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MODULI SPACES OF SL(r)-BUNDLES ON SINGULAR IRREDUCIBLE CURVES*

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Introduction. One of the problems in moduli theory, motivated by physics, is to study the degeneration of moduli spaces of semistable G-bundles on curves of genus $g \geq 2$. When a smooth curve Y specializes to a stable curve X, one expects that the moduli space of semistable G-bundles on Y specializes to a (nice) moduli space of generalized semistable G-torsors on X. It is well known ([Si]) that for any flat family $\mathcal{C} \to S$ of stable curves there is a family $\mathcal{U}(r,d)_S \to S$ of moduli spaces $\mathcal{U}_{\mathcal{C}_s}(r,d)$ of (s-equivalence classes of) semistable torsion free sheaves of rank r and degree d on curves $\mathcal{C}_s$ ($s \in S$). If we fix a suitable representation $G \to \mathrm{GL}(r)$, one would like to define a moduli space $\mathcal{U}_X(G)$ of sutiable G-sheaves on X with at least a morphism $\mathcal{U}_X(G) \to \mathcal{U}_X(r,d)$. Moreover, it should behave well under specialization, i.e. if a smooth curve Y specializes to X, then the moduli space of G-bundles on Y specializes to $\mathcal{U}_X(G)$. By my knowledge, the problem is almost completely open except for special case like $G = \mathrm{SO}(r)$ or $G = \mathrm{Sp}(r)$ ([Fa1], [Fa2]), where one has a generalisation of G-torsors which extends the case $G = \mathrm{GL}(r)$. It is open even for $G = \mathrm{SL}(r)$ (See [Fa1], [Fa2] for the introduction).

In this paper, we will consider the case $G = \mathrm{SL}(r)$ and X being irreducible (the case of a reducible curve with one node was studied in [Su2]). For any projective curve X, we will use $\mathcal{U}_X(r,d)$ to denote the moduli space of semistable torsion free sheaves of rank r and degree d on X. If X_η is a smooth curve and L_η is a line bundle of degree d on X_η, we use $\mathcal{U}_{X_\eta}(r,L_\eta)$ to denote the moduli space of semistable vector bundles of rank r with fixed determinant L_η on X_η, which is a closed subvariety of $\mathcal{U}_{X_\eta}(r,d)$. It is known that when X_η specializes to X the moduli space $\mathcal{U}_{X_\eta}(r,d)$ specializes to $\mathcal{U}_X(r,d)$. It is natural to expect that if L_η specializes to a torsion free sheaf L on X then $\mathcal{U}_{X_\eta}(r,L_\eta)$ specializes to a closed subscheme $\mathcal{U}_X(r,L) \subset \mathcal{U}_X(r,d)$ (or a scheme with a morphism to $\mathcal{U}_X(r,d)$). It is important that we should look for an intrinsic $\mathcal{U}_X(r,L)$ (i.e. independent of X_η) which should not be *too bad* and should represent a *moduli problem.*

Let $S = \mathrm{Spec}(A)$ where A is a discrete valuation ring, let $\mathcal{C} \to S$ be a proper flat family of curves with closed fibre $\mathcal{C}_0 \cong X$ and smooth generic fibre $\mathcal{C}_\eta$. Then we have a S-flat scheme $\mathcal{U}(r,d)_S \to S$ with generic fibre $\mathcal{U}_{\mathcal{C}_\eta}(r,d)$ and closed fibre being $\mathcal{U}_X(r,d)$. For any line bundle $\mathcal{L}_\eta$ of degree d on $\mathcal{C}_\eta$, there is a unique extension $\mathcal{L}$ on $\mathcal{C}$ such that $\mathcal{L}|_{\mathcal{C}_0} := L$ is torsion free of degree d (since X is irreducible). Then $\mathcal{U}_{\mathcal{C}_\eta}(r,\mathcal{L}_\eta) \subset \mathcal{U}(r,d)_S$ is an irreducible, reduced, locally closed subscheme. Let

$$f : \mathcal{U}(r,\mathcal{L})_S := \overline{\mathcal{U}_{\mathcal{C}_\eta}(r,\mathcal{L}_\eta)} \subset \mathcal{U}(r,d)_S \to S$$

be the Zariski closure of $\mathcal{U}_{\mathcal{C}_\eta}(r,\mathcal{L}_\eta)$ in $\mathcal{U}(r,d)_S$. Then $f : \mathcal{U}(r,\mathcal{L})_S \to S$ is flat and projective, but there is no reason that its closed fibre $f^{-1}(0)$ (even its support $f^{-1}(0)_{\mathrm{red}}$)

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is indpendent of the family $\mathcal{C} \to S$ and $\mathcal{L}_\eta$. However, there are conjectures ([NS]) that $f^{-1}(0)$ is intrinsic for irreducible curves X with only one node. To state them, we introduce the notation for any stable irreducible curves. Let X be an irreducible stable curve with δ nodes $\{x_1, ..., x_\delta\}$, and L a torsion free sheaf of rank one and degree d on X. A torsion free sheaf F of rank r and degree d on X is called with a determinant L if there exists a morphism $(\wedge^r F) \to L$ which is an isomorphism outside the nodes of X. The subset $\mathcal{U}_X(r, L) \subset \mathcal{U}_X(r, d)$ consists of s-equivalence classes $[F] \in \mathcal{U}_X(r, d)$ such that $[F]$ contains a sheaf with a fixed determinant L. Then D.S. Nagaraj and C.S. Seshadri made the following conjectures (See Conjecture (a) and (b) at page 136 of [NS]):

(1) If L is a line bundle on X and $\mathcal{U}_X(r, L)^0 \subset \mathcal{U}_X(r, L)$ is the subset of locally free sheaves, then $\mathcal{U}_X(r, L)$ is the closure of $\mathcal{U}_X(r, L)^0$ in $\mathcal{U}_X(r, d)$.

(2) Let $\mathcal{L}_\eta$ (resp. L) be a line bundle (resp. torsion free sheaf of rank one) of degree d on smooth curve Y (resp. X). Assume that $\mathcal{L}_\eta$ specializes to L as Y specializes to X. Then $\mathcal{U}_X(r, L)$ is the specialization of $\mathcal{U}_Y(r, \mathcal{L}_\eta)$.

We answer (1) completely. In fact, even if L is not locally free (thus $\mathcal{U}_X(r, L)$ contains no locally free sheaf), we prove that torsion free sheaves of type 1 (See Section 1) are dense in $\mathcal{U}_X(r, L)$.

THEOREM 1. *Let L be a torsion free sheaf of rank* 1 *and degree d. Define*

$$\mathcal{U}_X(r, L)^0 = \{F \in \mathcal{U}_X(r, L) \mid (\wedge^r F) \cong L\}$$

which coincides with the subset of locally free sheaves when L is locally free. Then

(1) *$\mathcal{U}_X(r, L)$ is the closure of $\mathcal{U}_X(r, L)^0$. If L is not locally free, $\mathcal{U}_X(r, L)^0$ is the subset of torsion free sheaves of type* 1.

(2) *There is a canonical scheme structure on $\mathcal{U}_X(r, L)^0$, which is reduced when L is locally free, such that when smooth curve $\mathcal{C}_\eta$ specializes to X and $\mathcal{L}_\eta$ specializes to L on X, the specialization $f^{-1}(0)$ of $\mathcal{U}_{\mathcal{C}_\eta}(r, \mathcal{L}_\eta)$ contains a dense open subscheme which is isomorphic to $\mathcal{U}_X(r, L)^0$. In particular,*

$$f^{-1}(0)_{\mathrm{red}} \cong \mathcal{U}_X(r, L).$$

If the specialization $f^{-1}(0)$ has no *embedded point*, then our theorem also proved Conjecture (2). Unfortunately, $\mathcal{U}_X(r, L)$ seems not represent a nice moduli functor, we can not say anything about the scheme structure of $\mathcal{U}_X(r, L)$. To remedy this, we consider the specialization of $\mathcal{U}_{\mathcal{C}_\eta}(r, \mathcal{L}_\eta)$ in the so called generalized Gieseker space $G(r, d)$ (See [NSe]). Let X be an irreducible stable curve with only one node p_0 and L be a line bundle of degrre d on X. Then, when $r = 2$, we show that there is a Cohen-Macaulay closed subscheme $G(r, L) \subset G(r, d)$ of pure dimension $(r^2 - 1)(g - 1)$, which represents a nice moduli functor (See Definition 3.2). Moreover, $G(r, L)$ satisfies the requirements in (2) for specializations. It is known ([NSe] that there is a canonical birational morphism $\theta : G(r, d) \to \mathcal{U}_X(r, d)$. We prove in Lemma 3.4 that the *set-theoretic image* of $G(r, L)$ is $\mathcal{U}_X(r, L)$. Thus we can endow $\mathcal{U}_X(r, L)$ a scheme structure by the *scheme-theoretic image* of $G(r, L)$. Then we have

THEOREM 2. *Let X be an irreducible curve of genus $g \geq 2$ with only one node p_0. Let L be a line bundle of degree d on X. Then, when $r = 2$ and $(2, d) = 1$, we have*

(1) *There is a Cohen-Macaulay projective scheme $G(2, L)$ of pure dimension*

$3(g-1)$, *which represents a moduli functor.*

(2) *Let* $\mathcal{C} \to S$ *be a proper family of curves over a discrete valuation ring, which has smooth generic fibre* $\mathcal{C}_\eta$ *and closed fibre* $\mathcal{C}_0 \cong X$. *If there is a line bundle* $\mathcal{L}$ *on* $\mathcal{C}$ *such that* $\mathcal{L}|_{\mathcal{C}_0} \cong L$. *Then there exists an irreducible, reduced, Cohen-Macaulay* S*-projective scheme* $f : G(2,\mathcal{L})_S \to S$, *which represents a moduli functor, such that* $f^{-1}(0) \cong G(2,L)$, $f^{-1}(\eta) \cong \mathcal{U}_{\mathcal{C}_\eta}(2,\mathcal{L}_\eta)$.

