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THE SERRE PROBLEM ON CERTAIN BOUNDED DOMAINS *

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Dedicated to Professor Siu's 60th birthday

Abstract. We give some new examples for which the Serre problem is solvable by using invariant pseudodistances.

1. Introduction. In 1953, Serre [14] raised the problem whether a holomorphic fiber bundle $\pi : E \to B$ with a Stein base B and a Stein fiber F is Stein. The answer is positive in the case of 0-dimensional fibers [19] and 1-dimensional fibers (cf. [7], [11], [15]). However, in high dimensional case, there are counterexamples (cf. [2], [17]). There are still some positive examples. Stehlé [18] solved the problem for hyperconvex Stein manifolds. Diederich-Fornaess [15] showed that any bounded C^2 pseudoconvex domains in $\mathbf{C}^n$ is hyperconvex, therefore, the Serre conjecture is true in this case. Siu [16] proved the case when the fiber is a bounded pseudoconvex domain in $\mathbf{C}^n$ with zero first Betti number. The purpose of this note is to show

THEOREM 1. *The answer to the Serre problem is positive if the fiber is either of the following:*

(i) *a bounded domain Ω in $\mathbf{C}^n$ which has a psh exhaustion function such that*

$$\psi \leq c \log\log 1/\delta_\Omega,$$

where δ_Ω denotes the Euclidean boundary distance;

(ii) *a Stein domain of the form $\Omega = \tilde{\Omega} \backslash S$, where $\tilde{\Omega}$ is a bounded domain in $\mathbf{C}^n$ which has a continuous bounded psh exhaustion function ρ with $-\rho \leq c\delta_{\tilde{\Omega}}^{\gamma}$ for suitable $c, \gamma > 0$, and S is a closed subset of $\tilde{\Omega}$ which is negligible w.r.t. to L^2 holomorphic functions, i.e., any L^2 holomorphic function on Ω extends holomorphically to $\tilde{\Omega}$.*

REMARK. a) We will show that any bounded hyperconvex domain together with some non-hyperconvex examples satisfy condition (i).

b) According to [3], any bounded C^2 pseudoconvex domain has a bounded psh exhaustion function $\rho = -(-r)^\alpha$ where $\alpha > 0$ and r is a defining function. On the other hand, there are obviously various examples whose boundary is not C^2, for example, the egg domain defined by $\{z \in \mathbf{C}^n : |z_1|^{\alpha_1} + \cdots + |z_n|^{\alpha_n} < 1\}$ where all $\alpha_i > 0$.

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2. Proof of Theorem 1. We recall the following criterion:

THEOREM 2. (cf. Stehlé [18], improved by Mok [11]) *Let $\pi : E \to B$ be a holomorphic fiber bundle with Stein base and fiber. If there exists a psh, not necessary continuous exhaustion function ψ on the fiber F such that $\psi \circ h - \psi$ is bounded for any $h \in \operatorname{Aut} F$, then E is Stein. Here $\operatorname{Aut} F$ denotes the automorphism group of F.*

In Stehlé's original criterion, one needs the hypothesis that ψ is continuous and that the assumption of continuity was removed in Mok [11].

PROPOSITION 3. *If there exists on F an upper semi-continuous function ϕ such that*

(i) ϕ is bounded from below by a psh exhaustion function on F;

(ii) $\phi \circ h - \phi$ is bounded above for any $h \in \operatorname{Aut} F$,

then the answer to the Serre problem is positive.

Proof. By Theorem 2, it suffices to construct a psh exhaustion function ψ on F such that $\psi \circ h - \psi$ is bounded above for any $h \in \operatorname{Aut} F$. We consider the following extremal function:

$$\psi(z) = \sup\{u(z) : u \in PSH(F), u \leq \phi\}$$

where $PSH(F)$ denotes all psh functions in F. We claim that ψ is the desired function. Since there exists an exhaustion function belonging to the above class, it follows that ψ is an exhaustion function on F. Since ϕ is upper semi-continuous, the upper envelope ψ^* of ψ is psh on F and satisfies $\psi^* \leq \phi$, which implies $\psi^* \leq \psi$. On the other hand, it is obvious that $\psi^* \geq \psi$. Hence $\psi = \psi^*$, which implies that ψ is a psh function on F. By (ii), we have for any $h \in \operatorname{Aut} F$

$$\begin{aligned} \psi \circ h(z) &= \sup\{u(h(z)) : u \in PSH(F),\ u \leq \phi\} \\ &\leq \sup\{v(z) : v \in PSH(F),\ v \leq \phi \circ h\} \\ &\leq \psi(z) + C_h. \end{aligned}$$

The proof is complete.

