ASIAN J. MATH.
Vol. 7, No. 1, pp. 115–132, March 2003

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PROPER AFFINE HYPERSPHERES WHICH FIBER OVER PROJECTIVE SPECIAL KÄHLER MANIFOLDS *

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Abstract. We show that the natural S^1-bundle over a projective special Kähler manifold carries the geometry of a proper affine hypersphere endowed with a Sasakian structure. The construction generalizes the geometry of the Hopf-fibration $S^{2n+1} \longrightarrow \mathbb{C}P^n$ in the context of projective special Kähler manifolds. As an application we have that a natural circle bundle over the Kuranishi moduli space of a Calabi-Yau threefold is a Lorentzian proper affine hypersphere.

Introduction. In a previous paper [BC], we proved that any simply connected special Kähler manifold admits a canonical immersion into affine space as a parabolic affine hypersphere. A particular important class of special Kähler manifolds are *conic* special Kähler manifolds. These are by definition special Kähler manifolds which are locally modelled on a complex cone over some complex projective manifold which is then called a *projective special Kähler manifold.* The purpose of this paper is to provide an understanding of the particular (affine) differential geometry which is canonically associated with projective special Kähler manifolds.

Whereas the conic special Kähler manifold M which is associated with a simply connected projective special Kähler manifold $\bar{M}$ carries the geometry of a *parabolic* (or improper) affine hypersphere, we show that the total space S of a natural circle bundle $S \to \bar{M}$ is a *proper* affine hypersphere. The S^1-action on S induces a Sasakian structure on S which is compatible with the affine differential geometry in a very specific sense. Moreover, all information about the conic special Kähler geometry on M is encoded in the *affine Sasakian geometry* on S.

Lu showed [L] that every *complete* affine special Kähler manifold with a positive definite metric is flat. Using a well known result of Calabi [Ca2] on complete convex affine hyperspheres we obtain an analogous result for *projective* special Kähler manifolds: We show that if $\bar{M}$ is a (simply connected) complete projective special Kähler manifold with a definite affine metric on S then $\bar{M}$ is isometric to $\mathbb{C}P^n$ with the canonical Fubini-Study metric.

The construction of the affine sphere S over a projective special Kähler manifold naturally relates to well known canonical data on the Kuranishi moduli space for Calabi-Yau three-manifolds. Thereby we show that a natural circle bundle over the Kuranishi moduli space admits a canonical structure of an affine hypersphere with affine metric of Lorentzian signature.

If $\bar{M}$ is complete, and the metric on S is not definite, as in the case of Kuranishi moduli spaces, then interesting complete models for projective special Kähler manifolds do exist. We describe all fibrations $S \to \bar{M}$ which admit a transitive semisimple group of automorphisms preserving the projective special Kähler structure on the base $\bar{M}$. These are particular examples of homogeneous Lorentzian affine hyperspheres fibering over Hermitian symmetric spaces.

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1. Preliminaries.

1.1. Affine hypersurfaces. For the convenience of the reader, and to fix the notation, we recall the basic definitions of affine differential geometry of hypersurfaces in $\mathbb{R}^{n+1}$ and the definition of *affine hyperspheres.* For more details, see for example [NS, Ca2]. Let det denote the standard volume form on $\mathbb{R}^{n+1}$, and ∇ the standard flat connection on $\mathbb{R}^{n+1}$. In the context of *affine immersions* we consider manifolds with a semi-Riemannian metric g and a torsionfree connection $\hat{\nabla}$ so that

i) the cubic tensor $\hat{\nabla}g$ is totally symmetric, and
ii) the metric volume form θ_g is $\hat{\nabla}$-parallel.

The data $(g, \hat{\nabla})$ are then said to satisfy the *compatibility condition* i) and the *equiaffine condition* ii). Every nondegenerate hypersurface immersion $\psi : M \to \mathbb{R}^{n+1}$ induces data $(\hat{\nabla}, g)$ on M which satisfy i) and ii) via the fundamental formula

$$\nabla_X Y = \hat{\nabla}_X Y + g(X,Y)E\ , \tag{1}$$

where X, Y denote vector fields on M, and E is the *affine normal* of the immersion. (Note that the notation identifies M as a submanifold of $\mathbb{R}^{n+1}$.) The affine normal E is a canonical normal vector field along ψ which is defined up to sign by the condition that the pair $(\hat{\nabla}, g)$ satisfies ii), and the normalizing condition

iii) $\det(E, \ldots) = \theta_g$ on M.

The metric g is then called the *Blaschke metric* and the immersion $\psi : (M, \hat{\nabla}, g) \to \mathbb{R}^{n+1}$ a *Blaschke immersion.* The tensor $A = -\nabla E$ is horizontal along ψ and is called the *shape tensor* of the immersion. The quantity $H = \frac{1}{n}\mathrm{tr}A$ is called the *affine mean curvature.* If $\hat{\nabla}$ is flat and $n > 1$ then, by the equation of Gauß, $A = 0$ and the affine normal is the restriction of a constant vector field. In this case, ψ is called a *parabolic* (or improper) *affine hypersphere.* If the shape tensor equals a constant multiple of the identity, $A = \kappa\, id$, where $\kappa \neq 0$, ψ is called a *proper affine hypersphere.* In this case, $\hat{\nabla}$ is projectively flat. An affine hypersphere has constant mean curvature $H = \kappa$.

Let M be a manifold with data $(\hat{\nabla}, g)$ which satisfy i) and ii). We put $\hat{\nabla}^*$ for the conjugate connection of $\hat{\nabla}$ with respect to g. It is torsionfree by the compatibility condition i). Then the fundamental theorem of affine differential geometry asserts that a simply connected manifold M with data $(\hat{\nabla}, g)$ arises from a Blaschke immersion ψ if and only if the *integrability condition*

iv) $\hat{\nabla}^*$ is projectively flat

is satisfied. The immersion ψ is determined by the data $(\hat{\nabla}, g)$ up to composition with an unimodular affine transformation. A special case arises if $\hat{\nabla}$ is flat. Then it is easily seen that $\hat{\nabla}^*$ is also flat. Hence, iv) is satisfied and M is a parabolic affine hypersphere. We also mention that the data $(\hat{\nabla}, g)$ arise from an immersion as an affine sphere if and only if the cubic tensor $C = \hat{\nabla}g$ has totally symmetric derivative $\hat{\nabla}C$. If $(M, \hat{\nabla}, g)$ is a manifold which satisfies the integrability conditions for a Blaschke immersion as an affine sphere we say that M has the structure of an affine sphere.