(3) *There exists a proper birational* S*-morphism* $\theta : G(2,\mathcal{L})_S \to \mathcal{U}(2,\mathcal{L})_S$ *which induces a morphism* $\theta : G(2,L) \to \mathcal{U}_X(2,L)$.

Theorem 1 is proved in Section 1. In Section 2, we introduce the objects which are used to define Gieseker moduli space. Then Theorem 2 is proved in Section 3.

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1. Torsion-free sheaves with fixed determinant on irreducible curves. Let X be a stable irreducible curve of genus g with δ nodes $x_1,\ldots, x_\delta$. Any torsion free sheaf $\mathcal{F}$ of rank r on X can be written into (locally at x_i)

$$\mathcal{F} \otimes \hat{\mathcal{O}}_{X,x_i} \cong \hat{\mathcal{O}}_{X,x_i}^{\oplus a_i} \oplus m_{x_i}^{\oplus(r-a_i)}.$$

We call that $\mathcal{F}$ has type $r-a_i$ at x_i. Let $\mathcal{U}_X(r,d)$ be the moduli space of s-equivalence classes of semistable torsion free sheaves of rank r and degree d on X. Inspired by [NS], we make the following definition.

DEFINITION 1.1. *Let* L *be a torsion free sheaf of rank one and degree* d *on* X. *A torsion free sheaf* $\mathcal{F}$ *of rank* r *and degree* d *on* X *is called with a determinant* L *if there exists a non-trivial morphism* $\wedge^r\mathcal{F} \to L$ *which is an isomorphism outside the nodes.*

LEMMA 1.2. *For any exact sequence* $0 \to \mathcal{F}_1 \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{F}_2 \to 0$ *of torsion free sheaves with rank* r_1, r, r_2 *respectively, we have a morphism*

$$(\wedge^{r_1}\mathcal{F}_1) \otimes (\wedge^{r_2}\mathcal{F}_2) \to \frac{\wedge^r\mathcal{F}}{torsion},$$

which is an isomorphism outside the nodes. In particular, if a semistable sheaf $\mathcal{F}$ *has a fixed determinant* L, *then the associated graded torsion free sheaf* $gr(\mathcal{F})$ *will also have the fixed determinant* L.

Proof. There is a morphism $\wedge^{r_2}\mathcal{F}_2 \to \mathcal{H}om(\wedge^{r_1}\mathcal{F}_1, \wedge^r\mathcal{F}/torsion)$, which locally is defined as follows: For any $\omega \in \wedge^{r_2}\mathcal{F}_2$, choose a preimage $\widetilde{\omega} \in \wedge^{r_2}\mathcal{F}$ with respect to $\wedge^{r_2}\beta$. Then the image of ω is defined to be the morphism

$$\wedge^{r_1}\mathcal{F}_1 \to \wedge^r\mathcal{F}/torsion,$$

which takes any $f \in \wedge^{r_1}\mathcal{F}_1$ to the section $(\wedge^{r_1}\alpha)(f) \wedge \widetilde{\omega} \in \wedge^r\mathcal{F}/torsion$, which does not depend on the choice of $\widetilde{\omega}$ since the image of $\wedge^{r_1+1}\alpha$ is a torsion sheaf. The morphism defined above is isomorphism outside the nodes (See Lemma 1.2 of [KW]). Thus we have the desired morphism

$$(\wedge^{r_1}\mathcal{F}_1) \otimes (\wedge^{r_2}\mathcal{F}_2) \to (\wedge^{r_1}\mathcal{F}_1) \otimes \mathcal{H}om(\wedge^{r_1}\mathcal{F}_1, \frac{\wedge^r\mathcal{F}}{torsion}) \to \frac{\wedge^r\mathcal{F}}{torsion}.$$

DEFINITION 1.3. *The subset $\mathcal{U}_X(r,L) \subset \mathcal{U}_X(r,d)$ and $\mathcal{U}_X(r,L)^0 \subset \mathcal{U}_X(r,L)$ are defined to be*

$$\mathcal{U}_X(r,L) = \left\{ \begin{array}{l} s\text{-equivalence classes } [\mathcal{F}] \in \mathcal{U}_X(r,d) \text{ such that} \\ [\mathcal{F}] \text{ contains a sheaf with a fixed determinant } L \end{array} \right\}$$

$$\mathcal{U}_X(r,L)^0 = \{\, [\mathcal{F}] \in \mathcal{U}_X(r,L) \mid \wedge^r \mathcal{F} \cong L\}$$

When L is a line bundle, $\mathcal{U}_X(r,L)^0$ consists of locally free sheaves with the fixed determinant L. When L is not a line bundle, $\mathcal{U}_X(r,L)^0$ consists of torsion free sheaves of type 1 at each node of X.

We first consider the case that L is a line bundle and X has only one node p_0. Let $\pi : \widetilde{X} \to X$ be the normalization with $\pi^{-1}(p_0) = \{p_1, p_2\}$. The normalization $\phi : \mathcal{P} \to \mathcal{U}_X(r,d)$ was studied in [Su1], where $\mathcal{P}$ is the moduli spaces of semistable generalized parabolic bundles (GPB) of degree d and rank r on $\widetilde{X}$. A GPB of degree d and rank r on $\widetilde{X}$ is a pair (E,Q) consisting of a vector bundle E of degree d and rank r on $\widetilde{X}$ and a r-dimensional quotient $E_{p_1} \oplus E_{p_2} \to Q$. There is a flat morphism (See Lemma 5.7 of [Su1])

$$Det : \mathcal{P} \to J^d_{\widetilde{X}}$$

sending (E,Q) to $det(E)$. Let $\widetilde{L} = \pi^*(L)$ and $\mathcal{P}^{\widetilde{L}} = Det^{-1}(\widetilde{L})$. Then $\mathcal{P}^{\widetilde{L}}$ is an irreducible projective variety (See the proof of Lemma 5.7 in [Su1]). Let $\mathcal{D}_i$ $(i = 1,2)$ be the divisor consisting of (E,Q) such that $E_{p_i} \to Q$ is not an isomorphism (See [Su1] for details). Let $\mathcal{D}_i^{\widetilde{L}} = \mathcal{D}_i \cap \mathcal{P}^{\widetilde{L}}$.

LEMMA 1.4. *The set $\mathcal{U}_X(r,L)$ is contained in the image $\phi(\mathcal{P}^{\widetilde{L}})$. Moreover,*

$$\mathcal{U}_X(r,L) \setminus \mathcal{U}_X(r,L)^0 \subset \phi(\mathcal{D}_1^{\widetilde{L}} \cap \mathcal{D}_2^{\widetilde{L}}).$$

Proof. Let $F \in \mathcal{U}_X(r,L)$ with $F \otimes \hat{\mathcal{O}}_{p_0} \cong \hat{\mathcal{O}}_{p_0}^{\oplus a} \oplus m_{p_0}^{\oplus(r-a)}$. Let $\widetilde{E} = \pi^* F/torsion$. Then, by local computions (See, for example, Remark 2.1, 2.6 of [NS]), we have

$$0 \to F \xrightarrow{d} \pi_* \widetilde{E} \to {}_{p_0}\widetilde{Q} \to 0 \tag{1.1}$$

where $dim(\widetilde{Q}) = a$ and the quotient $\pi_*\widetilde{E} \to {}_{p_0}\widetilde{Q}$ induces two surjective maps $\widetilde{E}_{p_i} \to \widetilde{Q}$ $(i = 1,2)$. Denote their kernel by K_i, we have

$$0 \to K_i \to \widetilde{E}_{p_i} \to \widetilde{Q} \to 0.$$

On the other hand, for $F \in \mathcal{U}_X(r,L)$, let $\mathcal{Q}$ be the cokernel of $\wedge^r F \to L$, then

$$0 \to det(\widetilde{E}) \to \widetilde{L} \to \pi^*\mathcal{Q} \to 0$$

where $\pi^*\mathcal{Q} = {}_{p_1}V_1 \oplus {}_{p_2}V_2$ and n_1, n_2 is respectively the dimension of V_1, V_2. Thus $det(\widetilde{E}) = \widetilde{L} \otimes \mathcal{O}_{\widetilde{X}}(-n_1p_1 - n_2p_2)$ where $n_i \geq 0$ and $n_1 + n_2 = r - a$.

