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GLOBAL SOLUTIONS OF EINSTEIN–DIRAC EQUATION *

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Abstract. The conformal space $\mathfrak{M}$ was introduced by Dirac in 1936. It is an algebraic manifold with a spin structure and possesses naturally an invariant Lorentz metric. By carefully studying the birational transformations of $\mathfrak{M}$, we obtain explicitly the transition functions of the spin bundle over $\mathfrak{M}$. Since the transition functions are closely related to the propagation in physics, we get a kind of solutions of the Dirac equation by integrals constructed from the propagation. Moreover, we prove that the invariant Lorentz metric together with one of such solutions satisfies the Einstein-Dirac combine equation.

1. The main results. In general relativity the 4-dimensional Lorentz manifold is used. It is Penrose [1] who began to apply 2-component spinor analysis for studying Einstein equation. It implied that the spin group $Spin(1,3)$ of a Lorentz spin manifold $\mathfrak{M}$ is locally isomorphic to the group $SL(2,\mathbb{C})$ such that there is a Lie group homeomorphism

$$\iota : SL(2,\mathbb{C}) \longrightarrow SO(1,3)$$

which is a two to one covering map. Then a two component Dirac operator $\mathfrak{D} : V_2(x) \to V_2^*(x)$ and $\mathfrak{D} : V_2^*(x) \to V_2(x)$ can be defined, where $V_2(x)$ is the vector space of spinors at $x \in \mathfrak{M}$ and $V_2^*(x)$ is the conjugate vector space of $V_2(x)$.

We will use the following lemma for studying the Dirac equation.

LEMMA 1. If ψ is a two component spinor field on $\mathfrak{M}$ and satisfies

$$\mathfrak{D}^2\psi = \mathfrak{D}\mathfrak{D}\psi = -m^2\psi \tag{1.1}$$

then

$$\Psi = \begin{pmatrix} \psi \\ \varphi^* \end{pmatrix}, \quad \varphi^* = \frac{i}{m}\mathfrak{D}\psi \tag{1.2}$$

is a 4-component spinor on $\mathfrak{M}$ and satisfies the Dirac equation

$$\not{\mathfrak{D}}\Psi = \begin{pmatrix} 0 & \mathfrak{D} \\ \mathfrak{D} & 0 \end{pmatrix}\Psi = -im\Psi. \tag{1.3}$$

The first purpose of this paper is to solve the equation (1.1) in the case that $\mathfrak{M}$ is the conformal space.

The conformal space $\mathfrak{M}$ was introduced by Dirac [2]. It is a quadratic algebraic 4-dimensional manifold defined by

$$\mathfrak{x}_1^2 + \mathfrak{x}_2^2 - \mathfrak{x}_3^2 - \mathfrak{x}_4^2 - \mathfrak{x}_5^2 - \mathfrak{x}_6^2 = 0,$$

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where $\mathfrak{x} = (\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_6)$ is the homogeneous coordinate of the real project space $\mathbb{RP}^5$, and it is the boundary of the 5-dimensional anti-de-Sitter space AdS_5:

$$\mathfrak{x}_1^2 + \mathfrak{x}_2^2 - \mathfrak{x}_3^2 - \mathfrak{x}_4^2 - \mathfrak{x}_5^2 - \mathfrak{x}_6^2 > 0.$$

So to study the field theory of the conformal space would be useful to study the problem of AdS/CFT corresponding, a research hot point in recent years (see the references in [3]). It should be noted that AdS is also introduced by Dirac [4] and is one kind of space-time studied in [5].

We use heavily the birational transformations of algebraic geometry to study in detail the transition functions of the Lorentz spin manifold $\mathfrak{M}$ so that the solutions Ψ of the Dirac equation can be expressed explicitly by integrals.

Let

$$ds^2 = g_{jk} ds^j ds^k = \sum_{j,k=0}^{3} g_{jk} dx^j dx^k = \eta_{ab} \omega^a \omega^b \tag{1.4}$$

be a Lorentz metric on $\mathfrak{M}$, where $(\eta_{ab}) = \{1, -1, -1, -1\}$ is a diagonal matrix and

$$\omega^a = e_j^{(a)} dx^j, \ (a = 0, 1, 2, 3); \quad \text{and} \quad X_a = e_{(a)}^j \frac{\partial}{\partial x^j} \quad (a = 0, 1, 2, 3) \tag{1.5}$$

are the Lorentz coframe and the dual frame respectively.

The second purpose of this paper is to find solutions of g_{jk} and Ψ which satisfy the Einstein-Dirac equation

$$R_{jk} - \frac{1}{2} R g_{jk} - \Lambda g_{jk} = \mathcal{X} T_{jk}, \quad \not{D}\Psi = -im\Psi \tag{1.6}$$

where Λ, $\mathcal{X}$ and $m(>0)$ are constants and T_{jk} is the energy-momentum tensor of Ψ such that

$$T_{jk} = \frac{i}{2} [\eta_{ab} \overline{\Psi}^{*\prime} \gamma^b (e_j^{(a)} \bigtriangledown_k \Psi + e_k^{(a)} \bigtriangledown_j \Psi) - \eta_{ab} (e_j^{(a)} \overline{\bigtriangledown_k \Psi^*}' + e_k^{(a)} \overline{\bigtriangledown_j \Psi^*}') \gamma^b \Psi]. \tag{1.7}$$

Here we denote $\overline{A}$ the complex conjugate of a matrix A and A' the transpose of A and

$$\Psi^* = \begin{pmatrix} \varphi^* \\ \psi \end{pmatrix}. \tag{1.8}$$

Besides, $\gamma^a (a = 0, 1, 2, 3,)$ are Dirac matrices and $\bigtriangledown_j$ is the covariant differentiation of 4-component spinor such that

$$\not{D} = \gamma^a e_{(a)}^j \bigtriangledown_j . \tag{1.9}$$

We at first map the conformal space $\mathfrak{M}$ by birational transformation into the compactized Minkowski space $\overline{M}$, which can be mapped by birational transformation [6] to the group manifold $U(2) \cong U(1) \times SU(2)$, and we will prove that $SU(2) \cong \overline{M} \cap \mathbf{P}_0$, where $\mathbf{P}_0$ is a hyperplane. It known that $U(1) \cong S^1$ and $SU(2) \cong S^3$. So we can introduce a Lorentz metric ds^2 on $\mathfrak{M}$ such that

$$ds^2 = ds_1^2 - ds_3^2 \tag{1.10}$$

where

$$ds_1^2 = (dx^0)^2 \quad \text{and} \quad ds_3^2 = \frac{\delta_{\alpha\beta}}{(1+xx')^2}dx^\alpha dx^\beta \tag{1.11}$$

are the Riemann metrics of $U(1)$ and $SU(2) \cong S^3$ respectively.

Since $SU(2) \cong \overline{M} \cap \mathbf{P}_0$ is a Riemann spin manifold, there is a principal bundle

$$Spin\{\overline{M} \cap \mathbf{P}_0, SU(2)\}$$

with base manifold $\overline{M} \cap \mathbf{P}_0$ and structure group $SU(2)$. The transition functions of this principal bundle can be written out explicitly.

Lemma 2. The isometric automorphism $T_u : \overline{M} \cap \mathbf{P}_0 \to \overline{M} \cap \mathbf{P}_0$ can be expressed by admissible local coordinates such that

$$y^\alpha \sigma_\alpha = U_0^{-1}\Phi(x,u)U_0, \quad \Phi(x,u) = (\sigma_0 + x^\mu u^\nu \sigma_\mu \sigma_\nu)^{-1}(x^\alpha - u^\alpha)\sigma_\alpha$$

where $U_0 \in SU(2)$, σ_0 is the 2×2 identity matrix and $\sigma_\alpha (\alpha = 1, 2, 3)$ are Pauli matrices. The transition function associated to T_u is

$$\mathfrak{A}_{T_u}(x) = U_0^{-1}U(x,u)^{-1},$$

where

$$U(x,u) = [(1+xu')^2 + xx'uu' - (xu')^2]^{-\frac{1}{2}}\left[(1+xu')\sigma_0 + ix^\mu u^\nu \delta^{123}_{\mu\nu\alpha}\sigma_\alpha\right],$$

which belongs to $SU(2)$ and $xu' = \delta_{\alpha\beta}x^\alpha u^\beta$.

U(x,u) is called the propagation.

