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HOLOMORPHIC EXTENSION OF SMOOTH CR-MAPPINGS BETWEEN REAL-ANALYTIC AND REAL-ALGEBRAIC CR-MANIFOLDS *

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1. Introduction and results. The classical Schwarz reflection principle states that a continuous map f between real-analytic curves M and M' in $\mathbb{C}$ that locally extends holomorphically to one side of M, extends also holomorphically to a neighborhood of M in $\mathbb{C}$. It is well-known that the higher-dimensional analog of this statement for maps $f\colon M \to M'$ between real-analytic CR-submanifolds $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ does not hold without additional assumptions (unless M and M' are totally real). In this paper, we assume that f is $\mathcal{C}^\infty$-smooth and that the target M' is *real-algebraic*, i.e. contained in a real-algebraic subset of the same dimension. If f is known to be locally holomorphically extendible to one side of M (when M is a hypersurface) or to a wedge with edge M (when M is a generic submanifold of higher codimension), then f automatically satisfies the tangential Cauchy-Riemann equations, i.e. it is CR. On the other hand, if M is *minimal*, any CR-map $f\colon M \to M'$ locally extends holomorphically to a wedge with edge M by TUMANOV's theorem [Tu88] and hence, in that case, the extension assumption can be replaced by assuming f to be CR.

Local holomorphic extension of a CR-map $f\colon M \to M'$ may clearly fail when M' contains an (irreducible) complex-analytic subvariety E' of positive dimension and $f(M) \subset E'$. Indeed, any nonextendible CR-function on M composed with a nontrivial holomorphic map from a disc in $\mathbb{C}$ into E' yields a counterexample. Our first result shows that this is essentially the only exception. Denote by $\mathcal{E}'$ the set of all points $p' \in M'$ through which there exist irreducible complex-analytic subvarieties of M' of positive dimension. We prove:

THEOREM 1.1. *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be respectively connected real-analytic and real-algebraic CR-submanifolds. Assume that M is minimal at a point $p \in M$. Then for any $\mathcal{C}^\infty$-smooth CR-map $f\colon M \to M'$, at least one of the following conditions holds:*

(i) f extends holomorphically to a neighborhood of p in $\mathbb{C}^N$;

(ii) f sends a neighborhood of p in M into $\mathcal{E}'$.

If M' is a real-analytic hypersurface, the set $\mathcal{E}'$ consists exactly of those points that are not of finite type in the sense of D'ANGELO [D'A82] (see LEMPERT [L86] for the proof) and, in particular, $\mathcal{E}'$ is closed. The same fact also holds if M' is any real-analytic submanifold or even any real-analytic subvariety (see [D'A91]). However, in general, $\mathcal{E}'$ may not even be a real-analytic subset (see Example 2.1). In case $\mathcal{E}' = V'$ is a subvariety, we have:

COROLLARY 1.2. *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be as in Theorem 1.1. Assume that M is minimal at a point $p \in M$ and that all positive-dimensional irreducible complex-analytic subvarieties in M' are contained in a fixed (complex-analytic) subvariety $V' \subset$*

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M'. Then any $\mathcal{C}^\infty$-smooth CR-map $f\colon M \to M'$ that does not send a neighborhood of p in M into V' extends holomorphically to a neighborhood of p in $\mathbb{C}^N$.

In view of an example due to EBENFELT [E96], the minimality assumption on M at p in Corollary 1.2 cannot be replaced by the assumption that M is minimal somewhere. On the other hand, if M is also real-algebraic, this replacement is possible:

THEOREM 1.3. *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be connected real-algebraic CR-submanifolds with $p \in M$ and let $V' \subset M'$ be as in Corollary* 1.2. *Then the conclusion of Corollary* 1.2 *holds provided M is minimal somewhere.*

In the setting of Theorem 1.3, any $\mathcal{C}^\infty$-smooth CR-map $f\colon M \to M'$ that does not send a neighborhood of p in M into V' extends even algebraically to a neighborhood of p in $\mathbb{C}^N$ by a result of [Z99] (see §7). Since the subset $\mathcal{E}' \subset M'$ is always closed, Corollary 1.2 and Theorem 1.3 imply:

COROLLARY 1.4. *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be respectively connected real-analytic and real-algebraic CR-submanifolds. Assume that M is minimal at a point $p \in M$ and that M' does not contain any irreducible complex-analytic subvariety of positive dimension through a point $p' \in M'$. Then any $\mathcal{C}^\infty$-smooth CR-map $f\colon M \to M'$ with $f(p) = p'$ extends holomorphically to a neighborhood of p in $\mathbb{C}^N$. The same conclusion holds at a point $p \in M$ if M is real-algebraic and only somewhere minimal.*

In the case when $M \subset \mathbb{C}^N$ is a real hypersurface, the first part of Corollary 1.4 is due to PUSHNIKOV [P90a, P90b] (see also [CPS00]).

A prototype of a target real-algebraic CR-submanifold with no nontrivial complex-analytic subvariety is given by the unit sphere $\mathbb{S}^{2N'-1} \subset \mathbb{C}^{N'}$. Even in that case, Corollary 1.4 seems to be new. Indeed, we have:

COROLLARY 1.5. *Let $M \subset \mathbb{C}^N$ be a connected real-analytic CR-submanifold, minimal at a point $p \in M$. Then any $\mathcal{C}^\infty$-smooth CR-map $f\colon M \to \mathbb{S}^{2N'-1}$ extends holomorphically to a neighborhood of p in $\mathbb{C}^N$. The same conclusion holds for any point $p \in M$ if M is real-algebraic and only somewhere minimal.*

For f of class $\mathcal{C}^\infty$, Corollary 1.5 extends results of WEBSTER [W79], FORSTNERIČ [F86, F89, F92], HUANG [H94] and BAOUENDI-HUANG-ROTHSCHILD [BHR96]. (On the other hand, in their setting, they prove holomorphic extension of f of class $\mathcal{C}^k$ for appropriate k.)

If we restrict ourselves to *submersive* CR-maps (i.e. maps for which the differential is surjective), a known obstruction to their holomorphic extension is the *holomorphic degeneracy* of the submanifolds. Recall that a real-analytic CR-submanifold M is holomorphically degenerate (see STANTON [S96]) at a point $p \in M$ if there is a nontrivial holomorphic vector field in a neighborhood of p in $\mathbb{C}^N$ whose real and imaginary parts are tangent to M. The existence of such a vector field and a nonextendible CR-function on M at p yields nonextendible local self CR-diffeomorphic maps of M near p (see [BHR96]). It is known (see [BER96]) that M is holomorphically degenerate at p if and only if it is holomorphically degenerate everywhere on the connected component of p. Our next result shows that for source minimal CR-submanifolds, holomorphic degeneracy is essentially the only obstruction for submersive CR-maps to be holomorphically extendible.

THEOREM 1.6. *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be respectively connected real-analytic and real-algebraic CR-submanifolds of the same CR-codimension with $p \in M$. Assume that M is everywhere minimal and M' is holomorphically nondegenerate.*

Then any $\mathcal{C}^\infty$-smooth CR-map $f\colon M \to M'$ which is somewhere submersive extends holomorphically to a neighborhood of p in $\mathbb{C}^N$.

In the case when $M, M' \subset \mathbb{C}^N$ are real hypersurfaces, a similar result is contained in [CPS00]. Example 2.2 below shows that the assumption that M is *everywhere minimal* cannot be replaced in Theorem 1.6 by the weaker assumption that M *is minimal at* p. On the other hand, if M is real-algebraic, a replacement with even weaker assumption on M is possible:

THEOREM 1.7. *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be connected real-algebraic CR-submanifolds of the same CR-codimension with $p \in M$. Then the conclusion of Theorem* 1.6 *holds provided M is somewhere minimal and M' is holomorphically nondegenerate.*

In the setting of Theorem 1.7, any $\mathcal{C}^\infty$-smooth CR-map $f\colon M \to M'$ extends in fact algebraically to a neighborhood of p in $\mathbb{C}^N$ by a result of [Z99] (see §7). Theorem 1.7 extends a result of [BHR96] who obtained the same conclusion for $M, M' \subset \mathbb{C}^N$ real-algebraic hypersurfaces and of KOJCINOVIC [K00] for $M, M' \subset \mathbb{C}^N$ generic submanifolds of equal dimension. For further related results and history on the analyticity problem for CR-mappings, the reader is referred to [F93, BER99, H01].