Let $h : \widetilde{E} \to E$ be the Hecke modifications at p_1 and p_2 such that $ker(h_{p_i}) \subset K_i$ has dimension n_i for $i = 1,2$. Then we have

$$0 \to \widetilde{E} \xrightarrow{h} E \to {}_{p_1}\widetilde{Q}_1 \oplus {}_{p_2}\widetilde{Q}_2 \to 0 \tag{1.2}$$

with $dim(\widetilde{Q}_i) = n_i$. Thus $det(E) = det(\widetilde{E}) \otimes \mathcal{O}_{\widetilde{X}}(n_1p_1 + n_2p_2) = \widetilde{L}$ and $\phi(E,Q) = F$ if we define Q by the exact sequence

$$0 \to F \xrightarrow{(\pi_* h)\cdot d} \pi_* E \to {}_{p_0}Q \to 0. \tag{1.3}$$

To describe the GPB $(E, E_{p_1} \oplus E_{p_2} \xrightarrow{q} Q \to 0)$, note that (1.3) induces

$$F_{p_0} \xrightarrow{d_{p_0}} \widetilde{E}_{p_1} \oplus \widetilde{E}_{p_2} \xrightarrow{h_{p_1} \oplus h_{p_2}} E_{p_1} \oplus E_{p_2} \xrightarrow{q} Q \to 0.$$

Then $d_{p_0}(F_{p_0}) \cap \widetilde{E}_{p_i} = K_i$ by (1.1) and $h_{p_i}(K_i) = ker(q_i)$ by the exactness of (1.3), where $q_i : E_{p_i} \to Q$ $(i = 1, 2)$ are projections induced by $E_{p_1} \oplus E_{p_2} \xrightarrow{q} Q \to 0$. Thus $dim(ker(q_i)) = r - a - n_i$ by the construction of h.

For any $F \in \mathcal{U}_X(r,L) \setminus \mathcal{U}_X(r,L)^0$, the cokernel $\mathcal{Q}$ of $\wedge^r F \to L$ must be non-trivial. This implies that both V_1 and V_2 in $\pi^*\mathcal{Q} = {}_{p_1}V_1 \oplus {}_{p_2}V_2$ are non-trivial since for any $i = 1, 2$, we have

$$Hom_{\mathcal{O}_{\widetilde{X}}}({}_{p_i}V_i, {}_{p_i}\mathbb{C}) = Hom_{\mathcal{O}_{\widetilde{X}}}(\pi^*\mathcal{Q}, {}_{p_i}\mathbb{C}) = Hom_{\mathcal{O}_X}(\mathcal{Q}, \pi_*({}_{p_i}\mathbb{C})) \neq 0.$$

Thus their dimensions n_1 and n_2 must be positive and $n_1 + n_2 = r - a$, which means that $ker(q_i) \neq 0$ $(i = 1, 2)$ and the GPB (E, Q) must be in $\mathcal{D}_1 \cap \mathcal{D}_2$. Thus

$$\mathcal{U}_X(r,L) \setminus \mathcal{U}_X(r,L)^0 \subset \phi(\mathcal{D}_1^{\widetilde{L}} \cap \mathcal{D}_2^{\widetilde{L}}).$$

Remark 1.5. This is also indicated in the following consideration. There is a $\mathbb{P}^1$-bundle $p : \mathbb{P} \to J^d_{\widetilde{X}}$ and the normalization map $\phi_1 : \mathbb{P} \to J^d_X$. The morphism $Det : \mathcal{P} \to J^d_{\widetilde{X}}$ can be lift to a rational morphism

$$\widetilde{Det} : \mathcal{P} \dashrightarrow \mathbb{P} \xrightarrow{\phi_1} J^d_X,$$

which is well-defined on $\mathcal{P} \setminus \mathcal{D}_1 \cap \mathcal{D}_2$. When L is a line bundle, $\widetilde{Det}^{-1}(L)$ is disjoint with $\mathcal{D}_i \setminus (\mathcal{D}_1 \cap \mathcal{D}_2)$.

LEMMA 1.6. *Let Λ be a discrete valuation ring and $T = Spec(\Lambda)$. Then, for any $F \in \mathcal{U}_X(r,L)$, there is a T-flat sheaf $\mathcal{F}$ on $X \times T$ such that*
(1) $\mathcal{F}_t = \mathcal{F}|_{X\times\{t\}}$ is locally free for $t \neq 0$ and $\mathcal{F}_0 = F$,
(2) $\wedge^r(\mathcal{F}|_{X\times(T\setminus\{0\})}) = p_X^* L$.
In particular, $\mathcal{U}_X(r,L)^0$ is dense in $\mathcal{U}_X(r,L)$.

Proof. Let $(E,Q) \in \mathcal{P}^{\widetilde{L}}$ be the GPB such that $\phi(E,Q) = F$ (Lemma 1.4). Then there exists a T-flat family of vector bundles $\mathcal{E}$ on $\widetilde{X} \times T$ with $det(\mathcal{E}) = p^*_{\widetilde{X}}\widetilde{L}$, and a T-flat quotient

$$\mathcal{E}_{p_1} \oplus \mathcal{E}_{p_2} \xrightarrow{q} \mathcal{Q} \to 0$$

such that $(\mathcal{E}_0, \mathcal{Q}_0) = (\mathcal{E}, \mathcal{Q})|_{\widetilde{X}\times\{0\}} = (E,Q)$. The quotient $\mathcal{E}_{p_1} \oplus \mathcal{E}_{p_2} \xrightarrow{q} \mathcal{Q} \to 0$ is determined by the two projections $q_i : \mathcal{E}_{p_i} \to \mathcal{Q}$ $(i = 1, 2)$, which can be choosen to be isomorphisms for $t \neq 0$ since $\mathcal{P}^{\widetilde{L}}$ is irreducible. The two maps q_i are given by two matrices

$$\begin{pmatrix} t^{a_1} & 0 & \dots & 0 \\ 0 & t^{a_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t^{a_r} \end{pmatrix}, \quad \begin{pmatrix} t^{b_1} & 0 & \dots & 0 \\ 0 & t^{b_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t^{b_r} \end{pmatrix}$$

where $0 \le a_1 \le a_2 \le \cdots \le a_r$ and $0 \le b_1 \le b_2 \le \cdots \le b_r$. When $t = 0$, they give the GPB (E, Q). We recall that when F is not locally free, the numbers n_1 and n_2 in the proof of Lemma 1.4 are positive. Thus the two projections $E_{p_i} \to Q$ are not isomorphism. Namely, there are k_1, k_2 such that $a_{k_1} > 0$, $b_{k_2} > 0$ but $a_j = 0$ $(j < k_1)$ and $b_j = 0$ $(j < k_2)$. It is clear now that we can change the positive numbers a_{k_1}, ..., a_r, b_{k_2}, ..., b_r freely such that the resulted family $(\mathcal{E}, \mathcal{Q})$ has the property that $(\mathcal{E}_0, \mathcal{Q}_0) = (E, Q)$. We modify the T-flat quotient $\mathcal{E}_{p_1} \oplus \mathcal{E}_{p_2} \xrightarrow{q} \mathcal{Q} \to 0$ by choosing a_{k_1}, ..., a_r, b_{k_2}, ..., b_r such that

$$\sum_{i=k_1}^{r} a_i - \sum_{i=k_2}^{r} b_i = 0.$$

Thus we get a T-flat sheaf $\mathcal{F}$ on $X \times T$ such that $\mathcal{F}_0 = F$. Moreover, on $T \setminus \{0\}$, $\mathcal{F}$ is obtained from $\mathcal{E}|_{\widetilde{X} \times (T \setminus \{0\})}$ by identifying $\mathcal{E}_{p_1}$ and $\mathcal{E}_{p_2}$ through the isomorphism

$$q_1 \cdot q_2^{-1} : \mathcal{E}_{p_1} \to \mathcal{E}_{p_2}.$$

$\wedge^r \mathcal{F}|_{X \times (T \setminus \{0\})}$ is obtained from $det(\mathcal{E})|_{\widetilde{X} \times (T \setminus \{0\})} = p_{\widetilde{X}}^* \widetilde{L}$ by identifying $\widetilde{L}_{p_1} \otimes K(T)$ and $\widetilde{L}_{p_2} \otimes K(T)$ through the isomorphism $\wedge^r (q_1 \cdot q_2^{-1})$, where $K(T)$ denote the field of rational functions on T. By the choice of a_{k_1}, ..., a_r, b_{k_2}, ..., b_r, we know that $\wedge^r (q_1 \cdot q_2^{-1})$ is the identity map. Thus

$$\wedge^r \mathcal{F}|_{X \times (T \setminus \{0\})} = (p_X^* L)|_{X \times (T \setminus \{0\})}.$$

LEMMA 1.7. *For any stable irreducible curve X, $\mathcal{U}_X(r, L)^0$ is dense in $\mathcal{U}_X(r, L)$.*

Proof. Let δ be the number of nodes of X, we will prove the lemma by induction to δ. When $\delta = 1$, it is Lemma 1.6. Assume that the lemma is true for curves with $\delta - 1$ nodes. Then we show that for any $F \in \mathcal{U}_X(r, L)$ there is a T-flat sheaf $\mathcal{F}$ on $X \times T$, where $T = \mathrm{Spec}(\Lambda)$ and Λ is a discrete valuation ring, such that

(1) $\mathcal{F}_t = \mathcal{F}|_{X \times \{t\}}$ is locally free for $t \ne 0$ and $\mathcal{F}_0 = F$,

(2) $\wedge^r (\mathcal{F}|_{X \times (T \setminus \{0\})}) = p_X^* L$.