COROLLARY. *The answer to the Serre problem is positive if there exists on the fiber a complete invariant pseudodistance relative to a fixed point which is bounded below by a psh exhaustion function.*

Proof. Let d denotes the invariant pseudodistance. We can take $\phi(z) = d(z_0, z)$ for some fixed point $z_0 \in F$. Clearly, for any $h \in \operatorname{Aut} F$, one has

$$\begin{aligned} \phi \circ h(z) - \phi(z) &= d(z_0, h(z)) - d(z_0, z) \\ &= d(h^{-1}(z_0), z) - d(z_0, z) \leq d(z_0, h^{-1}(z_0)). \end{aligned}$$

The result follows immediately from the above proposition.

Let us see some applications.

a) Siu's distance: Let $D \neq \mathbf{C}$ be a domain in $\mathbf{C}$. Siu [15] constructed an invariant distance on D satisfying

$$dist_S(z_0, z) \geq \frac{1}{4} \log \frac{\delta_D(z_0)}{\delta_D(z)}.$$

Note that the right side is naturally a subharmonic exhaustion function when D is bounded. To define a subharmonic exhaustion function when D is unbounded, one can consider at the same time the domain on the w−plane defined by $w = \frac{1}{z-p}$ for some point $p \in \Omega$.

b) Bergman distance: Let Ω be a bounded domain in $\mathbf{C}^n$, and let $K_\Omega(z,w)$ be the Bergman kernel and $K_\Omega(z) = K_\Omega(z,z)$. The Bergman metric is defined by

$$B_\Omega(z;X) = \left(\sum_{j.k=1}^{n} \frac{\partial^2 \log K_\Omega(z)}{\partial z_j \partial \bar{z}_k} X_j \bar{X}_k \right)^{1/2}$$

where $X = \sum_{j=1}^n X_j \partial/\partial z_j \in T^{1,0}(\mathbf{C}^n)$. The related distance is called the Bergman distance. We denote by $dist_B$.

Diederich-Ohsawa [4] showed that the Bergman distance satisfies the following estimate for bounded C^2 pseudoconvex domains

$$dist_B(z_0, z) \geq C \log\log 1/\delta_D(z).$$

On the other hand, there exists on Ω (cf. [9]) a negative psh function ρ satisfying

$$-\frac{A}{\log 1/\delta_\Omega(z)} \leq \rho(z) \leq -\frac{B}{\log 1/\delta_\Omega(z)}$$

for suitable positive constants A, B. This implies in particular

$$-\log(-\rho(z)) \leq C' \log\log 1/\delta_\Omega(z).$$

Note that the left side is also a psh exhaustion function.

c) Kähler-Einstein metric (proof of of Theorem 1): A Kähler-Einstein metric on a complex manifold is a Kähler metric for which the Ricci tensor coincides up to multiplication by a real constant with the metric tensor. Thanks to Mok-Yau [12], such a metric exists on any bounded pseudoconvex domain in $\mathbf{C}^n$. Moreover, it is complete, biholomorphically invariant, and the Kähler-Einstein distance $dist_{KE}$ satisfies

$$dist_{KE}(z_0, z) \geq C \log\log 1/\delta_\Omega(z).$$

Hence (i) of the proof of Theorem 1 follows immediately from Proposition 3.