We shall derive the above results in §7 from the following statement that relates analyticity properties of a smooth CR-map with geometric properties of its graph:

THEOREM 1.8. *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be respectively connected real-analytic and real-algebraic CR-submanifolds and $f\colon M \to M'$ a $\mathcal{C}^\infty$-smooth CR-map whose graph is denoted by Γ_f. Assume that M is minimal at a point $p \in M$ and that f does not extend holomorphically to any neighborhood of p. Then there exists an integer $1 \le n \le N' - 1$ and a real-analytic subset $A \subset M \times M'$ through $(p, f(p))$ containing a neighborhood Ω of $(p, f(p))$ in Γ_f and satisfying the following straightening property: for any point $(q, f(q))$ in a dense open subset of Ω, there exists a neighborhood U_q of $(q, f(q))$ in $\mathbb{C}^N \times \mathbb{C}^{N'}$ and a holomorphic change of coordinates in U_q of the form $(\tilde{z}, \tilde{z}') = (z, \varphi(z, z')) \in \mathbb{C}^N \times \mathbb{C}^{N'}$ such that*

$$A \cap U_q = \{(z, z') \in U_q : z \in M,\ \tilde{z}'_{n+1} = \cdots = \tilde{z}'_{N'} = 0\}. \tag{1.1}$$

Theorem 1.8 will follow from the more general Theorem 6.1, where the target $M' \subset \mathbb{C}^{N'}$ is assumed to be a real-algebraic subset and an estimate for the number n (in Theorem 1.8) is given. Our approach follows partially the techniques initiated in [P90a, P90b] and followed in [CPS00] in the case M is a hypersurface. A crucial point in the proof of Theorem 1.8 consists of showing (after possible shrinking M around p) that near a generic point of the graph Γ_f, the intersection of $M \times \mathbb{C}^{N'}$ with the local Zariski closure of Γ_f at $(p, f(p))$ (see §4 for the definition) is contained in $M \times M'$ (see Theorem 6.1 and Proposition 6.2). Here we have to proceed differently from [P90a, P90b, CPS00]. In §3 we give preliminary results based on a meromorphic extension property from [MMZ02]. In particular, Proposition 3.4 (ii) may be of independent interest. §4–6 are devoted to the proof of Theorem 1.8.

REMARK 1.9. *A preprint version of this paper was posted on the web on January, 28, 2002 (http://www.arxiv.org/abs/math.CV/0201267). Later the authors discovered the note by Coupet, Damour, Merker and Sukhov, "Sur l'analyticité des applications CR lisses à valeurs dans un ensemble algébrique réel",* C. R. Math. Acad. Sci. Paris, *334 (2002), no. 11, 953–956 (submitted on March 12, 2002), containing results and*

(sketched) proofs analogous to those in the posted preprint that were not contained in previous literature (in particular, the arguments of §3 there deviate from those in [CPS00] but follow §3 and §6 of this paper).

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2. Preliminaries and examples.

2.1. CR-submanifolds and CR-maps. A real submanifold $M \subset \mathbb{C}^N$ is called a CR-submanifold if the dimension of the complex tangent space $T^c_pM := T_pM \cap iT_pM$ is independent of $p \in M$. In this case $\dim_{\mathbb{C}} T^c_pM$ is called the *CR-dimension* and $\dim_{\mathbb{R}} T_pM - \dim_{\mathbb{R}} T^c_pM$ the *CR-codimension* of M. Furthermore, M is called *generic* if for any point $p \in M$, one has $T_pM + iT_pM = T_p\mathbb{C}^N$. For a CR-submanifold M we write $T^{0,1}M := T^{0,1}\mathbb{C}^N \cap \mathbb{C}TM$, where $T^{0,1}\mathbb{C}^N$ is the bundle of $(0,1)$ tangent vectors in $\mathbb{C}^N$. A function $h\colon M \to \mathbb{C}^{N'}$ of class $\mathcal{C}^1$ is called a CR-function if for any section L of the CR-bundle, $Lf = 0$. If h is merely continuous, h is still called CR if it is annihilated by all vector fields L as above in the sense of distributions. A continuous map $f\colon M \to M'$ between CR-submanifolds $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ is called a CR-map if all its components are CR-functions.

A CR-submanifold $M \subset \mathbb{C}^N$ is called *minimal* (in the sense of TUMANOV) at a point $p \in M$ if there is no real submanifold $S \subset M$ through p with $\dim S < \dim M$ and such that $T^c_qM \subset T_qS$, for all $q \in S$. It is well-known that if M is real-analytic, the minimality condition of M is equivalent to the finite type condition in the sense of KOHN and BLOOM-GRAHAM (see [BER99]).

A real (resp. complex) submanifold $M \subset \mathbb{C}^N$ is real-algebraic (resp. algebraic) if it is contained in a real-algebraic (resp. complex-algebraic) subvariety with the same real (resp. complex) dimension as that of M. A map $f\colon M \to M'$ between real submanifolds $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ is real-algebraic if its graph $\Gamma_f := \{(z, f(z)) : z \in M\}$ is a real-algebraic submanifold of $\mathbb{C}^N \times \mathbb{C}^{N'}$. Similarly, a holomorphic map between open subsets $\Omega \subset \mathbb{C}^N$ and $\Omega' \subset \mathbb{C}^{N'}$ is called algebraic if its graph is a complex-algebraic submanifold of $\Omega \times \Omega'$.

2.2. Examples. The following example shows that, even if $M' \subset \mathbb{C}^{N'}$ is a real-analytic hypersurface, the subset $\mathcal{E}' \subset M'$ of all points that are not of finite D'Angelo type is not real-analytic in general.

EXAMPLE 2.1. *Consider the tube real-analytic hypersurface $M' \subset \mathbb{C}^4$ given by*

$$(\operatorname{Re} z_1)^2 - (\operatorname{Re} z_2)^2 + (\operatorname{Re} z_3)^2 = (\operatorname{Re} z_4)^3 \tag{2.1}$$

near the point $(1,1,0,0) \in M'$. We claim that the subset $\mathcal{E}' \subset M'$ is given by $\operatorname{Re} z_4 \geq 0$ and is therefore not analytic. Indeed, every intersection of M' with $\{z_4 = \text{const}, \operatorname{Re} z_4 \geq 0\}$ contains complex lines through each point and is hence everywhere of D'ANGELO *infinite type. On the other hand, if $\operatorname{Re} z_4 < 0$, the coordinate $\operatorname{Re} z_2$ can be expressed as a strictly convex function of the other coordinates. Therefore, M' is strictly pseudoconvex at each such point and thus of* D'ANGELO *finite type.*

The following example shows that a somewhere submersive $\mathcal{C}^\infty$-smooth CR-map $f\colon M \to M'$ between connected real-analytic hypersurfaces in $\mathbb{C}^2$ can be real-analytic on some connected component of the set of minimal points of M and not real-analytic

in another component. In particular, the assumption of Theorem 1.6 that M is everywhere minimal cannot be replaced by the weaker assumption that M is minimal at p.