For $F \in \mathcal{U}_X(r, L)$, we can assume that F is not locally free. Let $p_0 \in X$ be a node at where F is not locally free. Let $\pi : \widetilde{X} \to X$ be the partial normalization at p_0 and $\pi^{-1}(p_0) = \{p_1, p_2\}$. Let $\widetilde{L} = \pi^* L$ and $\widetilde{E} = \pi^* F / torsion$, then by the same arguments of Lemma 1.4

$$0 \to F \xrightarrow{d} \pi_* \widetilde{E} \to {}_{p_0}\widetilde{Q} \to 0.$$

Note that $\wedge^r \widetilde{E} = \pi^*(\wedge^r F)/(\text{torsion at } \{p_1, p_2\})$ and the cokernel of $\wedge^r \widetilde{E} \to \widetilde{L}$ at $\{p_1, p_2\}$ is ${}_{p_1}\mathbb{C}^{n_1} \oplus {}_{p_2}\mathbb{C}^{n_2}$, we have the morphism

$$\wedge^r \widetilde{E} \to \widetilde{L} \otimes \mathcal{O}_{\widetilde{X}}(-n_1 p_1 - n_2 p_2)$$

which is an isomorphism outside the nodes of $\widetilde{X}$. As the same with proof of Lemma 1.4, we have the Hecke modification E of $\widetilde{E}$ at p_1 and p_2 such that

$$0 \to \widetilde{E} \xrightarrow{h} E \to {}_{p_1}\widetilde{Q}_1 \oplus {}_{p_2}\widetilde{Q}_2 \to 0$$

with $dim(\widetilde{Q}_i) = n_i$. Thus $\wedge^r E \cong (\wedge^r \widetilde{E}) \otimes \mathcal{O}_{\widetilde{X}}(n_1p_1 + n_2p_2) \to \widetilde{L}$ and the generalized parabolic sheaf (GPS) (E, Q) defines F by the exact sequence

$$0 \to F \xrightarrow{(\pi_* h)\cdot d} \pi_* E \to {}_{p_0}Q \to 0,$$

where Q is defined by requiring above sequence exact. The two projections $E_{p_i} \to Q$ $(i = 1, 2)$ are not isomorphism, thus, by choosing suitable bases of E_{p_1} and Q, they are given by matrices

$$P_1 = \begin{pmatrix} 1 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix}, \quad P_2 = A \cdot \begin{pmatrix} 1 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix} \cdot B$$

where A, B are invertable $r \times r$ matrices and $\text{rank}(P_i) = r_i < r$ $(i = 1, 2)$. Since $E \in \mathcal{U}_{\widetilde{X}}(r, \widetilde{L})$, by the assumption, there is a T-flat sheaf $\mathcal{E}$ on $\widetilde{X} \times T$ such that $\mathcal{E}_0 := \mathcal{E}|_{\widetilde{X}\times\{0\}} = E$ and $\mathcal{E}|_{\widetilde{X}\times(T\smallsetminus\{0\})}$ locally free with determinant $p^*_{\widetilde{X}}(\widetilde{L})$. Define the morphisms $q_i : \mathcal{E}_{p_i} := \mathcal{E}|_{\{p_i\}\times T} \to Q \otimes \mathcal{O}_T$ $(i = 1, 2)$ by using matrices

$$Q_1 = \begin{pmatrix} 1 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \\ 0 & \dots & 1 & 0\dots & 0 \\ 0 & \dots & 0 & t^{a_{r_1+1}}\dots & 0 \\ \vdots & \dots & \vdots & \ddots & \\ 0 & \dots & 0 & \dots & c\cdot t^{a_r} \end{pmatrix}, \quad Q_2 = A \cdot \begin{pmatrix} 1 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \\ 0 & \dots & 1 & 0\dots & 0 \\ 0 & \dots & 0 & t^{b_{r_2+1}}\dots & 0 \\ \vdots & \dots & \vdots & \ddots & \\ 0 & \dots & 0 & \dots & t^{b_r} \end{pmatrix} \cdot B$$

where t is the local parameter of Λ, a_{r_1+1}, ..., a_r, b_{r_2+1}, ..., b_r are positive integers satisfying $a_{r_1+1} + \cdots + a_r = b_{r_2+1} + \cdots + b_r$, and c is any constant. Then these morphisms q_i $(i = 1, 2)$ define a family $(\mathcal{E}, Q \otimes \mathcal{O}_T)$ of GPS, which induces a T-flat sheaf $\mathcal{F}$ on $X \times T$ such that $\mathcal{F}_0 = F$ and $\mathcal{F}_t$ $(t \neq 0)$ are locally free. The determinant $det(\mathcal{F}|_{X\times T^0})$, where $T^0 = T \smallsetminus \{0\}$, is defined by the sheaf $(det(\mathcal{E}|_{\widetilde{X}\times T^0}) = p^*_{\widetilde{X}}(\widetilde{L})$ through the isomorphism

$$det(q_2^{-1} \cdot q_1) : (det(\mathcal{E}|_{\widetilde{X}\times T^0})_{p_1} = (\wedge^r \mathcal{E}_{p_1})|_{T^0} \to (\wedge^r \mathcal{E}_{p_2})|_{T^0} = (det(\mathcal{E}|_{\widetilde{X}\times T^0})_{p_2},$$

which is a scale product by $det(Q_2^{-1} \cdot Q_1) = det(AB)^{-1} \cdot c$. Thus we can choose suitable constant c such that $det(\mathcal{F}|_{X\times T^0}) = p^*_X(L)$. We are done.

LEMMA 1.8. *When L is not locally free, $\mathcal{U}_X(r, L)^0$ consists of torsion free sheaves of type* 1 *at each node of X, which is dense in $\mathcal{U}_X(r, L)$.*

Proof. The proof follows the same idea. For simiplicity, we assume that X has only one node p_0. Let F be a torsion free sheaf of rank r and degree d on X with type $t(F) \geq 1$ at p_0. Then

$$deg(\wedge^r F/torsion) = d - t(F) + 1.$$

Thus $F \in \mathcal{U}_X(r, L)^0$ if and only if $t(F) = 1$.

For any $F \in \mathcal{U}_X(r, L)$ of type $t(F) > 1$, let $\widetilde{E} = \pi^* F/tosion$, then

$$0 \to F \xrightarrow{d} \pi_* \widetilde{E} \to {}_{p_0}\widetilde{Q} \to 0$$

where $dim(\widetilde{Q}) = r - t(F)$. Let $\widetilde{L} = \pi^* L/torsion$, then $deg(\widetilde{L}) = d-1$ and $L = \pi_*\widetilde{L}$. The condition $F \in \mathcal{U}_X(r, L)$ implies $det(\widetilde{E}) = \widetilde{L}(-n_1p_1 - n_2p_2)$ where $n_i \geq 0$ and $n_1 + n_2 = t(F) - 1$. As in the proof of Lemma 1.4, let $h : \widetilde{E} \to E$ be the Hecke modifications at p_1 and p_2 such that $dim(ker(h_{p_1})) = n_1 + 1$ and $dim(ker(h_{p_2})) = n_2$. Then we have $det(E) = det(\widetilde{E}) \otimes \mathcal{O}_{\widetilde{X}}((n_1+1)p_1 + n_2p_2) = \widetilde{L}(p_1)$, and there is an GPB $(E, E_{p_1} \oplus E_{p_2} \xrightarrow{q} Q \to 0)$ such that $\phi(E, Q) = F$, where $q_i : E_{p_i} \to Q$ $(i = 1, 2)$ satisfy $dim(ker(q_1)) = t(F) - n_1 - 1$ and $dim(ker(q_2)) = t(F) - n_2$.The two projections $q_i : E_{p_i} \to Q$ $(i = 1, 2)$ are given by matrices

$$P_1 = \begin{pmatrix} 1 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix}, \quad P_2 = A \cdot \begin{pmatrix} 1 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix} \cdot B$$

where $\text{rank}(P_1) = r - t(F) + n_1 + 1$, $\text{rank}(P_2) = r - t(F) + n_2$. Let $T = \text{Spec}(\mathbb{C}[t])$ and $\mathcal{E} = p_{\widetilde{X}}^* E$. Choose deformations $P_i(t)$ of P_i $(i = 1, 2)$ as following

$$\begin{pmatrix} 1 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & & & & \\ 0 & \dots & 1 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & t & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & t & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & t \end{pmatrix}, \quad A \cdot \begin{pmatrix} 1 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & & & & \\ 0 & \dots & 1 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & t & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & t & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \cdot B$$

where the number of t in $P_2(t)$ is $t(F) - n_2 - 1$. Then we get a family $(\mathcal{E}, Q \otimes \mathcal{O}_T)$ of GPB on $\widetilde{X} \times T$, which induces a T-flat sheaf $\mathcal{F}$ on $X \times T$ such that $\mathcal{F}_0 = F$ and $\mathcal{F}_t$ $(t \neq 0)$ are torsion free of type 1. To see that $\wedge^r \mathcal{F}_t \cong L$ $(t \neq 0)$, we note that $det(\mathcal{E}) = p_{\widetilde{X}}^* \widetilde{L}(p_1)$ and L is determined by the GPB

$$(det(E) = \widetilde{L}(p_1), G \subset det(E)_{p_1} \oplus det(E)_{p_2})$$

where G is the graph of zero map $det(E)_{p_2} \to det(E)_{p_1}$. Thus we have a non-trivial morphism $\wedge^r \mathcal{F}_t \to L$, which must be an isomorphism when $t \neq 0$.

