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A MODEL OF BRILL-NOETHER THEORY FOR RANK TWO VECTOR BUNDLES AND ITS PETRI MAP *

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Abstract. We study here the Brill-Noether theory for rank two vector bundles. First we construct a parameter space H_d for all base point free rank two vector bundles of degree d which generated by its sections. Then for each $E \in H_d$, we define a $2g \times d$ matrix W_E for which we call it the Brill-Noether matrix of E, it shares the same properties as the Brill-Noether matrix W_D for effective divisor D. By using W_E, the Brill-Noether variety $C^r_{2,d} = \{E \in H_d \mid dimH^0(C,E) \geq r+1\}$ could be given by $C^r_{2,d} = \{E \in H_d \mid rank(W_E) \leq d-r+1\}$, so $C^r_{2,d}$ is a determinant variety, we get its expected dimension is $4(g-1)+1-(r+1)(2(g-1)-d+r+1)+2r+1$. On the other hand, by using W_E, we define the Petri map to be $P: H^0(C,K(-E)) \otimes Im\{H^0(C,E) \mapsto H^0(C,[D])\} \mapsto H^0(C,K[D](-E))$, we show that $C^r_{2,d}$ has the expected dimension if and only if the Petri map is injective.

1. Introduction . Let C be a smooth irreducible complex projective curve of genus g(C a Compact Riemann surface), L a line bundle on C. We also use L to denote the sheaf of holomorphic sections of L. The Brill-Noether theory for line bundles is to study those bundles L for which both $H^0(C,L)$ and $H^1(C,L)$ are non-zero(L is then called special line bundle).

Let C_d be the d-fold symmetric product of C, C_d is a d-dimensional complex manifold. It is the space of all effective divisors of degree d. Since each line bundle L with $H^0(C,L) \neq 0$ is defined by an effective divisor, so C_d could be considered as a parameter space for all line bundles L with $deg(L) = d$ and $H^0(C,L) \neq 0$.

Define on C_d the Brill-Noether variety C^r_d to be

$$C^r_d = \{D \in C_d \mid dimH^0(C,[D]) \geq r+1\}.$$

Where $[D]$ is the line bundle defined by divisor D.

C^r_d could be considered as a parameter space for line bundles L with $deg(L) = d$ and $dimH^0(C,L) \geq r+1$. The key tool to study C^r_d is the Brill-Noether matrix.

Let $D = n_1p_1 + \cdots + n_kp_k$ be a given effective divisor with $d = deg(D) = n_1 + \cdots + n_k$. For $i = 1, \cdots, k$, let z_i be a local coordinate at p_i with $z_i(p_i) = 0$. Let $\{w_1, \cdots, w_g\}$ be a linear basis of the space of all holomorphic forms on C, for each i assume at p_i, $w_t(z_i) = f_{ti}(z_i)dz_i$ for $t = 1, \cdots, g$, let W_D be the matrix of the restrictions of $\{w_1, \cdots, w_g\}$ on D, that is

$$W_D = \begin{bmatrix} w_1 \mid_D \\ w_2 \mid_D \\ \vdots \\ w_g \mid_D \end{bmatrix}$$

$$= \begin{bmatrix} f_{11}(p_1) & \cdots & \frac{1}{(n_1-1)!}f_{11}^{(n_1-1)}(p_1) & f_{12}(p_2) & \cdots & \frac{1}{(n_2-1)!}f_{12}^{(n_2-1)}(p_2) & \cdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{g1}(p_1) & \cdots & \frac{1}{(n_k-1)!}f_{g1}^{(n_1-1)}(p_1) & f_{g2}(p_2) & \cdots & \frac{1}{(n_2-1)!}f_{g2}^{(n_2-1)}(p_2) & \cdots \end{bmatrix}.$$

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For a collection of Laurent tails $\mu = \{\mu_i = \sum_{k=-n_i}^{-1} b_{ik} z_i^k\}$, we denote it as a d-dimensional vector

$$\mu = (b_{1-1}, b_{1-2}, \cdots, b_{1-n_1}, b_{2-1}, \cdots, b_{2-n_2}, \cdots) \in C^d.$$

Then μ is the Laurent of a global meromorphic function if and only if $W_D \cdot \mu^t = 0$. From this one can get Riemann-Roch theorem easily.

The matrix W_D is called the Brill-Noether matrix of D.

Now let $[D]|_D$ be the skyscraper sheaf of the restriction of $[D]$ on D, then what we have above could be represented as

$$Ker(W_D) = \{\mu \mid \mu \in C^d, W_D \cdot \mu^t = 0\} \cong Im\{H^0(X, [D]) \mapsto H^0(X, [D] \mid_D)\} \qquad (*)$$

and in particular, we get

$$dimH^0(X, [D]) = deg(D) - rank(W_D) + 1. \qquad (**).$$

so C_d^r could be defined by

$$C_d^r = \{D \in C^d \mid Rank(W_D) \leq d - r\}.$$

It is a subvariety of C_d which locally is defined by the simultaneously vanishing of all $(d - r + 1) \times (d - r + 1)$ minors of W_D(Ref [ACGH] p159).

Now let $M(m, n) = M$ be the variety of all $m \times n$ complex matrices, and for $0 \leq k \leq min\{m, n\}$, denote by $M_k(m, n) = M_k$ the locus of matrices of rank at most k, that is

$$M_k = \{E \in M(m, n) \mid Rank(E) \leq k\}.$$

M_k is an irreducible subvariety of $M(m, n)$, and $codim(M_k) = (n - k)(m - k)$(Ref [ACGH] p67).

