y_I$. If take $(z,0)\in Y_I$, then $X=0$, thus $w\frac{\partial g}{\partial w}|_{w=0}=0$, therefore $Y(0)=\frac{\lambda}{N+1}$. We obtain $C=\frac{(\lambda-N-1)\lambda^N}{N(N+1)^{N+1}}$ from (9).

Suppose g is a solution of Monge-Ampère equation. Then from the above we have

$$g=\frac{1}{N+1}\log[(\frac{Y}{K})^{N-1}Y'\det(I-Z\overline{Z}^T)^{-(m+n+\frac{1}{K})}],$$

where Y are the solutions to the following problem:

$$\begin{cases} XY^{N-1}Y'=Y^{N+1}-\dfrac{\lambda-1}{N}Y^N+\dfrac{(\lambda-N-1)\lambda^N}{N(N+1)^{N+1}}, \\ Y(0)=\dfrac{\lambda}{N+1}. \end{cases} \tag{10}$$

The solutions of above problem are not unique. If g is the generating function of the complete Einstein-Kähler metric on Y_I, then g is the unique solution of the problem (1) and has the form as follows:

$$g=\frac{1}{N+1}\log[(\frac{Y}{K})^{N-1}Y'\det(I-Z\overline{Z}^T)^{-(m+n+\frac{1}{K})}],$$

where Y is a solution of problem (10). The unique solution g of (1), which includes the condition that $g(x)\to\infty$ as $x\to\partial Y_I$, determines a particular solution to (10).

For the problem (10), we obtain

$$C^*X=(Y-Y(0))\left(Y^N-\frac{\lambda-N-1}{N(N+1)}\sum_{k=1}^{N}Y(0)^{k-1}Y^{N-k}\right)e^{\varphi(Y)},$$

where

$$\varphi(Y)=-\int_{Y(0)}^{Y}\frac{(N+1)Y^{N-1}}{Y^N-\frac{\lambda-N-1}{N(N+1)}\sum_{k=1}^{N}Y(0)^{k-1}Y^{N-k}}dY.$$

C^* is a positive constant, and Y is the function in X. Therefore the problem (10) can be solved to give an implicit function in X. There is a unique choise of the positive constant corresponding to the unique solution g of problem (1).

3. Complete Einstein-Kähler metric with explicit formula on $\mathbf{Y_I}$. Let $K=\frac{mn+1}{m+n}$, $m\neq 1$ in Y_I. We have $\lambda=N+1, C=0$. The equation (10) becomes

$$\begin{cases} XY'=Y^2-Y, \\ Y(0)=1. \end{cases} \tag{11}$$

We obtained solution $Y=\frac{1}{1-C^*X}$, where C^* is a constant.

Let $C^*=1$. Then $Y=\frac{1}{1-X}$. Therefore

$$\begin{aligned} g &= \tfrac{1}{N+1}\log[(\tfrac{Y}{K})^{N-1}Y'\det(I-Z\overline{Z}^T)^{-(m+n+\frac{1}{K})}] \\ &= \log[\tfrac{1}{1-X}\det(I-Z\overline{Z}^T)^{-\frac{1}{K}}K^{\frac{1-N}{1+N}}]. \end{aligned}$$

Now, we will prove g is the solution of problem (1).

Obviously, $\det(g_{\alpha\overline{\beta}}(z,w))=e^{(N+1)g(z,w)}$ for any $(z,w)\in Y_I$.

If point $(\widetilde{z}, \widetilde{w}) \in \partial Y_I$ and $\widetilde{w} \neq 0$, when $(z,w) \in Y_I, (z,w) \longrightarrow (\widetilde{z}, \widetilde{w})$, we have $X \longrightarrow 1^-$. So $\frac{1}{1-X} \longrightarrow +\infty$ and $\det(I - Z\overline{Z}^T) \longrightarrow |\widetilde{w}|^{2K} > 0$. Therefore, $g(z,w) \longrightarrow +\infty$ as $(z,w) \longrightarrow \partial Y_I$.

If point $(\widetilde{z}, \widetilde{w}) \in \partial Y_I$, and $\widetilde{w} = 0$, when $(z,w) \in Y_I, (z,w) \longrightarrow (\widetilde{z}, 0)$, we have $\frac{1}{1-X} > 1$, and $\det(I - Z\overline{Z}^T) \longrightarrow 0, \det(I - Z\overline{Z}^T)^{-\frac{1}{K}} \longrightarrow +\infty$. That is $g(z,w) \to +\infty$ as $(z,w) \to \partial Y_I$.