EXAMPLE 2.2. *As in* EBENFELT*'s example [E96], let $M, M' \subset \mathbb{C}^2$ be connected real-analytic hypersurfaces through 0 given respectively by*

$$\operatorname{Im} w = \theta(\arctan|z|^2, \operatorname{Re} w), \quad \operatorname{Im} w = (\operatorname{Re} w)|z|^2,$$

where $t = \theta(\xi, s)$ is the unique solution of the algebraic equation $\xi(t^2+s^2)-t=0$ with $\theta(0,0)=0$ given by the implicit function theorem. Note that M and M' are minimal precisely outside the complex line $\{w=0\}$ and that M' is real-algebraic, but M is not. For every $\mathcal{C}^\infty$-smooth CR-function φ on M, define a map $f_\varphi\colon M \to \mathbb{C}^2$ by

$$f_\varphi(z,w) := \begin{cases} (z,\,0) & \operatorname{Re} w = 0 \\ (z,\,e^{-1/w}) & \operatorname{Re} w > 0 \\ (z+\varphi(z,w)\,e^{1/w},\,0) & \operatorname{Re} w < 0. \end{cases} \tag{2.2}$$

By similar arguments as in [E96] it follows that f_φ is always a $\mathcal{C}^\infty$-smooth CR-map sending M into M'. Suppose we can choose φ not holomorphically extendible to any neighborhood in $\mathbb{C}^2$ of a fixed minimal point $p_0 = (z_0, w_0) \in M$ with $\operatorname{Re} w_0 < 0$. Then it is easy to see that f_φ is somewhere submersive but does not extend holomorphically to any neighborhood of the minimal point $p_0 \in M$.

To show that the above choice of p_0 and φ is possible, observe that θ can be factored as $\theta(\xi,s) = s^2\xi(1+\widetilde{\theta}(\xi,s))$ with $\widetilde{\theta}$ analytic and vanishing at the origin. Hence $\operatorname{Im} w \geq 0$ for every sufficiently small $(z,w) \in M$. Then, for any real sufficiently small $x_0 \neq 0$, the point $p_0 := (0, x_0) \in M$ is minimal and a suitable branch of $e^{-1/(w-x_0)^{1/3}}$ extends to a $\mathcal{C}^\infty$-smooth CR-function φ on M that is not holomorphically extendible to any neighborhood of p_0.

3. A result on meromorphic extension and its applications. In what follows, for any subset $V \subset \mathbb{C}^k$, V^* denotes the set $\{\bar{z} : z \in V\}$ and, as usual, for any ring A, we denote by $A[X]$, $X = (X_1, \ldots, X_s)$, the ring of polynomials in s indeterminates with coefficients in A. An important role in the proof of Theorem 1.8 will be played by the following meromorphic extension result from [MMZ02, Theorem 2.6].

THEOREM 3.1. *Let $\Omega \subset \mathbb{C}^N$, $V \subset \mathbb{C}^k$ be open subsets, $M \subset \Omega$ a connected generic real-analytic submanifold, $G\colon M \to V$ a continuous CR-function and $\Phi, \Psi\colon V^* \times \Omega \to \mathbb{C}$ holomorphic functions. Assume that M is minimal at every point and that there exists a nonempty open subset of M where $\Psi(\overline{G(z)}, z)$ does not vanish and where the quotient*

$$H(z) := \frac{\Phi(\overline{G(z)}, z)}{\Psi(\overline{G(z)}, z)}$$

is CR. Then $\Psi(\overline{G(z)}, z)$ does not vanish on a dense open subset $\widetilde{M} \subset M$ and H extends from $\widetilde{M}$ meromorphically to a neighborhood of M in $\mathbb{C}^N$.

REMARK 3.2. *Results in the spirit of Theorem 3.1 have been important steps in proving regularity results for CR-mappings (see e.g. [P90a, P90b, BHR96, CPS99,*

CPS00, MMZ02, Mi03]). Also a special case of Theorem 3.1 with G being C^∞ and H CR on a dense open subset *is due to [Da01] but the stronger form of Theorem 3.1 is essential to complete the proofs in this paper and does not follow from the arguments of [Da01].*

For a generic real-analytic submanifold $M \subset \mathbb{C}^N$, denote by $\mathcal{C}^\infty(M)$ the ring of $\mathcal{C}^\infty$-smooth functions on M, by $\mathcal{O}(M)$ the ring of restrictions of holomorphic functions to M and by $\mathcal{O}_p(M)$ the corresponding ring of germs at a point $p \in M$. Similarly to [CPS99] (see also [P90a, P90b, CPS00, MMZ02]), define a subring $\mathcal{A}(M) \subset \mathcal{C}^\infty(M)$ as follows: *a function $\eta \in \mathcal{C}^\infty(M)$ belongs to $\mathcal{A}(M)$ if and only if, near every point $p \in M$, it can be written in the form $\eta(z) \equiv \Phi(\overline{G(z)}, z)$, where G is a $\mathbb{C}^k$-valued $\mathcal{C}^\infty$-smooth CR-function in a neighborhood of p in M for some k and Φ is a holomorphic function in a neighborhood of $(\overline{G(p)}, p)$ in $\mathbb{C}^k \times \mathbb{C}^N$.* Note that the ring $\mathcal{C}^\omega(M)$ of all real-analytic functions on M is a subring of $\mathcal{A}(M)$. We have the following known properties (see e.g. [MMZ02]):

LEMMA 3.3. *Let $M \subset \mathbb{C}^N$ be a connected generic real-analytic submanifold that is minimal at every point. Then for any $u \in \mathcal{A}(M)$ the following hold:*

(i) if u vanishes on a nonempty open subset of M, then it vanishes identically on M;

(ii) if L is a real-analytic $(0,1)$ vector field on M, then $Lu \in \mathcal{A}(M)$.

The following proposition is a consequence of Theorem 3.1 and will be essential for the proof of Theorem 1.8. In the proof we follow the approach of [P90b] (see also [CMS99, Proposition 5.1]).

PROPOSITION 3.4. *Let $M \subset \mathbb{C}^N$ be a connected generic real-analytic submanifold that is minimal at every point. Let $F_1, \dots, F_r$ be $\mathcal{C}^\infty$-smooth CR-functions on M satisfying one of the following conditions:*

(i) the restrictions of $F_1, \dots, F_r$ to a nonempty open subset of M satisfy a nontrivial polynomial identity with coefficients in $\mathcal{A}(M)$;

(ii) the restrictions of $F_1, \dots, F_r, \overline{F_1}, \dots, \overline{F_r}$ to a nonempty open subset of M satisfy a nontrivial polynomial identity with coefficients in $\mathcal{C}^\omega(M)$.

Then for any point $q \in M$, the germs at q of $F_1, \dots, F_r$ satisfy a nontrivial polynomial identity with coefficients in $\mathcal{O}_q(M)$.

Proof. We first observe that, for the rest of the proof, we can assume that the $(0,1)$ vector fields on M are spanned by global real-analytic vector fields on M. Indeed, suppose we have proved Proposition 3.4 under this additional assumption, then we claim that Proposition 3.4 follows from that case. For this, for fixed $F_1, ..., F_r$ as in Proposition 3.4 (i) (or (ii)), let $\Omega \subset M$ be the set of all points $q \in M$ for which the conclusion holds. Then Ω is clearly open. After shrinking M appropriately, we see that $\Omega \neq \emptyset$ by the above weaker supposed version of Proposition 3.4. Analogously, shrinking M around an accumulation point of Ω, we conclude that Ω is closed and therefore $\Omega = M$ as required.

Let now $\mathcal{R}(T)$ be a nontrivial polynomial in $T = (T_1, \dots, T_r)$ over $\mathcal{A}(M)$ such that

$$\mathcal{R}(F)|_U \equiv 0 \tag{3.1}$$

for some nonempty open subset $U \subset M$, where $F := (F_1, \dots, F_r)$. We write $\mathcal{R}(T)$ as

a linear combination

$$\mathcal{R}(T) = \sum_{j=1}^{l} \delta_j r_j(T), \tag{3.2}$$

where each $\delta_j \not\equiv 0$ is in $\mathcal{A}(M)$ and r_j is a monomial in T. By Lemma 3.3, each δ_j does not vanish on a dense open subset of M. By shrinking U, we may assume that δ_l does not vanish at every point of U. We prove the desired conclusion by induction on the number l of monomials in (3.2). For $l = 1$, (3.1) and (3.2) and the choice of U imply that $r_1(F)|_U = 0$. Since r_1 is a monomial and each component of $\overline{F}$ is in $\mathcal{A}(M)$, it follows from Lemma 3.3 that $F_j = 0$ for some j which yields the required nontrivial polynomial identity with coefficients in $\mathcal{O}(M)$ (even in $\mathbb{C}$).