By using the Brill-Noether matrix, locally we have a holomorphic map $BN : C_D \to M(m, n)$ with $BN(D) = W_D$ for each $D \in C_d$. C_d^r is then could be given by $C_d^r = BN^{-1}(M_{d-r})$. From the Theory of determinant variety, we get that if $C_d^r \neq \emptyset$, then $codim(C_d^r) \leq codim(M_{d-r}) = (g - (d - r))(d - (d - r))$. So if $C_d^r \neq \emptyset$, then

$$dimC_d^r \geq d - r(g - d + r) = g - (r + 1)(g - d + 1) = \rho(g, d, r) + r.$$

where $\rho(g, d, r) = g - (r + 1)(g - d + r)$ is the Brill-Noether number for line bundles. (Ref [ACGH] p215).

It was conjectured by Brill-Noether and Proved by Griffiths-Harris [GH] that for generic C, C_d^r do have the expected dimension $\rho(g, d, r) + r$.

On the other hand, by study the tangent map of $BN : C_D \to M(m, n), D \mapsto W_D$, Petri got that the variety C_d^r is smooth and has the "expected dimension" $\rho(g, d, r) + r$ at $D \in C_d^r - C_d^{r+1}$ if and only if the cup product homomorphism

$$\mu : H^0(C, [D]) \otimes H^0(C, K[-D]) \mapsto H^0(C, K)$$

is injective, where, K is the canonical line bundle of C(Ref [ACGH] p163).

The map μ is called the Petri map. Again, it was proved by Gieseker[G] that for generic C, the cup product homomorphism μ is indeed injective. This gives another prove of the result of Griffiths-Harris.

In this paper, we are trying to generalize those ideals to the study of rank two vector bundles.

First we will define a parameter space H_d for all base point free rank two vector bundles of degree d which generated by its sections(we called such vector bundles the effective vector bundles). H_d is a d-dimensional holomorphic vector bundle on C_d, so it is a 2d-dimensional complex manifold.

For each $E \in H_d$, we construct a $2g \times d$ matrix W_E for E which we call it the Brill-Noether matrix of E, it shares the same properties for E as the Brill-Noether matrix W_D for line bundle $[D]$. In particular, we have

$$dimH^0(C, E) = d - Rank(W_E) + 2.$$

From this, the Brill-Noether variety of rank two vector bundles

$$C^r_{2,d} = \{E \in H_d \mid dimH^0(C, E) \geq r + 1\}$$

could be given by

$$C^r_{2,d} = \{E \in H_d \mid Rank(W_E) \leq d - r + 1\}.$$

This defines $C^r_{2,d}$ as a subvariety of H_d.

Also by using W_E, locally we get a holomorphic map

$$BN : H_d \mapsto M(2d, g); BN(E) = W_E,$$

so $C^r_{2,d} = BN^{-1}(M_{d-r+1})$, and from the theory of determinant variety, we get that if $C^r_{2,d} \neq \emptyset$ then

$$codimC^r_{2,d} \leq (2g - (d - r + 1))(d - (d - r + 1))$$

so if $C^r_{2,d} \neq \emptyset$, then

$$dimC^r_{2,d} \geq 2d - (2g - (d - r + 1))(d - (d - r + 1)) = 2d - (r + 1)(2(g - 1) - d + r + 1) =$$

$$2d - (r + 1)(2(g - 1) - d + r + 1) + 2(2(g - 1) - d + r + 1) =$$

$$4(g - 1) + 1 - (r + 1)(2(g - 1) - d + r + 1) + 2r + 1 = \rho_2(g, d, r) + 2r + 1$$

here $\rho_2(g, d, r) = 4(g - 1) + 1 - (r + 1)(2(g - 1) - d + r + 1)$ is the Brill-Noether number for rank two vector bundles.

Also, by studying the tangent map of $BN : H_d \mapsto M(2g, d)$, we generalize the Petri map to rank two vector bundles. This is for each $E \in C^r_{2,d}$, we define a cup product homomorphism

$$P : H^0(C, K(-E)) \otimes Im\{H^0(C, E) \mapsto H^0(C, [D])\} \mapsto H^0(C, K[D](-E)).$$

Here $[D] = E/I$ is the quotient bundle of E with respect to the trivial line bundle I. We call P the Petri map for rank two vector bundles, and we show that $C^r_{2,d}$ has the "expected dimension" $\rho_2(g, d, r) + 2r + 1$ if and only if the Petri map P is injective.

2. The parameter space H_d.

DEFINITION 1. A point $p \in C$ is called a base point of vector bundle E if $s(p) = 0$ for all $s \in H^0(C, E)$. E is said to be base point free if E don't have base point.

DEFINITION 2 [A]. A rank two vector bundle E is said to be generated by its sections, if E has a splitting

$$0 \mapsto L_1 \mapsto E \mapsto L_2 \mapsto 0$$

such that both $H^0(C, L_1)$ and $Im\{H^0(C, E) \mapsto H^0(C, L_2)\}$ are not zero. Where L_1 is a line sub-bundle of E, and $L_2 = E/L_1$.

The Brill-Noether theory for rank two vector bundles is to study those bundles E with both $H^0(C,E)$ and $H^1(C,E)$ are non-zero. E is then called special rank two vector bundle. If E has a base point p, then $E\otimes[-p]$ is also special and we have $dimH^0(C,E\otimes[-p]) = dimH^0(C,E)$, $dimH^1(C,E\otimes[-p]) = dimH^1(C,E)+2$ and $deg(E\otimes[-p]) = deg(E)-2$. We can reduce the degree of E. If E is not generated by its sections, since $H^0(C,E)\neq 0$, let $s\in H^0(C,E)$ with $s\neq 0$, let L_1 be the line sub-bundle of E which generated by s, $L_2 = E/L_1$. Since E is not generated by its sections, so $H^0(C,E) = H^0(C,L_1)$, the study of $H^0(C,E)$ could be reduced to the study of $H^0(C,L_1)$, that is reduced to the study of Brill-Noether for line bundles. So to study the Brill-Noether for rank two vector bundles, we can restrict ourself to the study of base point free vector bundles which generated by its sections.