With the metric (1.10) the 2-component Dirac operator of $S^1 \times S^3$ is

$$\mathfrak{D} = \sigma_0 \frac{\partial}{\partial x^0} - \not{D}_{s^3} \tag{1.12}$$

where $\not{D}_{s^3}$ is the Dirac operator of the Riemann spin manifold of S^3 and x^0 the local coordinate of S^1 and $x = (x^1, x^2, x^3)$ the admissible local coordinate of S^3. Hence, if the spinor $\widehat{\psi}(x)$ satisfies the equation

$$\not{D}^2_{s^3}\widehat{\psi} = -(n^2 - m^2)\widehat{\psi} \tag{1.13}$$

then $e^{inx^0}\widehat{\psi}(x)$ is a solution of the equation

$$\mathfrak{D}^2[e^{inx^0}\widehat{\psi}(x)] = -m^2 e^{inx^0}\widehat{\psi}(x). \tag{1.14}$$

By Weitzenböck formula of S^3,

$$\not{D}^2_{s^3} = \Delta - \frac{1}{4}R_{S^3}\,\sigma_0 \tag{1.15}$$

where R_{S^3} is the scalar curvature of ds_3^2 and $\triangle$ is an elliptic differential operator. Hence to solve the equation (1.1) on $S^1 \times S^3$ is reduced to solve the equation on S^3,

$$\not{D}^2_{S^3}\widehat{\psi}(x) = -\lambda\widehat{\psi}(x) \tag{1.16}$$

where $\lambda = n^2 - m^2$ should be an eigen-value of $\not{D}^2_{S^3}$. The λ-eigen kernel is defined by

$$\mathcal{K}_\lambda(x,u) = \sum_{\xi=0}^{N_\lambda} \widehat{\psi}_\xi(x)\overline{\widehat{\psi}_\xi(x)}' \tag{1.17}$$

where $\{\widehat{\psi}_\xi(x)\}_{\xi=1,2,\cdots,N_\lambda}$ is an orthonormal basis of the vector space of λ-eigen functions of $\not{D}^2_{S^3}$. The eigen values of (1.16) and the corresponding dimensions N_λ are known(c.f. [7]) Then for any spinor $\widehat{\psi}_0$ on S^3,

$$\widehat{\psi}(x) = \int_{s^3} \mathcal{K}_\lambda(x,u)\widehat{\psi}_0(u)\dot{u}, \tag{1.18}$$

where $\dot{u}$ is the volume element associated to ds_3^2, is a solution of the equation (1.16). The problem to solve the Dirac equation on the conformal space $\mathfrak{M} \cong S^1 \times S^3$ is reduced to construct the λ-eigen kernel $\mathcal{K}_\lambda$ of $\not{D}^2_{S^3}$ on S^3 explicitly.

THEOREM 1. If we choose on $S^3 \cong SU(2)$ the metric

$$ds_3^2 = \frac{\delta_{\alpha\beta}}{(1+xx')^2} dx^\alpha dx^\beta, \tag{1.19}$$

then the λ-eigen kernel of $\not{D}^2_{S^3}$ is

$$\mathcal{K}_\lambda(x,u) = U(x,u)\big[f\big(\rho^2(x,u)\big)\sigma_0 + h\big(\rho^2(x,u)\big)\Phi(x,u)\big],$$

where $U(x,u)$ and $\Phi(x,u)$ are defined by Lemma 2,

$$\rho^2(x,u) = \frac{(x-u)(x-u)'}{1+2xu'+xx'uu'},$$

and $f(t) = \overline{f(t)}$ and $h(t) = -\overline{h(t)}$ are functions which satisfy respectively the following differential equations

$$4t(1+t)^2\frac{d^2f}{dt^2} + (1+t)[6(1+t)-4t)]\frac{df}{dt} - (2t+6)f = -\lambda f$$

and

$$4t(1+t)^2\frac{d^2h}{dt^2} + (1+t)[10(1+t)-4t)]\frac{dh}{dt} - 4h = -\lambda h.$$

In fact, the solutions of the equations are respectively

$$f(t) = c_0F_0(t) + c_1F_1(t) \quad \text{and} \quad h(t) = ic_2F_2(t) + ic_3F_3(t) \tag{1.20}$$

where $c_j(j=0,1,2,3)$ are real constants,

$$\begin{aligned} F_0(t) &= (1+t)^{1+\sqrt{\lambda}/2}F(\tfrac{\sqrt{\lambda}}{2}, \tfrac{3}{2}+\tfrac{\sqrt{\lambda}}{2}, 1+\sqrt{\lambda}, 1+t), \\ F_1(t) &= (1+t)^{1-\sqrt{\lambda}/2}F(-\tfrac{\sqrt{\lambda}}{2}, \tfrac{3}{2}-\tfrac{\sqrt{\lambda}}{2}, 1-\sqrt{\lambda}, 1+t) \end{aligned} \tag{1.21}$$

and

$$\begin{aligned} F_2(t) &= (1+t)^{1+\sqrt{\lambda}/2}F(\tfrac{3}{2}+\tfrac{\sqrt{\lambda}}{2}, 1+\tfrac{\sqrt{\lambda}}{2}, 1+\sqrt{\lambda}, 1+t), \\ F_3(t) &= (1+t)^{1-\sqrt{\lambda}/2}F(\tfrac{3}{2}-\tfrac{\sqrt{\lambda}}{2}, 1-\tfrac{\sqrt{\lambda}}{2}, 1-\sqrt{\lambda}, 1+t). \end{aligned} \tag{1.22}$$

Here $F(\alpha,\beta,\gamma,x)$ is the hypergeometric function. The constants $c_j (j = 0,1,2,)$ are determined from the equality

$$\int_{S^3} \mathcal{K}_\lambda(a,x)\mathcal{K}_\lambda(x,b)\dot{x} = \mathcal{K}_\lambda(a,b). \tag{1.23}$$

Since

$$\overline{\mathcal{K}_\lambda(a,b)}' = \mathcal{K}_\lambda(b,a), \tag{1.24}$$

there are four independent equations in (1.23) for determining the four constants $c_j (j = 0,1,2,3)$.

A spinor $\widehat{\psi}_0(x)$ on S^3 is said to be orthogonal invariant if $\widehat{\psi}_0(x\Gamma) = U\widehat{\psi}_0(x)$, where $\Gamma \in SO(3)$ and $U \in SU(2)$ such that Γ is the image of U by group homeomorphism ι restricted to the group $SU(2)$. The two component spinor

$$\psi(x_1) = e^{inx^0}\widehat{\psi}(x), \quad x_1 = (x^0, x), \tag{1.25}$$

where $\widehat{\psi}(x)$ defined by (1.18), is orthogonal invariant, provided that $\widehat{\psi}_0(x)$ is orthogonal invariant. By Lemma 1, the 4-component spinor on $S^1 \times S^3$

$$\Psi = \begin{pmatrix} \psi \\ \varphi^* \end{pmatrix}, \qquad \varphi^* = \frac{i}{m}\mathfrak{D}\psi \tag{1.26}$$

satisfies the Dirac equation and it is orthogonal invariant in the sense that $\varphi^*(x^0, x\Gamma) = U\varphi^*(x_1)$ whenever ψ is orthogonal. So

$$\Psi(x^0, x\Gamma) = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \Psi(x^0, x).$$

Theorem 2. If g_{ij} are defined by

$$g_{00} = 1, g_{0\alpha} = g_{\alpha 0} = 0, g_{\alpha\beta} = -\frac{\delta_{\alpha\beta}}{(1 + xx')^2}, \quad \alpha,\beta = 1,2,3,$$

and Ψ is defined by (1.26), in which ψ is given by (1.25) and $\widehat{\psi}$ by the integral (1.18), and is orthogonal invariant and the energy-momentum tensor T_{jk} of Ψ is not identically zero, then the pair $\{g_{jk}, \Psi\}$ satisfy the Einstein-Dirac equation with the constants

$$\Lambda = \frac{-T_{00}(0)}{T_{00}(0) + T_{11}(0)} R_{11}(0) - \frac{1}{2}R(0), \quad \mathcal{X} = \frac{1}{T_{00}(0) + T_{11}(0)} R_{11}(0)$$

and m is non-negative and satisfies

$$m^2 = n^2 - \lambda$$

where n is a positive integer and λ is an eigen value of the operator $\not{\mathcal{D}}^2_{s^3}$.

2. The relation between the Dirac operators of 2-component spinor and 4-component spinor. Let $\mathfrak{M}$ be a four-dimensional Lorentz spin manifold with the Lorentz metric

$$ds^2 = g_{ij}dx^j dx^j = \eta_{ab}\omega^a\omega^b \tag{2.1}$$

where $x = (x^0, x^1, x^2, x^3)$ is an admissible local coordinate of $\mathfrak{M}$, η_{ab} is a diagonal matrix with diagonal elements $\{1, -1, -1, -1\}$ and

$$\omega^a = e_j^{(a)}dx^j, \qquad a = 0, 1, 2, 3 \tag{2.2}$$

is a Lorentz co-frame. Let the dual frame of $\{\omega^a\}$ be

$$\mathbb{X}_a = e_{(a)}^j \frac{\partial}{\partial x^j}. \tag{2.3}$$

From the Christoffel symbol associated to ds^2

$$\left\{ \begin{array}{cc} & l \\ j & & k \end{array} \right\} = \frac{1}{2}g^{li}\left(\frac{\partial g_{ji}}{\partial x^k} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right), \tag{2.4}$$

which is an $\mathfrak{gl}(4, \mathbb{R})$-connection, there is a Lorentz connection

$$\Gamma_{bj}^a = e_k^{(a)}\frac{\partial e_{(b)}^k}{\partial x^j} + e_l^{(a)}\left\{ \begin{array}{ccc} & l & \\ k & & j \end{array} \right\} e_{(b)}^k. \tag{2.5}$$

We denote the matrix

$$\Gamma_j = \left(\Gamma_{bj}^a\right)_{0\le a,b\le 3}. \tag{2.6}$$

If we change the local coordinate $\widetilde{x}^\alpha = \widetilde{x}^\alpha(x)$ and the corresponding Lorentz co-frame as follows

$$\widetilde{\omega}^a(\widetilde{x}) = \ell_b^a(x)\omega^b(x), \qquad L(x) = (\ell_b^a(x))_{0\le a,b\le 3} \in O(1,3) \tag{2.7}$$

then the Lorentz connection $\widetilde{\Gamma}_j$ satisfies the relation

$$\widetilde{\Gamma}_j = \left(L\Gamma_k L^{-1} - \frac{\partial L}{\partial x^k}L^{-1}\right)\frac{\partial x^k}{\partial \widetilde{x}^j}. \tag{2.8}$$

