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COMPARING VIRTUAL FUNDAMENTAL CLASSES: GAUGE THEORETICAL GROMOV-WITTEN INVARIANTS FOR TORIC VARIETIES *

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0. Introduction. Perhaps one of the most important mathematical results obtained at the end of the last century is a principle which states that the algebraic geometric notion of *stability* is closely related to the global analytical notion *solution of a Hermite-Einstein type differential equation.* The first form of this principle, usually called the Kobayashi-Hitchin correspondence, was proved and applied in a spectacular way by S. Donaldson [D1], [D2], [DK] in the case of vector bundles on algebraic surfaces; later it was generalized to a large class of similar situations (see for instance [Bra], [Hi], [Mu1], [LT], [UY]). In general, a Kobayashi-Hitchin correspondence establishes an isomorphism between a moduli space of stable algebraic geometric objects and a moduli space of solutions of a certain (generalized) Hermite-Einstein equation.

Another fundamental concept introduced at the end of the last century was the notion of *virtual fundamental class.* Roughly speaking, this theory allows to endow *oversized* moduli spaces with a homology class with closed supports (or a class in its Chow ring) whose degree equals the expected dimension of the moduli space *in a canonical way.*

In the algebraic geometric framework this notion was introduced by Behrend-Fantechi [BF] generalizing ideas from Fulton [F]. Another version of this concept is due to Li-Tian [LiT2].

There is also an analogous concept in the differential geometric framework: the rigorous formalism developed by Brussee [Br] allows to endow every moduli space associated to a gauge theoretical problem of *Fredholm type* with a canonical Cech homology class with closed supports, whose degree equals the expected dimension of the moduli space.

We believe that the following general principle holds:

Let

$$\mathcal{M}^* \xrightarrow{KH} \mathcal{M}^{\mathrm{st}}$$

be <u>any</u> Kobayashi-Hitchin correspondence beween a moduli space of irreducible solutions of a Hermitian-Einstein type equation and the corresponding moduli space of stable complex geometric objects.

Assume that

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- *The gauge theoretical problem which defines $\mathcal{M}^*$ is of* Fredholm type, *i.e. this moduli space appears as the vanishing locus of a Fredholm section in a Banach bundle over a Banach manifold.*
- *All the data involved in the definition of $\mathcal{M}^{\mathrm{st}}$ are algebraic.*

Then $\mathcal{M}^{\mathrm{st}}$ has a natural perfect obstruction theory in the sense of Behrend-Fantechi, and the Kobayashi-Hitchin correspondence maps the gauge theoretical virtual fundamental class to the Behrend-Fantechi virtual fundamental class assc-ciated with this obstruction theory.

The second condition could probably be removed if one had a purely complex geometric version of the Behrend-Fantechi obstruction theory.

The importance of such a statement is obvious: it gives a universal comparison principle not only for moduli spaces, but also for a large class of *invariants* defined within the two categories.

The concept of "gauge theoretical problem of Fredholm type" is *much more general* than one would think. For instance, in [OT2] we showed that the vortex problem for line bundles on complex surfaces is of Fredholm type, although the elliptic deformation complex at a solution has a non-trivial degree 3 term.

Note that a rigorous proof of our conjecture cannot be easy: whereas the definition in gauge theory uses Sobolev completions, Fredholm operators and Cech homology with closed supports, the definition in the algebraic category uses sheaves in the étale topology, derived categories, Deligne-Mumford stacks, cotangent complexes and Chow rings!

The purpose of this paper is neither to give a precise general form of our conjecture, nor to speculate on a possible proof strategy. We will come back to the general case in future works.

Our purpose here is to illustrate this principle in an interesting concrete situation for which we will make a precise statement and give a complete proof.

Our concrete situation is the moduli problem considered in the definition of the *twisted gauge theoretical Gromov-Witten invariants associated with the symplectic factorization problem used to construct complete toric varieties.*

In our previous paper [OT2] we introduced a new type of gauge theoretical Gromov-Witten invariants which generalize the so called "Hamiltonian Gromov-Witten invariants" introduced independently in [Mu] and [CGS] (see also [CGMS]).

Our invariants are associated with triples (F, α, K), where (F, ω, J) is an almost Kähler manifold, α is a J-holomorphic action of a compact Lie group $\hat{K}$ on F, and K is a closed normal subgroup of $\hat{K}$ which leaves the symplectic form ω invariant.

In the quoted article we called such triples *symplectic factorization problems with additional symmetry*, because we only consider symplectic quotients with respect to the normal subgroup K, whereas the manifold F was endowed with the action of a larger Lie group $\hat{K}$, which will act (in general) non-trivially on the symplectic K-quotients of F.

The quotient group $K_0 := \hat{K}/K$ (called the parameter symmetry group or the *twisting* group) plays an important role in our approach [OT2]. The formalism developed in [CGS], [CGMS] corresponds to the case when K_0 is trivial.

Let us denote by $\pi : \hat{K} \to K_0$ the canonical projection. Our invariants are obtained by evaluating canonical cohomology classes on the virtual fundamental class of the moduli space of solutions of a certain generalized vortex equation, which depends on the choice of:

1. a system of (discrete) *topological parameters*, namely a triple $(Y, P_0, \mathfrak{c})$ consisting of a closed oriented real surface Y, a K_0-bundle P_0 on Y, an equivalence class $\mathfrak{c}$ of pairs $(\hat{P} \xrightarrow{\lambda} P_0, \hat{h})$ formed by a morphism of type π, and a homotopy class $\hat{h}$ of sections in the associated bundle $E := \hat{P} \times_{\hat{K}} F$.

2. a system of *continuous parameters*, namely a triple (μ, g, A^0) consisting of a Riemannian metric g on Y, a connection A^0 in P_0, and a $\hat{K}$-equivariant moment map μ for the K-action of F.

The first purpose of this paper is to apply this general set up in order to introduce the resulting invariants rigorously in the following important special case: $F = \mathbb{C}^r$, α is the natural action α_{can} of $\hat{K} = [S^1]^r$ on F, and K is the kernel K_w of an epimorphism $w : [S^1]^r \to K_0 = [S^1]^m$, hence a compact (but possibly non-connected) abelian group of dimension $r - m$. Therefore, the invariants we introduce and study should be called *K_w-equivariant, $[S^1]^m$-twisted gauge theoretical Gromov-Witten invariants of the affine space $\mathbb{C}^r$.*

If the adiabatic limit conjecture is true (see [G], [CGS], [GS]), these invariants should be related to the *twisted* Gromov-Witten invariants of toric varieties. These twisted Gromov-Witten invariants, which were introduced in [OT2], are natural generalizations of the Gromov-Witten invariants in the sense of Ruan [R]. They are obtained by replacing the moduli spaces of (pseudo)holomorphic morphisms $Y \to F$ in Ruan's definition of Gromov-Witten invariants by moduli spaces of (pseudo)holomorphic sections in F-bundles over Y with a fixed structure group K_0. In other words, we replace the gauged linear sigma models introduced by Witten [W3], and further investigated by Morrison und Plesser [MP], by *K_0-twisted gauged linear sigma models* [OT2].

Our first result is a Kobayashi-Hitchin correspondence which also gives an explicit complex geometric interpretation for the virtual fundamental class of the moduli space. More precisely:

Let $V \in M_{m,r}(\mathbb{Z})$ be an integer matrix of rank m, let $w : [S^1]^r \to [S^1]^m$ be the corresponding epimorphism, and put $K_w := \ker(w)$. Suppose that the columns of V are primitive and that

$$^tV(\mathbb{R}_m) \cap \{(t_1, \dots, t_r) \in \mathbb{R}_r |\ t_i \geq 0\} = \{0\}\ .^1$$

Let $-it \in \mathfrak{k}_w^\vee = i\text{coker}(^tV \otimes \text{id}_{\mathbb{R}})$ be a regular value of the standard moment map μ_w of the K_w-action on $\mathbb{C}^r$. Under these assumptions, the Kähler quotients

$$X_{\mathfrak{t}} := \mathbb{C}^r /\!\!/_{[\mu_w + i\mathfrak{t}]} K_w$$

is a compact toric variety with a natural orbifold structure.

Let Y be a closed oriented surface. We fix a $[S^1]^r$-bundle $\hat{P}$, a $[S^1]^m$-bundle P_0 and a w-morphism $\hat{P} \xrightarrow{\lambda} P_0$ over Y. Let g be a Riemannian metric on Y, and let A^0 be a connection on P_0. Then

[1]This condition assures that the symplectic quotients of $\mathbb{C}^r$ by K_w are compact.

THEOREM 0.1.

1. *(complex geometric interpretation) The moduli space $\mathcal{M}$ of solutions of the vortex equation associated with the data $(\lambda, (\mu_w + it, g, A^0))$ is a toric fibration over an abelian variety P of dimension $g(Y)(r-m)$.*
2. *(embedding theorem) The moduli space $\mathcal{M}$ can be identified with the vanishing locus of a section σ in an explicit holomorphic bundle $\mathcal{E}$ over the total space of a locally trivial holomorphic toric fibre bundle T over P.*

The standard fibre Φ of the toric fibre bundle T is a toric orbifold which can be obtained as a Kähler quotient of a suitable complex vector space by the same group K_w used to construct the toric variety X. In the case $g(Y) = 0$ this has already been observed in [MP]. This fibre Φ is a smooth manifold if the Kähler quotient X was smooth. In this case one gets an identification $\iota : \mathcal{M} \to Z(\sigma)$ between $\mathcal{M}$ and the subspace cut out by a holomorphic section σ in a holomorphic vector bundle $\mathcal{E}$ over a smooth complex manifold T.

This allows us to endow $\mathcal{M}$ with a distinguished homology class of degree $\dim(T) - \mathrm{rk}(\mathcal{E})$, namely the *algebraic geometric* virtual fundamental class of the triple $(T, \mathcal{E}, \sigma)$ in the sense of Fulton [Fu]. In the general case, T can be regarded as a Deligne-Mumford stack, so one can still endow $\mathcal{M}$ with a rational virtual fundamental class in the sense of Behrend-Fantechi [BF].

Unfortunately, the construction of T, $\mathcal{E}$, σ and of the embedding $\mathcal{M} \hookrightarrow T$ is <u>not</u> <u>canonical</u>: it depends on the choice of a system of sufficiently ample divisors $(D_j)_{1 \leq j \leq r}$ on Y, and it is not clear whether the virtual fundamental class obtained in this way is independent of this choice.

In order to prove this, one should check that different choices of systems of ample divisors lead to the same *perfect obstruction theory* in the sense of Behrend-Fantechi.

Our next result states:

THEOREM 0.2. *(comparison theorem) The algebraic geometric virtual fundamental class of $\mathcal{M}$ induced by the identification ι coincides with its gauge theoretical virtual fundamental class.*

Recall that the gauge theoretical virtual fundamental class is obtained by regarding $\mathcal{M}$ as the vanishing locus of a Fredholm section in a Banach bundle over a Banach manifold. The precise formalism was developed in [Br] (see also [OT2]). The obtained virtual class <u>*is*</u> <u>*canonical*</u>, i.e. it does not depend on any additional choices besides the parameters involved in the construction of the moduli space.

The above theorem is very important, because it allows the computation of the gauge theoretical (Hamiltonian) Gromov-Witten invariants of toric manifolds with purely algebraic geometric methods. Since the pair $(T, \mathcal{E})$ comes with a natural $[\mathbb{C}^*]^m$-action, and the section σ is equivariant, one can apply the localization theorem of Graber-Pandharipande [GP] for the Behrend-Fantechi virtual fundamental class. Explicit computations can be found in [Ha1], [Ha2].

In this way, one can compare – in the case of toric manifolds – the gauge theoretical (Hamiltonian) Gromov-Witten invariants with the standard (Kontsevich) Gromov-Witten invariants and check the adiabatic limit conjecture for this class of manifolds. Explicit computations of standard Gromov-Witten invariants for toric varieties can be found in [Sp].

And now a word about the proof of the comparison theorem, Theorem 0.2 (see section 5):

The configuration space of our gauge theoretical problem is the product of two factors: a space of connections (or semiconnections) and a space of sections. Our method is based on the following new idea: we complete the space of sections in the configuration space with respect to a *very weak Sobolev norm*, such that meromorphic sections with first order poles in finitely many simple points become elements in our Sobolev completion. The spaces of (semi)connections and the gauge group are completed as usual with respect to L^2_k Sobolev norms. This asymmetric Sobolev completion of the configuration space allows us to pass from the gauge theoretical to the algebraic geometric framework. Of course, one has to check that the new completed configuration space leads to the same gauge theoretical virtual fundamental class as the standard completion. The comparison Theorem 0.2 will follow from Brussee's *associativity principle* for virtual fundamental classes associated with Fredholm sections [OT2].

We believe that this method (weakening the Sobolev norm on the spaces of sections) can be adapted to a very large class of similar problems. It can be used to show that many Kobayashi-Hitchin correspondences (which relate gauge theoretical to algebraic geometric moduli spaces) map the gauge theoretical virtual fundamental class onto the algebraic geometric one.

Another interesting application of this technique will be considered in [DOT].

We mention that a similar comparison problem also occurs in connection with the classical Gromov-Witten invariants, which have been introduced in both the symplectic (see [FO], [LiT1] [R1], [R2], [Si1]) and the algebraic (see [BF], [KM], [LiT2]) framework. In this situation the comparison problem has already been considered by several authors [Si2], [LiT1], [LiT2].

1. Toric varieties as symplectic quotients. Any epimorphism $w:[S^1]^r\longrightarrow[S^1]^m$ is determined by the associated Lie algebra morphism, hence by a linear map $v:\mathbb{R}^r\to\mathbb{R}^m$ given by an integer matrix $V=(v^i_j)_{\substack{1\le i\le m\\ 1\le j\le r}}\in M_{m,r}(\mathbb{Z})$ of rank m; one has

$$w(e^{it^1},\dots,e^{it^r})=(e^{iv^1_jt^j},\dots,e^{iv^m_jt^j})\ .$$

We are not interested in *all* epimorphisms $[S^1]^r\to[S^1]^m$ as above, but only in those epimorphisms w with the property that the symplectic quotients of the form $\mathbb{C}^r/_{\mu}K_w$ are compact, because in this case, the corresponding invariants should be related to the *twisted Gromov-Witten invariants of toric varieties* [OT2].

Therefore we shall assume that w verifies the following properties:

P$_1$: *For every $j\in\{1,\dots,r\}$, the column $v_j\in\mathbb{Z}^m$ is primitive, i. e. it is a generator of the semigroup $\mathbb{Z}^m\cap\mathbb{R}_{\geq 0}v_j$.*

P$_2$: $\mathbb{R}_r^{\gneq 0}\cap\operatorname{im}(v^*)=\{0\}$ *in the dual space $\mathbb{R}_r$ of $\mathbb{R}^r$.*

Here we used the notation $\mathbb{R}_r^{\gneq 0}:=\{(t_1,\dots,t_r)\in\mathbb{R}_r|\ t_i\geq 0\}$.