Suppose now that the desired conclusion holds for any polynomial $\mathcal{R}$ whose number of monomials is strictly less than l. In view of (3.1) and (3.2) we have

$$r_l(F)|_U + \Big(\sum_{j<l} \frac{\delta_j}{\delta_l} r_j(F)\Big)|_U = 0. \tag{3.3}$$

Let L be any global CR vector field on M with real-analytic coefficients. Applying L to (3.3) and using the assumption that F_j is CR for any j, we obtain

$$\Big(\sum_{j<l} L\big(\frac{\delta_j}{\delta_l}\big) r_j(F)\Big)|_U = 0. \tag{3.4}$$

By Lemma 3.3 (ii), each coefficient $L(\delta_j/\delta_l)$ can be written as a ratio of two functions in $\mathcal{A}(M)$. From (3.4), we are led to distinguish two cases. If for some $j \in \{1, \dots, l-1\}$, $L(\delta_j/\delta_l)$ does not vanish identically in U, then the required conclusion follows from the induction hypothesis.

It remains to consider the case when

$$L(\delta_j/\delta_l) = 0, \quad \text{in } U, \tag{3.5}$$

for all j and for all choices of $(0,1)$ vector field L. Then (3.5) implies that each ratio δ_j/δ_l is CR on U by the assumption at the beginning of the proof. Hence, by Theorem 3.1, it follows that each δ_j/δ_l extends meromorphically to a neighborhood of M in $\mathbb{C}^N$ and therefore, (3.3) can be rewritten as

$$r_l(F)|_U + \Big(\sum_{j<l} m_j r_j(F)\Big)|_U = 0, \tag{3.6}$$

with $m_1, \dots, m_{l-1}$ being restrictions to M of meromorphic functions in a neighborhood of M. Since M is connected and minimal everywhere, the identity

$$r_l(F(z)) + \sum_{j<l} m_j(z) r_j(F(z)) = 0 \tag{3.7}$$

holds for every $z \in M$ outside the set S consisting of the poles of the m_j's. This proves the desired conclusion under the assumption (i).

For the statement under the assumption (ii), consider a nontrivial polynomial $\mathcal{P}(T, \widetilde{T}) \in \mathcal{C}^\omega(M)[T, \widetilde{T}]$ such that $\mathcal{P}(F, \overline{F})|_U = 0$ for a non-empty open subset $U \subset M$. We write

$$\mathcal{P}(T, \widetilde{T}) = \sum_{\nu \in \mathbb{N}^r, |\nu| \le l} \mathcal{P}_\nu(\widetilde{T}) T^\nu, \tag{3.8}$$

where each $\mathcal{P}_\nu(\widetilde{T}) \in \mathcal{C}^\omega(M)[\widetilde{T}]$ and at least one of the $\mathcal{P}_\nu$'s is nontrivial. If there exists $\nu_0 \in \mathbb{N}^r$ such that $\mathcal{P}_{\nu_0}(\overline{F})$ is not zero in the ring $\mathcal{A}(M)$, then it follows that the polynomial $\mathcal{Q}(T) := \mathcal{P}(T, \overline{F}) \in \mathcal{A}(M)[T]$ is nontrivial and satisfies $\mathcal{Q}(F)|_U = 0$. Then the condition (i) is fulfilled and the required conclusion is proved above.

It remains to consider the case when $\mathcal{P}_\nu(\overline{F}) = 0$ for any $\nu \in \mathbb{N}^r$. Fix any ν such that $\mathcal{P}_\nu(\widetilde{T})$ is nontrivial. Let $\overline{\mathcal{P}_\nu}(T)$ denote the polynomial in $\mathcal{C}^\omega(M)[T]$ obtained from $\mathcal{P}_\nu$ by taking the complex conjugates of its coefficients. Then $\overline{\mathcal{P}_\nu}(T)$ is a nontrivial polynomial in $\mathcal{A}(M)[T]$ and satisfies $\overline{\mathcal{P}_\nu}(F) = 0$ on M. Here again, condition (i) is fulfilled and the desired conclusion follows. The proof is complete. □

4. Zariski closure of the graph of a CR-map. Throughout this section, let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold, $p \in M$ a fixed point in M and $f\colon M \to \mathbb{C}^{N'}$ a $\mathcal{C}^\infty$-smooth CR-map. For $q \in \mathbb{C}^N$, denote by $\mathcal{O}_q$ the ring of germs at q of holomorphic functions in $\mathbb{C}^N$. The goal of this section is to define and give some basic properties of the local Zariski closure of the graph Γ_f at $(p, f(p))$ over the ring $\mathcal{O}_p[z']$.

4.1. Definition of the local Zariski closure. For M, f and p as above, define the (*local*) *Zariski closure* of Γ_f at $(p, f(p))$ with respect to the ring $\mathcal{O}_p[z']$ as the germ $\mathcal{Z}_f \subset \mathbb{C}^N \times \mathbb{C}^{N'}$ at $(p, f(p))$ of a complex-analytic set defined by the zero-set of all elements in $\mathcal{O}_p[z']$ vanishing on Γ_f. Note that since $\mathcal{Z}_f$ contains the germ of the graph of f through $(p, f(p))$, it follows that $\dim_{\mathbb{C}} \mathcal{Z}_f \geq N$. In what follows, we shall denote by $\mu_p(f)$ the dimension of the Zariski closure $\mathcal{Z}_f$.

REMARK 4.1. Observe that if M is furthermore assumed to be minimal at p, all the components of the map f extend to a wedge with edge M at p; in this case, it follows from unique continuation at the edge that $\mathcal{Z}_f$ is locally irreducible with respect to the ring $\mathcal{O}_p[z']$.

4.2. Dimension of the local Zariski closure and transcendence degree. In this section, we discuss a link between the dimension of the Zariski closure $\mu_p(f)$ defined above and the notion of transcendence degree considered in [P90a, P90b, CMS99, CPS00]. The reader is referred to [ZS58] for basic notions from field theory used here.

Since the ring $\mathcal{O}_p(M)$ is an integral domain, one may consider its quotient field that we denote by $\mathcal{M}_p(M)$. Recall that, by a theorem of TOMASSINI [To66], any germ in $\mathcal{O}_p(M)$ extends holomorphically to a neighborhood of p in $\mathbb{C}^N$. Hence an element belongs to $\mathcal{M}_p(M)$ if and only if it extends meromorphically to a neighborhood of p in $\mathbb{C}^N$. Note that if M is *moreover assumed to be minimal at* p, it follows that the ring of germs at p of $\mathcal{C}^\infty$-smooth CR-functions on M is an integral domain, which allows one to introduce its quotient field containing $\mathcal{M}_p(M)$. Therefore, for a generic submanifold M minimal at p, one may consider the finitely generated field extension $\mathcal{M}_p(M)(f_1, \ldots, f_{N'})$ over $\mathcal{M}_p(M)$ where $f_1, \ldots, f_{N'}$ are the components of f considered as germs at p. (In the hypersurface case such a field extension has been studied by PUSHNIKOV [P90a, P90b].) The transcendence degree $m_p(f)$ of the above field extension is called the *transcendence degree* of the CR-map f at p (see [CMS99, CPS00]). We have the following standard relation between $m_p(f)$ and $\mu_p(f)$:

LEMMA 4.2. *Let $M \subset \mathbb{C}^N$ be a generic real-analytic submanifold through some point $p \in M$ and $f\colon M \to \mathbb{C}^{N'}$ a $\mathcal{C}^\infty$-smooth CR-map. Assume that M is minimal at p. Then $\mu_p(f) = N + m_p(f)$.*

REMARK 4.3. The minimality of M is needed to guarantee that $\mathcal{M}_p(M)(f_1,\dots,f_{N'})$ is a field so that the transcendence degree is defined.