Before proving (ii), let us recall that the pluricomplex Green function of a bounded domain Ω with a pole at w is defined by

$$g_\Omega(z,w) = \sup\{u(z) : u < 0, u \in PSH(\Omega), \lim_{z \to w} \sup(u(z) - \log|z-w|) < +\infty\}.$$

It is well-known that $g_\Omega(\cdot, w)$ is a psh function and $g_\Omega(h(z), h(w)) = g_\Omega(z, w)$ for any $h \in \operatorname{Aut}\Omega$ (cf. [10]). Set

$$r_w(z) = \max\{g_\Omega(z, w), -1\}.$$

LEMMA 4. *Let $w, w' \in \Omega$. There exist positive constants C_1, C_2 depending only on w, w' such that*

$$C_1 \leq \frac{r_w(z)}{r_{w'}(z)} \leq C_2$$

for all $z \in \Omega$.

Proof. Without loss of generality, we assume $w \neq w'$. Set

$$\delta = \frac{1}{2}\min\{|w - w'|, \delta_\Omega(w), \delta_\Omega(w')\}$$

and

$$\eta(z) = \begin{cases} g(z, w') & \text{if } |z - w'| < \delta \\ \max\{C_3 r_w(z), g(z, w')\} & \text{if } |z - w'| \geq \delta \end{cases}$$

where $C_3 = C_3(w, w')$ is a positive constant which satisfies

$$C_3 \sup_{\{|z-w'|=\delta\}} r_w(z) \leq \inf_{\{|z-w'|=\delta\}} g(z, w')$$

because $g_\Omega(z, w') \geq \log|z - w'|/R$, $z \in \Omega$ and r_w is upper semi-continuous. Here R denotes the diameter of Ω and $0 < r < \delta_\Omega(w)$. Thus η is a well-defined negative psh function with a pole at w'. Hence $g_\Omega(z, w') \geq C_3 r_w(z)$ for $|z - w'| \geq \delta$. It follows that the inequality $r_w(z) \leq C_1 r_{w'}(z)$ holds on Ω for suitable constant $C_1 > 0$. The opposite inequality can be obtained in a similar way.

LEMMA 5. *Let $\tilde{\Omega}$ be a bounded domain in $\mathbf{C}^n$ such that there exists bounded psh exhaustion function ρ satisfying $-\rho(z) \leq c\delta^\gamma_{\tilde{\Omega}}(z)$ for suitable constants $c, \gamma > 0$. Suppose $\Omega \subset \tilde{\Omega}$. Then for any $z_0 \in \Omega$, there is a constant C_4 such that*

$$-r_{z_0}(z) \leq C_4 \delta^\gamma_{\tilde{\Omega}}(z), \ \forall\, z \in \Omega.$$

Proof. Let R denote the diameter of Ω. Similar as above, we set

$$\eta'(z) = \begin{cases} \log|z - z_0|/R & \text{if } |z - z_0| < \delta_\Omega(z_0)/2 \\ \max\{\log|z - z_0|/R, C_5\rho(z)\} & \text{if } |z - z_0| \geq \delta_\Omega(z_0)/2 \end{cases}$$

where C_5 satisfies

$$C_5 \sup_{\{|z-w'|=\delta_\Omega(z_0)/2\}} \rho(z) \leq \inf_{\{|z-w'|=\delta_\Omega(z_0)/2\}} \log|z - z_0|/R.$$

Therefore,

$$g_\Omega(z, z_0) \geq C_5\rho(z) \geq cC_5\delta^\gamma_{\tilde{\Omega}}(z)$$

for $|z - z_0| \geq \delta_\Omega(z)/2$. On the other hand, we note that $r_{z_0} \geq -1$ and $\delta_{\tilde{\Omega}}$ has a uniformly positive lower bound on $\{z \in \Omega : |z - z_0| \leq \delta_\Omega(z_0)/2\}$ because of the continuity of $\delta_{\tilde{\Omega}}$. Thus the desired inequality follows.

Proof of (ii) of Theorem 1. Set $\psi = -\log(-r_{z_0})$ for some fixed point z_0. Clearly, it is psh. Since $r_{z_0}(h(z)) = r_{h^{-1}(z_0)}(z)$ for any $h \in \operatorname{Aut}\Omega$, it follows from Lemma 4 that $\psi \circ h - \psi$ is bounded above; By Lemma 5, we also have $\psi(z) \geq C \log 1/\delta_{\tilde{\Omega}}(z)$, $\forall z \in \Omega$. Since Ω is Stein, according to Mok-Yau [12], if one writes the volume form of the Kähler-Einstein metric as

$$V_{KE}(z)(i/2)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n,$$

then

$$\begin{aligned} V_{KE}(z) &= V_{KE}(h(z))|\det h'(z)|^2, \ \forall\, h \in \operatorname{Aut}\Omega \\ V_{KE}(z) &\geq \frac{C}{\delta^2_{\tilde{\Omega}}(z)(\log \delta_\Omega(z))^2}. \end{aligned} \tag{1}$$