LEMMA 1. If E is a base point free rank two vector bundle which generated by its sections, then the trivial line bundle I is a line sub-bundle of E.

Proof. This is a special case of Lemma 1.1 of [TE].

Let E be a base point free rank two vector bundles which generated by its sections, assume $deg(E) = d$, by our Lemma, I is a line sub-bundle of E, so E has a splitting

$$0\mapsto I\mapsto E\mapsto L\mapsto 0$$

where $L = E/I$. Since E is generated by its sections, we have $Im\{H^0(C,E)\mapsto H^0(C,L)\}\neq 0$. Choose $s\in Im\{H^0(C,E)\mapsto H^0(C,L)\}$ with $s\neq 0$, let $D = div(s)$, then $D\geq 0$, and $L = [D]$. E is then an extension of $[D]$ by I, it is determined by an element $e\in H^1(C,[-D])$. Since $s\in H^0(C,[D])$ can be lift to a section of E, we get in particular that $s\cdot e = 0$, and from sequence

$$0\mapsto[-D]\mapsto^{\cdot s} I\mapsto I\mid_D\mapsto 0 \qquad (***)$$

we get an exact sequence

$$0\mapsto H^0(C,[-D])\mapsto H^0(C,I)\mapsto H^0(C,I\mid_D)\mapsto H^1(C,[-D])\mapsto\cdots$$

$s\cdot e = 0$ if and only if $e\in Im\{H^0(C,I\mid_D)\mapsto H^1(C,[-D])\}$. Let e be the image of some $f\in H^0(C,I\mid_D)$, f is then determined uniquely up to a constant. So from E we get a triple $\{I,D,f\}$.

Conversely, if we have a triple $\{I,D,f\}$, where D is an effective divisor of degree d, and $f\in H^0(C,I\mid_D)$, then let $e\in H^1(C,[-D])$ be the image of f in the map $H^0(C,I\mid_D)\mapsto H^1(C,[-D])$ which induced from sequence $(***)$, let E be the extension of $[D]$ by I which determined by e, then E has a splitting $0\mapsto I\mapsto E\mapsto[D]\mapsto 0$, and $s\in Im\{H^0(C,E)\mapsto H^0(C,[D])\}$, where s is the canonical section of D ($s\in H^0(C,[D])$, with $div(s) = D$). We get a base point free rank two vector bundle E of degree d which generated by its sections.

So to give a base point free rank two vector bundle of degree d which generated by its sections will be the same as to give a triple $\{I,D,f\}$, here $D\in C_d$ and $f\in H^0(C,[D]\mid_D)$, or the same the set of all base point free rank two vector bundle of degree d which generated by its sections could be represented by the set of all triples $\{I,D,f\}$. We will denote this as $E = \{I,D,f\}$.

Now let H_d be the vector bundle on C_d which for each $D\in C_d$, $H_d\mid_D = H^0(C,I\mid_D)$, by using local coordinate, it is easy to see that H_d is a holomorphic vector bundle of dimension d on C_d.

Each point of H_d could be represented as a triple $E = \{I,D,f\}$, and each triple $E = \{I,D,f\}$ could be represented as a point in H_d, so H_d could be considered as a parameter space for the set of all base point free rank two vector bundles of degree d which generated by its sections.

3. Brill-Neother matrix for $E = \{I, D, f\}$. Let L be a line bundle, $D = n_1p_1 + \cdots + n_kp_k \geq 0$ be a given effective divisor of degree d. For $i = 1, \cdots, k$, let z_i be a local coordinate at p_i with $z_i(p_i) = 0$. Then each $f \in H^0(C, L\mid_D)$ could be represented as a set of polynomials $f = \{f_i(z_i)\}_{i=1}^k$, where $f_i(z_i) = a_0^i + a_1^i z_i + \cdots + a_{n_i-1}^i z_i^{n_i-1}$ is a polynomial of z_i of degree less than n_i. So f could also be denoted as a d-dimensional vector $f = (a_0^1, a_1^1, \cdots, a_{n_1-1}^1; a_0^2, a_1^2, \cdots, a_{n_2-1}^2; \cdots\cdots)$. This gives $H^0(C, L\mid_D) \cong C^d$, where $d = deg(d)$.

DEFINITION 3. Let L_1, L_2 be two line bundles, $D = n_1p_1 + \cdots + n_kp_k \geq 0$ be a given effective divisor. For $f = \{f_i(z_i)\}_{i=1}^k \in H^0(C, L_1\mid_D)$ and $g = \{g_i(z_i)\}_{i=1}^k \in H^0(C, L_2\mid_D)$, we define $f * g \in H^0(C, L_1 \otimes L_2\mid_D)$ to be

$$f * g = \{f_i(z_i)g_i(z_i)(mod(z_i^{n_i}))\}_{i=1}^k.$$

LEMMA 2. $f * g = g * f$, and $(f * g) * h = f * (g * h)$.

Proof. Trivial.

LEMMA 3. For $E = \{I, D, f\}$, a section $s \in H^0(C, [D])$ could be lift to be a section of $H^0(C, E)$(which means $s \in Im\{H^0(C, E) \mapsto H^0(C, [D])\}$), if and only if

$$s\mid_D *f \in Im\{H^0(C, [D]) \mapsto H^0(C, [D]\mid_D)\}.$$

Proof. See [T].

Now let $(w_1, \cdots, w_g)$ be a linear basis of $H^0(C, K)$ of the space of all holomorphic forms on C. then for effective divisor D, the Brill-Noether matrix W_D for D could be defined by

$$W_D = \begin{bmatrix} w_1\mid_D \\ w_2\mid_D \\ \vdots \\ w_g\mid_D \end{bmatrix}.$$

An element $t \in H^0(C, [D]\mid_D)$ is in the image of map $H^0(C, [D]) \mapsto H^0(C, [D]\mid_D)$, if and only if

$$W_D * t = \begin{bmatrix} w_1\mid_D *t \\ w_2\mid_D *t \\ \vdots \\ w_g\mid_D *t \end{bmatrix} = 0.$$

That is $Im\{H^0(C, [D]) \mapsto H^0(C, [D]\mid_D)\} = Ker\{W_D\}$.