The following well-known proposition shows the relevance of $\mu_p(f)$ for the study of the holomorphic extension of f.

PROPOSITION 4.4. *Let $M \subset \mathbb{C}^N$ be a generic real-analytic submanifold through a point p and $f\colon M \to \mathbb{C}^{N'}$ a $\mathcal{C}^\infty$-smooth CR-map. Then the following are equivalent:*

(i) $\mu_p(f) = N$;

(ii) f is real-analytic near p.

(iii) f extends holomorphically to a neighborhood of p in $\mathbb{C}^N$.

Proposition 4.4 is a consequence of theorems of TOMASSINI [To66] and of MALGRANGE [Ma66].

5. Local geometry of the Zariski closure.

5.1. Preliminaries. We use the notation from §4 and assume that M is minimal at p and that

$$\mu_p(f) < N + N' \tag{5.1}$$

holds. By shrinking M around p if necessary, we may assume that M is connected and minimal at all its points. In what follows, for an open subset $\Omega \subset \mathbb{C}^k$, $\mathcal{O}(\Omega)$ will denote the ring of holomorphic functions in Ω.

In §4, we saw that $\mu_p(f) \geq N$ and $m := m_p(f) = \mu_p(f) - N$ coincides with the transcendence degree of the field extension $\mathcal{M}_p(M) \subset \mathcal{M}_p(M)(f_1,\dots,f_{N'})$, where $f = (f_1,\dots,f_{N'})$. This implies that there exist integers $1 \leq j_1 < \dots < j_m < N'$ such that $f_{j_1},\dots,f_{j_m}$ form a transcendence basis of $\mathcal{M}_p(M)(f)$ over $\mathcal{M}_p(M)$. After renumbering the coordinates $z' := (\zeta, w) \in \mathbb{C}^m \times \mathbb{C}^{N'-m}$ and setting $m' := N' - m$, we may assume that

$$f = (g,h) \in \mathbb{C}^m_\zeta \times \mathbb{C}^{m'}_w, \tag{5.2}$$

where $g = (g_1,\dots,g_m)$ forms a transcendence basis of $\mathcal{M}_p(M)(f)$ over $\mathcal{M}_p(M)$.

Since the components of the germ at p of the CR-map $h : M \to \mathbb{C}^{m'}$ are algebraically dependent over $\mathcal{M}_p(M)(g)$, there exist monic polynomials $P_j(T) \in \mathcal{M}_p(M)(g)[T]$, $j = 1,\dots,m'$, such that if $h = (h_1,\dots,h_{m'})$, then

$$P_j(h_j) = 0,\ j = 1,\dots,m',\ \text{in } \mathcal{M}_p(M)(f). \tag{5.3}$$

As a consequence, there exist non-trivial polynomials $\widehat{P_j}(T) \in \mathcal{O}_p(M)[g][T]$, $j = 1,\dots,m'$, such that

$$\widehat{P_j}(h_j) = 0,\ j = 1,\dots,m'. \tag{5.4}$$

For every $j = 1,\dots,m'$, we can write

$$\widehat{P_j}(T) = \sum_{\nu \leq k_j} q_{j\nu} T^\nu, \tag{5.5}$$

where each $q_{j\nu} \in \mathcal{O}_p(M)[g]$, $q_{jk_j} \not\equiv 0$ and $k_j \geq 1$. Since each $q_{j\nu}$ is in $\mathcal{O}_p(M)[g]$, we can also write

$$q_{j\nu} = q_{j\nu}(z) = R_{j\nu}(z, g(z)) \tag{5.6}$$

where $R_{j\nu}(z,\zeta) \in \mathcal{O}_p(M)[\zeta]$. Note that each $R_{j\nu}(z,\zeta)$ can also be viewed as an element of $\mathcal{O}_p[\zeta]$.

Let Δ_p^N be a polydisc neighborhood of p in $\mathbb{C}^N$ such that the analogues of (5.4) – (5.6) hold with germs replaced by their representatives in $M \cap \Delta_p^N$ (denoted by the same letters). Moreover, in view of Remark 4.1, we may assume that the Zariski closure $\mathcal{Z}_f$ can be represented by an irreducible (over the ring $\mathcal{O}_p[z']$) closed analytic subset of $\Delta_p^N \times \mathbb{C}^{N'}$ (also denoted by $\mathcal{Z}_f$). By shrinking M we may also assume that M is contained in Δ_p^N. Hence we have

$$\Gamma_f \subset \mathcal{Z}_f \subset \Delta_p^N \times \mathbb{C}^{N'}. \tag{5.7}$$

Define

$$\widetilde{P_j}(z,\zeta;T) := \sum_{\nu=0}^{k_j} R_{j\nu}(z,\zeta)T^\nu \in \mathcal{O}(\Delta_p^N)[\zeta][T],\ j=1,\dots,m'. \tag{5.8}$$

It follows from (5.4) – (5.6) that one has

$$\widetilde{P_j}(z,g(z);h_j(z)) = 0,\ z \in M,\ j=1,\dots,m'. \tag{5.9}$$

Here each $R_{j\nu}(z,\zeta) \in \mathcal{O}(\Delta_p^N)[\zeta]$, $k_j \geq 1$, and

$$R_{jk_j}(z,g(z)) \not\equiv 0, \quad z \in M. \tag{5.10}$$

Moreover, since $\mathcal{O}_p[\zeta][T]$ is a unique factorization domain (see e.g. [ZS58]) and since M is minimal at p, we may assume that the polynomials given by (5.8) are irreducible.

Consider the complex-analytic variety $\mathcal{V}_f \subset \mathbb{C}^N \times \mathbb{C}^{N'}$ through $(p,f(p))$ defined by

$$\mathcal{V}_f := \{(z,\zeta,w) \in \Delta_p^N \times \mathbb{C}^m \times \mathbb{C}^{m'} : \widetilde{P_j}(z,\zeta;w_j) = 0,\ j=1,\dots,m'\}. \tag{5.11}$$

By (5.9), $\mathcal{V}_f$ contains the graph Γ_f and hence the Zariski closure $\mathcal{Z}_f$. In fact, since by Lemma 4.2, $\dim_{\mathbb{C}} \mathcal{Z}_f = \mu_p(f) = N+m$, it follows from the construction that $\mathcal{Z}_f$ is the (unique) irreducible component of $\mathcal{V}_f$ (over $\mathcal{O}_p[z']$) containing Γ_f. Note that $\mathcal{V}_f$ is not irreducible in general and, moreover, can have a dimension larger than $\mu_p(f)$. (This may happen, e.g. if another component of $\mathcal{V}_f$ is of higher dimension than $\mu_p(f)$).