1.2. Special Kähler manifolds. We recall some basic notions and constructions from special Kähler geometry. For more details the reader can consult [ACD], and also [F]. A *special Kähler manifold* (M, J, g, ∇) is a (pseudo-) Kähler manifold (M, J, g) together with a flat torsionfree connection ∇ such that $\nabla\omega = 0$, where $\omega = g(\cdot, J\cdot)$ is the Kähler form, and such that ∇J is symmetric, i.e. $d^\nabla J(X,Y) := (\nabla_X J)Y - (\nabla_Y J)X = 0$ for all vector fields X and Y.

More precisely, one should speak of *affine* special Kähler manifolds since there is also the notion of a projective special Kähler manifold. In fact, there is a class of (affine) special Kähler manifolds (M, J, g, ∇), which are called *conic special Kähler manifolds* and which are characterized by the existence of a local holomorphic $\mathbb{C}^*$-action $\varphi_\lambda : M \to M$, $\lambda = re^{it} \in \mathbb{C}^*$, with the property:

$$(\varphi_\lambda)_* X = r \cos t X + r \sin t JX$$

for all ∇-parallel vector fields X on M. Under appropriate regularity assumptions on the action, the projection

$$\pi : M \longrightarrow \bar{M} = P(M)$$

onto the space of orbits $\bar{M} = P(M)$ is a holomorphic submersion onto a complex (Hausdorff-) manifold. Then $\bar{M}$ inherits a (pseudo-) Kähler metric $\bar{g}$ from (M, g), and the base $(\bar{M}, \bar{g})$ is called a *projective special Kähler manifold.* Although, strictly speaking, the fully fledged projective special Kähler geometry is encoded in the geometric data on the bundle $\pi : M \to \bar{M}$.

Special Kähler manifolds may also be characterized in terms of *complex Lagrangian immersions* (see [ACD]). In fact, any simply connected special Kähler manifold (M, J, g, ∇) has a canonical realization as a (pseudo-) Kählerian immersed Lagrangian submanifold of a pseudo-Hermitian, complex symplectic vector space (V, γ, Ω) with split signature. This means that there exists a holomorphic Lagrangian immersion $\lambda : M \to V$ so that $g = \lambda^* \gamma$ is the pull-back of the hermitian product γ. Moreover, the projection onto the subspace V^τ of real points for the real structure τ defined by the relation $\Omega = -i\gamma(\cdot, \tau\cdot)$ gives local flat coordinates on M which determine the flat connection ∇. The holomorphic Lagrangian immersion λ is determined by the data (g, ∇) up to a complex affine transformation which preserves γ and Ω. Conic special Kähler manifolds may be realized by immersions λ which are equivariant with respect to the natural $\mathbb{C}^*$-action on V. λ is then uniquely determined up to a complex linear transformation which preserves γ and Ω. We then call λ a *compatible* Lagrange immersion of the (conic) special Kähler manifold M.

2. The local geometry. It is well known that holomorphic Lagrangian immersions λ into a complex $2n$-dimensional symplectic vector space V are locally of the form $\lambda = \lambda_F := dF : U \to T^*\mathbb{C}^n \cong V$, where F is a holomorphic function defined on some domain $U \subset \mathbb{C}^n$. The Kähler condition for the holomorphic Lagrangian immersion λ_F is an open condition on the real 2-jet of F. Conic special Kähler manifolds correspond to potentials which are homogeneous of degree 2. Therefore the local geometry of (conic) special Kähler manifolds may be described in terms of a holomorphic potential F.

Special Kähler domains. Let $U \subset \mathbb{C}^n$ be a connected open domain and $F : U \to \mathbb{C}$ a holomorphic function which satisfies the condition that the matrix

$$\operatorname{Im}\left(\frac{\partial^2 F}{\partial z_i \partial z_j}\right)$$

is nondegenerate. Then the function

$$k = \frac{1}{2} \operatorname{Im}\left(\sum \frac{\partial F}{\partial z_i} \bar{z}_i\right)$$

defines a Kähler potential on U. With the corresponding Kähler form $\omega = i\,\partial\bar{\partial}k$, and metric $g = \omega(i\,\cdot,\cdot)$, the domain U is a (pseudo-) Kähler manifold[1]. Such a domain U will be called a *special Kähler domain.* On a special Kähler domain U there are flat coordinates, called *flat special coordinates,*

$$x_i = \mathrm{Re}(z_i)\ , \quad y_i = \mathrm{Re}(\frac{\partial F}{\partial z_i}) \tag{2}$$

which define on U a torsionfree flat connection ∇ so that ω is parallel. The complex manifold U with the data (g, ∇) is then a special Kähler manifold. Conversely, any special Kähler manifold is locally equivalent to a special Kähler domain (U, g, ∇).

Another peculiar feature of special Kähler domains is that the Kähler metric g is a *Hessian* metric with respect to the flat connection. This means that on U there exists a real potential function f so that $g = \nabla df$. (The fact that g is locally Hessian is well known. An explicit formula for f which is given in terms of the holomorphic function F, see [C2], shows that f exists globally on U.) Moreover, in the flat coordinates the smooth function f satisfies the Monge-Ampère equation

$$|\det \partial^2 f| = c\ , \tag{3}$$

where $c > 0$ is a constant. As a consequence, the data (g, ∇) give U the geometry of a parabolic affine hypersphere, see [BC]. Explicitly,

$$\lambda(u) = (x_1(u), \dots, x_n(u), y_1(u), \dots, y_n(u), f(u))$$

defines a Blaschke immersion $\lambda : U \to \mathbb{R}^{2n+1}$ into affine space $\mathbb{R}^{2n+1}$ which induces the data (∇, g).

2.1. The metric geometry of conic special Kähler domains. In this paper, we are mainly concerned with *conic special Kähler domains.* We call a special Kähler domain $U \subset \mathbb{C}^{n+1}\backslash\{0\}$ *conic,* if $\mathbb{C}^*U \subset U$ and if the holomorphic prepotential F is a homogeneous function of degree 2. Moreover, we require that the potential k does not vanish on a conic special Kähler domain. Locally, any conic special Kähler manifold is equivalent to a conic special Kähler domain $U \subset \mathbb{C}^{n+1}\backslash\{0\}$. To any conic domain $U \subset \mathbb{C}^{n+1}$ we let $\bar{U}$ denote its image in the projective space $\mathbb{C}\mathrm{P}^n$. We consider the projection map

$$\pi : U \longrightarrow \bar{U}$$

which is a submersion, and view U as a principal $\mathbb{C}^*$-bundle over $\bar{U}$. The special Kähler metric g on U naturally induces a Kähler metric $\bar{g}$ on $\bar{U}$ via the projection π. The metric $\bar{g}$ is defined by the formula

$$\bar{g}_{\pi(u)}(d\pi(X), d\pi(X)) = \frac{g_u(X,X)}{g_u(u,u)} - \left|\frac{g_u(X,u)}{g_u(u,u)}\right|^2\ , \quad X \in T_u\mathbb{C}^{n+1}\ . \tag{4}$$