Next we will prove that $\mathcal{U}_X(r, L)$ is the underlying scheme of specialization of the moduli spaces of semistable bundles with fixed determiant. This in particular implies that $\mathcal{U}_X(r, L) \subset \mathcal{U}_X(r, d)$ is a closed subset. Let $S = Spec(R)$ and R be a discrete valuation ring. Let $\mathcal{X} \to S$ be a flat proper family of curves with smooth generic fibre and closed fibre $\mathcal{X}_0 = X$. Let $\mathcal{L}$ be a relative torsion free sheaf on $\mathcal{X}$ of rank one and (relative) degree d such that $\mathcal{L}|_X = L$. It is well known that there exists a moduli scheme $f : \mathcal{U}(r, d)_S \to S$ such that for any $s \in S$ the fibre $f^{-1}(s)$ is the moduli space of semistable torsion free sheaves of rank r and degree d on $\mathcal{X}_s$ (where $\mathcal{X}_s$ denote the fibre of $\mathcal{X} \to S$ at s). Since $\mathcal{X}$ is smooth over $S^0 = S \setminus \{0\}$, there is a family $\mathcal{U}(r, \mathcal{L}|_{S^0})_{S^0} \to S^0$ of moduli spaces of semistable bundles with fixed determinant $\mathcal{L}|_{\mathcal{X}_s}$ on $\mathcal{X}_s$ $(s \in S^0)$. We have

$$\mathcal{U}(r, \mathcal{L}|_{S^0})_{S^0} \subset \mathcal{U}(r, d)_S.$$

Let Z be the Zariski closure of $\mathcal{U}(r,\mathcal{L}|_{S^0})_{S^0}$ inside $\mathcal{U}(r,d)_S$. We get a flat family

$$f : Z \to S$$

of projective schemes. For any $0 \neq s \in S$, the fibre Z_s is the moduli space of semistable bundles on $\mathcal{X}_s$ with fixed determinant $\mathcal{L}|_{\mathcal{X}_s}$.

LEMMA 1.9. *The fibre Z_0 of $f : Z \to S$ at $s = 0$ is contained in $\mathcal{U}_X(r,L)$ as a set.*

Proof. We can assume that for any $[F] \in Z_0$ there is a discrete valuation ring A and $T = Spec(A) \to S$ such that there is a T-flat family of torsion free sheave $\mathcal{F}$ on $\mathcal{X}_T = \mathcal{X} \times_S T \to T$, so that

$$\wedge^r \mathcal{F}_\eta \cong \mathcal{L}_\eta, \quad \mathcal{F}|_X \cong F.$$

By Proposition 5.3 of [Se] and its proof (see [Se], it deals with one node curve, but generalization to our case is straightforward since its proof is completely local), there is a birational morphism $\sigma : \Gamma \to \mathcal{X}_T$ and a vector bundle $\mathcal{E}$ on Γ such that $\sigma_*\mathcal{E} = \mathcal{F}$. Moreover, the morphism σ is an isomorphism over $\mathcal{X}_T \setminus \{x_1,...,x_k\}$. Since $(\wedge^r\mathcal{E})|_{\Gamma_\eta} \cong (\sigma^*\mathcal{L})^{\vee\vee}|_{\Gamma_\eta}$, note that $(\wedge^r\mathcal{E})^{-1} \otimes (\sigma^*\mathcal{L})^{\vee\vee}$ is torsion free (thus T-flat), we can extend the isomorphism into a morphism $\wedge^r\mathcal{E} \to (\sigma^*\mathcal{L})^{\vee\vee}$. Since σ_* and σ^* are adjoint functors, $\sigma_*\mathcal{O}_\Gamma = \mathcal{O}_{\mathcal{X}_T}$, we have $\sigma_*((\sigma^*\mathcal{N})^\vee) = \mathcal{N}^\vee$ for any coherent sheaf $\mathcal{N}$. Then, by using $\sigma^*(\mathcal{N}^\vee) = \sigma^*\sigma_*((\sigma^*\mathcal{N})^\vee) \to (\sigma^*\mathcal{N})^\vee$, we have a canonical morphism $\sigma_*((\sigma^*\mathcal{N})^{\vee\vee}) \to \mathcal{N}^{\vee\vee}$. In particular, there is a canonical morphism

$$\sigma_*((\sigma^*\mathcal{L})^{\vee\vee}) \to \mathcal{L}^{\vee\vee} \cong \mathcal{L}$$

which induce a morphism $\vartheta : \wedge^r\mathcal{F} = \wedge^r(\sigma_*\mathcal{E}) \to \sigma_* \wedge^r \mathcal{E} \to \mathcal{L}$. Modified by some power of the maximal ideal of A, we can assume the morphism ϑ being nontrivial on X, which means that ϑ is an isomorphism on $\mathcal{X}_T \setminus \{x_1,...,x_k\}$ since X is irreducible. Thus $[F] \in \mathcal{U}_X(r,L)$.

THEOREM 1.10. *$\mathcal{U}_X(r,L)$ is the closure of $\mathcal{U}_X(r,L)^0$ in $\mathcal{U}_X(r,d)$. When smooth curve $\mathcal{X}_s$ specializes to $\mathcal{X}_0 = X$ and $\mathcal{L}_s$ specializes to L, the moduli spaces $\mathcal{U}_{\mathcal{X}_s}(r,\mathcal{L}_s)$ of semistable bundles of rank r with fixed determinant $\mathcal{L}_s$ on $\mathcal{X}_s$ specializes to an irreducible scheme Z_0 with $(Z_0)_{\text{red}} \cong \mathcal{U}_X(r,L)$.*

Proof. Let $\mathcal{U}(r,d)^0_S \subset \mathcal{U}(r,d)_S$ be the open subscheme of torsion free sheaves of type at most 1. Then there is a well-defined S-morphism (taking determinant $det(\bullet) = \wedge^r(\bullet)$)

$$det : \mathcal{U}(r,d)^0_S \to \mathcal{U}(1,d)_S.$$

The given family of torsion free sheaves $\mathcal{L}$ on $\mathcal{X}$ of rank one and degree d gives a S-point $[\mathcal{L}] \in \mathcal{U}(1,d)_S$. It is clear that

$$Z^0 := (det)^{-1}([\mathcal{L}]) \subset Z$$

and the fibre of $f|_{Z^0} : Z^0 \to S$ at $s = 0$ is irreducible with support $\mathcal{U}_X(r,L)^0$ (it is also reduced when L is a line bundle). Thus $f^{-1}(0) = Z_0$ contains the closure $\overline{\mathcal{U}_X(r,L)^0}$ of $\mathcal{U}_X(r,L)^0$ in $\mathcal{U}_X(r,d)$. On the other hand, by Lemma 1.9, Lemma 1.8 and Lemma 1.7, we have

$$\overline{\mathcal{U}_X(r,L)^0} \subset (Z_0)_{\text{red}} \subset \mathcal{U}_X(r,L) \subset \overline{\mathcal{U}_X(r,L)^0}.$$

Hence $\mathcal{U}_X(r,L) = \overline{\mathcal{U}_X(r,L)^0} = (Z_0)_{\rm red}$. In particular, the fibre of $f: Z \to S$ at $s = 0$ is irreducible.

2. Stability and Gieseker functor. Let X be a stable curve with δ nodes $\{x_1, ..., x_\delta\}$. Any semistable curve with stable model X can be obtained from X by destabilizing the nodes x_i with chains R_i $(i = 1, ..., \delta)$ of projective lines. It will be denoted as $X_{\vec{n}}$, where $\vec{n} = (n_1, ..., n_\delta)$ and n_i is the length of R_i (See [NSe] for the example of $\delta = 1$). Then $X_{\vec{n}}$ are the curves which are semi-stably equivalent to X, we use $\pi: X_{\vec{n}} \to X$ to denote the canonical morphism contracting R_1, ..., R_δ to x_1, ..., x_δ respectively. A vector bundle E of rank r on a chain $R = \cup C_i$ of projective lines is called *positive* if $a_{ij} \geq 0$ in the decomposition $E|_{C_i} = \oplus_{j=1}^r \mathcal{O}(a_{ij})$ for all i and j. A *postive* E is called *strictly positive* if for each C_i there is at least one $a_{ij} > 0$. E is called *standard* (resp. *strictly standard*) if it is positive (resp. strictly positive) and $a_{ij} \leq 1$ for all i and j (See [NSe], [Se]).

For any semistable curve $X_{\vec{n}} = \cup X_{\vec{n}}^k$ of genus $g \geq 2$, let $\omega_{X_{\vec{n}}}$ be its canonical bundle and

$$\lambda_k = \frac{deg(\omega_{X_{\vec{n}}}|_{X_{\vec{n}}^k})}{2g-2},$$

it is easy to see that $\lambda_k = 0$ if and only if the irreducible component $X_{\vec{n}}^k$ is a component of the chains of projective lines.

DEFINITION 2.1. *A sheaf E of constant rank r on $X_{\vec{n}}$ is called (semi)stable, if for every subsheaf $F \subset E$, we have*

$$\chi(F) < (\leq) \frac{\chi(E)}{r} \cdot r(F) \quad \textit{when } r(F) \neq 0, r,$$

$$\chi(F) \leq 0 \quad \textit{when } r(F) = 0, \textit{ and } \chi(F) < \chi(E) \textit{ when } r(F) = r,\ F \neq E,$$

where, for any sheaf F, the rank $r(F)$ is defined to be $\sum \lambda_k \cdot rank(F|_{X_{\vec{n}}^k})$.

Let $C = X_{\vec{n}}$ and $C_0 = X_{(0,n_2,...,n_\delta)}$ (namly, C_0 is obtained from C by contracting the chain $R_1 = \bigcup_{k=1}^{n_1} \mathbb{P}_k^1$ of projective lines $\mathbb{P}_k^1 = \mathbb{P}^1$).