For $j=1,\dots,m'$, let $\widetilde{D_j}(z,\zeta) \in \mathcal{O}(\Delta_p^N)[\zeta]$ be the discriminant of the polynomial $\widetilde{P_j}(z,\zeta;T)$ (with respect to T). Consider the complex-analytic set

$$\widetilde{\mathcal{D}} := \cup_{j=1}^{m'} \{(z,\zeta) \in \Delta_p^N \times \mathbb{C}^m : \widetilde{D_j}(z,\zeta) = 0\}. \tag{5.12}$$

By the irreducibility of each polynomial $\widetilde{P_j}(z,\zeta;T)$, we have $\widetilde{D_j}(z,\zeta) \not\equiv 0$ in $\Delta_p^N \times \mathbb{C}^m$, for $j=1,\dots,m'$. Therefore from the algebraic independence of the components of the map g over $\mathcal{M}_p(M)$, it follows that the graph of g is not contained in $\widetilde{\mathcal{D}}$, i.e. that for $z \in M$,

$$\widetilde{D_j}(z,g(z)) \not\equiv 0,\ \text{for } j=1,\dots,m'. \tag{5.13}$$

By minimality of M as before, the sets

$$\Sigma_j := \{z \in M : \widetilde{D_j}(z,g(z)) = 0\},\ j=1,\dots,m',$$

are nowhere dense in M, and so is the set

$$\Sigma := \cup_{j=1}^{m'} \Sigma_j = \{z \in M : (z,g(z)) \in \widetilde{\mathcal{D}}\}. \tag{5.14}$$

5.2. Description of $\mathcal{Z}_f$ on a dense subset of the graph of f. By the implicit function theorem, for any point $z_0 \in M \setminus \Sigma$, there exist polydisc neighborhoods of z_0, $g(z_0)$ and $h(z_0)$, denoted by $\Delta^N_{z_0} \subset \Delta^N_p \subset \mathbb{C}^N$, $\Delta^m_{g(z_0)} \subset \mathbb{C}^m$, $\Delta^{m'}_{h(z_0)} \subset \mathbb{C}^{m'}$ respectively and a holomorphic map

$$\theta(z_0;\cdot) : \Delta^N_{z_0} \times \Delta^m_{g(z_0)} \to \Delta^{m'}_{h(z_0)} \tag{5.15}$$

such that for $(z,\zeta,w) \in \Delta^N_{z_0} \times \Delta^m_{g(z_0)} \times \Delta^{m'}_{h(z_0)}$,

$$(z,\zeta,w) \in \mathcal{V}_f \iff (z,\zeta,w) \in \mathcal{Z}_f \iff w = \theta(z_0;z,\zeta). \tag{5.16}$$

Since $\Gamma_f \subset \mathcal{Z}_f$ in view of (5.16), for every fixed $z_0 \in M \setminus \Sigma$, we have

$$h(z) = \theta(z_0;z,g(z)),\ z \in M \cap \Delta^N_{z_0}. \tag{5.17}$$

Let $Z_f \subset M \times \mathbb{C}^{N'}$ be the real-analytic subset given by

$$Z_f := \mathcal{Z}_f \cap (M \times \mathbb{C}^{N'}), \tag{5.18}$$

and, for every $z_0 \in M \setminus \Sigma$, consider the real-analytic submanifold $Z_f(z_0) \subset Z_f$ defined by setting

$$Z_f(z_0) := Z_f \cap (\Delta^N_{z_0} \times \Delta^m_{g(z_0)} \times \Delta^{m'}_{h(z_0)}). \tag{5.19}$$

Note that $Z_f(z_0)$ contains the graph of f over $M \cap \Delta^N_{z_0}$ and that, by making the holomorphic change of coordinates $(\widetilde{z},\widetilde{z}') = (z,\varphi(z,z')) \in \mathbb{C}^N \times \mathbb{C}^{N'}$ where $\varphi(z,z') = \varphi(z,(\zeta,w)) := (\zeta, w - \theta(z_0;z,\zeta))$, the submanifold $Z_f(z_0)$ is given in these new coordinates by

$$Z_f(z_0) = \{(\widetilde{z},\widetilde{z}') \in \Delta^N_{z_0} \times \Delta^m_{g(z_0)} \times \mathbb{C}^{m'} : \widetilde{z} \in M,\ \widetilde{z}'_{m+1} = \ldots = \widetilde{z}'_{N'} = 0\}, \tag{5.20}$$

where we write $\widetilde{z}' = (\widetilde{z}'_1,\ldots,\widetilde{z}'_{N'})$.

We summarize the above in the following proposition.

PROPOSITION 5.1. *Let $M \subset \mathbb{C}^N$ be a generic real-analytic submanifold through a point $p \in M$ and $f : M \to \mathbb{C}^{N'}$ a $\mathcal{C}^\infty$-smooth CR-map. Let $\mathcal{Z}_f$ be the local Zariski closure at $(p,f(p))$ of the graph of f as defined in §4.1. Assume that M is minimal at p and that $\mu_p(f) < N + N'$. Then after shrinking M around p, the following holds. For $z_0 \in M \setminus \Sigma$, where Σ is the nowhere dense open subset of M given by (5.14), there exists a holomorphic change of coordinates near $(z_0,f(z_0))$ of the form $(\widetilde{z},\widetilde{z}') = (z,\varphi(z,z')) \in \mathbb{C}^N \times \mathbb{C}^{N'}$ such that the real-analytic subset $\mathcal{Z}_f \cap (M \times \mathbb{C}^{N'})$ is given near $(z_0,f(z_0))$ by (5.20), with $m = \mu_p(f) - N$.*

For every $z_0 \in M \setminus \Sigma$, denote by Ω_{z_0} the (unique) connected component of $(M \cap \Delta^N_{z_0}) \times \Delta^m_{g(z_0)}$ passing through $(z_0,g(z_0))$. Since Ω_{z_0} is connected, it makes sense to consider the quotient field of the ring of real-analytic functions $\mathcal{C}^\omega(\Omega_{z_0})$ that we denote by $\mathcal{K}(z_0)$. Let

$$j\colon \mathcal{C}^\omega(M)[\zeta,\bar{\zeta}] \to \mathcal{C}^\omega(\Omega_{z_0})$$

be the restriction map and

$$\mathcal{D} := \operatorname{Im} j \subset \mathcal{C}^\omega(\Omega_{z_0})$$

be the image of j. Note that, since Ω_{z_0} is open in $M \times \mathbb{C}^m$, j is an injective ring homomorphism and hence, one can identify $\mathcal{D}$ with $\mathcal{C}^\omega(M)[\zeta, \bar{\zeta}]$ via j. Denote by $\mathcal{F}$ the quotient field of $\mathcal{D}$. The field $\mathcal{F}$ is naturally identified with the field of all rational functions in $(\zeta, \bar{\zeta})$ with coefficients that extend as real-analytic functions on M. We have the field extension $\mathcal{F} \subset \mathcal{K}(z_0)$.

The following lemma, which will be needed for the proof of Theorem 1.8, is a direct consequence of (5.16) (see also [CMS99] for a related argument):

LEMMA 5.2. *For every fixed $z_0 \in M \setminus \Sigma$, the restriction of the map $\theta(z_0; z, \zeta)$ (given by (5.16)) to Ω_{z_0} satisfies a nontrivial polynomial identity with coefficients in $\mathcal{O}(\Delta_p^N)[\zeta]$.*

6. Proof of Theorem 1.8. With all the tools defined in §4–§5 at our disposal, we are now ready to prove the following statement from which Theorem 1.8 will follow.

THEOREM 6.1. *Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold through a point $p \in M$. Let $f\colon M \to \mathbb{C}^{N'}$ be a $\mathcal{C}^\infty$-smooth CR-map and $\mathcal{Z}_f$ the local Zariski closure over $\mathcal{O}_p[z']$ at $(p, f(p))$ of Γ_f as defined in §4.1. Suppose that M is minimal at p and f maps M into M', where M' is a proper real-algebraic subset of $\mathbb{C}^{N'}$. Then, shrinking M around p and choosing an appropriate union $\widetilde{Z}_f$ of local real-analytic irreducible components of $\mathcal{Z}_f \cap (M \times \mathbb{C}^{N'})$ at $(p, f(p))$ if necessary, one has the following:*

(i) $\mu_p(f) < N + N'$ for $\mu_p(f) = \dim \mathcal{Z}_f$;

(ii) $\Gamma_f \subset \widetilde{Z}_f \subset M \times M'$;

(iii) $\widetilde{Z}_f$ satisfies the following straightening property: for any point q in a dense subset of M, there exists a neighborhood U_q of $(q, f(q))$ in $\mathbb{C}^N \times \mathbb{C}^{N'}$ and a holomorphic change of coordinates in U_q of the form $(\widetilde{z}, \widetilde{z}') = \Phi(z, z') = (z, \varphi(z, z')) \in \mathbb{C}^N \times \mathbb{C}^{N'}$ such that

$$\widetilde{Z}_f \cap U_q = \{(z, z') \in U_q : z \in M,\ \widetilde{z}'_{m+1} = \cdots = \widetilde{z}'_{N'} = 0\}, \tag{6.1}$$

where $m = \mu_p(f) - N$.