Now for $E = \{I, D, f\}$, we define its Brill-Noether matrix W_E to be

$$W_E = \begin{bmatrix} w_1\mid_D \\ w_2\mid_D \\ \vdots \\ w_g\mid_D \\ w_1\mid_D *f \\ w_2\mid_D *f \\ \vdots \\ w_g\mid_D *f \end{bmatrix} = \begin{bmatrix} W_D \\ W_D * f \end{bmatrix}.$$

THEOREM 1. $Ker\{W_E\} = \{v \in C^d \mid W_E \cdot v = 0\} \cong Im\{H^0(C, E) \mapsto H^0(C, [D]) \mapsto H^0(C, [D]\mid_D)\}$.

Proof. By $H^0(C,[D]\mid_D)\cong C^d$, each $v\in C^d$ could be identified to an element $v\in H^0(C,[D]\mid_D)$, let W_D be the Brill-Noether matrix for D, then $W_D\cdot v=W_D*v$, and $(W_D*f)*v=W_D*(f*v)$. So $W_E\cdot v=0$ if and only if $W_D*v=0$ and $W_D*(f*v)=0$. From $W_D*v=0$, we get that $v\in Im\{H^0(C,[D])\mapsto H^0(C,[D]\mid_D)\}$. Let it be the image of some $s\in H^0(C,[D])$, this is $v=s\mid_D$. Then from $(W_D*f)*v=0$, we get $(W_D*f)*s\mid_D=W_D*(f*s\mid_D)=0$. That means $f*s\mid_D\in Im\{H^0(C,[D])\mapsto H^0(C,[D]\mid_D)\}$. By our Lemma 3, s is then can be lift to a section of E.

Conversely, if $v\in Im\{H^0(C,E)\mapsto H^0(C,[D])\mapsto H^0(C,[D]\mid_D)\}$, let it be the image of some $s\in H^0(C,[D])$, so $W_D\cdot v=0$, and since s can be lift to a section of E, by our Lemma 3, $f*v\in Im\{H^0(C,[D])\mapsto H^0(C,[D]\mid_D)\}$, so $W_D*f*v=0$, we get $W_E\cdot v=0$. This completes the proof.

Now from the exact sequence

$$0\mapsto I\mapsto E\mapsto [D]\mapsto 0$$

we get exact sequence

$$0\mapsto H^0(C,I)\mapsto H^0(C,E)\mapsto H^0(C,[D])\mapsto H^1(C,I)\mapsto\cdots.$$

Since $dimH^0(C,I)=1$, so

$$dimH^0(C,E)=dimIm\{H^0(C,E)\mapsto H^0(C,[D])\}+1=$$

$$dimIm\{H^0(C,E)\mapsto H^0(C,[D])\mapsto H^0(C,[D]\mid_D)\}+2=$$

$$dimKer(W_E)+2=d-rank(W_E)+2.$$

That is

THEOREM 2. Let $E=\{I,D,F\}$ and W_E be its Brill-Noether matrix, then we have $Ker(W_E)\cong Im\{H^0(C,E)\mapsto H^0(C,[D])\mapsto H^0(C,[D]\mid_D)\}$, and in particular $dimH^0(C,E)=d-rank(W_E)+2$.

Now we define the Brill-Noether variety $C^r_{2,d}$ for rank two vector bundles to be

$$C^r_{2,d}=\{E\in H_d\mid dimH^0(C,E)\geq r+1\}.$$

By Theorem 2, $C^r_{2,d}$ could also be given by

$$C^r_{2,d}=\{E\in H_d\mid rank(W_E)\leq d-r+1\}.$$

This gives $C^r_{2,d}$ as a subvariety of H_d which $C^r_{2,d}$ is defined locally by the simultaneously vanishing of all $(d-r+2)\times(d-r+2)$ minors of W_E.

By using the Brill-Noether matrix W_E, locally, we get a holomorphic map $BN:H_d\mapsto M(2g,d)$ with $BN(E)=W_E$ for each $E\in H_d$, where $M(2g,d)$ is the variety of all $2g\times d$ complex matrices. Let

$$M_{d-r+1}=\{E\in M(2g,d)\mid rank(E)\leq d-r+1\}.$$

Then M_{d-r+1} is a subvariety of M(2g,d), and $codim(M_{d-r+1})=(2g-(d-r+1))\times(d-(d-r+1))$. By definition, we have $C^r_{2,d}=BN^{-1}(M_{d-r+1})$. So from the Theory of determinant variety, we get that if $C^r_{2,d}\neq\emptyset$, then

$$codimC^r_{2,d}\leq(2g-(d-r+1))\times(d-(d-r+1)).$$

This is

$$dimC^r_{2,d}\geq 2d-(2g-(d-r+1))\times(d-(d-r+1))=$$

$$4(g-1)+1-(r+1)(2(g-1)-d+r+1)+2r+1=\rho_2(g,d,r)+2r+1.$$

Here $\rho_2(g,d,r)=4(g-1)+1-(r+1)(2(g-1)-d+r+1)$ is the Brill-Noether number for rank two vector bundles. Same as the case of line bundles, we get that the expected dimension of $C^r_{2,d}$ is $\rho_2(g,d,r)+2r+1$, this is

THEOREM 3. If $C^r_{2,d}\neq\emptyset$, then each component of $C^r_{2,d}$ will have dimension at least $\rho_2(g,d,r)+2r+1$.