Since Γ_j for each j belongs to the of Lie algebra of $O(1,3)$ and this algebra is $\mathfrak{so}(1,3)$, we have

$$Tr(\Gamma_j) = 0. \tag{2.9}$$

There is a Lie group homeomorphism

$$\iota : SL(2,\mathbb{C}) \to SO(1,3) \tag{2.10}$$

which is defined by the following manner. Let

$$\begin{aligned} \sigma_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{aligned} \tag{2.11}$$

which form a base of the vector space of all 2×2 Hermitian matrices. For any $\mathfrak{A} \in SL(2, \mathbb{C})$ we denote the transpose matrix and the complex conjugate matrix of $\mathfrak{A}$ by $\mathfrak{A}'$ and $\overline{\mathfrak{A}}$ respectively. Each matrix $\mathfrak{A}\sigma_j\overline{\mathfrak{A}}'$ is a Hermitian matrix, so it can be expressed as a linear combination of σ_k. That is

$$\mathfrak{A}\sigma_j\overline{\mathfrak{A}}' = \ell_j^k \sigma_k. \tag{2.12}$$

It is proved (see [8] Th. 2.4.1) that the corresponding matrix

$$L = \left(\ell_k^j\right)_{0 \le j,k \le 3} \in SO(1,3) \tag{2.13}$$

and the homeomorphism ι is a two to one covering map and hence a local isomorphism. Especially, when $\mathfrak{A} \in SU(2)$, the corresponding L is of the form

$$L = \begin{pmatrix} 1 & 0 \\ 0 & K \end{pmatrix}, \qquad K \text{ is a } 3 \times 3 \text{ orthogonal matrix.} \tag{2.14}$$

Moreover, according to Th. 2.4.2 in [8], associated to the $\mathfrak{so}(1,3)$-connection Γ_j, there is locally a $\mathfrak{sl}(2, \mathbb{C})$-connection

$$\mathfrak{B}_j = \frac{1}{4}\eta^{cb}\Gamma_{cj}^a \sigma_a \sigma_b^*, \quad \sigma_b^* = \epsilon\overline{\sigma}_b\epsilon', \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.15}$$

This means that, when Γ_j suffers the transformation relation (2.8), the corresponding relation of $\mathfrak{B}$ is

$$\widetilde{\mathfrak{B}}_j = (\mathfrak{A}\mathfrak{B}_k\mathfrak{A}^{-1} - \frac{\partial \mathfrak{A}}{\partial x^k}\mathfrak{A}^{-1})\frac{\partial x^k}{\partial \widetilde{x}^j} \tag{2.16}$$

where $\mathfrak{A}$ corresponds to the matrix L defined by (2.12). When $\mathfrak{M}$ is a Lorentz spin manifold $\mathfrak{B}_j$ is globally defined on $\mathfrak{M}$. We call $\mathfrak{B}_j$ the 2-component spinor connection derived from the Lorentz connection of the spin manifold $\mathfrak{M}$.

A two component spinor ψ on a Lorentz spin manifold $\mathfrak{M}$ is a vector

$$\psi(x) = \begin{pmatrix} \psi^1(x) \\ \psi^2(x) \end{pmatrix}$$

on each admissible local coordinate neighborhood $\mathfrak{V}$ and x is the local coordinate of this neighborhood. Let $\widetilde{\psi}(\widetilde{x})$ is the vector defined on another admissible local coordinate neighborhood $\widetilde{\mathfrak{V}}$ and $\widetilde{x}$ is the corresponding local coordinate of $\widetilde{\mathfrak{V}}$. When $\mathfrak{V} \cap \widetilde{\mathfrak{V}} \neq \emptyset$, there a matrix $\mathfrak{A} \in SL(2, \mathbb{C})$ such that

$$\widetilde{\psi}(\widetilde{x}) = \mathfrak{A}(x)\psi(x). \tag{2.17}$$

The matrix $\mathfrak{A}(x)$ is the transition function of the spin manifold $\mathfrak{M}$.

A spinor ψ corresponds to a conjugate spinor

$$\psi^* = \epsilon\overline{\psi}. \tag{2.18}$$

Then under the coordinate transformation between two admissible local coordinates,

$$\widetilde{\psi}^*(\widetilde{x}) = \overline{\mathfrak{A}}'^{-1}\psi^*(x) \tag{2.19}$$

because for any 2×2 matrix A

$$A\epsilon A' = (detA)\epsilon. \tag{2.20}$$

Now we can define the covariant differential $\mathfrak{D}_j$ of a spinor ψ by the connection $\mathfrak{B}_j$ such that

$$\mathfrak{D}_j\psi = \frac{\partial\psi}{\partial x^j} + \mathfrak{B}_j\psi. \tag{2.21}$$

which satisfies

$$\widetilde{\mathfrak{D}}_j\widetilde{\psi} = \frac{\partial x^k}{\partial \widetilde{x}^j}\mathfrak{A}\mathfrak{D}_k\psi. \tag{2.22}$$

under admissible coordinate transformation. This means that $\mathfrak{D}_j\psi$ is still a spinor, but a covariant vector with respect to the index j. If we operate again to $\mathfrak{D}_j\psi$ by $\mathfrak{D}_k$ and wish $\mathfrak{D}_k\mathfrak{D}_j\psi$ still be covariant, then it needs in addition a $\mathfrak{gl}(4,\mathbb{R})$ connection to define the covariant differentiation of $\mathfrak{D}_j\psi$. In usual tensor calculus, a covariant differentiation ∇_j of a contravariant vector can be extended to operate on any mixed tensors. We can do the same to define $\mathfrak{D}_j$ such that it can operate on mixed tensors.

Since

$$\mathfrak{B}_j = \left(\mathfrak{B}^A_{Bj}\right)_{1\le A,B\le 2} \tag{2.23}$$

is derived from the $\mathfrak{so}(1,3)$-connection Γ^a_{bj} by (2.15) and Γ^a_{bj} is derived from the $\mathfrak{gl}(4,\mathbb{R})$-connection $\left\{ \begin{array}{c} l \\ j\,k \end{array} \right\}$ by (2.5) and (2.4). $\mathfrak{D}_j$ can be extended to operate on mixed tensor of $SL(2,\mathbb{C})$-,$SO(1,3)$- and $GL(4,\mathbb{R})$-type. For example, the components of the spinor ψ are $\psi^A \quad (A=1,2)$. (2.21) can be rewritten into

$$\mathfrak{D}_j\psi^A = \frac{\partial\psi^A}{\partial x^j} + \mathfrak{B}^A_{Bj}\psi^B \tag{2.24}$$

which is contravariant with respect to the spinor index A and covariant with respect to the index j. Then $\mathfrak{D}_k\mathfrak{D}_j\psi^A$ is defined as

$$\mathfrak{D}_k\mathfrak{D}_j\psi^A = \frac{\partial}{\partial x^k}\mathfrak{D}_j\psi^A + \mathfrak{B}^A_{Bk}\mathfrak{D}_j\psi^B - \left\{ \begin{array}{c} l \\ kj \end{array} \right\}\mathfrak{D}_l\psi^A, \tag{2.25}$$

which is still a mixed tensor, contravariant with respect to spin index A and $GL(2,\mathbb{R})$ covariant with respect to the indices j and k. Moreover, if

$$T^{jA\overline{C}}_{aB\overline{D}}$$

is a tensor $GL(4,\mathbb{R})$-contravariant w.r.t. j, $SO(1,3)$-covariant w.r.t. a, spin tensor w.r.t. A, B, C, D, then its covariant differentiation is defined as follows

$$\begin{aligned} \mathfrak{D}_k T^{jA\overline{C}}_{aB\overline{D}} &= \frac{\partial}{\partial x^k}T^{jA\overline{C}}_{aB\overline{D}} + \mathfrak{B}^A_{Ek}T^{jE\overline{C}}_{aB\overline{D}} - \mathfrak{B}^E_{Bk}T^{jA\overline{C}}_{aE\overline{D}} \\ &+ \overline{\mathfrak{B}}^C_{Ek}T^{jA\overline{E}}_{aB\overline{D}} - \overline{\mathfrak{B}}^E_{Dk}T^{jA\overline{C}}_{aB\overline{E}} - \Gamma^b_{ak}T^{jA\overline{C}}_{bB\overline{D}} + \left\{ \begin{array}{c} j \\ lk \end{array} \right\} T^{lA\overline{C}}_{aB\overline{D}} \end{aligned} \tag{2.26}$$

which is a mixed tensor of the same type plus $GL(4,\mathbb{R})$-covariant w.r.s. to the index k.