Note that the second property is equivalent to

$$\sum_{j=1}^{r} \mathbb{R}_{\geq 0} v_j = \mathbb{R}^m .$$

Applying the functor $\mathrm{Hom}(\ \cdot\ , S^1)$ to the exact sequence

$$0 \longrightarrow K_w \longrightarrow [S^1]^r \longrightarrow [S^1]^m \longrightarrow 0$$

we get the exact sequence

$$0 \longrightarrow \mathbb{Z}_m \xrightarrow{V^*} \mathbb{Z}_r \longrightarrow \mathrm{Hom}(K_w, S^1) \longrightarrow 0 .$$

This shows that one has natural identifications

$$K_w = \mathrm{Hom}(\mathrm{coker}(V^*), S^1)\ ,\ \mathfrak{k}_w = i\ker(v)\ ,\ \mathfrak{k}_w^\vee = i\mathrm{coker}(v^*)$$

The standard moment map $\hat{\mu} : \mathbb{C}^r \to i\mathbb{R}_r = \hat{\mathfrak{k}}^\vee$ of the canonical action of $[S^1]^r$ on $\mathbb{C}^r$ is given, with respect to the standard dual basis, by

$$\hat{\mu}(z^1, \dots, z^r) = -\frac{i}{2}(|z^1|^2, \dots, |z_r|^2) .$$

The standard moment map $\mu_w : \mathbb{C}^r \to \mathfrak{k}_w^\vee = i\mathrm{coker}(v^*)$ of the K_w-action on $\mathbb{C}^r$ is defined by

$$\mu_w(z^1, \dots, z^r) = -\frac{i}{2} p_v(|z^1|^2, \dots, |z_r|^2) ,$$

where p_v is the canonical projection $\mathbb{R}_r \to \mathrm{coker}(v^*)$. The image of μ_w is the convex set

$$A_v^+ = -ip_v(\mathbb{R}_r^{\geq 0}) \subset i\mathrm{coker}(v^*) .$$

An immediate consequence of the assumption $\mathbf{P}_2$ is

LEMMA 1.1.

i) There is a constant $c > 0$ such that

$$\| \mu_w(z) \|^2 \geq c \| z \|^4 .$$

ii) All symplectic quotients

$$\mu_w^{-1}(-it)/_{K_w}\ ,\quad -it \in A_v^+$$

are compact.

The symplectic quotients which correspond to regular values of the moment map μ_w are projective toric varieties with (at most) orbifold singularities.

Conversely, let $J \subset \{1, \dots, r\}$, and let Σ be a *complete, simplicial* fan of strictly convex rational polyhedral cones in $\mathbb{R}^m$ whose 1-skeleton $\Sigma(1)$ is

$$\Sigma(1) = \{\mathbb{R}_{\geq 0} v_j |\ j \in J\} .$$

Let $a = (a_1, \ldots, a_r) \in \mathbb{R}_r$. For every $\sigma \in \Sigma$ we define the functional $f_\sigma^a \in \langle\sigma\rangle^\vee$ by requiring

$$\langle f_\sigma^a, v_j\rangle = -a_j \quad \text{if } \mathbb{R}_{\geq 0} v_j \text{ is a face of } \sigma\ .$$

The system $(f_\sigma^a)_{\sigma\in\Sigma}$ depends only on $(a_j)_{j\in J}$ and it defines a continuous piecewise linear function f^a on the support $|\Sigma| = \mathbb{R}^m$ of Σ. We put:

$$K(\Sigma) := \{p_v(a) \mid a_i \geq 0,\ \ \langle f_\sigma^a, v_j\rangle \geq -a_j\ \forall\sigma\in\Sigma\ ,\ \forall j\in\{1,\ldots,r\}\}$$

$$K_0(\Sigma) := \{p_v(a)\in K(\Sigma) \mid \langle f_\sigma^a, v_j\rangle > -a_j\ \forall\sigma\in\Sigma\ ,\ \forall j\in\{1,\ldots,r\}$$

$$\text{for which } \mathbb{R}_{\geq 0}v_j \text{ is not a face of } \sigma\}$$

The conditions in the two definitions depend only on the class $[a]$ modulo $\mathrm{im}(v^*)$, because changing a by an element of the form $v^*(f)$, $f \in \mathbb{R}_m$ modifies all maps f_σ^a by the same linear functional f.

When $p_v(a) \in K(\Sigma)$ $(K_0(\Sigma))$ the piecewise linear map f^a is convex (strictly convex) on $|\Sigma|$ (see [Co]). We denote by $k(\Sigma)$ (respectively $k_0(\Sigma)$) the cone of (strictly) convex Σ-linear maps on $|\Sigma|$. One has obvious surjective maps $K(\Sigma) \to k(\Sigma)$, $K_0(\Sigma) \to k_0(\Sigma)$. Note that the cones $k(\Sigma)$, $k_0(\Sigma)$ depend only on the simplicial fan Σ, whereas $K(\Sigma)$, $K_0(\Sigma)$ also depend on the rays $\mathbb{R}_{\geq 0}v_j$ which are not faces of Σ.

Recall that every complete simplicial fan Σ in $\mathbb{R}^m$ with

$$\Sigma(1) \subset \{\mathbb{R}_{\geq 0}v_1, \ldots, \mathbb{R}_{\geq 0}v_r\}$$

defines an associated compact toric variety X_Σ as follows: Set

$$U(\Sigma) = \{z \in \mathbb{C}^r \mid \exists\sigma\in\Sigma \text{ such that } z^j \neq 0\ \forall j\in\{1,\ldots,r\} \text{ for which}$$

$$\mathbb{R}_{\geq 0}v_j \text{ is not a face of } \sigma\}\ .$$

Then there is a geometric quotient

$$X_\Sigma := {}^{U(\Sigma)}/\!\!/_{K_w^{\mathbb{C}}}$$

and this quotient is a compact algebraic (not necessarily projective) variety with a natural orbifold structure. The variety X_Σ is projective if and only if $k_0(\Sigma)$ (or equivalently $K_0(\Sigma)$) is non-empty. In this case one has a canonical epimorphism $\mathrm{coker}(v^*) \to H^2(X_\Sigma, \mathbb{R})$ and, under this epimorphism, $K_0(\Sigma)$ is mapped onto the Kähler cone of the orbifold X_Σ, which can be identified with $k_0(\Sigma)$. We refer to [Co], [Gi2] for more details and the following theorem

THEOREM 1.2. *Let Σ be a complete simplicial fan Σ in $\mathbb{R}^m$ with*

$$\Sigma(1) \subset \{\mathbb{R}_{\geq 0}v^1, \ldots, \mathbb{R}_{\geq 0}v^r\}\ .$$

i) For every $\mathfrak{t} \in K_0(\Sigma)$, the set of semistable points with respect to the moment map $\mu_w + i\mathfrak{t}$ coincides with the correponding set of stable points, and the symplectic quotient ${}^{\mu_w^{-1}(-i\mathfrak{t})}/_{K_w}$ can be identified as a complex orbifold with the projective toric variety

X_Σ.
ii) (The GKZ decomposition) The cone $p_v(\mathbb{R}_r^{\geq 0})$ *can be decomposed as a union of subcones whose interiors are pairwise disjoint:*

$$p_v(\mathbb{R}_r^{\geq 0}) = \bigcup_{\substack{\Sigma \text{ complete simplicial fan in } \mathbb{R}^m \\ \Sigma(1)\subset\{\mathbb{R}_{\geq 0}v_1,\dots,\mathbb{R}_{\geq 0}v_r\} \\ k_0(\Sigma)\neq\emptyset}} K(\Sigma)\ ,$$

iii) The nonempty open subcones $-iK_0(\Sigma)$ *are the connected components of the complement of the critical locus* $\mathrm{Crit}(\mu_w)$ *of* μ_w *in* $\mathrm{im}(\mu_w) = -ip_v(\mathbb{R}_r^{\geq 0})$. *Moreover, one has*

$$\mathrm{Crit}(\mu_w) = \bigcup_{\substack{J\subset\{1,\dots,r\} \\ |J|=m+1}} \mu_w(Z_I)\ ,$$

where $Z_J \subset \mathbb{C}^r$ *is the subspace defined by the equations* $z^j = 0$, $j \in J$.

2. Moduli spaces of toric vortices and the associated invariants.

2.1. Moduli spaces of toric vortices. Let Y be a closed connected oriented real surface. The data of a $[S^1]^m$-bundle P_0 on Y is equivalent to the data of a system $(L_i^0)_i$ of m Hermitian line bundles L_i^0 on Y.

Fix a rank m integer matrix $V \in M_{m,r}(\mathbb{Z})$ with primitive columns and let $w : [S^1]^r \longrightarrow [S^1]^m$ be the associated epimorphism. The data of a w-morphism $\lambda : \hat{P} \longrightarrow P_0$ of principal bundles is equivalent to the data of a system $(L_j)_j$ of r Hermitian line bundles and a system of m unitary isomorphisms

$$\lambda_i : \otimes_{j=1}^r [L_j^{\otimes v_j^i}] \longrightarrow L_i^0\ .$$

Following the general strategy explained in [OT2], we consider the following moduli problem:

Fix a Riemannian metric g on Y, a system of Hermitian connections $A^0 = (A_i^0)_{1\leq i\leq m}$ on $(L_i^0)_{1\leq i\leq m}$, and an element $\mathfrak{t} \in \mathrm{coker}(v^*)$. Classify all systems $(A_j,\varphi_j)_{1\leq j\leq r}$, consisting of

i) a connection A_j on L_j for every j, $1 \leq j \leq r$ such that

$$\lambda_i(\otimes_{j=1}^r [A_j^{\otimes v_j^i}]) = A_i^0\ ,\ \forall\ i \in \{1,\dots,m\}, \tag{1}$$

ii) an A_j-holomorphic section φ_j in L_j for every $j \in \{1,\dots,r\}$,

such that $(A_j,\varphi_j)_{1\leq j\leq r}$ solves the vortex-type equation

$$p_v\left[(i\Lambda_g F_{A_j} - 2\pi\deg(L_j) + \frac{1}{2}|\varphi_j|^2)_j\right] = \mathfrak{t}\ . \tag{$V_g^{\mathfrak{t}}$}$$

Two such systems are considered equivalent if they are in the same orbit with respect to the natural action of the gauge group $\mathcal{G} := \mathcal{C}^\infty(Y, K_w)$, i. e.

$$\mathcal{G} = \{(f_1,\dots f_r) \in \mathcal{C}^\infty(Y,S^1)^r \mid \prod_{j=1}^r f_j^{v_j^i} = 1\ \forall i \in \{1,\dots,m\}\}\ .$$

LEMMA 2.1.
i) If the equation $(V_g^{\mathfrak{t}})$ *has a solution* $(A_j, \varphi_j)_j$*, then*

$$\mathfrak{t} \in \bigcup_{\substack{\Sigma \text{ complete simplicial fan in } \mathbb{R}^m \\ \Sigma(1) \subset \{\mathbb{R}_{\geq 0} v^1, \dots, \mathbb{R}_{\geq 0} v^r\} \\ k_0(\Sigma) \neq \emptyset}} K(\Sigma) \ .$$

ii) If $\mathfrak{t} \in K_0(\Sigma)$ *and* $(A_j, \varphi_j)_j$ *is a solution of* $(V_g^{\mathfrak{t}})$*, then*

$$(\| \varphi_j \|_{L^2})_{1 \leq j \leq r} \in U(\Sigma) \ .$$

Proof. Indeed, integrating $(V_g^{\mathfrak{t}})$ over Y, we get

$$\mathfrak{t}\ \mathrm{Vol}_g(Y) = p_v \left[(\frac{1}{2} \| \varphi_j \|^2_{L^2})_j \right] \in p_v(\mathbb{R}_r^{\geq 0}) \ ,$$

which proves the first statement. For the second, the same argument gives

$$-\frac{i}{2} p_v((\| \varphi_j \|^2_{L^2})_j) = -i\mathfrak{t}\mathrm{Vol}_g(Y) \ ,$$

which implies that the system $(\| \varphi_j \|_{L^2})_{1 \leq j \leq r}$ is semistable with respect to the moment map $\mu_w + i\mathfrak{t}$. The result follows now from Theorem 1.2 □

The configuration space for our moduli problem is the product

$$\mathcal{A} := \left[\prod_{j=1}^{r} \mathcal{A}(L_j) \right]^{(\lambda, v)}_{A^0} \times \bigoplus_{j=1}^{r} A^0(L_j) \ ,$$

where the first factor denotes the affine subspace of $\prod_{j=1}^r \mathcal{A}(L_j)$ consisting of all systems of Hermitian connections $A = (A_j)_j$ satisfying the relation (1). Let $\mathcal{A}^{(H, V_g^{\mathfrak{t}})}$ be the subspace of $\mathcal{A}$ cut out by the integrabilty condition

$$\bar{\partial}_{A_j} \varphi_j = 0 \ , \ j = 1, \dots, r \qquad (H)$$

and the vortex equation $(V_g^{\mathfrak{t}})$. Our moduli space is the quotient

$$\mathcal{M}_{(\mathfrak{t}, g, A^0)}(\lambda) = {}^{\mathcal{A}^{(H, V_g^{\mathfrak{t}})}} / \mathcal{G} \ ,$$

and will be called *the moduli space of toric vortices* associated to the data $(\mathfrak{t}, g, A^0, \lambda)$.

PROPOSITION 2.2. *The moduli spaces* $\mathcal{M}_{(\mathfrak{t}, g, A^0)}(\lambda)$ *are compact.*

Proof. The proof uses the same argument as the demonstration of the compactness Theorem 2.12 in [OT2]. The crucial point is the properness of the moment map which, in the present case, is stated in Lemma 1.1, *i*). □

Let $\mathcal{A}^* \subset \mathcal{A}$ $(\mathcal{A}^{**} \subset \mathcal{A})$ be the space of points with finite (trivial) stabilizers in the configuration space, and put

$$\mathcal{B}^{**} := \mathcal{A}^{**} / \mathcal{G} \ , \ \mathcal{B}^* := \mathcal{A}^* / \mathcal{G} \ .$$

After suitable Sobolev completions $\mathcal{B}^{**}$ ($\mathcal{B}^{*}$) becomes a Banach manifold (Banach orbifold). In both cases, the local models are obtained in the usual way, by constructing local slices for the $\mathcal{G}$-action in the configuration space (see for instance [OT1]).