For the proof we shall need the following result.

PROPOSITION 6.2. *Under the assumptions of Theorem 6.1, shrinking M around p if necessary, one has the following:*

(i) $\mu_p(f) < N + N'$;

(ii) For any point $z_0 \in M \setminus \Sigma$, the real-analytic submanifold $Z_f(z_0)$ is contained in $M \times M'$, where Σ is the nowhere dense subset of M given by (5.14) and $Z_f(z_0) \subset \mathcal{Z}_f \cap (M \times \mathbb{C}^{N'})$ is given by (5.19).

Proof. [Proof of Proposition 6.2 (i)] We proceed by contradiction. Suppose that the dimension $\mu_p(f)$ of the local Zariski closure is $N + N'$. Since M' is a proper real-algebraic subset of $\mathbb{C}^{N'}$, there exists a nontrivial polynomial $\rho'(z', \overline{z'}) \in \mathbb{C}[z', \overline{z'}]$ vanishing on M'. Since f maps M into M', we have

$$\rho'(f(z), \overline{f(z)}) = 0 \tag{6.2}$$

for all $z \in M$. It follows from Proposition 3.4 (ii) (applied to $F := f = (f_1, \ldots, f_{N'})$) that the germs at p of the components $f_1, \ldots, f_{N'}$ satisfy a nontrivial polynomial identity with coefficients in $\mathcal{O}_p(M)$. This contradicts the assumption $\mu_p(f) = N + N'$. The proof of Proposition 6.2 (i) is complete. □

In view of Proposition 6.2 (i), we may now assume to be in the setting of §4–§5. Since M' is real-algebraic, it is given by

$$M' := \{z' \in \mathbb{C}^{N'} : \rho'_1(z', \overline{z'}) = \ldots = \rho'_l(z', \overline{z'}) = 0\}, \tag{6.3}$$

where each $\rho'_j(z', \overline{z'})$, for $j = 1, \ldots, l$, is a real-valued polynomial in $\mathbb{C}[z', \overline{z'}]$. For $j = 1, \ldots, l$, $z_0 \in M \setminus \Sigma$ and $(z, \zeta) \in \Omega_{z_0}$, define

$$r_j(z, \bar{z}, \zeta, \bar{\zeta}) := \rho'_j(\zeta, \theta(z_0; z, \zeta), \bar{\zeta}, \overline{\theta(z_0; z, \zeta)}) \in \mathcal{C}^\omega(\Omega_{z_0}), \tag{6.4}$$

where $\theta(z_0; \cdot)\colon \Omega_{z_0} \to \Delta^{m'}_{h(z_0)}$ is the restriction to Ω_{z_0} of the holomorphic map given by (5.16) and Ω_{z_0} is the open subset of $M \times \mathbb{C}^m$ given in §5.2. We need the following lemma.

LEMMA 6.3. *For every $z_0 \in M \setminus \Sigma$ and $j = 1, \ldots, l$, the real-analytic function r_j satisfies a nontrivial polynomial identity on Ω_{z_0} with coefficients in $\mathcal{C}^\omega(M)[\zeta, \bar{\zeta}]$.*

Proof. It follows from Lemma 5.2 that each component of the restriction to Ω_{z_0} of $\theta(z_0; \cdot)$, considered as an element of $\mathcal{C}^\omega(\Omega_{z_0})$, is algebraic over the field $\mathcal{F}$ defined in §5.2. Therefore, in view of the definition of $\mathcal{F}$, it is also the case for each component of the restriction to Ω_{z_0} of $\overline{\theta(z_0; \cdot)}$. Since for $j = 1, \ldots, l$, each $\rho'_j(z', \overline{z'})$ is a polynomial, it follows from (6.4) that each r_j belongs to the field generated by $\mathcal{F}$ and the components of the restriction to Ω_{z_0} of the maps $\theta(z_0; \cdot)$ and $\overline{\theta(z_0; \cdot)}$. Hence, by standard arguments from field theory (see e.g. [ZS58]), each r_j is also algebraic over $\mathcal{F}$ for $j = 1, \ldots, l$, which gives the desired statement of the lemma. □

Proof. [Proof of Proposition 6.2 (ii).] By contradiction, assume that there exists $z_0 \in M \setminus \Sigma$ such that the real-analytic submanifold $Z_f(z_0)$ given by (5.19) is not contained in $M \times M'$. In view of (5.16), (5.18), (5.19) and (6.4), this means that there exists $j_0 \in \{1, \ldots, l\}$ such that $r_{j_0} \not\equiv 0$ in Ω_{z_0}. By Lemma 6.3, there exists a nontrivial polynomial $Q(z, \bar{z}, \zeta, \bar{\zeta}; T) \in \mathcal{C}^\omega(M)[\zeta, \bar{\zeta}][T]$ such that

$$Q(z, \bar{z}, \zeta, \bar{\zeta}; r_{j_0}(z, \bar{z}, \zeta, \bar{\zeta})) \equiv 0, \quad \text{for } (z, \zeta) \in \Omega_{z_0}. \tag{6.5}$$

Moreover, since r_{j_0} does not vanish identically on Ω_{z_0} and M is connected, we may choose Q such that

$$Q(z, \bar{z}, \zeta, \bar{\zeta}; 0) \not\equiv 0 \text{ for } (z, \zeta) \in M \times \mathbb{C}^m. \tag{6.6}$$

Recall that we write $f = (g, h)$ as in (5.2) and that the graph of $g = (g_1, \ldots, g_m)$ over $M \cap \Delta^N_{z_0}$ is contained in Ω_{z_0}. Then (6.5) implies that for $z \in M \cap \Delta^N_{z_0}$,

$$Q(z, \bar{z}, g(z), \overline{g(z)}; r_{j_0}(z, \bar{z}, g(z), \overline{g(z)})) \equiv 0. \tag{6.7}$$

But since f maps M into M', we have for $j = 1, \ldots, l$,

$$\rho'_j(f(z), \overline{f(z)}) \equiv \rho'_j(g(z), h(z), \overline{g(z)}, \overline{h(z)}) \equiv 0, \quad z \in M. \tag{6.8}$$

Therefore, in view of (5.17), (6.4) and (6.8), we obtain that for all $z \in M \cap \Delta^N_{z_0}$,

$$r_{j_0}(z, \bar{z}, g(z), \overline{g(z)}) \equiv 0. \tag{6.9}$$

From (6.7) and (6.9), we conclude that for all $z \in M \cap \Delta^N_{z_0}$,

$$Q(z, \bar{z}, g(z), \overline{g(z)}; 0) \equiv 0. \tag{6.10}$$

In view of (6.6), condition (ii) in Proposition 3.4 is satisfied for the components $g_1, \dots, g_m$ of g that are $\mathcal{C}^\infty$-smooth CR-functions on M. By Proposition 3.4, the germs at p of $g_1, \dots, g_m$ satisfy a nontrivial polynomial identity with coefficients in $\mathcal{O}_p(M)$. This contradicts the fact that $g_1, \dots, g_m$ form a transcendence basis of $\mathcal{M}_p(M)(f)$ over $\mathcal{M}_p(M)$ (see §5.1). The proof is complete. □

Proof. [Proof of Theorem 6.1] We shrink M so that the conclusion of Proposition 6.2 holds. Define $\widetilde{Z}_f$ to be the union of those irreducible real-analytic components of $\mathcal{Z}_f \cap (M \times \mathbb{C}^{N'})$ that contain open pieces of Γ_f. Then the conclusions (i) and (ii) of Theorem 6.1 follow from Proposition 6.2 and the straightening property (iii) follows from Proposition 5.1. □