If ψ is a spinor,

$$\psi^* = \epsilon\overline{\psi} \tag{2.27}$$

is called the conjugate spinor of ψ. The covariant differentiation can be also extended to the conjugate spinor ψ^* such that

$$\mathfrak{D}_j\psi^* = \frac{\partial\psi^*}{\partial x^j} + \mathfrak{B}_j^*\psi^* \qquad \mathfrak{B}_j^* = \epsilon\overline{\mathfrak{B}}_j\epsilon'. \tag{2.28}$$

After this extension of the definition of covariant differentiation we can find its application. Since the following formula

$$\eta_{ab} = \eta_{cd}\ell_a^c\ell_b^d, \quad \text{for any} \quad L = (\ell_b^a)_{0\le a,b\le 3} \in SO(1,3)$$

means that η_{ab} is an $SO(1,3)$-covariant with respect to indices a and b, we have

$$\mathfrak{D}_j\eta_{ab} = \frac{\partial}{\partial x^j}\eta^{ab} - \Gamma_{aj}^c\eta_{cb} - \Gamma_{bj}^c\eta_{ac} = 0.$$

Similarly, let

$$\sigma_a = \left(\sigma_a^{A\overline{B}}\right)_{1\le A,B\le 2}, \quad a = 0,1,2,3, \quad \mathfrak{A} = \left(\mathfrak{A}_B^A\right)_{1\le A,B\le 2}.$$

(2.12) can be written as

$$\sigma_a^{A\overline{B}} = \sigma_b^{C\overline{D}}(L^{-1})_a^b\mathfrak{A}_C^A\overline{\mathfrak{A}}_D^B$$

which is $SO(1,3)$-covariant w.r.t. a, spin contravariant w.r.t. to A and complex conjugate spin contravariant w.r.t. $\overline{B}$. Then

$$\mathfrak{D}_j\sigma_a^{A\overline{B}} = \frac{\partial}{\partial x^j}\sigma_a^{A\overline{B}} - \Gamma_{aj}^b\sigma_b^{A\overline{B}} + \mathfrak{B}_{Cj}^A\sigma_a^{C\overline{B}} + \overline{\mathfrak{B}}_{Cj}^B\sigma_a^{A\overline{C}} = 0.$$

The 2-component Dirac operator is defined by

$$\mathfrak{D} = \eta^{ab}e_{(a)}^j\sigma_b^*\mathfrak{D}_j. \tag{2.29}$$

If ψ is a spinor on $\mathfrak{M}$, then according to the definition of σ_b^* and the formula (2.22), we have

$$\widetilde{\mathfrak{D}}\widetilde{\psi} = \overline{\mathfrak{A}}'^{-1}\mathfrak{D}\psi, \quad \widetilde{\mathfrak{D}}\widetilde{\psi}^* = \mathfrak{A}\mathfrak{D}\psi^*. \tag{2.30}$$

This means that $\mathfrak{D}$ is a map

$$\mathfrak{D} : V_2(x) \to V_2^*(x) \quad \text{and} \quad \mathfrak{D} : V_2^*(x) \to V_2(x)$$

where $V_2(x)$ is the vector space of 2-component spinors of $\mathfrak{M}$ at x and $V_2^*(x)$ the conjugate vector space. Obviously,

$$\mathfrak{D}^2 = \mathfrak{D}\mathfrak{D} : \quad V_2(x) \to V_2(x) \quad \text{and} \quad \mathfrak{D}^2 : V_2^*(x) \to V_2^*(x). \tag{2.31}$$

The equation

$$\mathfrak{D}^2\psi = -m^2\psi \tag{2.32}$$

is called the wave equation of spinor on $\mathfrak{M}$.

A solution ψ of the wave equation will give a solution of the 4-component Dirac equation. Before proving this assertion, we at first make clear the relation between the 2-component spinor and 4-component spinor.

Let

$$\gamma^a = \eta^{ab}\begin{pmatrix} 0 & \sigma_b \\ \sigma_b^* & 0 \end{pmatrix}, \qquad a, b = 0, 1, 2, 3. \tag{2.33}$$

According to the relation

$$\sigma_a\sigma_b^* + \sigma_b\sigma_a^* = 2\eta_{ab}\sigma_0. \tag{2.34}$$

we have the relation

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}I \tag{2.35}$$

where I is the 4×4 identity matrix and according to (2.12)

$$\gamma^a\mathcal{R}(\mathfrak{A}) = \ell^a_b(\mathfrak{A})\mathcal{R}(\mathfrak{A})\gamma^b \tag{2.36}$$

where $\ell^a_b(\mathfrak{A})$ is the element corresponding to $\mathfrak{A}$ by (2.12) and

$$\mathcal{R}(\mathfrak{A}) = \begin{pmatrix} \mathfrak{A} & 0 \\ 0 & \overline{\mathfrak{A}}'^{-1} \end{pmatrix} \tag{2.37}$$

is a representation of the group $SL(2,\mathbb{C})$. The relation (2.35) shows that $\{\gamma^0, \gamma^1, \gamma^2, \gamma^3\}$ is a set of Dirac matrices and the relation (2.36) means that the group

$$Spin(1,3) = \{\mathcal{R}(\mathfrak{A})\}_{\mathfrak{A}\in SL(2,\mathbb{C})} \tag{2.38}$$

is an 2 to 1 homeomorphism to the group $SO(1,3)$. The 4-component vector

$$\Psi = \begin{pmatrix} \psi \\ \varphi^* \end{pmatrix}, \tag{2.39}$$

where ψ is a 2-component spinor and φ^* a conjugate spinor, obviously satisfies the relation

$$\widetilde{\Psi} = \mathcal{R}(\mathfrak{A})\Psi \tag{2.40}$$

and conversely any $Spin(1,3)$ 4-component spinor must be of the form (2.39).

The Dirac operator $\not{\mathfrak{D}}$ is defined by

$$\not{\mathfrak{D}} = \begin{pmatrix} 0 & \mathfrak{D} \\ \mathfrak{D} & 0 \end{pmatrix} \quad \text{and} \quad \nabla_j = \begin{pmatrix} \mathfrak{D}_j & 0 \\ 0 & \mathfrak{D}_j \end{pmatrix} \tag{2.41}$$

and the Dirac equation is

$$\not{\mathfrak{D}}\Psi = -im\Psi. \tag{2.42}$$

If the 2-component spinor ψ is a solution of the wave equation (2.32), then we set

$$\varphi^* = \frac{i}{m}\mathfrak{D}\psi \tag{2.43}$$

and obtain

$$\mathfrak{D}\varphi^* = \frac{i}{m}\mathfrak{D}^2\psi = -im\psi \tag{2.44}$$

or

$$\psi = \frac{i}{m}\mathfrak{D}\varphi^* \tag{2.45}$$

and

$$\mathfrak{D}\psi = \frac{i}{m}\mathfrak{D}^2\varphi^* = \frac{-1}{m^2}\mathfrak{D}^3\psi = \mathfrak{D}\psi = -im\varphi^*. \tag{2.46}$$

Hence Ψ defined by (2.39) satisfies the Dirac equation

$$\not{\mathcal{D}}\Psi = -im\Psi. \tag{2.47}$$

This proves **Lemma 1** in §1.

It should be noted that

$$\begin{aligned}\not{\mathcal{D}}\Psi &= \begin{pmatrix} \mathfrak{D}\varphi^* \\ \mathfrak{D}\psi \end{pmatrix} = \begin{pmatrix} (\mathfrak{D}\varphi)^* \\ \mathfrak{D}\psi \end{pmatrix} = \begin{pmatrix} \eta^{ab}e^j_{(a)}\sigma_b\mathfrak{D}_j\varphi^* \\ \eta^{ab}e^j_{(a)}\sigma^*_b\mathfrak{D}_j\varphi \end{pmatrix} \\ &= \eta^{ab}e^j_{(a)}\begin{pmatrix} 0 & \sigma_b \\ \sigma^*_b & 0 \end{pmatrix}\begin{pmatrix} \mathfrak{D}_j\psi \\ \mathfrak{D}_j\varphi^* \end{pmatrix}.\end{aligned}$$

That is

$$\not{\mathcal{D}}\Psi = \gamma^a e^j_{(a)} \bigtriangledown_j \Psi \tag{2.48}$$

when we define the covariant differentiation of the 4-component spinor $\Psi = \begin{pmatrix} \psi \\ \varphi^* \end{pmatrix}$ by

$$\bigtriangledown_j \Psi = \begin{pmatrix} \mathfrak{D}_j\psi \\ \mathfrak{D}_j\varphi^* \end{pmatrix}. \tag{2.49}$$

3. The spin structure of S^3. It is well-known that S^3 is a Riemann spin manifold. For solving the Dirac equation on S^3 we need to describe the transition functions of the principal bundle $Spin\{S^3, SU(2)\}$ explicitly.

$$S^3 = \{(a,b) \in \mathbf{C}^2 | \quad |a|^2 + |b|^2 = 1\}$$

is equivalent to $SU(2)$ by the map

$$(a,b) \to \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}.$$

The unitary group $U(2)$ is the characteristic manifold of the classical domain

$$\mathfrak{R}_I(2,2) = \{W \in \mathbf{C}^{2\times 2} | \quad I - WW^\dagger > 0\}$$

where $W^\dagger = \overline{W}'$. Since $\mathfrak{R}_I(2,2)$ is a domain in the complex Grassmann manifold $\mathfrak{F}(2,2)$, $U(2)$ is a submanifold of $\mathfrak{F}(2,2)$. Since $SU(2)$ is a subgroup of $U(2)$, $SU(2)$ is also a submanifold of $\mathfrak{F}(2,2)$. The complex Grassmann manifold can be described by complex matrix homogeneous coordinate $\mathfrak{Z}$, which is a 2×4 complex matrix satisfying

$$\mathfrak{Z}\mathfrak{Z}^\dagger = I,$$

and two matrix homogeneous coordinates $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ represent a same point of $\mathfrak{F}(2,2)$ iff there is a 2×2 unitary matrix U such that $\mathfrak{Z}_1 = U\mathfrak{Z}_2$.