The maps $m_{\mathfrak{t}} : \mathcal{A} \to A^0(Y)^{\oplus m}$, $h : \mathcal{A} \to \oplus_j A^{01}(L_j)$ given by

$$m_{\mathfrak{t}}(A,\varphi) = p_v\left[(i\Lambda_g F_{A_j} - 2\pi\mathrm{deg}(L_j) + \frac{1}{2}|\varphi_j|^2)_j\right] - \mathfrak{t}\ ,\ h(A,\varphi) = (\bar\partial_{A_j}\varphi_j)_j$$

are $\mathcal{G}$ - equivariant, hence they descend to real analytic sections $\mathfrak{m}_{\mathfrak{t}}$, $\mathfrak{h}$ in Banach bundles (respectively Banach orbifold bundles) over $\mathcal{B}^{**}$ (respectively $\mathcal{B}^{*}$). Moreover, the section $(\mathfrak{m}_{\mathfrak{t}}, \mathfrak{h})$ in the product bundle is Fredholm.

PROPOSITION 2.3. *If K_w acts freely on $U(\Sigma)$ then the toric variety X_Σ is smooth, and the moduli space $\mathcal{M}_{(\mathfrak{t},g,A^0)}(\lambda)$ is a real analytic subspace of the Banach manifold $\mathcal{B}^{**}$ for every $\mathfrak{t} \in K_0(\Sigma)$. It can be identified with the vanishing locus of the Fredholm section $(\mathfrak{m}_{\mathfrak{t}}, \mathfrak{h})$.*

In the general case, the K_w-action on $U(\Sigma)$ has finite stabilizers and the moduli space $\mathcal{M}_{(\mathfrak{t},g,A^0)}(\lambda)$ is a real analytic suborbifold of the infinite dimensional orbifold $\mathcal{B}^{*}$. It can be identified with the real analytic suborbifold cut out by the section $(\mathfrak{m}_{\mathfrak{t}}, \mathfrak{h})$.

Proof. The point is that, if $\mathfrak{t} \in K_0(\Sigma)$, the stabilizer of a solution $(A_j, \varphi_j)_j$ of $(V_g^{\mathfrak{t}})$ with respect to the action of the gauge group $\mathcal{G}$ coincides with the stabilizer of the point $(\| \varphi_j \|_{L^2})_{1\leq j\leq r} \in U(\Sigma)$ with respect to the Lie group K_w (see Lemma 2.1).

Indeed, if a gauge transformation f leaves the connection system $(A_j)_j$ invariant, it must be a constant gauge transformation, i. e. an element of K_w. On the other hand, a constant gauge transformation leaves the section system $(\varphi_j)_j$ invariant if and only if it leaves the vector $(\| \varphi_j \|_{L^2})_{1\leq j\leq r} \in U(\Sigma)$ invariant.

This shows that the gauge group acts freely (with finite stabilizers) on the space of solutions of $(V_g^{\mathfrak{t}})$ if K_w acts freely (with finite stabilizers) on $U(\Sigma)$. Therefore $\mathcal{M}_{(\mathfrak{t},g,A^0)}(\lambda) \subset \mathcal{B}^{**}$ in the first case and $\mathcal{M}_{(\mathfrak{t},g,A^0)}(\lambda) \subset \mathcal{B}^{*}$ in the second.

In the second case, $\mathcal{M}_{(\mathfrak{t},g,A^0)}(\lambda)$ has a natural orbifold structure whose local models are finite dimensional real analytic spaces endowed with analytic actions of finite groups. □

This proposition allows us – at least in the case when the K_w-action on $U(\Sigma)$ is free – to endow the moduli space $\mathcal{M}_{(\mathfrak{t},g,A^0)}(\lambda)$ with a virtual fundamental class in Brussee's sense ([Br], [OT2]):

$$[\mathcal{M}_{(\mathfrak{t},g,A^0)}(\lambda)]^{\mathrm{vir}} \in H_{2e}(\mathcal{M}_{(\mathfrak{t},g,A^0)}(\lambda), \mathbb{Z})\ ,$$

where $e = \sum_j \chi(L_j) - (r-m)\chi(\mathcal{O}_Y)$ is the expected complex dimension of the moduli space. The obtained virtual fundamental class *does not depend on the chosen Sobolev completions* (see the proof of Proposition 5.4 and Remark 5.5). This is a rather non-trivial fact which holds for a large class of similar gauge theoretical moduli spaces.

The definition of virtual fundamental classes can be extended to the orbifold case and then yields a class in the rational homology of the moduli space. This generalization will be treated in a future work.

2.2. Canonical cohomology classes on the moduli spaces of toric vortices. As always in gauge theory, one defines invariants by evaluating *canonical cohomology classes* on the (virtual) fundamental class of the moduli space. This general principle was applied in [OT2] in the case of gauge theoretical Gromov-Witten invariants. In the general case, the canonical cohomology classes on the moduli space $\mathcal{M}$ associated with a triple (F, α, K) and a system of parameters $((Y, P_0, \mathfrak{c}), (\mu, g, A^0))$ are products of classes $\delta^\gamma(h)$ of the form

$$\delta^\gamma(h) = \Phi^*(\gamma)/h \ ,$$

where Φ is the universal section in the universal F-bundle over the product $\mathcal{M} \times Y$, $\gamma \in H^*_{\hat{K}}(F, \mathbb{Z})$ and $h \in H_*(Y, \mathbb{Z})$. The definition makes sense when the gauge group acts freely on the space of solutions. In the quoted article we showed that, in general, these classes satisfy a set of tautological identities, so that the relevant invariant can be regarded as a map on a quotient algebra $\mathbb{A}$ of the graded algebra generated by the symbols $\begin{pmatrix} \gamma \\ h \end{pmatrix}$. The algebra $\mathbb{A}$ depends only on the homotopy type of the topological parameters $(Y, P_0, \mathfrak{c})$.

In this section we describe these canonical cohomology classes and the corresponding tautological relations explicitly in the special case we are studying, namely in the case of the triple $(\mathbb{C}^r, \alpha_{\mathrm{can}}, K_w)$. Note first that the equivariant cohomology ring $H^*_{\hat{K}}(F, \mathbb{Z})$ can be identified with the polynomial ring $\mathbb{Z}[c_1, \dots, c_r]$ in r variables of degree 2.

Consider an element $\mathfrak{t} \in \mathrm{coker}(v^*)$, and suppose that there exists a complete simplicial fan Σ in $\mathbb{R}^m$ with $\Sigma(1) \subset \{\mathbb{R}_{\geq 0} v^1, \dots, \mathbb{R}_{\geq 0} v^r\}$ such that $\mathfrak{t} \in K_0(\Sigma)$.

<u>Case</u> 1. The action of K_w on $U(\Sigma)$ is free.

In this case the toric variety X_Σ is a smooth. We denote by $\hat{\mathcal{P}}$ the universal $[S^1]^r$-bundle

$$\hat{\mathcal{P}} := \mathcal{A}^{**} \times_{\mathcal{G}} \hat{P}$$

over $\mathcal{B}^{**} \times Y$, and by $\Phi : \hat{\mathcal{P}} \to \mathbb{C}^r$ the universal section in the associated bundle $\hat{\mathcal{P}} \times_{[S^1]^r} \mathbb{C}^r$. For every $\gamma \in \mathbb{Z}[c_1, \dots, c_r]$ and $h \in H_*(Y)$, we put as above

$$\delta^\gamma(h) = \Phi^*(\gamma)/h \ .$$

The image of $H^*(BK_0, \mathbb{Z})$ via the natural morphism

$$H^*(BK_0, \mathbb{Z}) \to H^*(B\hat{K}, \mathbb{Z}) \to H^*_{\hat{K}}(F, \mathbb{Z})$$

is the symmetric algebra $S^*(V^*(\mathbb{Z}_m))$. Using Proposition 1.1 in [OT2], we see that that $\delta^\gamma(h) = 0$ if $\deg(h) < \deg(\gamma)$ and $\gamma \in S^*(V^*(\mathbb{Z}_m))$. Therefore the assignment $(\gamma, h) \to \delta^\gamma(h)$ descends to a morphism

$$\delta : S^*(H) \otimes \Lambda^*(H \otimes H_1(Y, \mathbb{Z})) \to H^*(\mathcal{B}^{**}, \mathbb{Z})$$

of graded algebras, where $H := {}^{\mathbb{Z}_r}\!/_{V^*(\mathbb{Z}_m)}$.

Our invariant is the map

$$GGW(\mathbb{C}^r, \alpha_{\mathrm{can}}, K_w) : [S^*(H) \otimes \Lambda^*(H \otimes H_1(Y, \mathbb{Z}))]_{2e} \to \mathbb{Z} \ ,$$

$$u \mapsto \langle \delta(u), [\mathcal{M}_{(\mathfrak{t}, g, A^0)}(\lambda)]^{\mathrm{vir}} \rangle \ ,$$

where $e := \sum_j \chi(L_j) - (r-m)\chi(\mathcal{O}_Y)$ is the expected complex dimension of the moduli space, and $[\mathcal{M}_{(\mathfrak{t},g,A^0)}(\lambda)]^{\mathrm{vir}}$ is the virtual fundamental class of the zero locus of the Fredholm section $(\mathfrak{m}_\mathfrak{t}, \mathfrak{h})$.

Case 2. The action of K_w on $U(\Sigma)$ is not free.

In this case let $\mathcal{G}_0 \subset \mathcal{G}$ be the reduced gauge group in a fixed point $y_0 \in Y$, i. e. the kernel of the evaluation map $\mathrm{ev}_{y_0} : \mathcal{G} \to K_w$, and let $\mathcal{B}_0 := \mathcal{A}/\mathcal{G}_0$ be the corresponding quotient.

The same construction as above gives a morphism

$$\delta : S^*(H) \otimes \Lambda^*(H \otimes H_1(Y,\mathbb{Z})) \to H^*_{K_w}(\mathcal{B}_0, \mathbb{Z}) ,$$

and one has a natural restriction map $\rho : H^*_{K_w}(\mathcal{B}_0, \mathbb{Q}) \to H^*_{\mathrm{orb}}(\mathcal{B}^*, \mathbb{Q})$. On the other hand, one can generalize Brussee's method in the orbifold framework and construct a virtual fundamental class $[\mathcal{M}_{(\mathfrak{t},g,A^0)}(\lambda)]^{\mathrm{vir}} \in H^{\mathrm{orb}}_{2e}(\mathcal{M}_{(\mathfrak{t},g,A^0)}(\lambda), \mathbb{Q})$.

In this case, our invariant is the map

$$GGW(\mathbb{C}^r, \alpha_{\mathrm{can}}, K_w) : [S^*(H) \otimes \Lambda^*(H \otimes H_1(Y,\mathbb{Z}))]_{2e} \to \mathbb{Q} ,$$

$$u \mapsto \langle \rho \circ \delta(u), [\mathcal{M}_{(\mathfrak{t},g,A^0)}(\lambda)]^{\mathrm{vir}} \rangle .$$

3. A Kobayashi-Hitchin correspondence. In this section we introduce and study a complex geometric version of the moduli problem above, and we prove a Kobayashi-Hitchin type correspondence which relates the gauge theoretical and the complex geometric moduli spaces.

For $i \in \{1, \dots, m\}$, let $\mathcal{L}_i^0 = (L_i^0, \delta_i^0)$ be a holomorphic line bundle on Y, where δ_i^0 is a fixed semiconnection on the differentiable line bundle L_i^0. Let also $(L_j)_{1\leq j \leq r}$ be a system of differentiable complex line bundles, and λ a system of complex isomorphisms

$$\lambda_i : \otimes_{j=1}^r [L_j^{\otimes v_j^i}] \longrightarrow L_i^0 .$$

Our complex geometric moduli problem asks:

Classify all systems $((\delta_j)_{1\leq i\leq r}, (\varphi_j)_{1\leq i \leq r})$, where δ_j is a semiconnection in L_j such that

$$\lambda_i(\otimes_{j=1}^r [\delta_j^{\otimes v_j^i}]) = \delta_i^0$$

and φ_j is a δ_j-holomorphic section in L_j. Two systems are considered equivalent if they belong to the same orbit of the complex gauge group $\mathcal{G}^\mathbb{C} := \mathcal{C}^\infty(Y, K_w^\mathbb{C})$, i. e. of

$$\mathcal{G}^\mathbb{C} = \{(f_1, \dots f_r) \in \mathcal{C}^\infty(Y, \mathbb{C}^*)^r \mid \prod_{j=1}^r f_j^{v_j^i} = 1 \ \forall i \in \{1, \dots, m\}\} .$$

The configuration space is the product

$$\bar{\mathcal{A}} := \left[\prod_{j=1}^r \bar{\mathcal{A}}(L_j)\right]_{\delta^0}^{(\lambda,v)} \times \bigoplus_{j=1}^r A^0(L_j)$$

and the equation we consider is just the holomorphy condition

$$\delta_j \varphi_j = 0 \ , \ 1 \le j \le r \ . \tag{H}$$

DEFINITION 3.1. *A system $((\delta_j)_{1\le i\le r}, (\varphi_j)_{1\le i\le r})$ is called simple (strictly simple) if one of the following equivalent conditions is satisfied:*
i) its stabilizer is finite (respectively trivial),
ii) There is point $y \in Y$ such that the stabilizer of $(\varphi_1(y), \dots, \varphi_r(y))$ with respect to the natural action of $K_w^{\mathbb{C}}$ in the vector space $\oplus_{j=1}^r L_{j,y}$ is finite (trivial),
iii) There is point $y \in Y$ such that the stabilizer of $(\varphi_1(y), \dots, \varphi_r(y))$ with respect to the natural action of K_w in the vector space $\oplus_{j=1}^r L_{j,y}$ is finite (trivial).

REMARK 3.2. *A system $((\delta_j)_{1\le i\le r}, (\varphi_j)_{1\le i\le r})$ is simple if and only if*

$$\{\alpha \in \ker(v \otimes \mathrm{id}_{\mathbb{C}}) |\ \alpha_j \varphi_j = 0\ \forall j \in \{1, \dots, r\}\} = 0 \ .$$

The simple (strictly simple) systems form an open subspace of $\bar{\mathcal{A}}$ which we denote by $\bar{\mathcal{A}}^{\mathrm{simple}}$ ($\bar{\mathcal{A}}^{\mathrm{ssimple}}$).

An important role will be played by the moduli spaces

$$\mathcal{M}_{\delta^0}^{\mathrm{simple}}(\lambda) := {[\bar{\mathcal{A}}^{\mathrm{simple}}]^H}\big/_{\mathcal{G}^{\mathbb{C}}} \ , \ \mathcal{M}_{\delta^0}^{\mathrm{ssimple}}(\lambda) := {[\bar{\mathcal{A}}^{\mathrm{ssimple}}]^H}\big/_{\mathcal{G}^{\mathbb{C}}}$$

where $[\bar{\mathcal{A}}^{\mathrm{simple}}]^H$ stands for the space of simple solutions of the integrability equation (H). Note that the data V, L_j, L_i^0 can be deduced from λ, so the notation $\mathcal{M}_{\delta^0}^{\mathrm{simple}}(\lambda)$ makes sense.