Therefore g generates the complete Einstein-Kähler metric of Y_I in the case of $K = \frac{mn+1}{m+n}$, $m \neq 1$, and is given by the following explicit formula:

$$g = \log[\frac{1}{1-X} \det(I - Z\overline{Z}^T)^{-\frac{1}{K}} K^{\frac{1-N}{1+N}}].$$

where$X = X(Z,w) = |w|^2[\det(I - Z\overline{Z}^T)]^{-\frac{1}{K}}, N = mn+1$ and $(w,Z) \in Y_I$.

The Y_I in the case of $K = \frac{mn+1}{m+n}$, $m \neq 1$ is nonhomogeneous domain.

4. Complete Einstein-Kähler metric with explicit formulas on $\mathbf{Y_{II}}, \mathbf{Y_{III}}, \mathbf{Y_{IV}}$. Follow the above idea and similar procedure, we get the explicit formulas of complete Einstein-Kähler metrics on Y_{II}, Y_{III} and Y_{IV} as follows.

1. For Y_{II}.

Suppose g is a solution of Monge-Ampère equation on Y_{II}. Then we have

$$g = \frac{1}{N+1} \log[(\frac{Y}{K})^{N-1} Y' \det(I - Z\overline{Z}^T)^{-(p+1+\frac{1}{K})}],$$

where Y are the solutions to the following problem:

$$\begin{cases} XY^{N-1}Y' = Y^{N+1} - \dfrac{\lambda-1}{N} Y^N + \dfrac{(\lambda - N - 1)\lambda^N}{N(N+1)^{N+1}}, \\ Y(0) = \dfrac{\lambda}{N+1}. \end{cases} \tag{12}$$

Where $N = \frac{1}{2}p(p+1)+1, \lambda = K(p+1)+1$.

In the case of $K = 1/(p+1) + p/2, p > 1$, if g generates the complete Einstein-Kähler metric of Y_{II} then g is given by the following explicit formula:

$$g = \log[\frac{1}{1-X} \det(I - Z\overline{Z}^T)^{-\frac{1}{K}} K^{\frac{1-N}{1+N}}].$$

where$X = X(Z,w) = |w|^2[\det(I - Z\overline{Z}^T)]^{-\frac{1}{K}}$ and $(w,Z) \in Y_{II}$.

2. For Y_{III}.

Suppose g is a solution of Monge-Ampère equation on Y_{III}. Then we have

$$g = \frac{1}{N+1} \log[(\frac{Y}{K})^{N-1} Y' \det(I - Z\overline{Z}^T)^{-(q-1+\frac{1}{K})}],$$

where Y are the solutions to the following problem:

$$\begin{cases} XY^{N-1}Y' = Y^{N+1} - \dfrac{\lambda-1}{N} Y^N + \dfrac{(\lambda - N - 1)\lambda^N}{N(N+1)^{N+1}}, \\ Y(0) = \dfrac{\lambda}{N+1}. \end{cases} \tag{13}$$

Where $N = \frac{1}{2}q(q-1)+1, \lambda = K(q-1)+1$.

In the case of $K = 1/(q-1)+q/2, q > 1$, If g generates the complete Einstein-Kähler metric of Y_{III} then g is given by the following explicit formula:

$$g = \log[\frac{1}{1-X}\det(I - Z\overline{Z}^T)^{-\frac{1}{K}}(\frac{K}{2})^{\frac{1-N}{1+N}}].$$

where$X = X(Z,w) = |w|^2[\det(I - Z\overline{Z}^T)]^{-\frac{1}{K}}$ and $(w,Z) \in Y_{III}$.

3. For Y_{IV}.

Suppose g is a solution of Monge-Ampère equation on Y_{IV}. Then we have

$$g = \frac{1}{N+1}\log[(\frac{2Y}{K})^{N-1}Y'\beta(Z,Z)^{-(n+\frac{1}{K})}],$$

where Y are the solutions to the following problem:

$$\begin{cases} XY^{N-1}Y' = Y^{N+1} - \dfrac{\lambda-1}{N}Y^N + \dfrac{(\lambda-N-1)\lambda^N}{N(N+1)^{N+1}}, \\ Y(0) = \dfrac{\lambda}{N+1}. \end{cases} \tag{14}$$

Where $N = n+1, \lambda = Kn+1$.

In the case of $K = 1/n+1$, if g generates the complete Einstein-Kähler metric of Y_{IV} then g is given by the following explicit formula:

$$g = \log[\frac{1}{1-X}\beta(Z,Z)^{-\frac{1}{K}}(K/2)^{\frac{1-N}{1+N}}].$$

where$X = X(Z,w) = |w|^2[\beta(Z,Z)]^{-\frac{1}{K}}, (w,Z) \in Y_{IV}$, and $\beta(Z,Z) = 1 + ZZ^T\overline{ZZ^T} - 2Z\overline{Z}^T$.

In the above cases, Y_{II}, Y_{III} and Y_{IV} are non-homogeneous domains.

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