(Note that g is definite on the vertical spaces $\mathcal{V}_u = \mathbb{C}u \subset T_u\mathbb{C}^{n+1}$ of the fibration π by the condition that $k \neq 0$, see Lemma 2 below.) Let $\bar{\omega}$ denote the corresponding Kähler form on $\bar{U}$. Then it is easy to see that the pull-back $\pi^*\bar{\omega}$ on U is given by

$$\pi^*\bar{\omega} = i\,\partial\bar{\partial}\log k$$

[1] We do not require that the Kähler metric g is definite

on the horizontal space $\mathcal{H}_u = \mathcal{V}_u^{\perp}$. We call the domain $\bar{U} \subset \mathbb{C}P^n$ with the metric $\bar{g}$ a *projective special Kähler domain.* The simplest example of such a domain is projective space $\mathbb{C}P^n$ itself with the Fubini-Study metric:

EXAMPLE 2.1. Putting $U = \mathbb{C}^{n+1}\backslash\{0\}$ and $F(z_0, \ldots, z_n) = i\sum z_j^2$, formula (4), defines the Fubini-Study metric on $\bar{U} = \mathbb{C}P^n$. The famous *Hopf-fibration*

$$S^{2n+1} \longrightarrow \mathbb{C}P^n$$

exhibits the sphere $S^{2n+1} = \{u \in \mathbb{C}^{n+1} \mid |u|^2 = 1\}$ as a S^1-principal bundle over $\mathbb{C}P^n$. The Hopf fibration is also known to be a Riemannian submersion with respect to the standard metric on the sphere if the metric on $\mathbb{C}P^n$ is suitably normalized.

It is the content of our next proposition that the geometric construction of the Hopf-fibration generalizes in the context of projective special Kähler domains. To establish this result we consider now the Kähler potential k on U. We remark that k satisfies $k(\alpha u) = |\alpha|^2 k(u)$, for $\alpha \in \mathbb{C}^*$, and, by assumption, never vanishes on U. We put $M_c = \{u \in U \mid |k(u)| = c\}$. Then the level surface M_c is a real hypersurface in $U \subset \mathbb{C}^{n+1}$, and S^1 acts freely on M_c.

PROPOSITION 1. *The hypersurfaces $M_c \subset U$ are nondegenerate with respect to the metric g. Moreover, S^1 acts isometrically on (M_c, g), and M_c is a S^1-principal bundle over $\bar{U}$. If $k > 0$ then the projection map*

$$\pi_c : (M_c, g) \longrightarrow (\bar{U}, \bar{g})$$

is a semi-Riemannian submersion for $c = \frac{1}{2}$. (If $k < 0$ then π_c is an anti-isometry on horizontal vectors for $c = \frac{1}{2}$)

We will need a lemma. Let $h = g + i\omega$ denote the Hermitian product on U which is defined by g. We let $\xi(u) = u$ denote the position vector field on U.

LEMMA 2.
i) $h(\xi, \cdot) = 2\bar{\partial}k$
ii) $g(\xi, \cdot) = dk$
iii) $g(\xi, \xi) = 2k$

Proof. In the complex coordinates we have $\xi = \sum(z_j \frac{\partial}{\partial z_j} + \bar{z}_j \frac{\partial}{\partial \bar{z}_j})$ and

$$h = \sum \operatorname{Im}\left(\frac{\partial^2 F}{\partial z_i \partial z_j}\right) dz_i \otimes d\bar{z}_j \, . \tag{5}$$

Consequently,

$$\begin{aligned} h(\xi, \cdot) &= \sum \operatorname{Im}\left(\frac{\partial^2 F}{\partial z_i \partial z_j}\right) z_i d\bar{z}_j \\ &= -\frac{i}{2}\left(\sum \frac{\partial^2 F}{\partial z_i \partial z_j} z_i d\bar{z}_j - \sum \frac{\partial^2 \bar{F}}{\partial \bar{z}^i \partial \bar{z}^j} z_i d\bar{z}_j\right) \\ &= -\frac{i}{2}\left(\sum \frac{\partial F}{\partial z_j} d\bar{z}_j - \sum \frac{\partial^2 \bar{F}}{\partial \bar{z}^i \partial \bar{z}^j} z_i d\bar{z}_j\right) \\ &= 2\bar{\partial}k \end{aligned}$$

This proves i). Now ii) follows from i) by calculating

$$g(\xi, \cdot) = \operatorname{Re} h(\xi, \cdot) = (\bar{\partial}k + \partial k) = dk \, .$$

Equation iii) is implied by ii), taking into account that the function k is $\mathbb{R}^{>0}$-homogeneous of degree 2. □

Proof of Proposition 1. We consider the g-orthogonal decomposition $T_u\mathbb{C}^{n+1} = \mathcal{V}_u \oplus \mathcal{H}_u$ into vertical and horizontal space which is defined by the canonical submersion $\pi : U \to \bar{U}$. Then $\mathcal{V}_u$ is the real span of ξ and $J\xi$, and in fact $\mathcal{H}_u = \{X \in T_u\mathbb{C}^{n+1} \mid h(\xi, X) = 0\}$. In particular, $g(\xi, X) = 0$, for $X \in \mathcal{H}_u$. Therefore, by ii) from the lemma, it follows that $\mathcal{H}_u \subset \ker dk = TM_c$. We compute the pull back $\pi^*\bar{g}$ of the special Kähler metric $\bar{g}$ on $\bar{U}$ on the tangent space of M_c. Using (4) we get that $g_u(u,u)\,\pi^*\bar{g}_u = g_u$ on T_uM_c. Now, by iii) of the lemma, $g_u(u,u) = 2k(u)$. The proposition follows.

PROPOSITION 3. *The vector field ξ, which is the position vector field on the conic complex domain U, is also the position vector field in the affine coordinates x_i, y_i.*

Proof. To see this, we compute

$$\begin{aligned} dx_i(\xi) &= \operatorname{Re} dz_i(\xi) = \operatorname{Re} z_i = x_i \\ dy_i(\xi) &= \operatorname{Re} d\left(\frac{\partial F}{\partial z_i}\right)(\xi) = \operatorname{Re} \sum \frac{\partial^2 F}{\partial z_i \partial z_j} z_j = \operatorname{Re} \frac{\partial F}{\partial z_i} = y_i \,. \end{aligned}$$

Hence, $\xi = \sum x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i}$ as claimed. □

Metric cones. For any manifold M with a (pseudo-) Riemannian metric g, the manifold $C(M) = \mathbb{R}^{>0} \times M$ with the metric $dr^2 + r^2 g$ is called the *metric cone* over M. More generally, we consider cone metrics of the type $g_\kappa = \frac{1}{\kappa} dr^2 + r^2 g$, where $\kappa \neq 0$ is a constant. We denote the corresponding metric cone as $C_\kappa(M) = (C(M), g_\kappa)$. Let us put $\operatorname{sign} k = 1$ if $k > 0$ and $\operatorname{sign} k = -1$ if $k < 0$.