$\mathfrak{F}(2,2)$ is a complex spin manifold because for any $T \in SU(4)$ there is a holomorphic automorphism defined by

$$\mathfrak{W} = U_T \mathfrak{Z} T, \quad U_T \in U(2) \tag{3.1),}$$

where U_T is the transition function of the principal bundle $E\{\mathfrak{F}(2,2), U(2)\}$ (c.f.[9]), and the transition function of the reduced bundle $Spin\{\mathfrak{F}(2,2), SU(2)\}$ is

$$\mathfrak{A}_T = (detU_T)^{-\frac{1}{2}} U_T. \tag{3.2}$$

Without lose of generality we assume that in $\mathfrak{Z} = (Z_1, Z_2)$ and $\mathfrak{W} = (W_1, W_2)$ the submatrices Z_1 and W_1 are non-singular. We write

$$T = \begin{pmatrix} A & C \\ B & D \end{pmatrix}, \tag{3.3}$$

where A, B, C, D are 2×2 matrices satisfying

$$AA^\dagger + CC^\dagger = I, \quad AB^\dagger + CD^\dagger = 0, \quad BB^\dagger + DD^\dagger = I. \tag{3.4}$$

Comparing the submatrices of (3.1) we obtain

$$U_T = W_1(Z_1A + Z_2B)^{-1} = W_1(A + ZB)^{-1}Z_1^{-1}, \tag{3.5}$$

where

$$Z = Z_1^{-1}Z_2 \quad \text{and} \quad W = W_1^{-1}W_2 \tag{3.6}$$

are the local coordinates. From

$$\mathfrak{Z}\mathfrak{Z}^\dagger = Z_1Z_1^\dagger + Z_2Z_2^\dagger = Z_1(I + ZZ^\dagger)Z_1^\dagger = I$$

we have a unique positively definite Hermitian matrix $Z_1 = (I + ZZ^\dagger)^{-\frac{1}{2}}$ satisfies the above equation, so that the transition function

$$U_T = (I + WW^\dagger)^{-\frac{1}{2}}(A + ZB)^{-1}(I + ZZ^\dagger)^{\frac{1}{2}}. \tag{3.7}$$

When the transformation (3.1) is expressed in local coordinates

$$W = (A + ZB)^{-1}(C + ZD), \tag{3.8}$$

we have

$$I + WW^{\dagger} = (A + ZB)^{-1}(I + ZZ^{\dagger})(A + ZB)^{\dagger -1}. \tag{3.9}$$

The classical domain $\mathfrak{R}_I(2,2)$ can be transformed to the Siegel domain

$$\mathfrak{H}_I(2,2) = \{Z \in \mathbf{C}^{2\times 2} | \quad \frac{1}{2i}(Z - Z^{\dagger}) > 0\}$$

by the transformation

$$W = (I - iZ)^{-1}(I + iZ) \tag{3.10}$$

such that the characteristic manifold $U(2)$ is transformed to $\overline{M}$ by

$$U = (I - iH)^{-1}(I + iH), \quad H^{\dagger} = H. \tag{3.11}$$

Let $\mathcal{G}$ be the subgroup of $SU(4)$ such that the submatrices in (3.3) satisfy

$$C = -B, \quad D = A, \quad A^{\dagger}A + B^{\dagger}B = I, \quad B^{\dagger}A = A^{\dagger}B. \tag{3.12}$$

The transformation for $T \in \mathcal{G}$

$$K = (A + HB)^{-1}(-B + HA) \tag{3.13}$$

is an automorphism of $\overline{M}$ i.e., $K^{\dagger} = K$. This transformation must map a certain point, say $H = H_0$, to the point $K = 0$. Then the condition (3.12) becomes

$$B = H_0 A, \quad A = (I + H_0^2)^{-\frac{1}{2}}U_0, \quad U_0 \in SU(2) \tag{3.14}$$

and (3.13) can be written into

$$K = U_0^{-1}(I + H_0^2)^{\frac{1}{2}}(I + HH_0)^{-1}(H - H_0)(I + H_0^2)^{-\frac{1}{2}}U_0. \tag{3.15}$$

$SU(2)$ is a subgroup of $U(2)$. The transformation (3.11) must map $SU(2)$ into a submanifold of $\overline{M}$.

Lemma 3. The necessary and sufficient that $U \in SU(2)$ in transformation (3.11) is $Tr(H) = 0$.

Proof. Since the Hermitian matrix H can be written into $H = x^j \sigma_j$, the condition

$$Tr(H) = 0 \quad \text{equivalent} \quad x^0 = 0. \tag{3.16}$$

When the above condition is satisfied we write

$$H = H_x = x^{\alpha}\sigma_{\alpha}$$

which satisfies the relations

$$\det H_x = -xx' \quad \text{and} \quad H_x^2 = xx'\sigma_0, \quad x = (x^1, x^2, x^3). \tag{3.17}$$

The above relation implies that the characteristic roots of H_x are $\sqrt{xx'}$ and $-\sqrt{xx'}$ so that there is a $V \in SU(2)$ such that

$$H_x = \sqrt{xx'}V\sigma_3 V^{\dagger}. \tag{3.18}$$

According to (3.11)

$$\det U = \det[V(I + i\sqrt{xx'}\sigma_3)^{-1}(I - i\sqrt{xx'}\sigma_3)V^{\dagger}] = 1.$$

This means that $U \in SU(2)$. Conversely, if $U \in SU(2)$, then the inverse of (3.11) is

$$H = i(I+U)^{-1}(I-U) = \frac{i}{|1+a|^2+|b|^2}\begin{pmatrix} 1+\overline{a} & b \\ -\overline{b} & 1+a \end{pmatrix}\begin{pmatrix} 1-a & b \\ -\overline{b} & 1-\overline{a} \end{pmatrix} \quad (3.19)$$

so that $Tr(H) = 0$ because $|a|^2 + |b|^2 = 1$. The lemma is proved.

Since $x^0 = 0$ is a hyperplane $\mathbf{P}_0$ in $\overline{M}$, Lemma 3 implied that $SU(2) \cong \overline{M} \cap \mathbf{P}_0$ and we can use the admissible local coordinate of $\overline{M} \cap \mathbf{P}_0$ as the local coordinate of $SU(2) \cong S^3$. Consequently,

$$\mathfrak{M} \cong \overline{M} \cong U(2) \cong U(1) \times SU(2) \cong S^1 \times S^3 \cong U(1) \times \mathbf{M}_1$$

where we set

$$\mathbf{M}_1 = \overline{M} \cap \mathbf{P}_0. \quad (3.20)$$

Now we take in the transformation (3.15)

$$H_0 = H_a = a^{\alpha}\sigma_{\alpha}. \quad a = (a^1, a^2, a^3), \quad (3.21)$$

Since $H_0^2 = aa'\sigma_0$, the transformation becomes

$$K = U_0^{-1}(I + HH_a)^{-1}(H - H_a)U_0. \quad (3.22)$$

Lemma 4. The transformation (3.22) is an automorphism of $\overline{M}_1$, in other words, it transforms $Tr(H) = 0$ to $Tr(K) = 0$.

Proof. Since $Tr(H) = 0$, it can be written into $H_x = x^{\alpha}\sigma_{\alpha}$ and

$$H_xH_a = x^{\mu}a^{\nu}\sigma_{\mu}\sigma_{\nu} = \frac{1}{2}x^{\mu}a^{\nu}[(\sigma_{\mu}\sigma_{\nu} + \sigma_{\nu}\sigma_{\mu}) + (\sigma_{\mu}\sigma_{\nu} - \sigma_{\nu}\sigma_{\mu})]$$

$$= x^{\mu}a^{\nu}[\delta_{\mu\nu}\sigma_0 + i\delta^{123}_{\mu\nu\alpha}\sigma_{\alpha}] = xa'\sigma_0 + if^{\alpha}(x,a)\sigma_{\alpha}, \quad (3.23)$$

where

$$f^{\alpha}(x,a) = x^{\mu}a^{\nu}\delta^{123}_{\mu\nu\alpha}. \quad (3.24)$$

Since

$$(I + H_xH_a)((I + H_xH_a)^{\dagger} = [(1 + xa')I + iH_f][(1 + xa')I + iH_f]^{\dagger}$$

$$= (1 + xa')^2I + H_f^2 = [(I + xa')^2 + ff']I = \chi^2 I$$

where

$$\chi = \chi(x,a) = [(1 + xa')^2 + xx'aa' - xa'xa']^{\frac{1}{2}}, \quad (3.25)$$

the matrix

$$U(x,a) = \chi^{-1}(I + H_x H_a) \tag{3.26}$$

is a unitary matrix with $detU(x,a) = 1$ and

$$(I + H_x H_a)^{-1}(H_x - H_a) = \chi^{-2}((1 + xa')I - iH_f)H_{(x-a)}$$

$$= \chi^{-2}[(1 + xa')H_{(x-a)} - if(x,a)(x-a)'\sigma_0 + f^\alpha(f(x,a), x-a)\sigma_\alpha]. \tag{3.27}$$

Hence

$$Tr(K) = 0$$

because

$$f(x,a)(x-a)' = x^\mu a^\nu \delta^{123}_{\mu\nu\alpha}(x^\alpha - a^\alpha) = 0.$$

The lemma is proved.

By Lemma 4, we can write

$$K = H_y = y^\alpha \sigma_\alpha$$

and according to (3.27) the transformation (3.22) can be written into usual manner

$$y^\nu = \chi^{-2}\{x^\mu - a^\mu + xa'(x^\mu - a^\mu) + [x(x-a)'a^\mu - a(x-a)'x^\mu]\}\gamma^\nu_\mu, \tag{3.28}$$

where $(\gamma^\alpha_\beta) \in SO(3)$. Moreover all such transformations form a group, which is a group of automorphism of $\mathbf{M}_1$, or all the matrices of the form

$$T_a = (1 + aa')^{-\frac{1}{2}} \begin{pmatrix} I & -H_a \\ H_a & I \end{pmatrix} \begin{pmatrix} U_0 & 0 \\ 0 & U_0 \end{pmatrix} \tag{3.29}$$

form a group $\mathcal{G}_1$ which is a subgroup of $\mathcal{G}$. So when $T_a \in \mathcal{G}_1$ the transition function (3.7) becomes, according to (3.9) and (3.26),

$$U_{T_a} = [(A+H_xB)^{-1}(I+H_x^2)(A+H_xB)^{\dagger-1}]^{-\frac{1}{2}}(A+H_xB)^{-1}(I+H_x^2)^{\frac{1}{2}} = U_0^\dagger U(x,a)^{-1}, \tag{3.30}$$

and $detU_{T_a} = 1$. Hence

$$\mathfrak{A}_{T_a} = U_{T_a} = U_0^\dagger U(x,a)^{-1}. \tag{3.31}$$

This proves Lemma 2 in §1.