4. The Petri map. Since $C^r_{2,d}=BN^{-1}(M_{d-r+1})$, to get the dimension of $C^r_{2,d}$, analogous to the case of line bundles, we should consider the tangent map

$$BN^*: T_E\mapsto T_{BN(E)}$$

for each $E=\{I,f,D\}\in H_d$. Here T_E and $T_{BN(E)}$ are the tangent space of E and $BN(E)$ in H_d and $M(2g,d)$.

Now let $E=\{I,D,f\}$, then

$$BN(E)=W_E=\begin{bmatrix} W_D\\ W_D*f\end{bmatrix}.$$

Since for each $D\in C_d$, the tangent space of C_d at D is $T_D=H^0(C,[D]\mid_D)$ (Ref [ACGH] P160), so by definition we get that the tangent space of H_d at E is $T_E=H^0(C,[D]\mid_D)\oplus H^0(C,I\mid_D)$.

Now let $t=(-v,u)\in T_E=H^0(C,[D]\mid_D)\oplus H^0(C,I\mid_D)$, then by direct calculation, we have

$$BN^*(t)=\begin{bmatrix} \dot W_D*(-v)\\ \dot W_D*(-v)*f+W_D*u\end{bmatrix}.$$

Where $\dot W_D$ means the differential of W_D with respect to the local coordinates, and $\dot f=I$.

To get the dimension of $C^r_{2,d}$, we need to get the dimension of the space $V=\{t\in T_E\mid BN^*(t)\in T_{BN(E)}(M_{d-r+1})\}$. But from the theory of determinant variety(Ref [ACGH] p69), we know that $BN^*(t)\in T_{BN(E)}(M_{d-r+1})$ if and only if $Ker(W_E)\cdot BN^*(t)\subset Im(W_E)=C^{2g}\cdot W_E$. Here $Ker(W_E)=\{(b,e)=(b_1,\cdots,b_g;e_1,\cdots,e_g)\in C^d\mid (b,e)W_E=0\}$.

Now let $(b,e)=(b_1,\cdots,b_g;e_1,\cdots,e_g)\in Ker(W_E)$, this is $(b,e)\cdot W_E=b\cdot W_D+e\cdot W_D*f=0$. Choose an open cover $\{U_\alpha\}_{\alpha=1}^k$ of C, let $s=\{s_\alpha\}_{\alpha=1}^k\in H^0(C,[D])$ be the canonical section of $[D]$, this is $s\in H^0(C,[D])$ and $div(s)=D$. for the linear basis $\{w_1,\cdots,w_g\}$ of the holomorphic forms, let w_i be given with respect to the open cover by $w_i=\{w_{\alpha i}\}$, let $bw=b_1w_1+\cdots+b_gw_g=\{b_1w_{\alpha 1}+\cdot+b_gw_{\alpha g}\}=\{bw_\alpha\}\in H^0(C,K)$, and $ew=e_1w_1+\cdots+e_gw_g=\{e_1w_{\alpha 1}+\cdot+e_gw_{\alpha g}\}=\{ew_\alpha\}\in H^0(C,K)$, let $f=\{f_\alpha\}$ be a given representation for $f\in H^0(C,I\mid_D)$, where f_α is a holomorphic function on U_α.

LEMMA 4. $(b,e)\in Ker(W_E)$ if and only if

$$F=\{F_\alpha=\begin{bmatrix} e\cdot w_\alpha\\ -(b\cdot w_\alpha+e\cdot w_\alpha*f_\alpha)/s_\alpha\end{bmatrix}\}\in H^0(C,K(-E)).$$

Here $(-E)$ is the dual vector bundle of E.

Proof. For later using and also for making our notations easy to understand, we will give a proof of this Lemma in detail, and we will also use the proof to give a proof of Riemann-Roch Theorem for rank two vector bundles.

Let $\{U_\alpha\}_{\alpha=1}^k$ be the open cover of C. Then on $U_\alpha\cap U_\beta$, the transition matrix of $E=\{I,f,D\}$ can be given by

$$E_{\alpha\beta}=\begin{bmatrix} 1 & (f_\alpha-f_\beta)/s_\beta\\ 0 & s_\alpha/s_\beta\end{bmatrix}$$

where $e = \{e_{\alpha\beta} = (f_\alpha - f_\beta)/s_\beta\}$ is a representation of $e \in H^1(C, [-D])$.

From $E_{\alpha\beta}$, and by the definition of dual vector bundle, the transition matrix of $K(-E)$ can be given on $U_\alpha \cap U_\beta$ by

$$(K(-E))_{\alpha\beta} = \begin{bmatrix} k_{\alpha\beta} & 0 \\ -k_{\alpha\beta}(f_\alpha - f_\beta)/s_\beta & k_{\alpha\beta}s_\beta/s_\alpha \end{bmatrix}$$

where $\{k_{\alpha\beta}\}$ is the transition function of the canonical line bundle K.

By definition, $K(-E)$ is an extension of K by K[-D], which determined also by $f \in H^0(C, I\,|_D)$.

Now let $(b, e) \in Ker(W_E)$, that is $b \cdot W_D + e \cdot W_D * f = 0$, let $ew = e_1w_1 + \cdots + e_gw_g \in H^0(C, K)$, $bw = b_1w_1 + \cdots + b_gw_g \in H^0(C, K)$, then $b \cdot W_D + e \cdot W_D * f = 0$ means $ew\,|_D *f = -bw\,|_D$, by our Lemma 3(also Ref [T]), that means, ew can be lift to a section of $K(-E)$ and

$$F = \{F_\alpha = \begin{bmatrix} e \cdot w_\alpha \\ -(b \cdot w_\alpha + e \cdot w_\alpha * f_\alpha)/s_\alpha \end{bmatrix}\} \in H^0(C, K(-E)).$$

is one of the lift. This can also be proved by direct computation that $F_\alpha = K(-E)_{\alpha\beta} \cdot F_\beta$.