Using standard gauge theoretical methods (see [LO], [OT1], [Su]), one obtains

THEOREM 3.3. *The moduli space $\mathcal{M}_{\delta^0}^{\mathrm{simple}}(\lambda)$ ($\mathcal{M}_{\delta^0}^{\mathrm{ssimple}}(\lambda)$) has a natural structure of a complex analytic orbifold (complex analytic space).*

Note that this complex orbifold (complex space) is in general not smooth and not Hausdorff. The local models of the orbifold structure are (possibly singular) complex spaces endowed with finite group actions; they are obtained using *local slices* for the $\mathcal{G}^{\mathbb{C}}$-action.

There is also an *abstract (functorial)* formulation of our complex geometric classification problem, which does not use gauge theoretical methods, but only classical deformation theory:

Fix an integer matrix $V \in M_{m,r}(\mathbb{Z})$ as above, let $c = (c_j)_{1\le j\le r}$ be a system of integers, and let $\mathcal{L}^0 = (\mathcal{L}_i^0)_{1\le i\le m}$ be a system of holomorphic line bundles.

DEFINITION 3.4. *A holomorphic system of type $(V, c, \mathcal{L}^0)$ is a system*

$$((\mathcal{L}_j)_{1\le j\le r}, (\varepsilon_i)_{1\le i\le m}, (\varphi_j)_{1\le j\le r}) \ ,$$

where $\mathcal{L}_j$ is a holomorphic line bundle of degree c_j on Y, $\varepsilon_i : \bigotimes_{j=1}^r [\mathcal{L}_j^{\otimes v_j^i}] \to \mathcal{L}_i^0$ is a holomorphic isomorphism, and $\varphi_j \in H^0(\mathcal{L}_j)$.

An isomorphism between two such systems

$$((\mathcal{L}_j)_{1\le j\le r}, (\varepsilon_i)_{1\le i\le m}, (\varphi_j)_{1\le j\le r}) \ , \ ((\mathcal{L}'_j)_{1\le j\le r}, (\varepsilon'_i)_{1\le i\le m}, (\varphi'_j)_{1\le j\le r})$$

is a system of holomorphic isomorphisms $(u_j)_j$, $u_j : \mathcal{L}_j \to \mathcal{L}'_j$ such that $\varphi'_j = u_j(\varphi_j)$ and $\varepsilon'_i \circ [\otimes_j u_j^{\otimes v_j^i}] = \varepsilon'$.

A system of type $(V, c, \mathcal{L}^0)$ is called simple (strictly simple) if its group of automorphisms is finite (trivial).

Using standards techniques of deformation theory, one introduces the notion of families of simple systems of type $(V, c, \mathcal{L}^0)$ parameterized by a complex space Z and the notion of isomorphism of such families. The functor which associates to every complex space Z the set of isomorphism classes of families of simple (strictly simple) systems parameterized by Z, is represented by a complex space $\mathcal{M}^{\mathrm{simple}}_{\mathcal{L}^0}(V,c)$ $(\mathcal{M}^{\mathrm{ssimple}}_{\mathcal{L}^0}(V,c))$.

Choosing $c_i = \deg(L_i)$ and $\mathcal{L}^0_i = (L^0_i, \delta^0_i)$, one gets a natural embedding $\mathcal{M}^{\mathrm{simple}}_{\delta^0}(\lambda) \hookrightarrow \mathcal{M}^{\mathrm{simple}}_{\mathcal{L}_0}(V,c)$. Note, however, that this embedding is *not surjective*: the gauge group $\mathcal{C}^\infty(Y, \mathbb{C}^*)^r$ of $(L_j)_j$ acts in a natural way on the space Λ of systems $(\lambda_i)_i$ of complex isomorphisms $\lambda_i : \bigotimes_{j=1}^{r} [L_j^{\otimes v^i_j}] \to L^0_i$, and one can easily see that there is a canonical isomorphism:

$$\mathcal{M}^{\mathrm{simple}}_{\mathcal{L}_0}(V,c) = \coprod_{[\lambda]\in \Lambda/_{\mathcal{C}^\infty(Y,\mathbb{C}^*)^r}} \mathcal{M}^{\mathrm{simple}}_{\delta^0}(\lambda)\ .$$

The quotient $\Lambda/_{\mathcal{C}^\infty(Y,\mathbb{C}^*)^r}$ is a $H^1(Y, \pi_0(K_w))$-torsor, so it has $|\pi_0(K_w)|^{2g(Y)}$ elements.

Next we introduce some notations. Let Σ be a complete simplicial fan with $\Sigma(1) \subset \{\mathbb{R}_{\geq 0} v^1, \dots, \mathbb{R}_{\geq 0} v^r\}$, and let $T = (T_j)_{1\leq j\leq r}$ be a system of r complex vector spaces.

We put

$$U(\Sigma, T) := \{\tau \in \oplus_{j=1}^r T_j \mid \exists \sigma \in \Sigma \text{ such that } \tau_j \neq 0\ \forall j \in \{1,\dots,r\} \text{ for which}$$

$$\mathbb{R}_{\geq 0} v^j \text{ is not a face of } \sigma\}\ .$$

DEFINITION 3.5. *Let Σ be a complete simplicial fan in $\mathbb{R}^m$ such that*

$$\Sigma(1) \subset \{\mathbb{R}_{\geq 0} v^1, \dots, \mathbb{R}_{\geq 0} v^r\}\ .$$

A system $((\mathcal{L}_j)_{1\leq j\leq r}, (\varepsilon_i)_{1\leq i\leq m}, (\varphi_j)_{1\leq j\leq r})$ of type $(V, c, \mathcal{L}^0)$ will be called Σ-stable if one of the following equivalent conditions is satisfied:

1. *There exists a non-empty Zariski open set $Y_0 \subset Y$ such that for every $y \in Y_0$ one has $(\varphi_1(y), \dots, \varphi_r(y)) \in U(\Sigma, \mathcal{L}_y)$.*
2. *$\varphi \in U(\Sigma, H^0(\mathcal{L}))$.*

Here we denote by $H^0(\mathcal{L})$ the system $(H^0(\mathcal{L}_j))_{1\leq j\leq r}$. The stability condition is obviously an open condition, hence it defines open subspaces $\mathcal{M}^{\Sigma\text{-st}}_{\mathcal{L}_0}(V,c) \subset \mathcal{M}^{\mathrm{simple}}_{\mathcal{L}_0}(V,c)$, $\mathcal{M}^{\Sigma\text{-st}}_{\delta^0}(\lambda) \subset \mathcal{M}^{\mathrm{simple}}_{\delta_0}(\lambda)$.

THEOREM 3.6. *Let $\lambda = \left(\lambda_i : \otimes_{j=1}^r [L_j^{\otimes v^i_j}] \longrightarrow L^0_i\right)_i$ be a system of unitary isomorphisms, where L_j, L^0_i are Hermitian line bundles, and $V = (v^i_j) \in M_{m,r}(\mathbb{Z})$*

is an integer matrix with the properties $\mathbf{P}_1$, $\mathbf{P}_2$. *Let* $A^0 = (A^0_i)_i$, $A^0_i \in \mathcal{A}(L^0_i)$ *be a system of fixed Hermitian connections. Let* Σ *be a complete simplicial fan with* $\Sigma(1) \subset \{\mathbb{R}_{\geq 0}v^1, \ldots, \mathbb{R}_{\geq 0}v^r\}$, *and let* $\mathfrak{t} \in K_0(\Sigma)$. *Then there is a natural isomorphism of real analytic orbifolds*

$$\mathcal{M}_{(\mathfrak{t},g,A^0)}(\lambda) \simeq \mathcal{M}^{\Sigma\text{-st}}_{\bar{\partial}_{A^0}}(\lambda) \ .$$

Proof. By the universal Kobayashi-Hitchin correspondence for universal vortices [Mu1], [LT2], there is a natural isomorphism of real analytic orbifolds

$$\mathcal{M}_{(\mathfrak{t},g,A^0)}(\lambda) \simeq \mathcal{M}^{\mathrm{st}}_{\mu_w + i\mathfrak{t},\bar{\partial}_{A^0}}(\lambda) \ ,$$

where the right hand side denotes the moduli space of integrable pairs (δ, φ) which are analytically stable with respect to the moment map $\mu_w + i\mathfrak{t}$, $z \mapsto -\frac{i}{2}p_v(|z_1|^2, \ldots, |z_r|^2) + i\mathfrak{t}$.

Recall that a pair (δ, φ) is $(\mu_w + i\mathfrak{t})$-analytically stable if for every $\xi \in \ker v$ one has

$$\int\limits_Y \lambda_{\mu_w + i\mathfrak{t}}(\varphi, s(\xi)) vol_g > 0 \ .$$

But

$$\lambda_{\mu_w+i\mathfrak{t}}(\varphi, s(\xi))(y) = \lim_{t\to\infty}\left\langle p_v\begin{pmatrix} \frac{1}{2}e^{2\xi_1 t}|\varphi_1(y)|^2 \\ \cdot \\ \cdot \\ \cdot \\ \frac{1}{2}e^{2\xi_r t}|\varphi_r(y)|^2 \end{pmatrix} - \mathfrak{t}, \xi \right\rangle =$$

$$= \begin{cases} -\langle \mathfrak{t}, \xi\rangle & \text{if } \xi_j \leq 0 \ \forall j \ 1 \leq j \leq r \text{ for which } \varphi_j(y) \neq 0 \\ \infty & \text{if } \exists j \ 1 \leq j \leq r \text{ such that } \varphi_j(y) \neq 0 \text{ and } \xi_j > 0 \ . \end{cases}$$

This shows that (δ, φ) is $(\mu_w + i\mathfrak{t})$-analytically stable if and only if for every $\xi \in \ker v$ for which $\langle \mathfrak{t}, \xi \rangle \geq 0$ there exists a point $y \in Y$ and an index $j \in \{1, \ldots, r\}$ such that $\varphi_j(y) \neq 0$ and $\xi_j > 0$. It is easy to see, using the holomorphy of φ, that this happens if and only if there exists a point $y \in Y$ such that for every $\xi \in \ker v$ for which $\langle \mathfrak{t}, \xi\rangle \geq 0$ there is some $j \in \{1, \ldots, r\}$ such that $\varphi_j(y) \neq 0$ and $\xi_j > 0$. This holds if and only if there exists $y \in Y$ such that $\varphi(y)$ is analytically stable with respect to the natural action of K_w on $\oplus_{j=1}^r L_{j,y}$ and the same moment map $\mu_w + i\mathfrak{t}$. □

4. Complex geometric description of the moduli spaces.

4.1. Linear spaces associated with Poincaré line bundles. Let $\mathcal{E}$ be a holomorphic bundle over the product $X \times P$ of two complex manifolds, with X compact and finite dimensional. The disjoint union

$$H^0_X(\mathcal{E}) := \coprod_{p\in P} H^0(\mathcal{E}|_{\{p\}\times X})$$

has a natural structure of a holomorphic linear space over P. There are two ways to construct this linear space:

1. Using infinite dimensional analytic geometry:
We define $H^0_X(\mathcal{E})$ as the vanishing locus of the relative $\bar\partial$-equation equation as follows:
Fix Hermitian structures on X and $\mathcal{E}$. For every (p,q) and $k>0$ the union $A^{pq}_X(\mathcal{E})_k := \coprod_{p\in P} L^2_k(\mathcal{E}|_{\{p\}\times X}\otimes\Lambda^{pq}_X)$ is naturally a holomorphic locally trivial Banach bundle over P. The holomorphic sections in $A^{pq}_X(\mathcal{E})_k$ over an open set $Q\subset P$ are the sections in $[\mathrm{pr}^*_X(\Lambda^{pq}_X)\otimes\mathcal{E}]|_{Q\times X}$ which are L^2_k-Sobolev in the X-direction and holomorphic in the Q direction.
Then $H^0_X(\mathcal{E})$ is defined as the vanishing locus of the relative differential
$$\bar\partial_{\mathcal{E}/X} := A^{00}_X(\mathcal{E})_k \to A^{01}_X(\mathcal{E})_{k-1}$$
The result does not depend on the Sobolev index $k\in\mathbb{N}$.
2. Using the duality between linear spaces and coherent sheaves:
One defines $H^0_X(\mathcal{E})$ as the linear space which corresponds to the sheaf
$$R^{\dim(X)}[\mathrm{pr}_P]_*(\mathcal{E}^\vee\otimes\mathrm{pr}^*_X(K_X))\ .$$
Note that if $H^0(\mathcal{E}|_{\{p\}\times X})=0$ for generic $p\in P$, then $[\mathrm{pr}_P]_*(\mathcal{E})=0$, whereas the linear space $H^0_X(\mathcal{E})$ is non-trivial as soon $H^0(\mathcal{E}|_{\{p\}\times X})\neq 0$ for at least one $p\in P$.
The equivalence between the two definitions follows from the explicit description of the structure sheaf of the complex subspace defined by an equation with values in an infinite dimensional complex Banach space (see [Dou], [LT]).

Let now Y be a complex curve, and fix a point $y_0\in Y$. For every class $c\in H^2(Y,\mathbb{Z})$ denote by $\mathfrak{P}^c_{y_0}$ the Poincaré line bundle over $Y\times\mathrm{Pic}^c(Y)$ which is trivial on $\{y_0\}\times\mathrm{Pic}^c(Y)$. Denote by $\mathfrak{L}^c_{y_0}$ the corresponding linear space over $\mathrm{Pic}^c(Y)$, i. e.
$$\mathfrak{L}^c_{y_0} = H^0_Y(\mathfrak{P}^c_{y_0}).$$

The Poincaré line bundle $\mathfrak{P}^c_{y_0}$ has a gauge theoretical description, which we sketch here briefly:

Let L^c be a differentiable line bundle of Chern class c, and let $\bar{\mathcal{A}}(L^c)_k$ be the L^2_k completion of the affine space of semiconnections on L^c for a sufficiently large Sobolev index k. Denote by $\mathcal{C}_{y_0}(Y,\mathbb{C}^*)_{k+1}\subset\mathcal{C}(Y,\mathbb{C}^*)_{k+1}$ the kernel of the evaluation morphism ev_{y_0} on the $(k+1)$-Sobolev completion $\mathcal{C}(Y,\mathbb{C}^*)_{k+1}$ of the gauge group $\mathcal{C}^\infty(Y,\mathbb{C}^*)$. The group $\mathcal{C}_{y_0}(Y,\mathbb{C}^*)_{k+1}$ acts freely on $\bar{\mathcal{A}}(L^c)_k$.

The line bundle $\mathcal{T}:=\mathrm{pr}^*_Y(L^c)$ on the product $Y\times\bar{\mathcal{A}}(L^c)_k$ comes with a tautological integrable semiconnection, which agrees with δ over $Y\times\{\delta\}$ for every $\delta\in\bar{\mathcal{A}}(L_c)$, and which is the standard $\bar\partial$-operator over the infinite dimensional fibres $\{y\}\times\bar{\mathcal{A}}(L^c)_k$.