COROLLARY 4. *Let U be a conic special Kähler domain with Kähler potential k, and special Kähler metric g. Then (U, g) is isometric to the metric cone $C_{\operatorname{sign} k}(M_{\frac{1}{2}})$.*

Proof. Since $\mathbb{R}^{>0}$ acts freely on U, the map

$$\Phi : C(M_{\frac{1}{2}}) \to U \quad (r, u) \mapsto ru$$

is a diffeomorphism. Note that $d\Phi(r\frac{\partial}{\partial r}) = \xi$. The homogeneity of the holomorphic potential F implies that the second derivatives of F are constant on radial lines in U. Hence, by formula (5), we have $g_{ru}(rX, rX) = r^2 g_u(X, X)$, for $u \in M_{\frac{1}{2}}$, $X \in T_u M_{\frac{1}{2}}$. Moreover, by iii) of Lemma 2, $g_{ru}(\xi, \xi) = 2k(ru) = r^2 \operatorname{sign} k$. It is now immediate from ii) of Lemma 2 that Φ is an isometry. □

PROPOSITION 5. *The one-form $\eta := \omega(\xi, \cdot)$ defines a contact structure on $M_{\frac{1}{2}}$.*

Proof. $d\eta = L_\xi \omega = 2\omega$ is nondegenerate on $\ker \eta = J\xi^\perp$. □

2.2. The affine geometry of conic special Kähler domains. Having just seen that any conic special Kähler domain (U, g) has the geometry of a metric cone over the level surface $(M_{\frac{1}{2}}, g)$ of k, we consider now the question how the flat affine connection ∇ on U interacts with the cone structure of (U, g). The flat affine geometry on U is determined by the coordinate change (2) which embeds U as a domain in $\mathbb{R}^{2n+2}$. Since the symplectic form ω is ∇-parallel, so is the volume form

$$\theta = \theta_g = \frac{1}{(n+1)!}\, \omega^{\wedge n+1} \,.$$

Using the flat special coordinates we may view

$$M_{\frac{1}{2}} \longrightarrow \mathbb{R}^{2n+2}$$

immersed into affine space as a real hypersurface. In the light of Corollary 4, the next result shows that the metric structure on (U,g) is determined by the affine geometry of the hypersurface $M_{\frac{1}{2}}$.

THEOREM 6. *In the flat special coordinates of the special Kähler domain* $U \subset \mathbb{C}^{n+1}$ *the hypersurface* $M_{\frac{1}{2}} \subset U$ *immerses as a non-degenerate hypersurface in* $\mathbb{R}^{2n+2}$. *The transversal field* $E = -\operatorname{sign} k\, \xi$ *is a Blaschke-normal for* $M_{\frac{1}{2}}$ *with respect to the volume form* θ *on* $\mathbb{R}^{2n+2}$, *and the corresponding Blaschke-metric on* $M_{\frac{1}{2}}$ *coincides with the metric* g *induced from* (U,g). *Moreover,* $M_{\frac{1}{2}}$ *is an affine hypersphere of affine mean curvature* $\operatorname{sign} k$.

We start the proof of the theorem with a lemma. Any vector field X on $M_{\frac{1}{2}}$ with values in $\mathbb{C}^{n+1}$ has a natural extension $\tilde{X}$ on U which is defined by $\tilde{X}(ru) = rX(u)$, for $u \in M_{\frac{1}{2}}$.

LEMMA 7.
i) $\xi \cdot g(\tilde{X},\tilde{Y}) = 2g(\tilde{X},\tilde{Y})$,
ii) $(\nabla_\xi g)(\tilde{X},\tilde{Y}) = 0$,
iii) $g(\xi, \nabla_{\tilde{X}}\tilde{Y}) = -g(\tilde{X},\tilde{Y})$, *if* Y *is tangent to* $M_{\frac{1}{2}}$.

Proof. Using Proposition 3, i) follows since the function $g(\tilde{X},\tilde{Y})$ is $\mathbb{R}^{>0}$-homogeneous of degree 2. Also from $\tilde{X}(ru) = r\tilde{X}(u)$, for all $u \in U$, we deduce that $\nabla_\xi \tilde{X} = \nabla_{\tilde{X}}\xi = \tilde{X}$. Therefore, $(\nabla_\xi g)(\tilde{X},\tilde{Y}) = \xi \cdot g(\tilde{X},\tilde{Y}) - g(\nabla_\xi\tilde{X},\tilde{Y}) - g(\tilde{X},\nabla_\xi\tilde{Y}) = \xi \cdot g(\tilde{X},\tilde{Y}) - 2g(\tilde{X},\tilde{Y})$. Hence, ii) follows from i).

Now, if Y is tangent to $M_{\frac{1}{2}}$ then $g(\xi,\nabla_{\tilde{X}}\tilde{Y}) + g(\tilde{X},\tilde{Y}) = -(\nabla_{\tilde{X}}g)(\xi,\tilde{Y}) = -(\nabla_\xi g)(\tilde{X},\tilde{Y})$, by the symmetry of ∇g. Hence, iii) follows from ii). □

Proof of Theorem 6. Let X, Y denote vector fields tangent to $M_{\frac{1}{2}}$, and put $\kappa = \operatorname{sign} k$. Then, by ii),iii) of Lemma 2, and Lemma 7 the Gauß-formula (1) for the hypersurface $M_{\frac{1}{2}}$, with respect to ξ reads

$$\nabla_X Y = \hat{\nabla}_X Y - \kappa g(X,Y)\,\xi\,,$$

where $\hat{\nabla}$ defines the induced connection on $M_{\frac{1}{2}}$. Therefore the affine metric on $M_{\frac{1}{2}}$ with respect to the transversal vector field $E = -\kappa\,\xi$ coincides with the metric g. Let $\theta_{\frac{1}{2}}$ denote the metric volume form of the pseudo-Riemannian manifold $(M_{\frac{1}{2}}, g)$. To show that E is a Blaschke normal, we note that (for an appropriate choice of orientation of $M_{\frac{1}{2}}$) the metric volume form $\theta = \theta_g$ of the ambient space (U,g) is given by $\theta = -\kappa\, dr \wedge r^{(2n+1)}\theta_{\frac{1}{2}}$ in the conic product coordinates Φ from the proof of Corollary 4. And, therefore, $\theta(E,\ldots) = \theta(-\kappa r\frac{\partial}{\partial r},\ldots) = \theta_{\frac{1}{2}}$ along $M_{\frac{1}{2}}$. Hence, E is a Blaschke-normal. Since $A = -\nabla E = \kappa\,\mathrm{Id}$, $M_{\frac{1}{2}}$ is an affine hypersphere of affine mean curvature $H = \kappa$.