Conversely, let

$$F = \{F_\alpha = \begin{bmatrix} e \cdot w_\alpha \\ v_\alpha \end{bmatrix}\} \in H^0(C, K(-E)).$$

then $ew = e_1w_1 + \cdots + e_gw_g = \{ew_\alpha = e_1w_1\,|_{U_\alpha} + \cdots + e_gw_g\,|_{U_\alpha}\}$, is a section of K, here $e = (e_1, \cdots, e_g)$, and F is a lift of ew. $ew \in H^0(C, K)$ can be lift to a section of $H^0(C, K(-E))$, by our Lemma 3 , there exists an $bw = b_1w_1 + \cdots + b_gw_g \in H^0(C, K)$, such that $ew\,|_D *f = -bw\,|_D$, or the same, $ew\,|_D *f + bw\,|_D = 0$, that is $(b, e) \cdot W_E = 0$, so $(b, e) \in Ker(W_E)$.

Now if $ew = 0$, that is $e = 0$, then $F = \{F_\alpha = \begin{bmatrix} 0 \\ v_\alpha \end{bmatrix}\} \in H^0(C, K(-E))$ means $v = \{v_\alpha\} \in H^0(C, K \otimes [-D])$, but we know that $H^0(C, K \otimes [-D]) = \{w \in H^0(C, K) \mid w\,|_D = 0\}$. Assume $v = b_1w_1 + \cdots + b_gw_g = bw$, here $b = (b_1, \cdots, b_g)$, then $bw\,|_D = 0$ means $bW_D = 0$, so $(b, 0)W_E = 0$, this is $(b, 0) \in Ker(W_E)$. That completes the proof.

From the proof, we get

COROLLARY 1. $H^0(C, K(-E)) \cong Ker(W_E)$, and in particular

$$dimH^0(C, K(-E)) = 2g - rank(W_E).$$

But from the definition of W_E, we know that

$$dimH^0(C, E) = d - rank(W_E) + 2.$$

We get the Riemann-Roch Theorem for base point free rank two vector bundle which generated by its sections:

RIEMANN-ROCH THEOREM. If E is a base point free rank two vector bundle which generated by its sections, then

$$dimH^0(C, E) - dimH^0(C, K(-E)) = deg(E) - 2(g - 1).$$

Same as the case of line bundles, Riemann-Roch Theorem for all rank two vector bundles could be derived easily from this, we will not give it here.

Now, let $t \in T_E$, to get the dimension of $C^r_{2,d}$, we need to get the dimension of space $V = \{t \in T_E \mid BN^*(t) \in T_{BN(E)}(M_{d-r+1})\}$. So we need to know under what condition $BN^*(t) \in T_{BN(E)}(M_{d-r+1})$. From the theory of determinant variety, we know this same that

$$(b,e)BN^*(t) = (b,e)\begin{bmatrix} \dot{W}_D * (-v) \\ \dot{W}_D * (-v) * f + W_D * u \end{bmatrix} \in Im(W_E).$$

for all $(b,e) \in Ker(W_E)$. For this, we will first define a short exact sequence of sheaves.

Let V be a vector bundle on C, we will use V itself to denote the sheaf of holomorphic sections of V. For $E = \{I, f, D\}$, let $\{U_\alpha\}_{\alpha=1}^k$ be the given open cover of C, and $s = \{s_\alpha\}_{\alpha=1}^k \in H^0(C,[D])$ be the canonical section of $[D]$, this is $s \in H^0(C,[D])$ and $div(s) = D$. Let $f = \{f_\alpha\}$ be a given representation for $f \in H^0(C, I \mid_D)$, where f_α is a holomorphic function on U_α. Then by by using the transition matrix $E_{\alpha\beta}$ given in the proof of Lemma 4, one can check directly that

$$F = \{F_\alpha = \begin{bmatrix} f_\alpha \\ s_\alpha \end{bmatrix}\} \in H^0(C,E).$$

is the lift of the canonical section s. Now let $P_1 : K(-E) \mapsto K$ be the projective map which induced from sequence $0 \mapsto K[-D] \mapsto K \otimes [-E] \mapsto K \mapsto 0$, then from F and P_1, we define a map of sheaves $K(-E) \mapsto K \oplus K$ by

$$x \mapsto (P_1(x), -(x,F))$$

here $x \in K(-E)$, and $(\ ,\ )$: $K(-E) \otimes E \mapsto K$ is the duality map. We also define a map of sheaves $K \oplus K \mapsto K \mid_D$ to be $(s,t) \mapsto (s \mid_D *f + t \mid_D)$ for $(s,t) \in K \oplus K$.

Locally, let $\{U_\alpha\}$ be the given open cover of C, if $\begin{bmatrix} a \\ b \end{bmatrix} \in K(-E) \mid_{U_\alpha}$, then $K(-E) \mapsto K$ is defined by $\begin{bmatrix} a \\ b \end{bmatrix} \mapsto (a, -af_\alpha - bs_\alpha)$, and the map $K \oplus K \mapsto K_D$ could be given by $(c,d) \mapsto (c \mid_D *f + d \mid_D)$.

LEMMA 5. The sequence $0 \mapsto K(-E) \mapsto K \oplus K \mapsto K \mid_D \mapsto 0$ is a short exact sequence of sheaves on C.

Proof. We will use the local representation to give the proof.

If $\begin{bmatrix} a \\ b \end{bmatrix} \in K(-E)$, and $\begin{bmatrix} a \\ b \end{bmatrix} \mapsto (a, (-af - bs)) = 0$, then $a = 0$, and since $s \neq 0$ so $bs = 0$ means $b = 0$, the map $K(-E) \mapsto K \oplus K$ is injective.