The free action of $\mathcal{C}_{y_0}(Y,\mathbb{C}^*)_{k+1}$ on $Y\times\bar{\mathcal{A}}(L^c)_k$ admits a tautological holomorphic linearization in the line bundle $\mathcal{T}$. Therefore, as in the finite dimensional framework, one can regard the $\mathcal{C}_{y_0}(Y,\mathbb{C}^*)_{k+1}$-quotient of $\mathcal{T}$ as a holomorphic line bundle over
$$Y\times\bar{\mathcal{A}}(L^c)_k\big/_{\mathcal{C}_{y_0}(Y,\mathbb{C}^*)_{k+1}} = Y\times\mathrm{Pic}^c(Y)\ .$$
This quotient line bundle is just the Poincaré bundle $\mathfrak{P}^c_{y_0}$.

REMARK 4.1. *Using this gauge theoretical description of the Poincaré line bundle, one gets a corresponding gauge theoretical description of the associated linear space*

$\mathfrak{L}^c_{y_0}$. *This linear space can be identified with the finite dimensional subspace cut out by the holomorphy condition* $\delta\varphi = 0$ *in the quotient*

$$\bar{\mathcal{A}}(L^c)_k \times A^0(L^c)_k \Big/ \mathcal{C}_{y_0}(Y, \mathbb{C}^*)_{k+1} \ .$$

Consider now the divisor $P^c_y := \{y\} \times \mathrm{Pic}^c(Y)$ of $Y \times \mathrm{Pic}^c(Y)$. Any point $y \in Y$ defines a topologically trivial holomorphic line bundle

$$\mathfrak{P}^c_{y_0 y} := \mathfrak{P}^c_{y_0}|_{P^c_y}$$

over $\mathrm{Pic}^c(Y)$. Let $\pi^c : Y \times \mathrm{Pic}^c(Y) \to \mathrm{Pic}^c(Y)$ be the natural projection.

We have the exact sequence of sheaves on $Y \times \mathrm{Pic}^c(Y)$:

$$0 \longrightarrow [\mathfrak{P}^c_{y_0 y}]^\vee \otimes \mathrm{pr}^*_Y(K_Y) \longrightarrow [\mathfrak{P}^c_{y_0 y}]^\vee \otimes \mathrm{pr}^*_Y(K_Y)(P^c_y) \longrightarrow$$

$$\longrightarrow [\mathfrak{P}^c_{y_0 y}]^\vee \otimes \mathrm{pr}^*_Y(K_Y)(P^c_y)_{P^c_y} \longrightarrow 0 \ ,$$

and a canonical isomorphism

$$\mathrm{pr}^*_Y(K_Y)(P^c_y)_{P^c_y} = \mathcal{O}_{P^c_y} \ ,$$

which is induced by the obvious isomorphism $[K_Y](\{y\})_{\{y\}} \simeq \mathbb{C}$. Therefore we get a morphism

$$[\mathfrak{P}^c_{y_0 y}]^\vee = \pi^c_*([\mathfrak{P}^c_{y_0}]^\vee_{P^c_y}) \longrightarrow R^1\pi^c_*([\mathfrak{P}^c_{y_0}]^\vee \otimes (\mathrm{pr}_Y)^* K_Y) \ ,$$

which induces a morphism of linear spaces

$$\mathrm{ev}^c_{y_0 y} : \mathfrak{L}^c_{y_0} \to \mathfrak{P}^c_{y_0 y}$$

over $\mathrm{Pic}^c(Y)$. As a map of sets, $\mathrm{ev}^c_{y_0 y}$ is just the evaluation map associated with the point y.

PROPOSITION 4.2. *Let* $\iota_y : \mathrm{Pic}^c(Y) \to \mathrm{Pic}^{c+1}(Y)$ *be the isomorphism defined by* $\iota_y([\mathcal{L}]) := [\mathcal{L}(\{y\})]$.

1. There is a canonical isomorphism

$$(\mathrm{id}_Y \times \iota_y)^*(\mathfrak{P}^{c+1}_{y_0}(-P^{c+1}_y)) \simeq \mathfrak{P}^c_{y_0} \ .$$

2. There is an exact sequence of linear spaces over $\mathrm{Pic}^c(Y)$

$$0 \longrightarrow \mathfrak{L}^c_{y_0} \longrightarrow \iota^*_y(\mathfrak{L}^{c+1}_{y_0}) \xrightarrow{\iota^*_y(\mathrm{ev}^{c+1}_{y_0 y})} \iota^*_y(\mathfrak{P}^{c+1}_{y_0 y}) \ .$$

Proof. The first property follows from the universal property of the Poincaré line bundle. The second property can be obtained as follows:

Using the exact sequence of line bundles over $Y \times \mathrm{Pic}^{c+1}(Y)$

$$0 \longrightarrow [\mathfrak{P}^{c+1}_{y_0}]^\vee \otimes \mathrm{pr}^*_Y(K_Y) \longrightarrow [\mathfrak{P}^{c+1}_{y_0}]^\vee \otimes \mathrm{pr}^*_Y(K_Y)(P^{c+1}_y) \longrightarrow$$

$$\longrightarrow [\mathfrak{P}^{c+1}_{y_0}]^\vee \otimes \mathrm{pr}^*_Y(K_Y)(P^{c+1}_y)_{P^{c+1}_y} \longrightarrow 0$$

one gets an exact sequence of sheaves on $\mathrm{Pic}^{c+1}(Y)$:

$$\pi_*^{c+1}([\mathfrak{P}_{y_0}^{c+1}]^\vee \otimes \mathrm{pr}_Y^*(K_Y)(P_y^{c+1})) \longrightarrow \pi_*^{c+1}([\mathfrak{P}_{y_0}^{c+1}]^\vee \otimes \mathrm{pr}_Y^*(K_Y)(P_y^{c+1})_{P_y^{c+1}}) \longrightarrow$$

$$\longrightarrow R^1\pi_*^{c+1}([\mathfrak{P}_{y_0}^{c+1}]^\vee \otimes \mathrm{pr}_Y^*(K_Y)) \longrightarrow R^1\pi_*^{c+1}([\mathfrak{P}_{y_0}^{c+1}]^\vee \otimes \mathrm{pr}_Y^*(K_Y)(P_y^{c+1})) \longrightarrow 0 .$$

The claim follows now from the first statement and the definition of the morphism $\mathrm{ev}_{y_0 y}^{c+1}$. □

COROLLARY 4.3. *Let $\{y_1, \dots, y_k\}$ be a finite set of Y and let ι_D be the isomorphism $\mathrm{Pic}^c(Y) \to \mathrm{Pic}^{c+k}(Y)$ defined by the divisor $D = \sum_i y_i$. Then there is a canonical isomorphism*

$$(\iota_D)_*(\mathfrak{L}_{y_0}^c) \simeq \ker \left[\bigoplus_{i=1}^k \mathrm{ev}_{y_0 y_i}^{c+k} : \mathfrak{L}_{y_0}^{c+k} \longrightarrow \bigoplus_{i=1}^k \mathfrak{P}_{y_0 y_i}^{c+k} \right] .$$

Note that the linear space $\mathfrak{L}_{y_0}^{c+k}$ becomes a vector bundle if $c+k > 2(g-1)$. This remark will play an important role in the following section.

4.2. Moduli spaces of stable systems as subspaces of locally trivial toric fibre bundles. Let B be a complex space, let $\mathcal{T} = (\mathcal{T}_j)_{1\leq j\leq r}$ be a system of holomorphic linear spaces over B, and let Σ be a complete simplicial fan with

$$\Sigma(1) \subset \{\mathbb{R}_{\geq 0} v_1, \dots, \mathbb{R}_{\geq 0} v_r\} .$$

Note that the union $U_B(\Sigma, \mathcal{T}) := \bigcup_{b\in B} U(\Sigma, \mathcal{T}_b)$ is an open subspace of the fibre product $\prod_{j=1}^{r}{}_B \mathcal{T}_j$. Since the natural $K_w^{\mathbb{C}}$-action on $U_B(\Sigma, \mathcal{T})$ admits local slices, the quotient

$$X_{\Sigma,B}(\mathcal{T}) := {U_B(\Sigma, \mathcal{T})}/{K_w^{\mathbb{C}}}$$

has a natural complex analytic (in general singular) orbifold structure. It comes with a natural proper morphism to B.

DEFINITION 4.4. *The morphism $X_{\Sigma,B}(\mathcal{T}) \longrightarrow B$ will be called the toric fibration associated with the data $(w, \Sigma, \mathcal{T})$.*

Consider the map

$$\mathrm{Pic}(V) : \prod_{j=1}^r \mathrm{Pic}^{c_j}(Y) \longrightarrow \prod_{i=1}^m \mathrm{Pic}^{c_1(L_i^0)}(Y)$$

defined by the matrix V. The complex manifold

$$P := \left[\prod_{j=1}^r \bar{\mathcal{A}}(L_j)\right]_{\delta^0}^{(\lambda,v)} {\Big/}_{\mathcal{G}^{\mathbb{C}}}$$

can be identified with a connected component of the fibre $\mathrm{Pic}(V)^{-1}([L^0, \delta^0])$ of $\mathrm{Pic}(V)$; it is an abelian variety of dimension $g(Y)(r-m)$.

Denote by $\underline{c}$ the system $(c_j)_{1\leq j\leq r}$, by $\mathfrak{L}^{\underline{c}}$ the system $(\mathfrak{L}^{c_j}_{y_0}))_{1\leq j\leq r}$, and by $p_{\underline{c}}$ the system of projections $p_{c_j} : \prod_{j=1}^r \mathrm{Pic}^{c_j}(Y) \to \mathrm{Pic}^{c_j}(Y)$.

THEOREM 4.5. *Choose $\mathfrak{t} \in K_0(\Sigma)$ and fix $y_0 \in Y$. There is a canonical isomorphisms of complex analytic orbifolds*

$$\mathcal{M}^{\Sigma\text{-st}}_{\delta^0}(\lambda) \simeq X_{\Sigma,P}(p^*_{\underline{c}}(\mathfrak{L}^{\underline{c}}_{y_0})|_P) \ .$$

Proof. We denote by $\mathcal{G}^{\mathbb{C}}_{y_0} \subset \mathcal{G}^{\mathbb{C}}$ the kernel of the evaluation morphism $ev_{y_0} : \mathcal{G}^{\mathbb{C}} \to K^{\mathbb{C}}_w$ associated with y_0. $\mathcal{G}^{\mathbb{C}}_{y_0}$ acts freely on the configuration space $\left[\prod_{j=1}^r \bar{\mathcal{A}}(L_j)\right]^{(\lambda,v)}_{\delta^0} \times \oplus_{j=1}^r A^0(L_j)$.

Denote by $\left[\prod_{j=1}^r \bar{\mathcal{A}}(L_j)\right]_P$ the space of systems of semiconnections $\delta = (\delta_j)_j \in \prod_{j=1}^r \bar{\mathcal{A}}(L_j)$ such that $([\delta_j])_j \in P$. The larger configuration space

$$\left[\prod_{j=1}^r \bar{\mathcal{A}}(L_j)\right]_P \times \bigoplus_{j=1}^r A^0(L_j)$$

comes with a free action of $\mathcal{C}^\infty_{y_0}(Y,\mathbb{C}^*)^r$, and using the gauge theoretical interpretation of the linear spaces associated with Poincaré line bundles (Remark 4.1) , one sees easily that the complex subspace cut out by the integrability condition $\delta\varphi = 0$ in the quotient

$$\left[\prod_{j=1}^r \bar{\mathcal{A}}(L_j)\right]_P \times \bigoplus_{j=1}^r A^0(L_j) \Big/ \mathcal{C}^\infty_{y_0}(Y,\mathbb{C}^*)^r$$

is precisely

$$\left[\prod_{j=1}^r \mathfrak{L}^{c_j}_{y_0}\right]\Bigg|_P = \prod_{j=1}^r {}_P((p^*_{c_j}(\mathfrak{L}^{c_j}_{y_0})|_P)_j) \ .$$

On the other hand, after suitable Sobolev completions, the natural morphism

$$\left[\prod_{j=1}^r \bar{\mathcal{A}}(L_j)\right]^{(\lambda,v)}_{\delta^0} \times \bigoplus_{j=1}^r A^0(L_j) \Big/ \mathcal{G}^{\mathbb{C}}_{y_0} \to \left[\prod_{j=1}^r \bar{\mathcal{A}}(L_j)\right]_P \times \bigoplus_{j=1}^r A^0(L_j) \Big/ \mathcal{C}^\infty_{y_0}(Y,\mathbb{C}^*)^r$$

becomes an isomorphism of Banach manifolds, and induces an isomorphism of complex spaces

$$\left\{ [\delta,\varphi] \in \left[\prod_{j=1}^r \bar{\mathcal{A}}(L_j)\right]^{(\lambda,v)}_{\delta^0} \times \bigoplus_{j=1}^r A^0(L_j) \Big/ \mathcal{G}^{\mathbb{C}}_{y_0} \ \middle|\ \delta\varphi = 0 \right\} \xrightarrow{\simeq} \prod_{j=1}^r {}_P((p^*_{c_j}(\mathfrak{L}^{c_j}_{y_0})|_P)_j) \ .$$

According to Theorem 3.6, a class $[\delta, \varphi]$ is $\mu_w + it$-stable if and only if the corresponding element in this fibre product belongs to $U_P(\Sigma, p_{\underline{c}}^*(\mathfrak{L}_{y_0}^{\underline{c}})|_P)$. Therefore one obtains an isomorphism

$$\mathcal{M}_{\delta^0}^{\Sigma\text{-st}}(\lambda) \simeq {}^{U_P(\Sigma, p_{\underline{c}}^*(\mathfrak{L}_{y_0}^{\underline{c}})|_P)}\Big/_{K_w^{\mathbb{C}}} \ .$$

□

Now fix effective divisors D_j on Y of sufficiently large degrees d_j such that $h^1(\mathcal{L}') = 0$ for every holomorphic line bundle $\mathcal{L}'$ of degree $c_j' := c_j + d_j$. We assume that D_j is a set of d_j distinct simple points y_j^i, $1 \leq i \leq d_j$. Denote by $\underline{D}$ the system of divisors D_j and by

$$\iota_{\underline{D}} : \prod_{j=1}^{r} \mathrm{Pic}^{c_j}(Y) \to \prod_{j=1}^{r} \mathrm{Pic}^{c_j'}(Y)$$

the isomorphism defined by the system of maps $\otimes \mathcal{O}(\underline{D})$. Set $P' := \iota_{\underline{D}}(P)$.