Now it is easy to find a ∇-potential for g.

COROLLARY 8. *The Kähler potential* k *is also a* ∇*-potential for the special Kähler metric* g, *i.e.* $g = \nabla dk$ *on* U.

Proof. For homogeneous vector fields $\tilde{X}$ and $\tilde{Y}$, we compute

$$(\nabla_{\tilde{X}} dk)(\tilde{Y}) = \tilde{X} \cdot dk(\tilde{Y}) - dk(\nabla_{\tilde{X}}\tilde{Y})\,.$$

If Y is tangent to $M_{\frac{1}{2}}$ then, using ii) of Lemma 2 $dk(\nabla_{\tilde{X}}\tilde{Y}) = g(\xi, \nabla_{\tilde{X}}\tilde{Y})$, and iii) of Lemma 7 implies $(\nabla_{\tilde{X}}dk)(\tilde{Y}) = g(\tilde{X}, \tilde{Y})$. If $\tilde{Y} = \xi$ then we get $(\nabla_{\tilde{X}}dk)(\xi) = \tilde{X} \cdot dk(\xi) - dk(\tilde{X}) = \tilde{X} \cdot 2k - dk(\tilde{X}) = dk(\tilde{X}) = g(\xi, \tilde{X})$. □

3. Affine Sasakian hyperspheres. In Riemannian geometry a manifold (S, g) is called Sasakian if the corresponding metric cone $(C(S), g_1)$ is a Kähler manifold, see e.g. [BG]. More generally, we call a (pseudo-) Riemannian manifold Sasakian if the metric cone $C_\kappa(S)$ is a (pseudo-) Kähler manifold. Let U be a a conic special Kähler domain, and $S = M_{\frac{1}{2}} \subset U$ the affine sphere which is associated to U by Theorem 6. By Corollary 4, the affine hypersphere S is a Sasakian manifold. However, the concept of Sasakian manifold does not take into account the presence of the affine connection $\hat{\nabla}$ on S. Let $(S, g, \hat{\nabla})$ be a proper affine sphere. We show below that the metric cone $C_\kappa(S)$ admits, as the natural affine differential geometric structure induced from S, the geometry of a parabolic affine hypersphere $(C(S), g_\kappa, \nabla)$. This parabolic sphere is called the *parabolic cone* over S. In [BC] it was remarked that the geometric data of a special Kähler manifold are in fact the geometric data of a parabolic sphere (M, ∇, g) whose Blaschke metric is Kähler, and whose Kähler form ω is ∇-parallel. This motivates the following

DEFINITION 9. A proper affine hypersphere $(S, g, \hat{\nabla})$ is called an *affine Sasakian hypersphere* if the parabolic cone $(C(S), g, \nabla)$ over S is Kähler, and the corresponding Kähler form ω is ∇-parallel.

Equivalently, a proper affine sphere S is affine Sasakian, if and only if the parabolic cone over S is special Kähler.

3.1. The parabolic cone over a proper affine sphere. We show here that every proper affine hypersphere may be naturally realized as a hypersurface in a conic parabolic affine sphere. We already encountered this phenomenon, however in the particular context of conic special Kähler domains.

Proper spheres embed into conic parabolic spheres. Let (M, g) be a pseudo-Riemannian manifold. We view $M = \{1\} \times M$ in a canonical way as a submanifold of $C_\kappa(M)$ with the metric g induced from the cone metric g_κ on $C(M)$. Note also that the multiplicative group $\mathbb{R}^{>0}$ acts on $C(M)$.

PROPOSITION 10. *Let $\psi : S \longrightarrow \mathbb{R}^{n+1}$ be a proper affine hypersphere of affine mean curvature κ, and with induced Blaschke data $(\hat{\nabla}, h)$. Then the metric cone $C_\kappa(S)$ admits a torsionfree, flat, $\mathbb{R}^{>0}$-invariant connection ∇ so that the data (h_κ, ∇) satisfy the integrability conditions for a parabolic affine hypersphere.*

Proof. We consider the local diffeomorphism $\Phi : C(S) \to \mathbb{R}^{n+1}$ given by $(r, u) \mapsto r\psi(u)$ and let ∇ be the pullback of the canonical flat connection on $\mathbb{R}^{n+1}$. To simplify the notation we view S as a hypersurface in $\mathbb{R}^{n+1}$. Also we may then assume that $E = -\kappa\xi$ is the affine normal of S, where $\xi(x) = x$ is the position vector field on $\mathbb{R}^{n+1}$. For a vector field X on S, let $\bar{X}$ denote the constant extension of X to the product manifold $C(S) = \mathbb{R}^{>0} \times S$. Also we define the vector field $\tilde{X}$ on $U = \Phi(C(S))$ by $\tilde{X}(ru) = rX(u)$, where $u \in S$ and $r > 0$. We let $\bar{\xi} = r\frac{\partial}{\partial r}$ denote the position vector field on the cone $C(S)$. Then $\bar{X} = \Phi^*\tilde{X}$, and $\bar{\xi} = \Phi^*\xi$.

We show first that the metric volume form θ_{h_κ} is ∇ parallel. Note first that, for the right choice of orientation of $C(S)$,

$$\theta_{h_\kappa} = |\kappa|^{-\frac{1}{2}} dr \wedge r^n \theta_h \, .$$

We choose a (local) basis of vector fields $X_1, \ldots, X_n$ on S. Since $\theta_h(X_1, \ldots, X_n) = \det(E, X_1, \ldots, X_n)$ along S, we get on $C(S)$:

$$\theta_{h_\kappa}(\bar{\xi}, \bar{X}_1, \ldots, \bar{X}_n) = |\kappa|^{-\frac{1}{2}} r^{n+1} \theta_h(X_1, \ldots, X_n) = \pm|\kappa|^{\frac{1}{2}} \det(\xi, \tilde{X}_1, \ldots, \tilde{X}_n) .$$

Therefore $\theta_{h_\kappa} = \pm|\kappa|^{\frac{1}{2}} \Phi^* \det$, and hence the equiaffine condition ii) is satisfied with respect to ∇.