If $(c,d) \in K \oplus K$, and $(c,d) \mapsto (c \mid_D *f + d \mid_D) = 0$, we then get $c \mid_D *f = -d \mid_D$, by our Lemma 3, c can be lift locally to section of $K(-E)$ and same as Lemma 4, $\begin{bmatrix} c \\ -(cf+d)/s \end{bmatrix} \in K(-E)$ is one of the lift. But $\begin{bmatrix} c \\ -(cf+d)/s \end{bmatrix} \mapsto (c, -cf + (cf+d)) = (c,d)$. This shows that the sequence is exact at $K \oplus K$.

Also it is easy to see that the map $K \oplus K \mapsto K \mid_D$ is an onto map. This completes the proof.

From this short exact sequence, we get a long exact sequence

$$0 \mapsto H^0(C, K(-E)) \mapsto H^0(C, K \oplus K) \mapsto H^0(C, K \mid_D) \mapsto H^1(C, K(-E)) \mapsto \cdots$$

$a \in H^0(C, K \mid_D)$ is in the image of map $H^0(C, K \oplus K) = H^0(C,K) \oplus H^0(C,K) \mapsto H^0(C, K \mid_D)$ if and only if $\delta(a) = 0$, here $\delta : H^0(C, K \mid_D) \mapsto H^1(C, K(-E))$ is the co-boundary map. But from Serra duality, we know that for $\delta(a) \in H^1(C, K(-E))$, $\delta(a) = 0$ if and only if for any $f \in H^0(C,E)$, we have $(\delta(a), f) = 0$. Here $(,)$: $H^1(C, K(-E)) \otimes H^0(C,E) \mapsto H^1(C,K)$ is the duality map.

Now assume, for open cover $\{U_\alpha\}$, a is given by $a = \{a_\alpha\}$, where $a_\alpha \in H^0(U_\alpha, K \mid_{U_\alpha})$ and $a_\alpha \mid_{D \cap U_\alpha} = a \mid_{D \cap U_\alpha}$. Then by direct calculation, we get $\delta(a) \in H^1(C, K(-E))$, could be represented as

$$\delta(a) = \{\begin{bmatrix} 0 \\ k_{\alpha\beta}(-a_\alpha + a_\beta)/s_\alpha \end{bmatrix}\} = \begin{bmatrix} 0 \\ \tilde{\delta}(a) \end{bmatrix},$$

where $\tilde{\delta} : H^0(C, K\mid_D) \mapsto H^1(C, K[-D])$ is the co-boundary map from the following sequence

$$0 \mapsto H^0(C, K[-D]) \mapsto^{\cdot s} H^0(C, K) \mapsto H^0(C, K\mid_D) \mapsto H^1(C, K[-D]) \mapsto \cdots.$$

So for any $f = \{\begin{bmatrix} y_\alpha \\ x_\alpha \end{bmatrix}\} \in H^0(C, E)$, the dual map could be given by

$$\begin{aligned}(\delta(a), f) &= (\{\begin{bmatrix} 0 \\ (-a_\alpha + a_\beta)/s_\alpha \end{bmatrix}\}, \{\begin{bmatrix} y_\alpha \\ x_\alpha \end{bmatrix}\}) \\ &= \{(-a_\alpha + a_\beta)/s_\alpha) \cdot x_\alpha\} = (\tilde{\delta}(a), \{x_\alpha\}).\end{aligned}$$

but $\delta(a) = 0$ if and only if $(\delta(a), f) = 0$ for all $f \in H^0(C, E)$, from what we get above, this is same that $\delta(a) = 0$ if and only if for any $x = \{x_\alpha\} \in Im\{H^0(C, E) \mapsto H^0(C, [D])\}$, $(\tilde{\delta}(a), x) = 0$. We get the following Lemma.

LEMMA 6. For $a \in H^0(C, K\mid_D)$, $\delta(a) \in H^1(C, K(-E))$, with $\delta(a) = 0$ if and only if for any $x = \{x_\alpha\} \in Im\{H^0(C, E) \mapsto H^0(C, [D])\}$, $(\tilde{\delta}(a), x) = 0$.

Now go back to the tangent map of $BN : H_d \mapsto M(2g, d)$.

For $E = \{I, f, D\} \in C^r_{2,d}$, we know

$$BN(E) = W_E = \begin{bmatrix} W_D \\ W_D * f \end{bmatrix},$$

if $t = (u, -v) \in T_E = H^0(C, [D]\mid_D) \oplus H^0(C, I\mid_D)$, then

$$BN^*(t) = \begin{bmatrix} \dot{W_D} * (-v) \\ \dot{W_D} * (-v) * f + W_D * u \end{bmatrix}.$$

But we know that $BN^*(t) \in T_{BN(E)}(M_{d-r+1})$ if and only if $Ker(W_E) \cdot BN^*(t) \in Im(W_E)$. Since $Im(W_E) = C^{2g} \cdot W_E = \{(c, d) \begin{bmatrix} W_D \\ W_D * f \end{bmatrix} \mid (c, d) \in C^{2g}\}$. If we identify $C^{2g} = C^g \oplus C^g \cong H^0(C, K) \oplus H^0(C, K)$, then we get

$$Im(W_E) = Im\{H^0(C, K) \oplus H^0(C, K) \mapsto H^0(C, K\mid_D)\}.$$

Where the map $H^0(C, K) \oplus H^0(C, K) \mapsto H^0(C, K\mid_D)$ is induced from above exact sequence.

From this we get $BN^*(t) \in T_{BN(E)}(M_{d-r+1})$ if and only if for any $(b, e) \in Ker(W_E)$, $(b, e)BN^*(t) \in Im(W_E)$. This is $\delta((b, e)BN^*(t)) = 0$. By Lemma 6, we get

LEMMA 7. let $t \in T_E$, then $BN^*(t) \in T_{BN(E)}(M_{d-r+1})$ if and only if for any $(b, e) \in Ker(W_E)$, we have $(\tilde{\delta}((b, e)BN^*(t)), x) = 0$ for all $x \in Im\{H^0(C, E) \mapsto H^0(C, [D])\}$.