LEMMA 2.2. *Let $\pi: C \to C_0$ be the canonical morphism, let E be a torsion free sheaf that is locally free on R_1. If $E|_{R_1}$ is positive and π_*E is stable (semistable) on C_0, then E is stable (semistable) on C. In particular, a vector bundle on $X_{\vec{n}}$ is stable (semistable) if $E|_{R_i}$ $(1 \leq i \leq \delta)$ are positive and π_*E is stable (semistable) on X, where $\pi: X_{\vec{n}} \to X$ is the canonical morphism contracting R_1, ..., R_δ to x_1, ..., x_δ.*

Proof. Let $C = \widetilde{C}_0 \cup R_1$ and $\widetilde{C}_0 \cap R_1 = \{p_1, p_2\}$, where $\pi: \widetilde{C}_0 \to C_0$ is the partial normalization of C_0 at x_1. Let $\widetilde{E} = E|_{\widetilde{C}_0}$, $E' = E|_{R_1}$. Then we have exact sequence

$$0 \to E'(-p_1 - p_2) \to E \to \widetilde{E} \to 0. \tag{2.1}$$

If $E|_{R_1}$ is positive and π_*E stable (semistable), then $\pi_*E'(-p_1-p_2) = 0$. For any $E_1 \subset E$, consider the sequence (2.1), let $\widetilde{E}_1 \subset \widetilde{E}$ be the image of E_1 in $\widetilde{E}$ and $K \subset E'(-p_1-p_2)$ be the kernel of $E_1 \to \widetilde{E}_1$, then we have

$$0 \to \pi_*E_1 \to \pi_*\widetilde{E}_1 \to R^1\pi_*K = {}_{x_1}H^1(K),$$

and $\chi(E_1) = \chi(\widetilde{E}_1) + \chi(K) = \chi(\pi_*\widetilde{E}_1) - h^1(K) \le \chi(\pi_*E_1)$. Since $r(E_1) = r(\pi_*E_1)$,

$$\chi(E_1) - \frac{\chi(E)}{r} r(E_1) \le \chi(\pi_*E_1) - \frac{\chi(\pi_*E)}{r} r(\pi_*E_1).$$

Thus we will be done if we can check that $\chi(E_1) < \chi(E)$ when $r(E_1) = r(E)$ and $E_1 \ne E$. In this case, the quotient $E_2 = E/E_1$ is torsion outside the chains $\{R_i\}$. If $E_2|_R = 0$, where $R = \cup R_i$, then E_2 is a nontrivial torsion and we are done. If $E_2|_R \ne 0$, then $\chi(E_2) \ge \chi(E_2|_R)$. Since $E|_R$ is positive and the surjective map

$$E|_R = \bigoplus_{j=1}^{r} \mathcal{L}_j \to E_2|_R \to 0,$$

we have $H^1(E_2|_R) = 0$ and there is at least one line bundle $\mathcal{L}_j$ such that $\mathcal{L}_j \hookrightarrow E_2|_R$ on a sub-chain. Thus $\chi(E_2) \ge \chi(E_2|_R) = h^0(E_2|_R) > 0$ and $\chi(E_1) < \chi(E)$.

Remark 2.3. It is easy to show that if E is semistable on $X_{\vec{n}}$, then E is *standard* on the chains and π_*E is torsion free. It is expected that (semi)stability of E also implies the (semi)stability of π_*E.

DEFINITION 2.4. *Let $\mathcal{C} \to S$ be a flat family of stable curves of genus $g \ge 2$. The associated functor $\mathcal{G}_S$ (called the Gieseker functor) is defined as follows:*

$$\mathcal{G}_S : \{S - schemes\} \to \{sets\},$$

where $\mathcal{G}_S(T)$ = set of closed subschemes $\Delta \subset \mathcal{C} \times_S T \times_S Gr(m,r)$ such that

(1) *the induced projection map $\Delta \to T \times_S Gr(m,r)$ over T is a closed embedding over T. Let $\mathcal{E}$ denote the rank r vector bundle on Δ which is induced by the tautological rank r quotient bundle on $Gr(m,r)$.*

(2) *the projection $\Delta \to T$ is a flat family of semistable curves and the projection $\Delta \to \mathcal{C} \times_T T$ over T is the canonical morphism $\pi : \Delta \to \mathcal{C} \times_S T$ contracting the chains of projective lines.*

(3) *the vector bundles $\mathcal{E}_t = \mathcal{E}|_{\Delta_t}$ on Δ_t $(t \in T)$ are of rank r and degree $d = m + r(g-1)$. The qoutients $\mathcal{O}^m_{\Delta_t} \to \mathcal{E}_t$ induce isomorphisms*

$$H^0(\mathcal{O}^m_{\Delta_t}) \cong H^0(\mathcal{E}_t).$$

LEMMA 2.5 ([GI],[NSE],[SE]). *The functor $\mathcal{G}_S$ is represented by a $PGL(m)$-stable open subscheme $\mathcal{Y} \to S$ of the Hilbert scheme. The fibres $\mathcal{Y}_s$ $(s \in S)$ are reduced, and the singularities of $\mathcal{Y}_s$ are products of normal crossings. A point $y \in \mathcal{Y}_s$ is smooth if and only if the corresponding curve Δ_y is a stable curve, namely all chains in Δ_y are of length* 0.

Let *Quot* be the Quot-scheme of rank r and degree d quotiens of $\mathcal{O}^m_{\mathcal{C}}$ on $\mathcal{C} \to S$ (we choose the canonical polarization on any flat family $\mathcal{C} \to S$ of stable curves of genus $g \ge 2$). There is a universal quotient

$$\mathcal{O}^m_{\mathcal{C}\times_S Quot} \to \mathcal{F} \to 0$$

on $\mathcal{C} \times_S Quot \to Quot$. Let $\mathcal{R} \subset Quot$ be the $PGL(m)$-stable open subscheme consisting of $q \in Quot$ such that the quotient map $\mathcal{O}^m_{\mathcal{C}\times_S\{q\}} \to \mathcal{F}_q \to 0$ induces an isomorphism $H^0(\mathcal{O}^m_{\mathcal{C}\times_S\{q\}}) \cong H^0(\mathcal{F}_q)$ (thus $H^1(\mathcal{F}_q) = 0$). We can assume that d is large enough so that all semistable torsion free sheaves of rank r and degree d

on $\mathcal{C} \to S$ can be realized as points of $\mathcal{R}$. Let $\mathcal{R}^s$ ($\mathcal{R}^{ss}$) be the open set of stable (semistable) quotients, and let $\mathcal{W}$ be the closure of $\mathcal{R}^{ss}$ in $Quot$. Then there is an ample $PGL(m)$-line bundle $\mathcal{O}_{\mathcal{W}}(1)$ on $\mathcal{W}$ such that $\mathcal{R}^s$ (resp. $\mathcal{R}^{ss}$) is precisely the set of GIT stable (resp. GIT semistable) points. The moduli scheme $\mathcal{U}(r,d) \to S$ is the GIT quotient of $\mathcal{R}^{ss} \to S$.

Let $\Delta \subset \mathcal{C} \times_S \mathcal{Y} \times_S Gr(m,r)$ be the universal object of $\mathcal{G}_S(\mathcal{Y})$, and

$$\mathcal{O}_{\Delta}^m \to \mathcal{E} \to 0$$

be the induced quotient on Δ by the universal quotient on Grassmannian over $\mathcal{Y}$. Then there is a commutative diagram over S

$$\begin{array}{ccc} \Delta & \xrightarrow{\pi} & \mathcal{C} \times_S \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{Y} & = & \mathcal{Y} \end{array}$$

LEMMA 2.6. *If S is a smooth scheme, then $\pi_* \mathcal{O}_{\Delta} = \mathcal{O}_{\mathcal{C} \times_S \mathcal{Y}}$ and there is a birational S-morphism*

$$\theta : \mathcal{Y} \to \mathcal{R}$$

such that pullback of the universal quotient $\mathcal{O}^m_{\mathcal{C} \times_S \mathcal{R}} \to \mathcal{F} \to 0$ (by $id \times \theta$) is

$$\mathcal{O}^m_{\mathcal{C} \times_S \mathcal{Y}} \to \pi_* \mathcal{E} \to 0.$$

Proof. Similar with Proposition 6 and Proposition 9 of [NSe] (See also [Se]).

LEMMA 2.7. *Let $\mathcal{Y}^s = \theta^{-1}(\mathcal{R}^s)$ and $\mathcal{Y}^0 = \theta^{-1}(\mathcal{R}^{ss})$. Then*

$$\theta : \mathcal{Y}^s \to \mathcal{R}^s, \quad \theta : \mathcal{Y}^0 \to \mathcal{R}^{ss}$$

are proper birational morphisms.

Proof. The proof in [NSe] and [Se] for irreducible one node curves is completely local. Thus can be generalied to general stable curves.