Proof. [Proof of Theorem 1.8] Without loss of generality, we may assume that M is generic. Since f does not extend holomorphically to any neighborhood of p in $\mathbb{C}^N$, we have $n := \mu_p(f) - N > 0$ by Proposition 4.4. Then Theorem 1.8 follows immediately from Theorem 6.1. □

7. Proofs of Theorems 1.1, 1.3, 1.6, 1.7.

Proof. [Proof of Theorem 1.1] We need to prove that if f does not extend holomorphically to any neighborhood of p in $\mathbb{C}^N$, then necessarily f maps a neighborhood of p in M into $\mathcal{E}'$. By Theorem 1.8, there exists a neighborhood U of p in M such that for all points q in a dense open subset of U, one has $f(q) \in \mathcal{E}'$. Since the set $\mathcal{E}'$ is closed in M' (see §1), it follows that $f(U) \subset \mathcal{E}'$. This completes the proof of Theorem 1.1. □

Proof. [Proof of Theorem 1.3] We may assume that M is generic. Since M is real-algebraic, connected and minimal somewhere, it is minimal outside a proper real-algebraic subset S. In view of Corollary 1.2, we may assume that $p \in S$.

If W is a connected component of $M \setminus S$, then we claim that either f is real-algebraic on W or $f(W) \subset V'$. Indeed, if $f(W) \not\subset V'$, then f extends holomorphically to a neighborhood in $\mathbb{C}^N$ of some point $q \in W$ by Corollary 1.2. Therefore, it is real-algebraic by a result of the third author [Z99], i.e. every component of f satisfies a nontrivial polynomial identity in a neighborhood of q in M. Then by Tumanov's theorem and unique continuation, it follows that the same polynomial identities for the components of f hold everywhere on W and hence f is real-algebraic on W.

By repeating the arguments from [DF78, §6] one can show that, near $p' := f(p)$, the set V' (which may be empty) is complex-algebraic, i.e. given by the vanishing of a vector-valued holomorphic polynomial $P(z')$, $z' \in \mathbb{C}^{N'}$. Then, by the above claim, $P \circ f$ is real-algebraic on each connected component of $M \setminus S$. It is known (see e.g. [BR90]) that some neighborhood of p in M intersects only finitely many connected components of $M \setminus S$. Hence $P \circ f$ is real-algebraic in a neighborhood of p in M and therefore, since $P \circ f$ is $\mathcal{C}^\infty$-smooth, it is real-analytic by MALGRANGE's theorem (see [Ma66]).

If f does not send a connected neighborhood of p in M into V', the real-analytic map $P \circ f$ does not vanish identically on each of the components of $M \setminus S$ intersecting this neighborhood. Hence, by the above claim, f is real-algebraic on each such component and therefore in a neighborhood of p. Then the required holomorphic extension of f at p follows from MALGRANGE's and TOMASSINI's theorems. □

For the proof of Theorem 1.6, it will be convenient to derive the following corollary from Theorem 1.8.

COROLLARY 7.1. *Under the assumptions of Theorem* 1.8 *suppose furthermore that* M *and* M' *have the same CR-codimension and* f *is submersive at points arbitrarily close to* p. *Then there exists an integer* $1 \leq n \leq N' - 1$ *and a point* $q' \in M'$ *arbitrarily close to* $f(p)$ *such that an open neighborhood of* q' *in* M' *is biholomorphically equivalent to a product* $Y \times \omega$, *where* $Y \subset \mathbb{C}^{N'-n}$ *is a real-analytic submanifold and* ω *is an open subset in* $\mathbb{C}^n$.

Proof. [Proof of Corollary 7.1] Let A, n be given by Theorem 1.8, $\pi' \colon \mathbb{C}^N_z \times \mathbb{C}^{N'}_{z'} \to \mathbb{C}^{N'}_{z'}$ the natural projection and fix a sufficiently small open neighborhood U' of $f(p)$ in $\mathbb{C}^{N'}$. Choose a point $q \in M$ such that f is submersive at q and $f(q) \in U'$. Let U_q, φ be as in Theorem 1.8 and set $\Phi(z, z') := (z, \varphi(z, z'))$. Shrinking U_q if necessary, we may assume that $\pi'(A \cap U_q) \subset U'$ and $\pi'|_{A \cap U_q} : A \cap U_q \to M' \cap U'$ is submersive. By the conclusion of Theorem 1.8, there exists a neighborhood $M_q \subset M$ of q and an open subset $\omega \subset \mathbb{C}^n$ such that $\Phi(A \cap U_q) = M_q \times \omega \times \{0\}$. Since $\pi'|_{A \cap U_q}$ is a submersion on M', its extension $\pi' \circ \Phi^{-1}$ defines a holomorphic submersion between intrinsic complexifications $\mathcal{A}$ and $\mathcal{M}'$ of A and M' respectively near corresponding points. In particular, there exists a complex submanifold $V \subset \mathbb{C}^N$ through q such that $\psi := \pi' \circ \Phi^{-1}|_{V \times \omega \times \{0\}}$ is biholomorphic onto $\mathcal{M}'$. From the equality of the CR-codimensions of M and M', it follows that $Y := M \cap V$ is a CR-submanifold of $\mathbb{C}^N$ and that the biholomorphism ψ sends $Y \times \omega \times \{0\}$ onto a neighborhood of $f(q)$ in M'. The proof is complete. □

Proof. [Proof of Theorem 1.6] Let M, M' and f be as in Theorem 1.6. Then f is submersive at a point $p_0 \in M$. If f were not holomorphically extendible to a neighborhood of p_0 in $\mathbb{C}^N$, there would exist an open holomorphically degenerate submanifold $Y' \subset M'$ by Corollary 7.1. This would contradict the assumption that M' is holomorphically nondegenerate. Hence f is real-analytic at p_0. Now define $\Omega \subset M$ to be the maximal connected open subset containing p_0 where f is real-analytic. Then f is submersive on a dense subset of Ω. We claim that $\Omega = M$. Otherwise there would exist $p \in \overline{\Omega}$ where f is not real-analytic that would contradict Corollary 7.1 as before. Hence f is real-analytic everywhere on M and the proof is complete. □

Proof. [Proof of Theorem 1.7] As in the proof of Theorem 1.3, we may assume that M is generic and we let $S \subset M$ be the real-algebraic subset of all nonminimal points. By assumption, f is submersive at a point $p_0 \in M$ which can be assumed minimal without loss of generality. By Theorem 1.6, f extends holomorphically to a neighborhood in $\mathbb{C}^N$ of the connected component W_0 of p_0 in $M \setminus S$. Since both M and M' are real-algebraic, f is real-algebraic on W_0 by a result of [Z99]. The same argument shows that, for every connected component W of $M \setminus S$, either f is real-algebraic or it is nowhere submersive on W.

As in the proof of Theorem 1.6, define $\Omega \subset M$ to be the maximal open connected subset containing p_0 where f is real-analytic. Then f is submersive on a dense subset of Ω. Assume by contradiction that f is not real-analytic everywhere on M and hence that there exists a point $p_1 \in \overline{\Omega}$ where f is not real-analytic. Fix local real-algebraic coordinates in M and M' near p_1 and $f(p_1)$ respectively and denote by Δ a minor of the Jacobian matrix of f of the maximal size that does not vanish identically in any neighborhood of p_1. By the first part of the proof, we conclude that Δ is real-algebraic and hence real-analytic in a connected neighborhood U of p_1 in M. In particular, f is submersive on a dense subset of U. Hence f is real-algebraic on every component of $U \setminus S$ by the first part of the proof again, and hence, by MALGRANGE's theorem, it follows that f is real-analytic near p_1, which is a contradiction. This shows that

$\Omega = M$ and hence concludes the proof of the theorem. □

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