Next we show that ∇h_κ is totally symmetric. It is enough to verify that

$$(\nabla_X h_\kappa)(Y, Z) = (\nabla_Y h_\kappa)(X, Z) ,$$

for all vector fields X, Y and Z on $C(S)$. We remark that if X, Y are vector fields on S the following formulas hold on $C(S)$:

$$\nabla_{\bar{X}} \bar{Y} = \overline{\hat{\nabla}_X Y} - \kappa r^{-2} h_\kappa(\bar{X}, \bar{Y}) \xi , \tag{6}$$

$$\nabla_{\bar{X}} \bar{\xi} = \bar{X} , \quad \nabla_{\bar{\xi}} \bar{X} = \bar{X} \tag{7}$$

Therefore

$$\begin{aligned}(\nabla_{\bar{X}} h_\kappa)(\bar{Y}, \bar{Z}) &= \bar{X} \cdot h_\kappa(\bar{Y}, \bar{Z}) - h_\kappa(\nabla_{\bar{X}} \bar{Y}, \bar{Z}) - h_\kappa(\bar{Y}, \nabla_{\bar{X}} \bar{Z}) \\ &= r^2 X \cdot h(Y, Z) - r^2 h(\hat{\nabla}_X Y, Z) - r^2 h(Y, \hat{\nabla}_X Z) .\end{aligned}$$

Hence, for vector fields $\bar{X}, \bar{Y}, \bar{Z}$ the compatibility condition i) for h_κ is implied by i) for h. Next we compute

$$\begin{aligned}(\nabla_{\bar{\xi}} h_\kappa)(\bar{Y}, \bar{Z}) &= \bar{\xi} \cdot h_\kappa(\bar{Y}, \bar{Z}) - h_\kappa(\nabla_{\bar{\xi}} \bar{Y}, \bar{Z}) - h_\kappa(\bar{Y}, \nabla_{\bar{\xi}} \bar{Z}) \\ &= 2h_\kappa(\bar{Y}, \bar{Z}) - h_\kappa(\bar{Y}, \bar{Z}) - h_\kappa(\bar{Y}, \bar{Z}) = 0 .\end{aligned}$$

But also

$$\begin{aligned}(\nabla_{\bar{Y}} h_\kappa)(\bar{\xi}, \bar{Z}) &= -h_\kappa(\nabla_{\bar{Y}} \bar{\xi}, \bar{Z}) - h_\kappa(\bar{\xi}, \nabla_{\bar{Y}} \bar{Z}) \\ &= -h_\kappa(\bar{Y}, \bar{Z}) + \kappa r^{-2} h_\kappa(\bar{\xi}, \bar{\xi}) h_\kappa(\bar{Y}, \bar{Z}) = 0 .\end{aligned}$$

Finally, we easily see that $(\nabla_{\bar{\xi}} h_\kappa)(\bar{X}, \bar{\xi}) = (\nabla_{\bar{X}} h_\kappa)(\bar{\xi}, \bar{\xi}) = 0$. Hence, it follows that ∇h_κ is totally symmetric. □

Note that (h_κ, ∇) satisfies the integrability condition for *parabolic* spheres since ∇ is flat. Hence, $(C(S), h_\kappa, \nabla)$ has the structure of a parabolic affine sphere. As a consequence of the fundamental theorem of affine differential geometry, if $C(S)$ is simply connected, the data (h_κ, ∇) are obtained from a Blaschke immersion $\Phi : C(S) \to \mathbb{R}^{n+2}$ as a parabolic affine hypersphere. Thus, the affine sphere $(S, h, \hat{\nabla})$ is realized in a canonical way as a submanifold of a parabolic affine sphere $(C(S), h_\kappa, \nabla)$, and the Blaschke metric on S, with respect to $(C(S), \nabla)$, coincides with the metric h, induced from h_κ. We call the parabolic affine sphere $(C(S), h_\kappa, \nabla)$ the *parabolic cone* over S.

Completeness of affine spheres. We recall an important fact about parabolic spheres. Calabi [Ca1] and Pogorelov [Po] proved that if the affine metric g of a parabolic affine hypersphere (M, g, ∇) is definite and complete, then M must be a paraboloid. The case that a proper affine sphere $(S, h, \hat{\nabla})$ has a definite metric is also of particular interest. The Blaschke normal of S may be chosen so that the affine mean curvature $H = \kappa$ is positive. If (with this choice of normal) the metric h is positive definite, then S is called an *elliptic* affine sphere, if h is negative definite then S

is called *hyperbolic*. Therefore S is elliptic, if and only if the metric cone $C_\kappa(S)$ carries a definite metric h_κ. In the hyperbolic case the metric h_κ has Lorentzian signature $(1,n)$. There is the following result of Calabi [Ca2] on complete elliptic hyperspheres:

THEOREM 11. *Let S be an elliptic affine hypersphere with complete Blaschke metric h. Then S is an ellipsoid.*

Let S be an elliptic affine hypersphere with complete metric h. Then the parabolic sphere $(C(S), h_\kappa, \nabla)$ has definite metric h_κ. However, clearly the metric cone $C(S)$ is not complete. But Calabi's theorem implies that if S is complete then $C(S) = U \subset \mathbb{R}^{n+1}$ may be completed in $0 \in \mathbb{R}^{n+1}$ to $\bar{U} = \mathbb{R}^{n+1}$, so that the metric h_κ smoothly extends to $\mathbb{R}^{n+1}$. We deduce:

COROLLARY 12. *Let S be an elliptic affine hypersphere with complete metric h and affine mean curvature κ. Then the parabolic cone $(C_\kappa(S), h_\kappa, \nabla)$ is obtained by deleting a point in an elliptic paraboloid.*

3.2. Characterization of affine Sasakian hyperspheres. Let (S,g) be a (pseudo-) Riemannian manifold, D the Levi-Civita connection on S. Then a *Sasakian structure* on S is provided by a Killing vector field σ of constant length $g(\sigma,\sigma) = \kappa^{-1}$ so that the covariant derivative $\Phi = D\sigma$ satisfies

$$(D_X\Phi)(Y) = \kappa(g(\sigma,Y)X - g(X,Y)\sigma)\ .$$

The Killing vector field σ and the one-form $\eta = \kappa\, g(\sigma,\cdot)$ are called the *characteristic vector field* and the *characteristic one-form* of the Sasakian structure on S. Let $C_\kappa(S)$ be a metric cone over S, and let $\xi = r\frac{\partial}{\partial r}$ denote the Euler field on $C(S)$. We define a complex structure J on $C(S)$ by the formulas

$$J\bar{X} = \overline{\Phi X} - \eta(X)\xi\ ,\ J\xi = \sigma\ .$$

It is straightforward to verify that in fact $J^2 = -\mathrm{Id}$, and that the cone metric g_κ is J-invariant. Moreover J is parallel with respect to the Levi-Civita connection. Hence, J is integrable and $C_\kappa(S)$ is Kähler. Conversely, if $C_\kappa(S)$ is Kähler with respect to the complex structure J then $\sigma = J\xi$ defines the characteristic vector field of a Sasakian structure on S.