There is a $PGL(m)$-equivariant factorisation (See [NSe], [Se], [Sch])

$$\begin{array}{ccccc} \mathcal{Y}^s & \xrightarrow{\imath} & \mathcal{Y}^0 & \xrightarrow{\imath} & \mathcal{H} \\ {\scriptstyle\theta}\downarrow & & {\scriptstyle\theta}\downarrow & & {\scriptstyle\lambda}\downarrow \\ \mathcal{R}^s & \xrightarrow{\imath} & \mathcal{R}^{ss} & \xrightarrow{\imath} & \mathcal{W} \end{array}$$

and linearisation $\mathcal{O}_{\mathcal{H}}(1)$, where $\imath$ is open embedding. Let $L_a = \lambda^*(\mathcal{O}_{\mathcal{W}}(a)) \otimes \mathcal{O}_{\mathcal{H}}(1)$. Then, for a large enough, the set $\mathcal{H}(L_a)^{ss}$ ($\mathcal{H}(L_a)^s$) of GIT-semistable (stable) points satisfies: (i) $\mathcal{H}(L_a)^{ss} \subset \lambda^{-1}(\mathcal{R}^{ss})$, (ii) $\mathcal{H}(L_a)^s = \lambda^{-1}(\mathcal{R}^s)$. By Lemma 2.7, θ is proper, we have $\lambda^{-1}(\mathcal{R}^{ss}) = \mathcal{Y}^0$ and $\lambda^{-1}(\mathcal{R}^{ss}) = \mathcal{Y}^s$. Thus

$$\mathcal{H}(L_a)^s = \mathcal{Y}^s = \theta^{-1}(\mathcal{R}^s), \quad \mathcal{H}(L_a)^{ss} \subset \mathcal{Y}^0 = \theta^{-1}(\mathcal{R}^{ss}).$$

NOTATION 2.8. *$\mathcal{G}(r,d)_S = \mathcal{H}(L_a)^{ss}//PGL(m)$ is called (according to [NSe]) the generalized Gieseker semistable moduli space (or Gieseker space for simplicity). It is intrinsic by recent work [Sch].*

Let $y = (\Delta_y, \mathcal{O}^m_{\Delta_y} \to \mathcal{E}_y \to 0) \in \mathcal{Y}^0$. Obviously, for $y \in \mathcal{H}(L_a)^{ss} \setminus \mathcal{H}(L_a)^s$, we have to add extra conditions besides the semistability of $\pi_*\mathcal{E}_y$. Alexander Schmitt ([Sch]) recently figure out a sheaf theoretic condition (H_3) (See Definition 2.2.10 in [Sch]) for $\pi_*\mathcal{E}_y$, which is a sufficient and necessary condition for $y \in \mathcal{H}(L_a)^{ss}$. The pair (C, E) of a semstable curve C with a vector bundle E is called H-(semi)stable (See [Sch]) if E is *strictly positive* on the chains of projective lines, and the direct image (on stable model of C) π_*E is semistable satisfying the condition (H_3).

THEOREM 2.9. *The projective S-scheme $\mathcal{G}(r,d)_S \to S$ universally corepresents the moduli functor $\mathcal{G}(r,d)^\sharp_S : \{S\text{-schemes}\} \to \{sets\}$,*

$$\mathcal{G}(r,d)^\sharp_S(T) = \left\{ \begin{array}{l} \textit{Equivalence classes of pairs } (\Delta_T, \mathcal{E}_T), \textit{ where } \Delta_T \to T \\ \textit{is a flat family of semistable curves with stable model} \\ \mathcal{C} \times_S T \to T \textit{ and } \mathcal{E}_T \textit{ is an } T\textit{-flat sheaf such that for} \\ \textit{any } t \in T,\ (\mathcal{E}_T)|_{\Delta_t} \textit{ is } H\textit{-(semi)stable vector bundle of} \\ \textit{rank } r \textit{ and degree } d. \end{array} \right\}$$

We call that $(\Delta_T, \mathcal{E}_T)$ is equivalent to $(\Delta'_T, \mathcal{E}'_T)$ if there is an T-automorphism $g : \Delta_T \to \Delta'_T$, which is identity outside the chains, such that $\mathcal{E}_T$ and $g^*\mathcal{E}'_T$ are fibrewisely isomorphic.

3. A Gieseker type degeneration for rank two. Let $\mathcal{C} \to S$ be a flat family of irreducible stable curves and $\mathcal{L}$ be a line bundle on $\mathcal{C}$ of relative degree d. We simply call the families in $\mathcal{G}(r,d)^\sharp_S(T)$, the families of semistable Gieseker bundles parametrized by T.

DEFINITION 3.1. *The subfunctor $\mathcal{G}_\mathcal{L} : \{S\text{-schemes}\} \to \{sets\}$ of $\mathcal{G}$ is defined to be*

$$\mathcal{G}_\mathcal{L}(T) = \left\{ \begin{array}{l} \Delta \in \mathcal{G}(T) \textit{ such that for any } t \in T \textit{ there is} \\ \textit{a morphism } det(\mathcal{E}|_{\Delta_t}) \to \pi^*\mathcal{L}_t \textit{ on } \Delta_t \textit{ which} \\ \textit{is an isomorphism outside the chain of } \mathbb{P}^1 s \end{array} \right\}.$$

DEFINITION 3.2. *The moduli functor $\mathcal{G}(r,\mathcal{L})^\sharp_S$ of semistable Gieseker bundles with a fixed determinant is defined to be*

$$\mathcal{G}(r,\mathcal{L})^\sharp_S(T) = \left\{ \begin{array}{l} (\Delta_T, \mathcal{E}_T) \in \mathcal{G}(r,d)^\sharp_S(T) \textit{ such that for any } t \in T \\ \textit{there exists a morphism } det(\mathcal{E}_T|_{\Delta_t}) \to \pi^*\mathcal{L}_t \textit{ on } \Delta_t \\ \textit{which is an isomorphism outside the chain of } \mathbb{P}^1 s \end{array} \right\}.$$

When $S = Spec(\mathbb{C})$, the above defined functor is denoted by $\mathcal{G}(r,L)^\sharp$.

Let $S = \mathrm{Spec(D)}$ where D is a discrete valuation ring. Let $\mathcal{C} \to S$ be a family of curves with smooth generic fibre and closed fibre $\mathcal{C}_0 = X$. Assume that X is irreducible with only one node p_0. Then we have the following result that is similar with Lemma 1.19 of [Vi].

LEMMA 3.3. *When $r = 2$, the moduli functor $\mathcal{G}(r,\mathcal{L})^{\sharp}_S$ is a closed subfunctor of $\mathcal{G}(r,d)^{\sharp}_S$. More precisely, for any family $(\Delta_T, \mathcal{E}_T) \in \mathcal{G}(r,d)^{\sharp}_S(T)$, there exists a closed subscheme $T' \subset T$ such that a morphism $T_1 \to T$ of schemes factors through $T_1 \to T' \hookrightarrow T$ if and only if*

$$(\Delta_T \times_T T_1, pr_1^*\mathcal{E}_T) \in \mathcal{G}(r,\mathcal{L})^{\sharp}_S(T_1).$$

Similarly, $\mathcal{G}_{\mathcal{L}}$ is a closed subfunctor of $\mathcal{G}$.

Proof. Let $\pi : \Delta_T \to \mathcal{C} \times_S T$ be the birational morphism contracting the chain of rational curves and $\mathcal{L}_T$ be the pullback $\pi^*\mathcal{L}$ to Δ_T. Let $f : \Delta_T \to T$ be the family of semistable curves (thus $f_*(\mathcal{O}_{\Delta_T}) = \mathcal{O}_T$). Then the condition that defines the subfunctor is equivalent to the existence of a global section of $det(\mathcal{E}_T|_{\Delta_t})^{-1} \otimes \pi^*\mathcal{L}_t$ which is nonzero outside the chain $R_t \subset \Delta_t$ of $\mathbb{P}^1$s. But, in the case $r = 2$, any non-trivial section of $det(\mathcal{E}_T|_{\Delta_t})^{-1} \otimes \pi^*\mathcal{L}_t$ is automatically nonzero outside the chain $R_t \subset \Delta_t$ of $\mathbb{P}^1$s. There is a complex

$$\mathcal{K}^{\bullet}_T : \mathcal{K}^0_T \xrightarrow{\delta_T} \mathcal{K}^1_T \tag{3.1}$$

of locally free sheaves on T such that for any base change $T_1 \to T$ the pullback of $\mathcal{K}^{\bullet}_T$ to T_1 computes the direct image of $det(\mathcal{E}_{T_1})^{-1} \otimes \mathcal{L}_{T_1}$ (which equals to the kernel of $\delta_{T_1} : \mathcal{K}^0_{T_1} \to \mathcal{K}^1_{T_1}$). There is a canonical closed subscheme $T' \subset T$ (defined locally by some minors of δ_T) where δ_T is not injective.

For simiplicity, we assume that r and d are coprime $(r,d) = 1$. In this case, the functor $\mathcal{G}(r,d)^{\sharp}_S$ is representable by an irreducible Cohen-Macaulay S-scheme $\mathcal{G}(r,d)_S \to S$ (See [NSe]), whose fibres are reduced, irreducible projective schemes with at most normal crossing singularities. Moreover, there is a canonical proper birational S-morphism

$$\theta : \mathcal{G}(r,d)_S \to \mathcal{U}(r,d)_S, \tag{3.2}$$

where $\mathcal{U}(r,d)_S \to S$ is the family (associated to $\mathcal{C} \to S$) of moduli spaces of semistable torsion free sheaves with rank r and degree d.

For general r, let $\mathcal{G}(r,\mathcal{L})_S \subset \mathcal{G}(r,d)_S$ be the subset of Gieseker bundles satisfying the conditions of functor, then we can show it being a closed subset. But we do not know how to define the correct *subscheme* structure on it.