By the well-known translation formula of the Bergman kernel function, the ratio V_{KE}/K_Ω is a function which is invariant under $\operatorname{Aut}\Omega$. Since S is negligible w.r.t. L^2 holomorphic functions, we have $K_\Omega(z) = K_{\tilde{\Omega}}(z)$ for any $z \in \Omega$. By (1), the function $\phi = \log V_{KE}/K_\Omega + N\psi$ satisfies the conditions of Proposition 3 provided the constant N large enough, since

$$K_{\tilde{\Omega}}(z) \leq K_{B(z,\delta_{\tilde{\Omega}}(z))}(z) \leq C_1' \delta_{\tilde{\Omega}}^{-2n}(z). \tag{2}$$

Here $B(p,r) \subset \Omega$ denotes the Euclidean ball with centre p and radius r and $C_1' > 0$ is a constant depending only on n.

The class of domains in (i) is quite large since we have the following

PROPOSITION 6. *Let Ω be a bounded hyperconvex domain in $\mathbf{C}^n$. Then there exists a continuous psh exhaustion function ψ on Ω such that*

$$\psi(z) \leq C \log\log 1/\delta_\Omega(z).$$

Proof. We proceed the proof with the help of the Bergman kernel function $K_\Omega(z)$. Take a cut-off function $\chi : \mathbf{R} \to [0,1]$ such that $\chi|_{(-\infty,-2\log 2]} = 1$ and $\chi|_{[-\log 2,+\infty)} = 0$. Set

$$\varphi_z = 2n g_\Omega(\cdot, z) - \log(-g_\Omega(\cdot, z) + 1).$$

By a standard limiting procedure, we can solve, according to Lemma 4.4.1 in [8], the equation

$$\bar{\partial} u = \bar{\partial}\chi(-\log(-g_\Omega(\cdot, z) + 1))$$

in the weak sense together with the estimate

$$\begin{aligned} \int_\Omega |u|^2 e^{-\varphi_z} dV &\leq \int_\Omega \left|\bar{\partial}\chi(-\log(-g_\Omega(\cdot, z) + 1))\right|^2_{\sqrt{-1}\partial\bar{\partial}\varphi_z} e^{-\varphi_z} dV \\ &\leq C_2' \operatorname{vol}\left(\{g_\Omega(\cdot, z) < -1\}\right) \end{aligned}$$

because

$$\sqrt{-1}\partial\bar{\partial}\varphi_z \geq \frac{\sqrt{-1}g_\Omega(\cdot,z)\bar{\partial}g_\Omega(\cdot,z)}{(-g_\Omega(\cdot,z)+1)^2}.$$

Here $|\cdot|_{\partial\bar{\partial}\varphi_z}$ denotes the pointwise norm with respect to the (singular) metric $\partial\bar{\partial}\varphi_z$ and C_2' depends only on n and the choice of χ. Set

$$f = \chi(-\log(-g_\Omega(\cdot,z)+1)) - u.$$

Clearly, f is a holomorphic function on Ω which satisfies $f(z)=1$ and

$$\begin{aligned}\int_\Omega |f|^2 dV &\leq 2\int_\Omega |\chi(-\log(-g_\Omega(\cdot,z)+1))|^2 dV + 2\int_\Omega |u|^2 dV \\ &\leq C_3' \mathrm{vol}\left(\{g_\Omega(\cdot,z) < -1\}\right)\end{aligned}$$

since $\varphi_z < 0$ and $\varphi_z \leq 2n\log|\cdot - z| + O(1)$. It follows that

$$K_\Omega(z) \geq (C_3'\mathrm{vol}\left(\{g_\Omega(\cdot,z) < -1\}\right))^{-1}. \tag{3}$$