In S^3 there is a natural Riemann metric

$$ds_3^2 = \frac{1}{4}(|da|^2 + |db|^2) = \frac{1}{8}Tr(dUdU^\dagger), \tag{3.32}$$

where

$$U = \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}, \quad |a|^2 + |b|^2 = 1.$$

Differentiating (3.11) and substituting dU into (3.32) we have

$$ds_3^2 = \frac{1}{2}Tr[(I+H_x^2)^{-1}dH_x(I+H_x^2)^{-1}dH_x] = \frac{\delta_{\mu\nu}}{(1+xx')^2}dx^\mu dx^\nu. \tag{3.33}$$

Differentiating (3.13) we have

$$dH_y = (A+H_xB)^{-1}dH_x(A+H_xB)^{-1}. \tag{3.34}$$

Applying (3.9) and (3.34) we obtain

$$ds_3^2 = \frac{\delta_{\mu\nu}}{(1+yy')^2}dy^\mu dy^\nu = \frac{1}{2}Tr[(I+H_y^2)^{-1}dH_y(I+H_y^2)^{-1}dH_y]$$

$$= \frac{1}{2}Tr[(I+H_x^2)^{-1}dH_x(I+H_x^2)^{-1}dH_x] = \frac{\delta_{\mu\nu}}{(1+xx')^2}dx^\mu dx^\nu. \qquad (3.35).$$

This means that the ds_3^2 is invariant under the group $\mathcal{G}_1$. When we set

$$a = \xi^0 + i\xi^3, \quad b = \xi^1 + i\xi^2 \tag{3.36}$$

and use (3.11),

$$\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} = (I-iH_x)^{-1}(I+iH_x) = (1+xx')^{-1}\begin{pmatrix} 1-xx'+2ix^3 & -2x^2+2ix^1 \\ 2x^2+2ix^1 & 1-xx'-2ix^3 \end{pmatrix},$$

we obtain the coordinate transformation

$$\xi^0 = \frac{1-xx'}{1+xx'}, \quad \xi^\alpha = \frac{2x^\alpha}{1+xx'}, \quad \alpha = 1,2,3 \tag{3.37}$$

such that

$$ds_3^2 = \frac{1}{4}\delta_{jk}d\xi^j d\xi^k = \frac{\delta_{\mu\nu}}{(1+xx')^2}dx^\mu dx^\nu. \tag{3.38}$$

4. The harmonic analysis of Dirac spinors on $S^1 \times S^3$. Now we discuss the case that $\mathfrak{M} \cong S^1 \times S^3$ with the metric (1.4) as its Lorentz metric. It is obvious that $S^1 \times S^3$ is a Lorentz spin manifold and S^3 a Riemann spin manifold with the metric

$$ds_3^2 = \frac{\delta_{\mu\nu}}{(1+xx')^2}dx^\mu dx^\nu. \tag{4.1}$$

Since in S^1

$$ds_1^2 = (dx^0)^2 \tag{4.2}$$

the tensor g_{jk} in (1.4) is of the form

$$\begin{cases} g_{00} = 1, \quad g_{0\mu} = g_{\mu 0} = 0, \quad \mu = 1,2,3, \\ g_{\mu\nu} = \dfrac{-1}{[1+r^2(x_1)]^2}\delta_{\mu\nu}, \quad \mu,\nu = 1,2,3 \end{cases} \tag{4.3}$$

and the Christoffel symbol is

$$\left\{ \begin{matrix} l \\ j\,k \end{matrix} \right\} = 0, \quad \text{when one of the indices } l, j, k \text{ equals to } 0 \tag{4.4}$$

and

$$\left\{ \begin{matrix} \lambda \\ \mu\,\nu \end{matrix} \right\}, \qquad \lambda, \mu, \nu = 1, 2, 3$$

is the Christoffel symbol of ds_3^2. The coefficients of the Lorentz coframe of ds^2 are

$$e_0^{(0)} = 1, \quad e_\mu^{(0)} = 0, \quad \mu = 1, 2, 3$$

and

$$e_\nu^{(\alpha)} = (1 + xx')^{-1}\delta_\nu^\alpha, \quad (\alpha, \nu = 1, 2, 3). \tag{4.5}$$

The later ones are the coefficients of the Riemann co-frame of ds_3^2. Since $g_{\mu\nu}$ do not depend on the coordinate x^0, the Lorentz connection

$$\Gamma_{bj}^a = 0 \quad \text{when one of the indices } a, b, j \text{ equal to } 0$$

and is a $\mathfrak{s}o(1,3)$-connection. So the connection defined by (2.15) is

$$\mathfrak{B}_j = \frac{1}{4}\sigma_\alpha\sigma_\beta\Gamma_{\beta j}^\alpha \quad \text{because} \quad \sigma_\alpha^* = -\sigma_\alpha, \tag{4.7}$$

and

$$\mathfrak{B}_0 = 0, \qquad \mathfrak{B}_\mu = \frac{1}{4}\sigma_\alpha\sigma_\beta\Gamma_{\beta\mu}^\alpha. \tag{4.8}$$

Then the covariant differentiation defined by (2.21) is

$$\mathfrak{D}_0\psi = \frac{\partial\psi}{\partial x^0}, \qquad \mathfrak{D}_\mu\psi = \frac{\partial\psi}{\partial x^\mu} + \mathfrak{B}_\mu\psi \tag{4.9}$$

where $\mathfrak{B}_\mu$ is an $su(2)$-connection on S^3, so

$$\mathfrak{D} = \sigma_0\frac{\partial}{\partial x^0} - \not{\mathfrak{D}}_{S^3}, \qquad \not{\mathfrak{D}}_{S^3} = e_{(\alpha)}^\mu\sigma_\alpha\mathfrak{D}_\mu \tag{4.10}$$

where $\not{\mathfrak{D}}_{S^3}$ is the Dirac operator of the Riemann spin manifold of S^3. Hence

$$\mathfrak{D}^2\psi = \frac{\partial^2\psi}{(\partial x^0)^2} - \not{\mathfrak{D}}_{S^3}^2\psi. \tag{4.11}$$

where $\not{\mathfrak{D}}_{S^3}^2$ does not depend on the coordinate x^0. So we use the method of separating variables to solve (1.1). Let

$$\psi^{(n)}(x_1) = e^{inx^0}\widehat{\psi}(x) \tag{4.12}$$

where $\widehat{\psi}$ is a spinor on S^3 and e^{inx^0} is defined on S^1, then e^{inx^0} should be a periodic function with n being an integer and $\widehat{\psi}$ should satisfy

$$\not{D}^2_{S^3}\widehat{\psi} = -(n^2 - m^2)\widehat{\psi} \tag{4.13}$$

if ψ satisfies (1.1). Since the eigen value of $\not{D}^2_{S^3}$ is known[7] to be of the form

$$n^2 - m^2 = (l + \frac{1}{2})^2 \tag{4.14}$$

where l is a positive integer. So the integer n must be sufficiently large so that

$$n^2 - m^2 > 0. \tag{4.15}$$

Using Weitzenböck formulae for Riemann spin manifold S^3, we have

$$\not{D}^2_{S^3}\widehat{\psi} = \triangle\widehat{\psi} - \frac{1}{4}R_{S^3}\widehat{\psi} \tag{4.16}$$

where

$$\begin{aligned}\triangle\widehat{\psi} = g^{\mu\nu}\left(\frac{\partial^2\widehat{\psi}}{\partial x^\mu \partial x^\nu} - \left\{ \begin{array}{c}\lambda \\ \mu\,\nu\end{array} \right\} \frac{\partial\widehat{\psi}}{\partial x^\lambda}\right) + g^{\mu\nu}\left(\frac{\partial \mathfrak{B}_\mu}{\partial x^\nu} - \left\{ \begin{array}{c}\lambda \\ \mu\,\nu\end{array} \right\} \mathfrak{B}_\lambda\right)\widehat{\psi} \\ + g^{\mu\nu}\left(\mathfrak{B}_\mu \frac{\partial\widehat{\psi}}{\partial x^\nu} + \mathfrak{B}_\nu \frac{\partial\widehat{\psi}}{\partial x^\mu}\right) + g^{\mu\nu}\mathfrak{B}_\mu\mathfrak{B}_\nu\widehat{\psi}\end{aligned} \tag{4.17}$$

and R_{S^3} is the scalar curvature of S^3. It is known $R_{S^3} = 24$. Hence, to solve the equation (1.1) is reduced to solve the following equation

$$\not{D}^2_{S^3}\widehat{\psi} = -(n^2 - m^2)\widehat{\psi}. \tag{4.18}$$

Since $\not{D}^2_{S^3}$ is an elliptic differential operator and S^3 is compact, there is, in general, no solution of (4.18) for arbitrary $m > 0$ unless $\lambda = n^2 - m^2$ is an eigenvalue of the operator $\not{D}^2_{S^3}$. In this case the linear independent solutions of (4.18) is finite. Let

$$\widehat{\psi}_\xi(\lambda, x^1, x^2, x^3), \quad \xi = 1, 2, \cdots N_\lambda \tag{4.19}$$

be an orthonormal base of the λ-eigen function space such that

$$\int_{S^3} \overline{\widehat{\psi}}'_\xi \widehat{\psi}_\eta \sqrt{-g}dx^1dx^2dx^3 = \delta_{\xi\eta}, \tag{4.20}$$

where $g = det(g_{ij})_{0\le i,j\le 3} = -det(g_{\alpha\beta})_{1\le\alpha,\beta\le 3}$.