The toric fibration $p' : X_{\Sigma,P'}(p_{\underline{c}'}^*(\mathfrak{L}_{y_0}^{\underline{c}'})|_{P'}) \to P'$ is a locally trivial fibre bundle over P'. Let q_j be the natural map

$$q_j : \prod_s \mathfrak{L}_{y_0}^{c_s'} \to \mathrm{Pic}^{c_j'}(Y) \ .$$

The maps $\mathrm{ev}_{y_0, y_j^i}^{c_j'} : \mathfrak{L}_{y_0}^{c_j'} \to \mathfrak{P}_{y_0 y_j^i}^{c_j'}$ are equivariant with respect to the natural actions of the group $K_w^{\mathbb{C}}$, so they define sections e_j^i in the orbifold line bundles

$$\mathcal{L}_j^i := {}^{[q_j]^*(\mathfrak{P}_{y_0 y_j^i}^{c_j'})\big|_{U_{P'}(\Sigma, p_{\underline{c}'}^*(\mathfrak{L}_{y_0}^{\underline{c}'}))}}\Big/_{K_w^{\mathbb{C}}}$$

over the *locally trivial* toric fibre bundle $X_{\Sigma,P'}(p_{\underline{c}'}^*(\mathfrak{L}_{y_0}^{\underline{c}'})|_{P'})$.

Combining Corollary 4.3 and Theorem 4.5, we get the following description of our moduli space as the vanishing locus of a system of sections in line bundles over a locally trivial toric fibre bundle.

THEOREM 4.6. *(Embedding theorem) There are canonical isomorphisms*

$$\mathcal{M}_{\delta^0}^{\Sigma\text{-st}}(\lambda) \simeq X_{\Sigma,P}(p_{\underline{c}}^*(\mathfrak{L}_{y_0}^{\underline{c}})|_P) \simeq Z((e_j^i)_{\substack{1 \leq i \leq d_j \\ 1 \leq j \leq r}}) \ .$$

Using these isomorphisms and the methods developed by Fulton ([Fu], p. 244) and Behrend-Fantechi, one can endow $\mathcal{M}_{\delta^0}^{\Sigma\text{-st}}(\lambda)$ with a distinguished homology class, namely the virtual fundamental class associated with the section $\oplus_{i,j}(e_j^i)$ in the bundle $\oplus_{i,j}\mathcal{L}_j^i$ over the smooth orbifold $X_{\Sigma,P'}(p_{\underline{c}'}^*(\mathfrak{L}_{y_0}^{\underline{c}'})|_{P'})$. We will call this class *the algebraic geometric virtual fundamental class.* Note that the construction of the embedding in Theorem 4.6 depends on the (non-canonical) parameters y_0 and D_j, hence the obtained virtual fundamental class might a priori depend also on these parameters.

5. Identifying virtual fundamental classes. The purpose of the this chapter is to show that, at least in the smooth case, the algebraic geometric virtual fundamental class obtained in the previous section coincides with the correct virtual fundamental class of our moduli space, namely with the virtual fundamental class associated with its initial gauge theoretical construction (see section 2.1). Consequently, the algebraic geometric virtual fundamental class does not depend on the non-canonical parameters y_0 and D_j which occur in the construction of the embedding of the moduli space in the smooth toric fibre bundle $X_{\Sigma,P'}(p_{\underline{c}'}^*(\mathfrak{L}_{y_0}^{\underline{c}'})|_{P'})$.

We come back to the gauge theoretical construction of the moduli spaces $\mathcal{M}_{\delta^0}^{\Sigma\text{-st}}(\lambda)$:

The configuration space of our moduli problem is

$$\bar{\mathcal{A}} = \left[\prod_{j=1}^r \bar{\mathcal{A}}(L_j)\right]_{\delta^0}^{(\lambda,v)} \times \bigoplus_{j=1}^r A^0(L_j) = [\bar{\mathcal{A}}(L)]_{\delta^0}^{(\lambda,v)} \times A^0(L) ,$$

where $L := \oplus_j L_j$ is regarded as a rank r vector bundle with structure group $[\mathbb{C}^*]^r$.

The space

$$\left[A^{pq}(Y)^{\oplus r}\right]^v := \{(\alpha_1,\dots,\alpha_r) \in A^{pq}(Y)^{\oplus r} |\ \sum_j v_j^i \alpha_j = 0\}$$

is the model vector space of the affine space $[\bar{\mathcal{A}}(L)]_{\delta^0}^{(\lambda,v)}$.

In order to complete our configuration space, we fix a large integer $k \gg 0$ and a real number $s \in (1,2)$.

We denote by $[A^{pq}(Y)^{\oplus r}]_k^v$ ($[\bar{\mathcal{A}}(L)]_{\delta^0,k}^{(\lambda,v)}$) the completion of the vector space $[A^{pq}(Y)^{\oplus r}]^v$ (affine space $[\bar{\mathcal{A}}(L)]_{\delta^0}^{(\lambda,v)}$) with respect to the Sobolev norm L_k^2 (Sobolev L_k^2-topology), and by $L^s(L)$ the completion of $A^0(L)$ with respect to the Sobolev norm L^s.

Thus our completed configuration space is

$$\bar{\mathcal{A}}_k^s := [\bar{\mathcal{A}}(L)]_{\delta^0,k}^{(\lambda,v)} \times L^s(L) .$$

The reason for completing the configuration space in this odd way will become clear later: the completed configuration space contains in particular pairs whose section component is meromorphic with simple poles.

Note that, if we fix a Hermitian metric on L, then the function $|\varphi|^2$ associated to an L^s-section φ is not integrable, so the vortex equation is not well defined on the completed configuration space. In particular, one *cannot* use such a Sobolev completion to prove a Kobayashi-Hitchin correspondence.

As usual we let $\mathcal{G}_{k+1}^{\mathbb{C}}$ be the completion of the gauge group with respect to the L_{k+1}^2 norm.

The left hand term $\delta\varphi$ in the integrability equation

$$\delta\varphi = 0 \tag{H}$$

should be regarded as an element of the distribution space $L_{-1}^s(\Lambda_Y^{01} \otimes L)$.

One can show that the completion procedure above does not introduce new orbits in the moduli space, hence

PROPOSITION 5.1. *Any weak solution $(\delta_1, \varphi_1) \in \bar{\mathcal{A}}^s_k$ of the integrability equation (H) is $\mathcal{G}^{\mathbb{C}}_{k+1}$-equivalent to a smooth solution, which is unique up to $\mathcal{G}^{\mathbb{C}}$-equivalence.*

Proof. To prove this, one brings δ_1 in Coulomb gauge with respect to a smooth semiconnection δ_0 using a gauge transformation of the type $\exp(u)$, $u \in \left[A^{00}(Y)^{\oplus r}\right]^v_{k+1}$. The new semiconnection $\delta'_1 := \delta \cdot \exp(u)$ will be smooth, since the $(0,1)$ form $\delta'_1 - \delta_1$ solves an elliptic equation with smooth coefficients. The L^s-section $\varphi'_1 := \varphi_1 \cdot \exp(u)$ will also be smooth, by elliptic regularity. □

COROLLARY 5.2. *If $(\delta, \varphi) \in \bar{\mathcal{A}}^s_k$ is an integrable pair, then $\varphi \in L^2_{k+1}(L)$.*

We suppose for simplicity that the $K^{\mathbb{C}}_w$-action on $U(\Sigma)$ is free, so that all (weak) solutions of the system (H, V^t_g) have trivial stabilizers. Therefore they belong to the subspace $[\bar{\mathcal{A}}^s_k]^{**}$ of weak pairs with trivial stabilizer.

Since we use a very weak Sobolev topology on the second factor of our configuration space, the general procedure ([DK], [LT]) to endow the quotient

$$\bar{\mathcal{B}}^{**} := {[\bar{\mathcal{A}}^s_k]^{**}} \big/ {\mathcal{G}^{\mathbb{C}}_{k+1}}$$

with the structure of a Banach manifold must be adapted carefully. The usual (formal) L^2-adjoint of the infinitesimal action

$$D^0_p(f) = (\bar{\partial} f, -f\varphi)$$

at a point $p = (\delta, \varphi)$ is given by the formula

$$[D^0_p]^*(\alpha, \psi) = \bar{\partial}^* \alpha - \sum_j (\psi_j, \varphi_j) ,$$

where the pairing $(\cdot,\cdot)$ stands for the pointwise Hermitian product. The difficulty comes from the fact that, when φ is only an L^s-section, $[D^0_p]^*$ does not necessarily extend to a bounded operator $\left[A^{01}(Y)^{\oplus r}\right]^v_k \times L^s(L) \to \left[A^{00}(Y)^{\oplus r}\right]^v_{k-1}$.

However, one can modify this adjoint operator to get bounded operators with the desired properties. The restriction of D^0_p to the Lie algebra

$$\mathrm{Lie}(K^{\mathbb{C}}_w) = \ker v \otimes \mathbb{C}$$

of the subgroup of constant gauge transformations is given by

$$f \mapsto (0, -f\varphi) .$$

The operator $m_\varphi : \ker v \otimes \mathbb{C} \to L^s(L)$ defined by multiplication with φ is injective, since (δ, φ) is an irreducible pair. The image of this operator is a finite dimensional subspace of the Banach space $L^s(L)$, hence it is closed and has a closed complement. Therefore, m_φ admits a continuous left inverse, say $q_\varphi : L^s(L) \to \ker v \otimes \mathbb{C}$. We set

$$q_p(\alpha, \psi) := \bar{\partial}^* \alpha - q_\varphi(\psi) \ , \ q_p : \left[A^{01}(Y)^{\oplus r}\right]^v_k \oplus L^s(L) \to \left[A^{00}(Y)^{\oplus r}\right]^v_{k-1} .$$

Note that, since the operators $\bar{\partial}^*$ and q_φ take values in direct summands, one has

$$\ker q_p = \ker \bar{\partial}^* \oplus \ker q_\varphi . \tag{3}$$

PROPOSITION 5.3. *Put*

$$V_{p,\varepsilon} := \{p + (\alpha, \psi) |\ q_p(\alpha, \psi) = 0,\ \| \alpha \|_{L^2_k} < \varepsilon,\ \| \psi \|_{L^s} < \varepsilon\}\ ,$$

and define $f_{p,\varepsilon} : V_{p,\varepsilon} \times \mathcal{G}^{\mathbb{C}}_{k+1} \to [\bar{\mathcal{A}}^s_k]$ *by the formula* $(p', \gamma) \mapsto p' \cdot \gamma$.

1. *Let* $p = (\delta, \varphi) \in [\bar{\mathcal{A}}^s_k]^{**}$ *be a weak strictly irreducible pair. Then there exists* $\varepsilon_p > 0$ *such that* f_{p,ε_p} *defines a diffeomorphism from* $V_{p,\varepsilon} \times \mathcal{G}^{\mathbb{C}}_{k+1}$ *onto an open neighbourhood of the orbit* $p \cdot \mathcal{G}^{\mathbb{C}}_{k+1}$ *in* $[\bar{\mathcal{A}}^s_k]^{**}$.
2. *There is a unique Banach manifold structure on* $\bar{\mathcal{B}}^{**}$ *such that the natural maps* $V_{p,\varepsilon_p} \longrightarrow \bar{\mathcal{B}}^{**}$ *become smooth parametrizations.*

Proof. We first seek ε sufficiently small such that $f_{p,\varepsilon}$ becomes injective. Let $(\alpha_i, \psi_i) \in \ker q_p$, $g_i \in \mathcal{G}^{\mathbb{C}}_{k+1}$ such that

$$(p + (\alpha_1, \psi_1)) \cdot g_1 = (p + (\alpha_2, \psi_2)) \cdot g_2\ .$$

Put $\gamma := g_1 g_2^{-1}$. It follows that

$$\gamma^{-1}\bar{\partial}\gamma = \alpha_2 - \alpha_1\ ,\ (\gamma^{-1} - 1)\varphi = \psi_2 - \psi_1\ . \tag{4}$$

Write $\gamma = k + \gamma_0$, where $k \in [\mathbb{C}]^r$ and $\gamma_0 \in \bar{\partial}^*(A^{01}(Y, \mathbb{C}^r)_{k+2})$. The first relation can be written as

$$\bar{\partial}\gamma_0 = k(\alpha_2 - \alpha_1) + \gamma_0(\alpha_2 - \alpha_1)\ . \tag{5}$$

Taking into account that $(\alpha_i, \varphi_i) \in \ker q_p$ and using (3), we get $\bar{\partial}^*(\alpha_2 - \alpha_1) = 0$, so (5) yields

$$\Delta\gamma_0 = \bar{\partial}^*[\gamma_0(\alpha_2 - \alpha_1)]\ .$$

This gives an estimate of the form

$$\| \gamma_0 \|_{L^2_{k+1}} \leq c \| \gamma_0(\alpha_2 - \alpha_1) \|_{L^2_k} \leq c' \| \gamma_0 \|_{L^2_{k+1}} \| (\alpha_2 - \alpha_1) \|_{L^2_k}\ .$$

If we choose $\varepsilon < \frac{1}{2c'}$, this inequality implies $\gamma_0 = 0$, hence $k = \gamma \in K^{\mathbb{C}}_w$. The second relation in (4) can be written as $m_\varphi(\gamma - 1) = \psi_2 - \psi_1$. Using $(\alpha_i, \psi_i) \in \ker q_p$ and (3), one obtains

$$(\gamma - 1) = q_\varphi m_\varphi(\gamma - 1) = q_\varphi(\psi_2 - \psi_1) = 0\ .$$

This shows that $g_1 = g_2$, and $(\alpha_1, \psi_1) = (\alpha_2, \psi_2)$.

Now we want to prove that, for sufficiently small ε, $f_{p,\varepsilon}$ is étale. It suffices to prove this property at the points of the form (p', e). Indeed, since $f_{p,\varepsilon}(p', h\gamma) = f_{p,\varepsilon}(p', h) \cdot \gamma$, it follows that $f_{p,\varepsilon}$ is étale in (p', γ) if and only if it is étale in (p', e). Write $p' = p + (\alpha', \psi')$. One has

$$d_{(p',e)} f_{p,\varepsilon}(\alpha, \psi, u) = (\alpha + \bar{\partial}u, \psi - u(\varphi + \psi'))\ .$$

For $\psi' = 0$, this operator is invertible. Its inverse

$$r : \left[A^{01}(Y)^{\oplus r}\right]^v_k \oplus L^s(L) \longrightarrow \ker q_p \oplus \left[A^{00}(Y)^{\oplus r}\right]^v_{k+1}$$

is given by

$$r(a,v) = \left(a - \bar\partial G\bar\partial a, \left[G\bar\partial a - q_\varphi\left((G\bar\partial a)\varphi + v\right)\right]\varphi + v, G\bar\partial a - q_\varphi\left((G\bar\partial a)\varphi + v\right)\right) ,$$

where G denotes the Green operator of the Laplacian $\Delta = \bar\partial^*\bar\partial$.