But by direct calculation, we get

$$\begin{aligned}(b, e)BN^*(t) = (b, e)\begin{bmatrix} \dot{W_D} * (-u) \\ \dot{W_D} * (-u) * f + W_D * v \end{bmatrix} &= b\dot{W_D} * u + e\dot{W_D} * u * f + eW_D * v \\ &= \begin{bmatrix} eW_D \\ -(b\dot{W_D} + e\dot{W_D} * f) \end{bmatrix} * \begin{bmatrix} u \\ -v \end{bmatrix}.\end{aligned}$$

Notice that by using local coordinate, it is easy to see that

$$\begin{bmatrix} eW_D \\ -(b\dot{(W_D)} + e\dot{W_D} * f \end{bmatrix} = \begin{bmatrix} eW_D \\ -(bW_D + eW_D * f)/s \end{bmatrix}.$$

Since $(b, e) \in Ker(W_E)$, by Lemma 4, we get

$$\begin{bmatrix} eW_D \\ -(bW_D + eW_D * f)/s \end{bmatrix} \in Im\{H^0(C, K(-E)) \mapsto H^0(C, K(-E)\mid_D)\}.$$

let it be the image of some $F \in H^0(C, K(-E))$. Now notice that $E \mid_D = I \mid_D \oplus [D] \mid_D = T_E$ and $K(-E) \mid_D = K \mid_D \oplus K[D] \mid_D = (I \mid_D \oplus [D] \mid_D)^* = T_E^*$ then follow the proof of Lemma 1.5 p162 [ACGH] step by step, for $x \in Im\{H^0(C,E) \mapsto H^0(C,[D])\}$, we have

$$(\tilde{\delta}((b,e)BN^*(t)), x) = (\tilde{\delta}(F * t), x) = (\delta_1(t), (F \otimes x)) = (t, (F \otimes x) \mid_D)$$

Where $\delta_1 : (I \mid_D \oplus [D] \mid_D) \mapsto H^1(C, E[-D])$ is the co-boundary map follow from sequence $0 \mapsto E[-D] \mapsto^{\cdot s} E \mapsto E \mid_D \mapsto 0$. So $t \in V = \{t \in T_E \mid BN^*(t) \in T_{BN(E)}(M_{d-r+1})\}$ if and only if for any $F \in H^0(C, K(-E))$ and $x \in Im\{H^0(C,E) \mapsto H^0(C,[D])\}$, we have $(t, (F \otimes x) \mid_D) = 0$. We get

LEMMA 8. $t \in V = \{t \in T_E \mid BN^*(t) \in T_{BN(E)}(M_{d-r+1})\}$, if and only if

$$t \in \{Im\{H^0(C,K(-E)) \otimes Im\{H^0(C,E) \mapsto H^0(C,[D])\}$$

$$\mapsto H^0(C, K(-E)[D]) \mapsto H^0(C, K(-E)[D] \mid_D)\}^{\perp}.$$

Now assume $E \in C^r_{2,d} - C^{r+1}_{2,d}$,From what we get above, the expected dimension of $C^r_{2,d}$ at E could be given by

$$dim(C^r_{2,d}) = dim(V) =$$

$$2d - dim\{Im\{H^0(C,K(-E)) \otimes Im\{H^0(C,E) \mapsto H^0(C,[D])\}$$

$$\mapsto H^0(C, K(-E)[D]) \mapsto H^0(C, K(-E)[D] \mid_D)\}\} =$$

$$2d - (2(g-1) - d + r + 1)r + 2(g-1) - d + r + 1 + dimW.$$

where $(2(g-1) - d + r + 1)r = dim[H^0(C,K(-E)) \otimes Im\{H^0(C,E) \mapsto H^0(C,[D])\}] = dimH^0(C,K(-E)) \times dimIm\{H^0(C,E) \mapsto H^0(C,[D])\}$, and $2(g-1) - d + r + 1 = dimKer\{H^0(C,K(-E)[D]) \mapsto H^0(C,K(-E)[D] \mid_D)\}$, $W = Ker\{H^0(C,K(-E)) \otimes Im\{H^0(C,E) \mapsto H^0(C,[D])\} \mapsto H^0(C,K(-E)[D])\}$.

We then get

$$dim(C^r_{2,d}) = 4(g-1) + 1 - (r+1)(2(g-1) - d + r + 1) + 2r + 1 + dimW$$

$$= \rho(2,d,r) + 2r + 1 + dimW.$$

THEOREM 3. $C^r_{2,d}$ has the expected dimension $\rho(2,d,r) + 2r + 1$ at $E \in C^r_{2,d} - C^{r+1}_{2,d}$, if and only if for all $E \in C^r_{2,d}$, $W = \{0\}$.

This is the same that $C^r_{2,d}$ has the expected dimension $\rho(2,d,r) + 2r + 1$, if and only if for all $E \in C^r_{2,d}$, the map

$$H^0(C,K(-E)) \otimes Im\{H^0(C,E) \mapsto H^0(C,[D])\} \mapsto H^0(C,K(-E)[D])$$

is injective.

Compare with the case of line bundles, we then called the map

$$H^0(C,K(-E)) \otimes Im\{H^0(C,E) \mapsto H^0(C,[D])\} \mapsto H^0(C,K(-E)[D])$$

the Petri map for rank two vector bundles. We have

THEOREM 4. $C^r_{2,d}$ has the expected dimension $\rho(2,d,r) + 2r + 1$, if and only if for all $E \in C^r_{2,d}$, the Petri map is injective.

This is a generalization of Lemma 1.6 of [ACGH] P163.

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