Now we let

$$x_1 = (x^0, x) \quad \text{and} \quad H_{x_1} = x^j\sigma_j$$

and construct the kernel of λ-eigen space

$$\mathcal{H}_\lambda(H_{x_1}, H_{y_1}) = \sum_{\xi=1}^{N_\lambda} \psi^{(n)}_\xi(\lambda, x_1)\overline{\psi^{(n)}_\xi(\lambda, y_1)}' \tag{4.21}$$

which is an 2×2 matrix of matrix variables H_{x_1} and H_{y_1}. We set

$$\psi_\xi^{(n)}(\lambda, x_1) == \frac{1}{\sqrt{2\pi}} e^{inx^0}\widehat{\psi}_\xi(\lambda, x). \tag{4.22}$$

It should be noted that

$$\lambda = n^2 - m^2 \tag{4.23}$$

is positive.

According to the (3.22) given in §3, the transformation T_{a_1}

$$y^0 = x^0 - a^0, \quad H_y = (A + H_x B)^{-1}(-B + H_x A), \quad B = H_a A, \tag{4.24}$$

is an automorphism of $S^1\times S^3$ and it transforms the point $x_1 = a_1$ to $y_1 = 0$. Since ds^2 is invariant under the transformation, the co-frame is changed as follows:

$$\omega^0 = 1, \quad \omega^\alpha(y) = \omega^\alpha(x)\ell_\alpha^\beta(x), \quad \left(\ell_\beta^\alpha(x)\right)_{1\le\alpha,\beta\le 3} \in SO(3)$$

and the spinor

$$\psi_{T_{a_1}}(y_1) = \mathfrak{A}_{T_{a_1}}(x_1)\psi(x_1) \tag{4.25}$$

where $\mathfrak{A}_{T_{a_1}}(x_1) = \mathfrak{A}_{T_a}(x)$ is defined by (3.31) and belongs to $SU(2)$. Let

$$\psi_{T_{a_1},\xi}(y_1) = \mathfrak{A}_{T_{a_1}}(x_1)\psi_\xi(x_1).$$

Since

$$\overline{\psi^{(n)}_{T_{a_1},\xi}(\lambda, y_1)}'\,\psi^{(n)}_{T_{a_1},\eta}(\lambda, y_1) = \overline{\psi^{(n)}_\xi(\lambda, x_1)}'\,\psi^{(n)}_\eta(\lambda, x_1), \tag{4.26}$$

the

$$\left\{\psi^{(n)}_{T_{a_1},\eta}(\lambda, y_1)\right\} \tag{4.27}$$

is a base of spinors of λ-eigenvalue in $S^1\times S^3$. If $u_1\in S^1\times S^3$ is another point which is mapped to the point v_1 under the same transformation T_{a_1} , we have

$$\mathcal{H}_\lambda(H_{y_1}, H_{v_1}) = \mathfrak{A}_{T_{a_1}}(x_1)\mathcal{H}_\lambda(H_{x_1}, H_{u_1})\mathfrak{A}_{T_{a_1}}(u_1)^{-1}. \tag{4.28}$$

According to the definition (4.21), we have

$$\mathcal{H}_\lambda(H_{x_1}, H_{u_1}) = e^{in(x^0-u^0)}\mathcal{K}_\lambda(H_x, H_u) \tag{4.29}$$

where

$$\mathcal{K}_\lambda(H_x, H_u) = \sum_{\xi=1}^{N_\lambda}\widehat{\psi}_\xi(\lambda, x)\overline{\widehat{\psi}_\xi(\lambda, u)}' \tag{4.30}$$

is the kernel of λ-eigen functions of the operator $\not{\mathcal{D}}^2_{S^3}$ of the Riemann manifold S^3 with the metric ds_3^2. Under the transformation (4.24),

$$\mathcal{K}_\lambda(H_y, H_v) = \mathfrak{A}_{T_a}(x)\mathcal{K}_\lambda(H_x, H_u)\mathfrak{A}_{T_a}(u)^{-1}. \tag{4.31}$$

Since $\not{D}^2_{S^3}$ is a covariant differentiation, we have

$$\not{D}^2_{S^3}(y)\mathcal{K}_\lambda(H_y, H_v) = \mathfrak{A}_{T_a}(x)\not{D}^2_{S^3}(x)\mathcal{K}_\lambda(H_x, H_u)\mathfrak{A}_{T_a}(u)^{-1}, \tag{4.32}$$

where $\not{D}_{S^3}(x)$ means that $\not{D}_{S^3}$ operates with respect to the variable x.

Since

$$T_a: \quad H_x \to \quad H_y = U_0^\dagger(I + H_x H_a)^{-1}(H_x - H_a)U_0 \tag{4.33}$$

we have

$$\begin{aligned}\left[\not{D}^2_{S^3}(y)\mathcal{K}_\lambda(H_y, H_v)\right]_{v=0} &= \mathfrak{A}_{T_a}(x)\left[\not{D}^2_{S^3}(x)\mathcal{K}_\lambda(x,a)\right]\mathfrak{A}_{T_a}(a)^{-1} \\ &= -\lambda\mathfrak{A}_{T_a}(x)\mathcal{K}_\lambda(x,a)\mathfrak{A}_{T_a}(a)^{-1}.\end{aligned} \tag{4.34}$$

Since $\mathfrak{A}_{T_a}(x)$ is known explicitly by (3.31) and (3.26), it remains to calculate $\not{D}^2_{S^3}(x)\mathcal{K}_\lambda(H_x, 0)$ in (4.34).

According to (2.12), (2.14) and (4.31),

$$\mathcal{K}_\lambda(UH_xU^\dagger, 0) = \mathcal{K}_\lambda(H_{xK}, 0) = U\mathcal{K}_\lambda(H_x, 0)U^\dagger$$

for any $U \in SU(2)$, $\mathcal{K}_\lambda(H_x, 0)$ can be expanded into power series of the matrix variable H_x such that

$$\begin{aligned}\mathcal{K}_\lambda(H_x, 0) &= \sum_{n=0}^{\infty} C_n H_x^n = \sum_{n=0}^{\infty} C_{2n} H_x^{2n} + \sum_{n=0}^{\infty} C_{2n+1} H_x^{2n+1} \\ &= \sum_{n=0}^{\infty} C_{2n} r^{2n}(x) I + \sum_{n=0}^{\infty} C_{2n+1} r^{2n}(x) H_x = f(r^2(x))I + h(r^2(x))H_x,\end{aligned} \tag{4.35}$$

where C_n are complex constants $r^2(x) = xx'$ and f and h are functions of $r^2(x)$ but not real values in general.

We set $u = a$ in (4.31) and have by Lemma 2

$$\mathcal{K}_\lambda(H_x, H_a) = \mathfrak{A}_{T_a}(x)^{-1}\mathcal{K}_\lambda(H_y, 0)U_0^{-1} = U(x,a)[fI + h\Phi(x,a,)], \tag{4.36}$$

where we have written in (3.22) that

$$H = H_x \quad \text{and} \quad K = H_y$$

so that (3.22) becomes

$$H_y = U_0^{-1}\Phi(x,a)U_0, \quad \Phi(x,a) = (I + H_x H_a)^{-1} H_{x-a}. \tag{4.37}$$

By the definition of $\mathcal{K}_\lambda$,

$$\mathcal{K}_\lambda(H_x, H_a)^\dagger = \mathcal{K}_\lambda(H_a, H_x) \tag{4.38}$$

and, by (3.26) and $H_x^\dagger = H_x$,

$$U(x,a)^\dagger = U(a,x). \tag{4.39}$$

So from (4.36) we have the equality

$$\overline{f}U(a,x) + \overline{h}\Phi(x,a)^\dagger U(a,x) = fU(a,x) + hUa,x)\Phi(a,x)$$

or

$$\overline{f}I + \overline{h}\Phi(x,a)^{\dagger} = fI + hU(a,x)\Phi(a,x)U(a,x)^{-1}. \tag{4.40}$$

According to Lemma 4 $\Phi(x,a)$ is Hermitian and $Tr[\Phi(x,a)] = 0$. So the trace of (4.40) implies

$$\overline{f} = f \tag{4.41}$$

and then

$$\overline{h}\Phi(x,a)^{\dagger} = hU(a,x)\Phi(a,x)U(a,x)^{-1}. \tag{4.42}$$

We let $x = 0$ in (4.42) and have

$$\overline{h}(aa')H_{-a} = h(aa')H_a$$

or

$$\overline{h} = -h. \tag{4.43}$$

Moreover, we have the following formulas

$$\frac{\partial \mathcal{K}_\lambda(H_x,0)}{\partial x^\mu} = 2f'x^\mu I + 2h'x^\mu H_x + h\sigma_\mu \tag{4.44}$$

and

$$\frac{\partial^2 \mathcal{K}_\lambda(H_x,0)}{\partial x^\mu x^\nu} = (4f''x^\mu x^\nu + 2f'\delta_{\mu\nu})I + (4h''x^\mu x^\nu + 2h'\delta_{\mu\nu})H_x + 2(h'x^\mu\sigma_\nu + h'x^\nu\sigma_\mu). \tag{4.45}$$