Since the linear bounded operator $d_{(p',e)}f_{p,\varepsilon}$ depends continuously on the parameter $\psi' \in L^s(L)$, it must remain invertible for ψ' sufficiently small.

The second statement follows easily from the first. □

The map $\bar h : [\bar{\mathcal{A}}^s_k]^{**} \to L^s_{-1}(\Lambda^{01}_Y \otimes L)$, given by $(\delta,\varphi) \mapsto \delta\varphi$, is equivariant with respect to the action of the gauge group $\mathcal{G}^{\mathbb{C}}_{k+1}$, hence it descends to a holomorphic Fredholm section $\bar{\mathfrak{h}}$ in the Banach bundle

$$\bar{\mathfrak{E}} := [\bar{\mathcal{A}}^s_k]^{**} \times_{\mathcal{G}^{\mathbb{C}}_{k+1}} L^s_{-1}(\Lambda^{01}_Y \otimes L)$$

over $\bar{\mathcal{B}}^{**}$.

Using the regularity result given by Proposition 5.1, it follows that the moduli space $\mathcal{M}^{\mathrm{ssimple}}_{\delta^0}(\lambda)$ is the vanishing locus of the holomorphic Fredholm section $\bar{\mathfrak{h}}$. Under our assumptions, the moduli space $\mathcal{M}^{\Sigma\text{-st}}_{\delta^0}(\lambda)$ of Σ-stable pairs is a compact subspace of this vanishing locus.

PROPOSITION 5.4.
i) There exists a Hausdorff open neighbourhood $\bar{\mathcal{B}}(\Sigma)$ of $\mathcal{M}^{\Sigma\text{-st}}_{\delta_0}(\lambda)$ in $\bar{\mathcal{B}}^{**}$ such that $Z(\mathfrak{h}|_{\bar{\mathcal{B}}(\Sigma)}) = \mathcal{M}^{\Sigma\text{-st}}_{\delta_0}(\lambda)$. In particular, $\mathcal{M}^{\Sigma\text{-st}}_{\delta^0}(\lambda)$ is an open subspace of $\mathcal{M}^{\mathrm{ssimple}}_{\delta^0}(\lambda)$.
ii) The virtual fundamental class of $\mathcal{M}^{\Sigma\text{-st}}_{\delta^0}(\lambda)$ defined by the Fredholm section $\bar{\mathfrak{h}}|_{\bar{\mathcal{B}}(\Sigma)}$ coincides with the virtual fundamental class obtained using the usual Sobolev completion

$$\bar{\mathcal{A}}_k := [\bar{\mathcal{A}}(L)]^{(\lambda,v)}_{\delta^0,k} \times L^2_k(L)$$

and the corresponding Fredhom section $\bar{\mathfrak{h}}_k$ in the Banach bundle

$$\bar{\mathfrak{E}}_k := [\bar{\mathcal{A}}^{**}_k] \times_{\mathcal{G}^{\mathbb{C}}_{k+1}} L^2_{k-1}(\Lambda^{01}_Y \otimes L)$$

over the Banach manifold

$$\bar{\mathcal{B}}^{**}_k := {\bar{\mathcal{A}}^{**}_k}/{\mathcal{G}^{\mathbb{C}}_{k+1}} \ .$$

Proof.
i) We define (compare with Definition 3.5)

$$U^s_k(\Sigma) := \{(\delta,\varphi) \in \bar{\mathcal{A}}^s_k |\ \exists\sigma \in \Sigma \ \text{ such that } \varphi_j \neq 0\ \forall j \in \{1,\dots,r\} \text{ for which}$$

$$\mathbb{R}_{\geq 0}v^j \text{ is not a face of } \sigma\} \ .$$

The set U^s_k is obviously open in $\bar{\mathcal{A}}^s_k$ and $\mathcal{G}^{\mathbb{C}}_{k+1}$-invariant. Since $K^{\mathbb{C}}_w$ acts freely on $U(\Sigma) \subset \mathbb{C}^r$, it follows that $\mathcal{G}^{\mathbb{C}}_{k+1}$ acts freely on $U^s_k(\Sigma)$, hence $U^s_k(\Sigma) \subset [\bar{\mathcal{A}}^s_k]^{**}$. One the other hand, an integrable pair (δ,φ) is obviously Σ-stable. Put

$$\bar{\mathcal{B}}(\Sigma) := U^s_k(\Sigma)/\mathcal{G}^{\mathbb{C}}_{k+1} \ .$$

ii) Since the proof uses the explicit definition of the virtual fundamental class associated with a Fredholm section, we recall briefly this construction. Let s be a Fredholm section in a Banach bundle E over a Banach manifold B, and suppose, for simplicity, that the vanishing locus $Z(s) \subset B$ is compact.

The virtual fundamental class $[Z(s)]^{\rm vir}_s$ is obtained in two steps:

a) one shows that, for a sufficiently small open neighbourhood B_0 of $Z(s)$ in B, there exists a finite rank subbundle E' of $E|_{B_0}$ such that the section s_0 induced by s in the quotient bundle $E_0 := E|_{B_0}/E'$ is regular.

b) The restriction of s to the finite dimensional manifold $Z(s_0)$ takes values in the bundle of finite rank $E'|_{Z(s_0)}$; let s' be the induced section in $E'|_{Z(s_0)}$. One has $Z(s) = Z(s')$, and one defines

$$[Z(s)]^{\rm vir}_s := [Z(s')]^{\rm vir}_{s'} ,$$

where $[Z(s')]^{\rm vir}_{s'}$ stands for the cap product

$$[Z(s')]^{\rm vir}_{s'} := e(E'|_{Z(s_0)}, s') \cap [Z(s_0)] \in H_d(Z(s'), \mathbb{Z}) .$$

Here $e(E'|_{Z(s_0)}, s') \in H^{\mathrm{rk}(E')}(Z(s_0), Z(s_0) \setminus Z(s'), \mathbb{Z})$ is the localized Euler class of $E'|_{Z(s_0)}$ with respect to the section s' (see [Br], [OT2] for details). The class $[Z(s')]^{\rm vir}_{s'}$ is a priori an element in the Cech homology of $Z(s')$ (which coincides with the usual homology in our case), and it does not depend on the chosen finite rank subbundle E'.

Now we come back to the proof. The idea is very simple: we show that, applying step a) in a suitable way to the sections $\bar{\mathfrak{h}}$, $\bar{\mathfrak{h}}_k$ in the two bundles $\bar{\mathfrak{E}}$, $\bar{\mathfrak{E}}_k$, one gets the same finite dimensional manifold $Z(s_0)$, the same bundle over $Z(s_0)$, and the same section s'. Therefore, the above two-step procedure yields the same virtual fundamental class, because the final step b) will be actually the same.

Consider the Banach bundle

$$\bar{\mathfrak{F}} := [\bar{\mathcal{A}}^s_k]^{**} \times_{\mathcal{G}^{\mathbb{C}}_{k+1}} L^2_{k-1}(\Lambda^{01}_Y \otimes L) .$$

One has an obvious continuous injective bundle morphism $\bar{\mathfrak{F}} \hookrightarrow \bar{\mathfrak{E}}$. Note also that

$$\bar{\mathfrak{F}}|_{\bar{\mathcal{B}}^{**}_k} = \bar{\mathfrak{E}}_k .$$

By Corollary 5.2, every integrable pair $p = (\delta, \varphi) \in \bar{\mathcal{A}}^s_k$ belongs to the standard completed configuration space $\bar{\mathcal{A}}_k$. Therefore, the differential $d_p\bar{h}$ maps

$$T_p(\bar{\mathcal{A}}_k) = [A^{01}(Y)^{\oplus r}]^v_k \times L^2_k(L)$$

onto $L^2_{k-1}(\Lambda^{01}_Y \otimes L)$. Let K_p be the (finite dimensional) cokernel of the restriction

$$d_p\bar{h}|_{[A^{01}(Y)^{\oplus r}]^v_k \times L^2_k(L)} : [A^{01}(Y)^{\oplus r}]^v_k \times L^2_k(L) \longrightarrow L^2_{k-1}(\Lambda^{01}_Y \otimes L) .$$

The family of cokernels $(K_p)_{\bar{h}(p)=0}$ descend to a linear space $\mathfrak{K}$ over $\mathcal{M}^{\rm ssimple}_{\delta 0}(\lambda)$, which is a quotient linear space of the restriction $\bar{\mathfrak{F}}|_{\mathcal{M}^{\rm ssimple}_{\delta 0}(\lambda)}$.

Since $\bar{\mathcal{M}}^{\Sigma\text{-st}}_{\delta 0}(\lambda)$ is compact, there exists a finite system $\underline{s} = (s_l)_{1 \leq l \leq n}$ of sections of $\bar{\mathfrak{F}}$ which span $\mathfrak{K}_{[p]}$, for every $[p] \in \mathcal{M}^{\Sigma\text{-st}}_{\delta 0}(\lambda)$. A generic perturbation $\underline{s}' = (s'_l)_{1 \leq l \leq n}$ of $\underline{s}$, which is sufficiently close to $\underline{s}$, will have the following properties:

1. It still spans $\mathfrak{K}_{[p]}$, for every $[p] \in \mathcal{M}^{\Sigma\text{-st}}_{\delta 0}(\lambda)$.
2. It is fibrewise linear independent in every point $[p] \in \mathcal{M}^{\Sigma\text{-st}}_{\delta 0}(\lambda)$.

The system $\underline{s}'$ spans a rank n subbundle $\bar{\mathfrak{E}}'$ of the restriction of $\bar{\mathfrak{F}}$ to a sufficiently small neighbourhood of $\mathcal{M}^{\Sigma\text{-st}}_{\delta 0}(\lambda)$.

But the inclusion

$$L^2_{k-1}(\Lambda^{01}_Y \otimes L) \longrightarrow L^s_{-1}(\Lambda^{01}_Y \otimes L)$$

induces an isomorphism

$$L^2_{k-1}(\Lambda^{01}_Y \otimes L)\Big/_{d_p\bar{h}([A^{01}(Y)^{\oplus r}]^v_k \times L^2_k(L))} \simeq L^s_{-1}(\Lambda^{01}_Y \otimes L)\Big/_{\mathrm{im}(d_p\bar{h})}$$

for every integrable pair p. This follows using the duality $L^s_{-1} = [L_1^{\frac{s}{s-1}}]^*$ and standard L^p - theory. Therefore, the system $\underline{s}'$ also spans the cokernels of the absolute differentials of the section $\bar{\mathfrak{h}}$ in the points of $\mathcal{M}^{\Sigma\text{-st}}_{\delta 0}(\lambda)$. The section $\mathfrak{h}_0$ induced by $\bar{\mathfrak{h}}$ in the quotient bundle $\bar{\mathfrak{E}}/\bar{\mathfrak{E}}'$ is therefore regular on a sufficiently small neighbourhood of $\mathcal{M}^{\Sigma\text{-st}}_{\delta 0}(\lambda)$. Hence one can use the finite dimensional manifold $Z(\bar{\mathfrak{h}}_0)$ and the induced section $\bar{\mathfrak{h}}'$ in $\bar{\mathfrak{E}}'|_{Z(\mathfrak{h}_0)}$ to compute the virtual fundamental class of $[\mathcal{M}^{\Sigma\text{-st}}_{\delta 0}(\lambda)]^{\mathrm{vir}}_{\bar{\mathfrak{h}}}$.

On the other hand, the restriction $\bar{\mathfrak{E}}'|_{\mathcal{B}^{**}_k}$ is a rank n subbundle of $\bar{\mathfrak{E}}_k$, and the induced section $[\bar{\mathfrak{h}}_k]_0$ in the quotient bundle is regular around $\mathcal{M}^{\Sigma\text{-st}}_{\delta 0}(\lambda)$.

The proof is now completed since $\bar{\mathfrak{E}}' \subset \bar{\mathfrak{F}}$, so

$$Z([\bar{\mathfrak{h}}_k]_0) = Z(\bar{\mathfrak{h}}_0)$$

by elliptic regularity. □

REMARK 5.5. *The method used in the proof of the proposition above can be used to show that the virtual fundamental class obtained using the standard Sobolev completion $\bar{A}_k$, does not depend on the (sufficiently large) index k.*

The analogous statement should be true for all gauge theoretical problems of Fredholm-type.

PROPOSITION 5.6. *The Kobayashi-Hitchin correspondence gives an identification*

$$\mathcal{M}_{(\mathfrak{t},g,A^0)}(\lambda) \xrightarrow{\simeq} \mathcal{M}^{\Sigma\text{-st}}_{\bar{\partial}_{A^0}}(\lambda)$$

which maps $[\mathcal{M}_{(\mathfrak{t},g,A^0)}(\lambda)]^{\mathrm{vir}}_{(\mathfrak{m}_\mathfrak{t},\mathfrak{h})}$ onto $[\mathcal{M}^{\Sigma\text{-st}}_{\bar{\partial}_{A^0}}(\lambda)]^{\mathrm{vir}}_{\bar{\mathfrak{h}}}$.

Proof. By the Proposition above, it suffices to show that the Kobayashi-Hitchin isomorphism maps $[\mathcal{M}_{(\mathfrak{t},g,A^0)}(\lambda)]^{\mathrm{vir}}_{(\mathfrak{m}_\mathfrak{t},\mathfrak{h})}$ onto $[\mathcal{M}^{\Sigma\text{-st}}_{\bar{\partial}_{A^0}}(\lambda)]^{\mathrm{vir}}_{\bar{\mathfrak{h}}_k}$.

But this follows by exactly same argument as in the proof of Theorem 3.2 in [OT2]. The main ingredient is the technique developed in [T] (see also [LT]) to prove that the Kobayashi-Hitchin correspondence is a global isomorphism of real analytic spaces. □

Let E be a Hermitian vector bundle on Y. The distribution space $L^s_{-1}(E)$ is the topological dual of the Banach space $L^t_1(E^\vee)$, where t is related to s by the formula

$$\frac{1}{t} + \frac{1}{s} = 1 \ .$$

Since $s < 2$, it follows that $t > 2$, hence on has a bounded embedding $L^t_1(E^\vee) \subset \mathcal{C}^0(E^\vee)$. In particular, for every point $y \in Y$ one has a well defined continuous evaluation map $\mathrm{ev}_y : L^t_1(E^\vee) \to E^\vee_y$. Let $L^t_1(E^\vee)_y$ be the kernel of this map.

The exact sequence of Banach spaces

$$0 \longrightarrow L^t_1(E^\vee)_y \longrightarrow L^t_1(E^\vee) \xrightarrow{\mathrm{ev}_y} E^\vee_y \longrightarrow 0$$

splits topologically, so one gets an exact sequence of dual spaces

$$0 \longrightarrow E_y \longrightarrow L^s_{-1}(E) \xrightarrow{\rho^E_y} [L^t_1(E^\vee)_y]^* \longrightarrow 0\ .$$

The monomorphism on the left is the embedding of the space of Dirac distributions concentrated in y in the space $L^s_{-1}(E)$.