PROPOSITION 13. *Let $(S,g,\hat{\nabla})$ be a proper affine hypersphere with Sasakian structure σ. Then the parabolic cone over $(S,g,\hat{\nabla})$ is special Kähler with respect to the complex structure J induced from σ if and only if $\Phi = \hat{\nabla}\sigma$.*

Proof. Let us first recall the formulas (6), (7) from the proof of Proposition 10, which are satisfied by the flat connection ∇ on $C(S)$. Note also that the same (warped product) relations hold for the metric connections D and $\bar{D}$, where $\bar{D}$ is the Levi-Civita connection of the cone metric g_κ. Next we remark that the parabolic cone $(C(S), g_\kappa, \nabla)$ is special Kähler if and only if the special Kähler condition

$$d^\nabla J = 0 \tag{8}$$

is satisfied. For a vector field Y on S, we compute $(\nabla_\xi J)\bar{Y} = 0$, and $(\nabla_{\bar{Y}} J)\xi = \nabla_{\bar{Y}}\bar{\sigma} - J\bar{Y}$, where $J\bar{Y} = \overline{D_Y\sigma} - \eta(Y)\xi$ and $\nabla_{\bar{Y}}\bar{\sigma} = \overline{\hat{\nabla}_Y\sigma} - \kappa g(Y,\sigma)\xi$. Therefore if (8) is satisfied $(\nabla_{\bar{Y}}J)\xi = 0$, and hence $\hat{\nabla}.\sigma = D.\sigma = \Phi$. Conversely, from $\Phi = \hat{\nabla}.\sigma$ we deduce that $J\bar{Y} = \nabla_{\bar{Y}}\bar{\sigma}$ and hence, since ∇ is flat, it follows (8) along S. Moreover, from the above equations $d^\nabla J(\bar{Y},\xi) = (\nabla_{\bar{Y}}J)\xi = 0$ follows immediately. Therefore, the parabolic cone $(C(S), g_\kappa, \nabla)$ is special Kähler. □

Consequently, if the Sasakian structure σ satisfies $\Phi = \hat{\nabla}\sigma$ we call σ an *affine Sasakian structure* on the hypersphere $(S,g,\hat{\nabla})$.

4. Applications.

4.1. The Canonical circle bundle. Let $\pi : M \to \bar{M}$ be a projective special Kähler manifold, where the conic manifold M carries the data (J, g, ∇). Let $(\tilde{M}, J, g, \nabla)$ be the universal covering space of M, and $\lambda : \tilde{M} \to V$ a compatible Lagrangian embedding into a pseudo-Hermitian, symplectic vector space (V, γ, Ω). Since the embedding λ is unique up to isometry of (V, γ, Ω), the function

$$k(p) = \frac{1}{2}\gamma(\lambda(p), \lambda(p)) , \quad p \in \tilde{M}$$

is invariant under deck-transformations of the covering, and hence defines a function $k : M \to \mathbb{R}^{>0}$. Note that, by iii) of Lemma 2 and by Corollary 8, (M, ∇, g) is a Hessian-manifold with potential k. We define a family of hypersurfaces $M_c = \{p \in M \mid k(p) = c\}$ in M. Then the hypersurfaces M_c are invariant by the natural isometric $S^1 \subset \mathbb{C}^*$ action on the conic manifold M. We call

$$S := M_{\frac{1}{2}} \longrightarrow \bar{M}$$

the *canonical circle bundle* over the projective special Kähler manifold $\bar{M}$.

THEOREM 14. *Let $\bar{M}$ be a projective special Kähler manifold and $S \to \bar{M}$ its canonical circle bundle. Then S has a canonical structure of a proper affine hypersphere. Moreover, S carries an affine Sasakian structure which determines the projective special Kähler geometry on $\bar{M}$.*

Proof. It is enough to prove the theorem locally. Therefore we assume $M_{\frac{1}{2}} \subset U$, where U is a special Kähler domain with data (g, J, ∇). By Theorem 6, $S = M_{\frac{1}{2}} \subset U$ is a proper affine sphere, so that (U, g) is the metric cone over S. Since the flat coordinates on U are conic, i.e. $\mathbb{R}^{>0}$-equivariant, the flat connection ∇ on $U = C(S)$ coincides with the flat connection on $C(S)$ which is constructed in Proposition 10. Hence, (U, g, ∇) is the parabolic cone over S, and the parabolic cone is special Kähler. In particular, the sphere S is affine Sasakian, and, by Proposition 13, the Sasakian structure σ on S induced from J is affine Sasakian. □

4.2. Projective special Kähler domains with a definite metric. Let $\bar{U}$ be a projective special Kähler domain with a definite metric $\bar{g}$ and F the potential function of the corresponding special Kähler domain $U \subset \mathbb{C}^{n+1}\backslash\{0\}$ which carries the special Kähler metric g defined by formula (5). Note that by formula (4) the function $-F$ induces the same metric $\bar{g}$ on $\bar{U}$, however the signature of the metric g on U is inverted.

DEFINITION 15. A projective special Kähler domain $\bar{U}$ with a definite metric $\bar{g}$ is called of *elliptic type* if the metric g on U is definite.

We remark that if $\bar{U}$ is an elliptic projective special Kähler domain, then by formula (4) the metric $\bar{g}$ on $\bar{U}$ must be positive definite. Moreover the affine hypersphere $S \subset U$ which is associated to $\bar{U}$ by Theorem 6 has a definite metric, and S is an elliptic affine hypersphere. Conversely, if $\bar{U}$ is a projective special Kähler domain with a negative definite metric $\bar{g}$, then the associated affine hypersphere S has an affine metric with Lorentzian signature.

Characterization of complex projective space. In [L] it was proved that a special Kähler manifold M with a (positive) definite complete metric is flat. In fact, it may also be deduced from this result that any complete special Kähler domain $U \subset \mathbb{C}^n$ with a definite metric is just $\mathbb{C}^n$ with a Hermitian inner product. In the case of projective special Kähler domains there are many (homogeneous) examples with a definite and complete metric known, for instance, the examples given in the section 5. Among elliptic special Kähler domains though, the projective space $\mathbb{C}\mathrm{P}^n$ is characterized by its completeness property:

THEOREM 16. *Let $\bar{U} \subset \mathbb{C}\mathrm{P}^n$ be a projective special Kähler domain of elliptic type with a complete metric $\bar{g}$. Then $\bar{U} = \mathbb{C}\mathrm{P}^n$ and $\bar{g}$ is homothetic to the Fubini-Study metric on $\mathbb{C}\mathrm{P}^n$.*

Proof. We may choose F on $U \subset \mathbb{C}^{n+1}$ so that g is positive definite. Therefore the Kähler potential k on U is positive. By Theorem 6, the associated affine hypersphere S is of elliptic type with a positive definite metric and, since $S \longrightarrow \bar{U}$ is a Riemannian submersion with a complete base and compact fibre S^1, S has a complete metric as well. Hence S is an ellipsoid by Thm 11. Recall that, by Corollary 4, the special Kähler domain $U \subset \mathbb{C}^{n+1}$ over $\bar{U}$ is the parabolic cone over S and, by Corollary 12, $U = \mathbb{C}^{n+1}\backslash\{0\}$. Also by Corollary 12, the metric g on U has a quadratic potential with respect to the flat connection ∇ on U. Since, by Corollary 8, k is a ∇-potential for g, k must be a homogeneous quadratic function in the affine coordinates. Hence, it follows that the cone metric g is parallel with respect to ∇. Therefore $\nabla = D$, which is possible only if F is a quadratic function and g is just a Hermitian inner product on $\mathbb{C}^{n+1}$. In this case, $\bar{g}$ is homothetic to the Fubini-Study metric. □