The Christoffel symbol associated to ds_3^2 is

$$\left\{ \begin{array}{c} \alpha \\ \beta\mu \end{array} \right\} = -\frac{2}{1+xx'}(x^\mu\delta^\alpha_\beta + x^\beta\delta^\alpha_\mu - x^\alpha\delta_{\beta\mu}), \tag{4.46}$$

and

$$g^{\beta\mu}\left\{ \begin{array}{c} \alpha \\ \beta\mu \end{array} \right\} = 2(1+xx')x^\alpha. \tag{4.47}$$

The Riemann connection is

$$\Gamma^\alpha_{\beta\mu} = \frac{2}{1+xx'}(x^\alpha\delta^\beta_\mu - x^\beta\delta^\alpha_\mu) \tag{4.48}$$

And the spin connection is

$$\mathfrak{B}_\mu = \frac{1}{2(1+xx')}[H_x,\sigma_\mu] = \frac{1}{2(1+xx')}(H_x\sigma_\mu - \sigma_\mu H_x). \tag{4.49}$$

We have the following formulae:

(i)

$$g^{\mu\nu}\frac{\partial^2\mathcal{K}_\lambda(H_x,0)}{\partial x^\mu x^\nu}=(1+r^2)^2[(4f''r^2+6f')I+(4h''r^2+10h')H_x];$$

(ii)

$$g^{\mu\nu}\left\{\begin{array}{c}\alpha\\ \mu\,\nu\end{array}\right\}\frac{\partial\mathcal{K}_\lambda(H_x,0)}{\partial x^\alpha}=2(1+r^2)[2f'r^2I+(2h'r^2+h)H_x];$$

(iii)

$$g^{\mu\nu}\frac{\partial\mathfrak{B}\mu}{\partial x^\nu}=0;$$

(iv)

$$-g^{\mu\nu}\left\{\begin{array}{c}\alpha\\ \mu\,\nu\end{array}\right\}\mathfrak{B}_\alpha\mathcal{K}_\lambda(H_x,0)=0;$$

(v)

$$g^{\mu\nu}\left[\mathfrak{B}_\mu\frac{\partial}{\partial x^\nu}\mathcal{K}_\lambda(H_x,0)+\mathfrak{B}_\nu\frac{\partial}{\partial x^\mu}\mathcal{K}_\lambda(H_x,0)\right]=4(1+r^2)hH_x;$$

(vi)

$$g^{\mu\nu}\mathfrak{B}_\mu\mathfrak{B}_\nu\mathcal{K}_\lambda(H_x,0)=-2r^2fI-2hr^2H_x.$$

From (i) to (vi) and the Weizenböck formula we have

$$\begin{aligned}&\not{D}^2_{S^3}\mathcal{K}_\lambda(H_x,0)=\Delta\mathcal{K}_\lambda(H_x,0)-6(fI+hH_x)\\&=\left\{4r^2(1+r^2)^2f''+(1+r^2)[6(1+r^2)-4r^2]f'-(2r^2+6)f\right\}I\\&+\left\{4r^2(1+r^2)^2h''+(1+r^2)[10(1+r^2)-4r^2]h'+[2(1+r^2)-2r^2-6]h\right\}H_x\\&=-\lambda(fI+hH_x).\end{aligned}$$

This means that $f(t)$ and $h(t)$ $(t=r^2)$should satisfy the following differential equations respectively

$$4t(1+t)^2f''+(1+t)[6(1+t)-4t]f'-(2t+6)f=-\lambda f,\tag{4.50}$$

and

$$4t(1+t)^2h''+(1+t)[10(1+t)-4t]h'-4h=-\lambda h.\tag{4.51}$$

For simplicity we write $\mathcal{K}_\lambda(x,a)=\mathcal{K}_\lambda(H_x,H_a)$.

Theorem 1 in §1 is proved.

5. The solution of the Einstein-Dirac equation. Let $\widehat{\psi}_0(x)$ be a spinor in S^3 which is orthogonal invariant. Obviously, the spinor

$$\widehat{\psi}(x) = \int_{S^3} \mathcal{K}_\lambda(x,u)\widehat{\psi}_0(u)\dot{u}, \quad \dot{u} = \sqrt{-g}du^1du^2du^3 \tag{5.1}$$

is orthogonal invariant. This spinor satisfies

$$\not{\mathfrak{D}}^2_{S^3}\widehat{\psi}(x) = -\lambda\widehat{\psi}(x) \tag{5.2}$$

where λ is a eigenvalue of $\not{\mathfrak{D}}^2_{S^3}$, and the spinor

$$\psi(x) = e^{inx^0}\widehat{\psi}(x_1) \tag{5.4}$$

satisfies

$$\mathfrak{D}^2\psi(x) = -m^2\psi(x) \tag{5.4}$$

when m is taken as $\lambda = n^2 - m^2$. Moreover, according to **Theorem 1**, the 4-component spinor obtained by the following formula

$$\Psi = \begin{pmatrix} \psi \\ \varphi^* \end{pmatrix}, \qquad \varphi^* = \frac{i}{m}\mathfrak{D}\psi \tag{5.5}$$

satisfies the Dirac equation

$$\not{\mathfrak{D}}\Psi = -im\Psi \tag{5.6}$$

If the energy-momentum tensor T_{jk} of Ψ is not identically zero, then the tensor at $x = 0$ must be of the form

$$(T_{jk}(0)) = \begin{pmatrix} c_0 & 0 \\ 0 & c_1 I \end{pmatrix}. \tag{5.7}$$

In fact, since the metric ds^2 is invariant under $\mathcal{G}_1$, the tensor T_{jk} must be invariant under $\mathcal{G}_1$. That is

$$T_{jk}(y_1) = T_{pq}(x_1)\frac{\partial x^p}{\partial y^j}\frac{\partial x^q}{\partial y^k} \tag{5.8}$$

where $y^0 = x^0 - a^0$ and y^μ is defined by (3.28). Especially, if we choose $a_1 = (a^0, a) = 0$, we have

$$T_{jk}(0) = T_{pq}(0)\ell^p_j\ell^q_k \tag{5.9}$$

where

$$L = \left(\ell^j_k\right) = \begin{pmatrix} 1 & 0 \\ 0 & \Gamma \end{pmatrix}, \quad \Gamma \in SO(3).$$

Therefore, (5.9) can be written into matrix form

$$(T_{jk}(0)) = \begin{pmatrix} 1 & 0 \\ 0 & \Gamma \end{pmatrix}(T_{jk}(0))\begin{pmatrix} 1 & 0 \\ 0 & \Gamma' \end{pmatrix}$$

for arbitrary Γ. Hence $T_{jk}(0)$ must be the form (5.7).

We assert $c_0+c_1 \neq 0$. In fact, c_0 and c_1 can be not zero simultaneously, otherwise $T_{jk}(x_1) \equiv 0$ according to (5.8), because $\mathcal{G}_1$ acts transitively on $S^1 \times S^3$. Moreover, according to the definition of T_{jk}, we have

$$\begin{aligned} g^{jk}T_{jk} &= \tfrac{i}{2}\left[\overline{\Psi}^{*\prime}\eta_{ab}\gamma^b\left(\eta^{ac}e^k_{(c)}\bigtriangledown_k\Psi+\eta^{ac}e^j_{(c)}\bigtriangledown_j\Psi\right)\right] \\ &\quad -\tfrac{i}{2}\left[\eta_{ab}\left(\eta^{ac}e^k_{(c)}\overline{\bigtriangledown_k\Psi^{*\prime}}+\eta^{ac}e^j_{(c)}\overline{\bigtriangledown_j\Psi^{*\prime}}\right)\gamma^b\Psi\right] \\ &= i\left[\overline{\Psi^{*\prime}}\not{\mathcal{D}}\Psi-\overline{(\not{\mathcal{D}}\Psi^*)}'\Psi\right]=-m\left[\overline{\Psi^{*\prime}}\Psi-\overline{\Psi^{*\prime}}\Psi\right]=0. \end{aligned} \tag{5.10}$$

Especially,

$$\left(g^{jk}T_{jk}\right)_{x=0}=c_0-3c_1=0, \quad \text{or} \quad c_0=3c_1. \tag{5.11}$$

So $c_0+c_1=4c_1\neq 0$.

Hence the Einstein equation at $x=0$ is

$$R_{jk}(0)-\frac{1}{2}g_{jk}(0)R(0)-\Lambda g_{jk}(0)=\mathcal{X}T_{jk}(0) \tag{5.12}$$

According to the orthogonal invariant of $R_{jk}(0)$ and $R_{0j}=R_{j0}=0$, we have (5.12) in form of matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & R_{11}(0)I \end{pmatrix}-\frac{1}{2}R(0)\begin{pmatrix} 1 & 0 \\ 0 & -I \end{pmatrix}-\Lambda\begin{pmatrix} 1 & 0 \\ 0 & -I \end{pmatrix}=\mathcal{X}\begin{pmatrix} c_0 & 0 \\ 0 & c_1 I \end{pmatrix}$$

or

$$\begin{cases} -\frac{1}{2}R(0)-\Lambda=\mathcal{X}c_0 \\ R_{11}(0)+\frac{1}{2}R(0)+\Lambda=\mathcal{X}c_1 \end{cases}$$

If we choose

$$\mathcal{X}=\frac{1}{c_0+c_1}R_{11}(0), \qquad \Lambda=\frac{-c_0}{c_0+c_1}R_{11}(0)-\frac{1}{2}R(0)$$

then (5.12) is satisfied and the Einstein equation is also satisfied at any point of $S^1\times S^3$ because it is invariant under $\mathcal{G}_1$.

Theorem 2 given in §1 is proved.

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