More generally, if $D = \{y_1, \ldots, y_k\} \subset E$ is a finite set, one has an exact sequence

$$0 \longrightarrow \bigoplus_{y\in D} E_y \longrightarrow L^s_{-1}(E) \xrightarrow{\rho^E_D} [L^t_1(E^\vee)_D]^* \longrightarrow 0\ ,$$

where $L^t_1(E^\vee)_D$ is the closed subspace of $L^t_1(E^\vee)$ consisting of L^t_1-sections of $E^\vee$ which vanish on D.

LEMMA 5.7. *Let E be a $\mathcal{C}^\infty$ vector bundle on Y, δ a semiconnection on E, and let $\mathcal{E}$ be the corresponding holomorphic bundle. Let $D \subset Y$ be a finite set. Consider the bounded operators*

$$L^s(E) \xrightarrow{\delta} L^s_{-1}(\Lambda^{01}_Y \otimes E) \xrightarrow{\rho_D^{[\Lambda^{01}_Y\otimes E]}} [L^t_1(\Lambda^{01}_Y \otimes E)^\vee_D]^*\ :$$

1. One has

$$\ker\left[\rho_D^{[\Lambda^{01}_Y\otimes E]} \circ \delta\right] = H^0(\mathcal{E}(D))\ .$$

2. When $\deg(E^\vee \otimes K_Y(-D)) < 0$, the map $\rho_D^{[\Lambda^{01}_Y\otimes E]} \circ \delta$ is surjective.

Proof.
1. A section $\varphi \in L^s(E)$ belongs to $\ker\left[\rho_D^{[\Lambda^{01}_Y\otimes E]} \circ \delta\right]$ if and only if $\delta\varphi$ is a linear combination of Dirac distributions concentrated in the points of D. This means that φ is meromorphic with poles of order at most 1 in the points of D.
2. The composition $\rho_D^{[\Lambda^{01}_Y\otimes E]} \circ \delta$ is the adjoint of the composition

$$L^t_1([\Lambda^{01}_Y]^\vee \otimes E^\vee)_D \overset{i_D}{\hookrightarrow} L^t_1([\Lambda^{01}_Y]^\vee \otimes E^\vee) \xrightarrow{\delta'} L^t_1(E^\vee)\ ,$$

where the operator δ' is the adjoint of δ, i. e. it satisfies the identity

$$\int_Y \langle \delta'(\alpha), \beta\rangle vol_g = \int_Y \langle \alpha, \delta(\beta)\rangle vol_g$$

for all smooth sections $\alpha \in A^0([\Lambda^{01}_Y]^\vee \otimes E^\vee)$, $\beta \in A^0(E)$.

The crucial observation is that δ' is just the tensor product semiconnection $\delta_{\mathrm{can}} \otimes \delta^\vee$ on the bundle $\Lambda^{10}_Y \otimes E^\vee$, where δ_{can} is the canonical holomorphic structure on $\Lambda^{10}_Y = K_Y$ and $\delta^\vee$ is the dual semiconnection on $E^\vee$.

On the other hand, the composition $\delta' \circ i_D$ is Fredholm, because the space $L^t_1([\Lambda^{01}_Y]^\vee \otimes E^\vee)_D$ is closed and has finite codimension in $L^t_1([\Lambda^{01}_Y]^\vee \otimes E^\vee)$.

Therefore the surjectivity of the composition $\rho_D^{[\Lambda^{01}_Y \otimes E]} \circ \delta$ is equivalent to the injectivity of $\delta' \circ i_D$. Let $\mathcal{E}$ be the holomorphic bundle defined by δ.

By elliptic regularity we have

$$\ker(\delta' \circ i_D) = \{s \in H^0(K_Y \otimes \mathcal{E}^\vee) |\ s|_D = 0\} = H^0(K_Y \otimes \mathcal{E}^\vee(-D)) \ ,$$

which vanishes, when $\deg(E^\vee \otimes K_Y(-D)) < 0$. □

Now we come back to our gauge theoretical problem. For $1 \leq j \leq r$ let $D_j \subset Y$ be a finite set of points, and denote by D the system $(D_j)_j$. The same argument applied to the line bundles $\Lambda^{01}_Y \otimes L_j$ yields the exact sequence

$$0 \longrightarrow \bigoplus_j \bigoplus_{y \in D_j} (\Lambda^{01}_Y \otimes L_j)_y \longrightarrow \bigoplus_j L^s_{-1}(\Lambda^{01}_Y \otimes L_j) \xrightarrow{\rho_D} \bigoplus_j [L^t_1(\Lambda^{01}_Y \otimes L_j)^\vee_{D_j}]^* \longrightarrow 0 \ ,$$

where $L^t_1(\Lambda^{01}_Y \otimes L_j)_{D_j}$ is the space of L^t_1-sections of $\Lambda^{01}_Y \otimes L_j$ which vanish on D_j.

We denote by $\bar{\mathfrak{E}}_0$ the associated bundle of the principal bundle $[\bar{\mathcal{A}}^s_k]^{**} \to \bar{\mathcal{B}}^{**}$ with standard fibre

$$L^s_{-1}(\Lambda^{01}_Y \otimes L)_D := \bigoplus_j [L^t_1(\Lambda^{01}_Y \otimes L_j)^\vee_{D_j}]^* \ ,$$

and by $\bar{\mathfrak{E}}'$ the associated bundle with fibre

$$(\Lambda^{01}_Y \otimes L)_D := \bigoplus_j \bigoplus_{y \in D_j} (\Lambda^{01}_Y \otimes L_j)_y \ .$$

One obtains an exact sequence of Banach bundles over $\bar{\mathcal{B}}^{**}$:

$$0 \longrightarrow \bar{\mathfrak{E}}' \longrightarrow \bar{\mathfrak{E}} \xrightarrow{\mathfrak{r}_D} \bar{\mathfrak{E}}_0 \longrightarrow 0 \ , \qquad (*)$$

where $\mathfrak{r}_D$ is the bundle epimorphism induced by ρ_D. Let $\bar{\mathfrak{h}}_0$ be the induced section $\bar{\mathfrak{h}}_0 := \mathfrak{r}_D \circ \bar{\mathfrak{h}}$.

PROPOSITION 5.8. *Suppose that $\#D_j + \deg(L_j) > 2g_Y - 2$ for all $j \in \{1, \ldots, r\}$. Then the section $\bar{\mathfrak{h}}_0 := \mathfrak{r}_D \circ \bar{\mathfrak{h}}$ is regular at every point $[\delta, \varphi]$ of its vanishing locus.*

Proof. It suffices to show that the differential at $p = (\delta, \varphi)$ of the restriction of $\rho_D \circ \bar{h}$ to the slice V_{p,ε_p} provided by Proposition 5.3. is surjective.[2]

On the other hand, the differential of $\bar{h}$ at p vanishes on the tangent space at the orbit $p \cdot \mathcal{G}^{\mathbb{C}}_{k+1}$, so it suffices to show that the differential of $\rho_D \circ \bar{h}$ at p is surjective. But

$$\left.\frac{\partial}{\partial \varphi}\right|_p (\rho_D \circ \bar{h}) = \rho_D \circ \delta \ ,$$

so the result follows from the previous lemma. □

[2]In the infinite dimensional framework the surjectivity of the differential at a point does not suffices to assure that the map is a submersion at that point. One also has to check that the kernel of the differential has a closed complement. But this condition is obviously satisfied in our case, since this kernel has finite dimension.

We recall Brussee's associativity principle for virtual fundamental classes associated with Fredholm sections ([Br], [OT2]) in the special case of sections with compact vanishing locus.

THEOREM 5.9. *Let*

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E_0 \longrightarrow 0$$

be an exact sequence of Banach bundles over a Banach manifold B, and let s be a Fredholm section in E with compact vanishing locus $Z(s)$. Suppose that the induced section s_0 in the bundle E_0 is regular in every point of its vanishing locus $Z(s_0)$. The restriction $s|_{Z(s_0)}$ can be regarded as a section s' in $E'|_{Z(s_0)}$. Via the obvious identification $Z(s) = Z(s')$ one has

$$[Z(s)]^{\rm vir}_s = [Z(s')]^{\rm vir}_{s'} \ .$$

Using this we can prove our main result

THEOREM 5.10. *The identifications given by the Kobayashi-Hitchin correspondence (Theorem 3.6) and by the embedding theorem (Theorem 4.6) map the virtual fundamental class induced by the Fredholm description of the moduli space $\mathcal{M}_{(\mathfrak{t},g,A^0)}(\lambda)$ onto the algebraic geometric virtual fundamental class defined by the system of sections e^i_j on the smooth algebraic variety $X_{\Sigma,P'}(p^*_{\underline{c}'}(\mathfrak{L}^{\underline{c}'}_{y_0})|_{P'})$.*

Proof. First note that, by Proposition 5.6, it suffices to compare the gauge theoretical virtual fundamental class $[\mathcal{M}^{\Sigma\text{-st}}_{\delta^0}(\lambda)]^{\rm vir}_{\bar{\mathfrak{h}}}$ with the algebraic geometric virtual fundamental class defined by the sections e^i_j on the smooth algebraic variety $X_{\Sigma,P'}(p^*_{\underline{c}'}(\mathfrak{L}^{\underline{c}'}_{y_0})|_{P'})$.

We apply the associativity principle to the restriction of the exact sequence (*) to the open set $\bar{\mathcal{B}}(\Sigma)$ of $\bar{\mathcal{B}}^{**}$. The hypothesis of this principle is verified by Proposition 5.8.

Put $Z_0 := Z(\bar{\mathfrak{h}}_0|_{\bar{\mathcal{B}}(\Sigma)})$, and let $\bar{\mathfrak{h}}'$ be the induced section in $\mathfrak{E}'|_{Z_0}$. It follows that

$$[\mathcal{M}^{\Sigma\text{-st}}_{\delta^0}(\lambda)]^{\rm vir}_{\bar{\mathfrak{h}}} = [Z(\bar{\mathfrak{h}}')]^{\rm vir}_{\bar{\mathfrak{h}}'} \ .$$

We claim that:

a) With the notations of section 4, there is a natural identification

$$Z_0 = [\iota_{\underline{D}}|_P]^*(X_{\Sigma,P'}(p^*_{\underline{c}'}(\mathfrak{L}^{\underline{c}'}_{y_0})|_{P'})) \ ,$$

where, on the right $X_{\Sigma,P'}(p^*_{\underline{c}'}(\mathfrak{L}^{\underline{c}'}_{y_0})|_{P'})$ was considered as a toric fibre bundle over $P' := \iota_{\underline{D}}(P)$.
b) Via the identification above, the sections $[\iota_{\underline{D}}|_P]^*(e^i_j)$ coincide with the components of $\bar{\mathfrak{h}}'$.

The proof of the two claims is easy: for a) one uses the first statement in Lemma 5.7 to get a set theoretical bijection; then the method used in the proof of Theorem 4.5 gives the needed isomorphism. Note that this time, one just has to identify two smooth complex manifolds, so it is not necessary to take ringed space structures into account. For b) it suffices to notice that the components of $\bar{\mathfrak{h}}'$ and the sections $[\iota_{\underline{D}}|_P]^*(e^i_j)$ are induced by evaluation maps associated with the points of D_j.

On the other hand, by the Lemma below, Brussee's virtual fundamental class associated with a regular section in an algebraic vector bundle over a smooth algebraic variety coincides with the corresponding Fulton virtual fundamental class via the cycle map (see [Fu], ch. 19.1 - 19.2). □

LEMMA 5.11. *Let $E \to X$ be an algebraic vector bundle of rank r over a smooth n-dimensional algebraic variety X, and let s be a section in E.*

Then the Fulton virtual fundamental class $\mathbb{Z}(s) \in A_{n-r}(Z(s))$ is mapped onto the Brussee virtual fundamental class $[Z(s)]^{\rm vir}_s$ via the cycle map

$$cl_{Z(s)} : A_{n-r}(Z(s)) \longrightarrow H^{\rm BM}_{2(n-r)}(Z(s), \mathbb{Z}) \ ,$$

where $A_*(Z(s))$ denotes the Chow groups of the complex scheme $Z(s)$ and $H^{\rm BM}_*(Z(s), \mathbb{Z})$ the Borel-Moore homology groups of $Z(s)$.

Proof. We will give the proof in the case when $Z(s)$ is compact. This case is sufficient for our purposes, and the general case follows the same idea.

Note first that the Cech homology of $Z(s)$, to which $[Z(s)]^{\rm vir}_s$ belongs, coincides with the standard singular homology ([Br], Ch. 2). Let $s_E : X \to E$ be the zero-section of E. By definition, one has

$$[Z(s)]^{\rm vir}_s = s^*(\tau_E) \cap [X] \in H_{2(n-r)}(Z(s), \mathbb{Z}) \ ,$$

where $\tau_E \in H^{2r}(E, E \setminus \mathrm{im}(s_E), \mathbb{Z})$ is the Thom class of E, and $[X]$ is the fundamental class of X in Borel-Moore homology.

Since E is a smooth algebraic variety, the class $\mathbb{Z}(s)$ can be identified with the *refined intersection product* of algebraic cycles $\mathrm{im}(s_E) \cdot \mathrm{im}(s) \in A_{n-r}(\mathrm{im}(s_E) \cap \mathrm{im}(s))$ via the obvious identification $\mathrm{im}(s_E) \cap \mathrm{im}(s) = Z(s)$ (see [Fu] ch. 8.1, Corollary 8.1.1, Ch. 14.1). Therefore, by Corollary 19.2 [Fu], we get

$$cl_{Z(s)}(\mathbb{Z}(s)) = cl_{Z(s)}(\mathrm{im}(s_E) \cdot \mathrm{im}(s)) = cl_{Z(s)}(\mathrm{im}(s_E)) \cdot cl_{Z(s)}(\mathrm{im}(s)) =$$

$$= [\mathrm{im}(s_E)] \cdot [\mathrm{im}(s)] \ ,$$

where the dot in the last two term stands for the refined topological intersection ([Fu], p. 378). Note now that the Poincaré dual

$$PD([\mathrm{im}(s_E)]) \in H^{2r}(E, E \setminus \mathrm{im}(s_E), \mathbb{Z})$$

of $[\mathrm{im}(s_E)]$ in E is just the Thom class τ_E. Therefore,

$$[\mathrm{im}(s_E)] \cdot [\mathrm{im}(s)] = \tau_E \cap s_*[X] = (s|_{Z(s)})_* \left(s^*(\tau_E) \cap [X]\right)$$

in $H_*(\mathrm{im}(s_E) \cap \mathrm{im}(s), \mathbb{Z})$. This shows that

$$cl_{Z(s)}(\mathbb{Z}(s)) = s^*(\tau_E) \cap [X] = [Z(s)]^{\rm vir}_s \ .$$

□

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