In [1], Blocki-Pflug proved that there exists a bounded continuous psh exhaustion function ρ on Ω such that

$$\int_\Omega (-g_\Omega(\cdot,z))^n dV \leq n!(2\pi)^n \, \|\rho\|_{L^\infty(D)}^{n-1} \, |\rho(z)|,$$

which implies

$$\mathrm{vol}\left(\{g_\Omega(\cdot,z) < -1\}\right) \leq C_4'|\rho(z)|. \tag{4}$$

By (2)–(4), we obtain

$$-\rho \geq C_5\delta_\Omega^{2n}.$$

To complete the proof, we only need to set $\psi = \log(1-\log(-\rho))$ (without loss of generality, we may assume $-\rho < 1$ on Ω).

We have also some *Non-hyperconvex examples*:

1) Consider the Hartogs domain defined as follows

$$\Omega = \{(z,w) \in D \times \mathbf{C}^m : |w| < \exp(-\exp\varphi(z))\}$$

where D is a bounded pseudoconvex domain in $\mathbf{C}^n$ and φ is an continuous psh exhaustion function of D. Set

$$\psi(z,w) = \max\left\{\varphi(z), \log(1-\log(1-|w|\exp\exp\varphi(z)))\right\}.$$

Clearly, ψ is a psh exhaustion function of Ω. Note that

$$\delta_\Omega((z,w)) \leq \exp(-\exp\varphi(z)) - |w|,$$

which implies

$$\varphi(z) \leq \log\log 1/\delta_\Omega((z,w)).$$

We also have

$$\begin{aligned}1-|w|\exp\exp\varphi(z) &= \exp\exp\varphi(z)\,(\exp(-\exp\varphi(z))-|w|)\\ &\geq \exp(-\exp\varphi(z))-|w|\\ &\geq \delta_\Omega((z,w)).\end{aligned}$$

It follows that

$$\psi(z,w)\leq C\log\log 1/\delta_\Omega((z,w)).$$

It is well known that Ω is hyperconvex iff D is hyperconvex. Hence we can obtain various non-hyperconvex examples.

2) Herbort's example (cf. [6]):

$$\Omega=\left\{(z_1,z_2)\in\mathbf{C}^2 : z_1\in\Delta^*,\ |z_2|^2e^{1/|z_1|^2}<1\right\}$$

where Δ^* denotes the punctured unit disk. By a similar argument as above, one can show that

$$\psi(z)=\max\left\{-\log|z_1|,\log(1-\log(1-|z_2|^2e^{1/|z_1|^2}))\right\}$$

satisfies the condition (i) of Theorem 2.

REMARK. If Ω_1,Ω_2 satisfy condition (i) of Theorem 2, then $\Omega=\Omega_1\times\Omega_2$ also satisfies this condition: it suffices to take $\psi(z',z'')=\max\{\psi_1(z'),\psi_2(z'')\}$ for the slow growth psh exhaustion functions ψ_j relative to $\Omega_j, j=1,2$.

d) Kobayashi pseudodistance: Let M be a complex manifold and let Δ denote the unit disk in $\mathbf{C}$. The Kobayashi-Royden pseudometric is defined by

$$F_{KR}(z;X):=\inf\{|a|^{-1}:\exists\ f:\Delta\to M \text{ holomorphic with } f(0)=z, f'(0)=aX\}.$$

The related pseudodistance is called the Kobayashi pseudodistance which is denoted by $dist_K$.

According to Wu's theorem, any complete simply-connected Kähler manifold of nonpositive sectional curvature is Stein, namely, $\log(1+\rho^2)$ is a strictly psh exhaustion function. Here ρ denotes the distance function relative to some fixed point of M. If furthermore, the holomorphic sectional curvature is bounded from above by $-\frac{A}{1+\rho^2}$, then M is complete hyperbolic [5], moreover, the Kobayashi distance satisfies

$$dist_K(z_0,z)\geq C\log(1+\rho^2(z)).$$

Thus we obtain the following

THEOREM 7. *The answer to the Serre problem is affirmative if the fiber is a complete simply-connected Kähler manifold of nonpositive sectional curvature such that the holomorphic sectional curvature is bounded above by* $-\frac{A}{1+\rho^2}$.

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