LEMMA 3.4. *$\mathcal{G}(r,\mathcal{L})_S \subset \mathcal{G}(r,d)_S$ is a closed subset of $\mathcal{G}(r,d)_S$. In fact, for the closed fibre $\mathcal{C}_0 = X$, we have*

$$\mathcal{G}(r,\mathcal{L})^{\sharp}_S(\{0\}) = \theta^{-1}(\mathcal{U}_X(r,\mathcal{L}_0)). \tag{3.3}$$

Proof. It is enough to prove (3.3). For any $(\Delta, E) \in \mathcal{G}(r,d)^{\sharp}_S(\{0\})$, let

$$\pi : \Delta \to X$$

be the morphism contracting the chain R of $\mathbb{P}^1$s. Then, by definition of θ,

$$\theta((\Delta, E)) = \pi_*(E) := F \in \mathcal{U}_X(r,d).$$

Note that F has type of $t(F) = \deg(E|_R)$ (See [NSe]), then $\pi_*(det(E))$ has torsion of dimension $t(F) - 1$ supported at the node $p_0 = \pi(R)$. There is a natural morphism

$$\wedge^r F = \wedge^r(\pi_* E) \to \pi_*(\wedge^r E) = \pi_*(det(E)),$$

which is an isomorphism outside p_0. Thus we have an isomorphism

$$\wedge^r F/torsion \cong \pi_* det(E)/torsion$$

since $\deg(\wedge^r F/torsion) = \deg(\pi_* det(E)/torsion) = d - t(F) + 1$. By using this isomorphism, it is clear that

$$(\Delta, E) \in \mathcal{G}(r, \mathcal{L})^{\sharp}_S(\{0\}) \Longleftrightarrow \theta((\Delta, E)) \in \mathcal{U}_X(r, \mathcal{L}_0).$$

$\mathcal{G}(2, \mathcal{L})_S$ is in fact a degeneracy loci of a map of vector bundles. To study it, we recall some standard results (See [FP] for example). Let $\varphi : F \to E$ be a morphism of vector bundles on a variety M with $rk(F) = m$ and $rk(E) = n$. The closed subsets of M

$$D_r(\varphi) = \{x \in M \mid \operatorname{rank}(\varphi_x) \le r\}$$

are the so called degeneracy locus of φ. We collect the results into

LEMMA 3.5. *The codimension of each irreducible component of $D_r(\varphi)$ is at most $(n-r)(m-r)$. If M is Cohen-Macaulay and the codimension of each irreducible of $D_r(\varphi)$ equals to $(n-r)(m-r)$, then $D_r(\varphi)$ is Cohen-Macaulay.*

In (3.1), $rk(\mathcal{K}^1_T) - rk(\mathcal{K}^0_T) = g - 1$ since $det(\mathcal{E}_T)^{-1} \otimes \mathcal{L}_T$ has relative degree 0. Then one sees that

$$T' = D_{k_0}(\delta_T), \quad k_0 = rk(\mathcal{K}^0_T) - 1.$$

In what follows, we will use Codim($\bullet$) to denote: codimension of each irreducible component of $\bullet$. Thus $\operatorname{Codim}(T') \le g$, and it is Cohen-Macaulay if

$$\operatorname{Codim}(T') = g.$$

In particular, let X be the singular fibre of $\mathcal{C} \to S$ and $L = \mathcal{L}|_X$. The closed fibre $G(r, d)$ of $\mathcal{G}(r, d)_S \to S$ is the so called generalized Gieseker moduli space (associated to X) of [NSe], which has normal crossing singularities. The closed fibre of $T' \to S$, denoted by $G(r, L)$, is the degeneracy loci

$$D_{k_0}(\delta_{T_0}) \subset T_0 \subset G(r, d)$$

of $\delta_{T_0} : \mathcal{K}^0_{T_0} \to \mathcal{K}^1_{T_0}$, where T_0 is the closed fibre of $T \to S$. Thus

$$\operatorname{Codim}(G(r, L)) \le g$$

and $G(r, L)$ is Cohen-Macaulay if $\operatorname{Codim}(G(r, L)) = g$. When $r = 2$, $G(r, L) \subset G(r, d)$ is a closed subscheme that represents a moduli functor (See Theorem 3.7 for definition).

LEMMA 3.6. *When $r = 2$, Codim$(G(r, L)) = g$. In particular, $\mathcal{G}(r, \mathcal{L})_S \subset \mathcal{G}(r, d)_S$ is an irreducible, reduced, Cohen-Macaulay subscheme of codimension g.*

Proof. Assume that $\operatorname{Codim}(G(r, L)) = g$. Note that there is a unique irreducible component of $\mathcal{G}(r, \mathcal{L})_S$ with codimension g dominates S since $\mathcal{C} \to S$ has smooth generic fibre. Thus other irreducible components (if any) of $\mathcal{G}(r, \mathcal{L})_S$ will fall in $G(r, L)$ and their codimension in $G(r, d)$ are at most $g - 1$ since $\mathcal{G}(r, d)_S \to S$ is flat over S. This contradicts $\operatorname{Codim}(G(r, L)) = g$. Hence $\mathcal{G}(r, \mathcal{L})_S \subset \mathcal{G}(r, d)_S$ is an irreducible, Cohen-Macaulay subscheme of codimension g. It has to be reduced since it is Cohen-Macaulay and has a reduced open subscheme.

Now we prove that $\mathrm{Codim}(G(r,L)) = g$ in $G(r,d)$. Let J_X^0 be the Jacobian line bundles of degree 0 on X. Consider a morphism

$$\phi : G(r,L) \times J_X^0 \to G(r,d)$$

that sends any $\{(\Delta, E), \mathcal{N}\} \in G(r,L) \times J_X^0$ to $(\Delta, E \otimes \pi^*\mathcal{N}) \in G(r,d)$, where $\pi : \Delta$ X is the morphism contracting the chain R of $\mathbb{P}^1$s. We claim that

$$\dim \phi^{-1}((\Delta, E_0)) \leq 1, \quad \text{for any } (\Delta, E_0) \in G(r,d).$$

Let $\sigma : J_X^0 \to J_{\widetilde{X}}^0$ be the morphism induced by pulling back line bundles on X its normalization $\widetilde{X}$. The fibres of σ are of dimension 1. On the other hand, it is ea to see that the projection $G(r,L) \times J_X^0 \to J_X^0$ induces an injective morphism

$$\rho : \phi^{-1}((\Delta, E_0)) \to J_X^0.$$

To prove the claim, it is enough to show that the image $\mathrm{Im}(\rho)$ falls in a finite numb of fibres of σ. Note that, for any $\{(\Delta, E), \mathcal{N}\} \in \phi^{-1}((\Delta, E_0))$, we have

$$det(E) \otimes \pi^*(\mathcal{N}^{\otimes r}) = det(E_0)$$

on Δ. Recall that, by definition of $G(r,L)$, there is a morphism $det(E) \to \pi^*L$ whi is an isomorphism outside the chain R of $\mathbb{P}^1$s. We have

$$det(E)|_{\widetilde{X}} = \pi^*L|_{\widetilde{X}}(-n_1p_1 - n_2p_2) = \widetilde{L}(-n_1p_1 - n_2p_2),$$

where $\widetilde{L}$ is the pullback of L to $\widetilde{X}$, n_1, n_2 are nonnegative integers such that

$$n_1 + n_2 = \deg(E_0|_R) = t(F_0), \quad F_0 := \pi_*(E_0).$$

Thus $\sigma \circ \rho(\{(\Delta, E), \mathcal{N}\}) = \sigma(\mathcal{N}) = \widetilde{\mathcal{N}} \in J_{\widetilde{X}}^0$ falls in the set

$$\{\widetilde{\mathcal{N}} \in J_{\widetilde{X}}^0 \mid \widetilde{\mathcal{N}}^{\otimes r} = det(E_0)|_{\widetilde{X}} \otimes \widetilde{L}^{-1}(n_1p_1 + n_2p_2)\},$$

which is clearly a finite set. This proves that fibres of ϕ are at most dimension 1.

There is a unique irreducible component $G(r,L)^0$ of $G(r,L)$ containing $\Delta \cong$ which has codimension g. For any other irreducible component (if any), say $G(r,L)$ all of Δs in $G(r,L)^+$ must have chain (with positive length) of $\mathbb{P}^1$s. Then the ima $\phi(G(r,L)^+ \times J_X^0)$ has to fall in a subvariety of $G(r,d)$, which has codimension at lea 1. Thus $\dim(G(r,L)^+ \times J_X^0) \leq \dim G(r,d)$, that is,

$$\mathrm{Codim}(G(r,L)^+) \geq g.$$

By Lemma 3.5, $G(r,L)$ is Cohen-Macaulay of pure codimension g.

THEOREM 3.7. *Let X be an irreducible curve of genus $g \geq 2$ with only one no p_0. Let L be a line bundle of degree d on X. Then, when $r = 2$ and $(2,d) = 1$, have*

(1) *There is a Cohen-Macaulay projective scheme $G(r,L)$ of pure dimension $(r^2$ $1)(g-1)$, which represents the moduli functor*

$$\mathcal{G}(r,L)^\sharp : (\mathbb{C} - schemes) \to (sets)$$

which is defined in Definition 3.2.

(2) *Let $\mathcal{C} \to S$ be a proper family of curves over a discrete valuation ring, whi has smooth generic fibre $\mathcal{C}_\eta$ and closed fibre $\mathcal{C}_0 \cong X$. If there is a line bundle $\mathcal{L}$*

$\mathcal{C}$ such that $\mathcal{L}|_{\mathcal{C}_0} \cong L$. Then there exists an irreducible, reduced, Cohen-Macaulay S-projective scheme $f : G(r, \mathcal{L})_S \to S$ such that

$$f^{-1}(0) \cong G(r, L), \quad f^{-1}(\eta) \cong \mathcal{U}_{\mathcal{C}_\eta}(r, \mathcal{L}_\eta).$$

Moreover $G(r, \mathcal{L})_S$ represents the moduli functor $\mathcal{G}(r, \mathcal{L})^\sharp_S$ in Definition 3.2.

(3) *There exists a proper birational S-morphism $\theta : G(r, \mathcal{L})_S \to \mathcal{U}(r, \mathcal{L})_S$ which induces a morphism $\theta : G(r, L) \to \mathcal{U}_X(r, L)$.*

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