4.3. Calabi-Yau moduli space. We recall that a *Calabi-Yau m-fold* (of general type) is an oriented compact Riemannian manifold (X, g) with holonomy group $\mathrm{Hol}(X, g) = \mathrm{SU}(m)$. This implies that X admits a unique complex structure J compatible with the orientation such that (X, J, g) is a Kähler manifold and a parallel J-holomorphic $(m, 0)$-form vol (a holomorphic volume form), which is unique up to constant scale. In particular, (X, J) is a complex manifold of (complex) dimension m with trivial canonical bundle $\wedge^{m,0}T^*X$. Let $\bar{M}$ be the Kuranishi moduli space of (X, J), i.e. the (local) moduli space of complex structures I on X. There is a natural holomorphic line bundle over $\bar{M}$ whose fibre at $I \in \bar{M}$ is $\Gamma_{hol}(\wedge_I^{m,0}T^*X) = H^{m,0}(X, I)$ (Γ_{hol} stands for *holomorphic* sections). Let $\pi : M \to \bar{M}$ be the corresponding holomorphic $\mathbb{C}^*$-bundle: $\pi^{-1}(I) = H^{m,0}(X, I) - \{0\}$. The one-dimensional complex vector spaces $H^{m,0}(X, I)$ have a natural norm: $\|\mathrm{vol}\|^2 := (\sqrt{-1})^{-m} \int_X \mathrm{vol} \wedge \overline{\mathrm{vol}}$. Let $S \subset M$ be the unit circle bundle with respect to that norm.

THEOREM 17. *Let $S \to \bar{M}$ be the above circle bundle over the Kuranishi moduli space of a Calabi-Yau threefold. Then S has naturally the structure of a Lorentzian affine Sasakian hypersphere. In particular, S is a proper affine hypersphere.*

Proof. It is known that $\bar{M}$ has the structure of a projective special Kähler manifold. We briefly recall the construction of that structure. (For more details, see [C1]). The cup product defines a complex symplectic form Ω on $V := H^3(X, \mathbb{C})$ and $\gamma = \sqrt{-1}\Omega(\cdot, \bar{\cdot})$ is a pseudo-Hermitian form of (complex) signature $(n+1, n+1)$, where $n = h^{1,2} = \dim \bar{M}$. The map

$$\bar{M} \ni I \mapsto H^{3,0}(X, I) \in P(V)$$

is a holomorphic immersion and is induced by a conic holomorphic immersion $\phi : M \to V - \{0\}$, with the following properties: $\phi^*\Omega = 0$ (ϕ is Lagrangian) and $g = \mathrm{Re}\, \phi^*\gamma$

is a Kähler metric of complex signature $(1, n)$ on the complex manifold M. These properties correspond to the first and second Hodge-Riemann bilinear relations for the underlying variation of Hodge structure of weight 3. As explained in section 1.2 the conic immersion ϕ induces on M the structure of a conic special Kähler manifold such that the corresponding projective special Kähler metric on $\bar{M}$ is negative definite (according to the conventions of this paper). Moreover, the circle bundle S defined above coincides with the canonical circle bundle $S = M_{\frac{1}{2}}$ of the projective special Kähler manifold $\bar{M}$ (notice that $(\sqrt{-1})^{-m} = \sqrt{-1}$ for $m = 3$ and hence $\|u\|^2 = \gamma(u, u)$ for $u \in H^{3,0}(X, I)$). Now we can apply Theorem 14. □

5. Homogeneous examples. The basic example of an affine Sasakian hypersphere S is provided by the total space of the Hopf fibration

$$S = S^{2n+1} = \mathrm{SU}(n+1)/\mathrm{SU}(n) \longrightarrow \mathbb{C}\mathrm{P}^n = \mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(n)\mathrm{U}(1)) .$$

In the Lagrangian picture the corresponding conic affine special Kähler manifold (M, J, g, ∇) is given as a linear Lagrangian subspace $M \subset V = T^*\mathbb{C}^{n+1}$ for which the restriction of the Hermitian metric γ is positive definite. Since M is a linear subspace the flat connection ∇ coincides with the Levi-Civita connection D of $g = \mathrm{Re}\,\gamma$. The group $\mathrm{SU}(n+1)$ acts transitively on $\bar{M} = \mathbb{C}\mathrm{P}^n$ by holomorphic isometries of the special Kähler metric (Fubini-Study metric). The action is induced from the canonical linear symplectic action of $\mathrm{SU}(n+1)$ on $V = T^*\mathbb{C}^{n+1}$ which preserves the Hermitian metric γ and the Lagrangian subspace $M \subset V$. This action preserves also the affine Sasakian hypersphere $S^{2n+1} \subset M$ and induces a transitive action on S^{2n+1} preserving the affine geometric and Sasakian structures.

More generally, one can consider Lagrangian subspaces $M \subset V = T^*\mathbb{C}^{n+1}$ of arbitrary Hermitian signature (p, q), $p + q = n + 1$. They correspond to fibrations

$$S = \mathrm{SU}(p, q)/\mathrm{SU}(p, q-1) \longrightarrow \mathrm{SU}(p, q)/\mathrm{S}(\mathrm{U}(p, q-1)\mathrm{U}(1)) = \bar{M} .$$

The case $q = 1$ is of particular interest. In that case the projective special Kähler metric is negative definite (as for the Calabi-Yau moduli space and as for the target manifolds of N=2 D=4 supergravity theories with vector multiplets) and hence the metric of the affine Sasakian hypersphere has Lorentzian signature: $\bar{M} = \mathbb{C}\mathrm{H}^n$ is the complex hyperbolic space and S is the real hyperbolic (2n+1)-space of Lorentzian signature (anti de Sitter space).

The Classification. A projective special Kähler manifold $\bar{M} = P(M)$ will be called *homogeneous* if it admits a transitive group of isometries G whose action is induced by a G-action on the conic manifold M preserving the data (g, J, ∇). Homogeneous projective special Kähler manifolds

$$\bar{M} = P(M) = G/K$$

with K compact have been classified in [AC] under the assumption that G is a real semisimple Lie group. We recall the result only in the most interesting case of negative definite metric on $\bar{M}$. It turns out that in this case the manifolds $\bar{M} = G/K$ are Hermitian symmetric spaces of non-compact type and are in one-to-one correspondence with the complex simple Lie algebras $\mathfrak{l}$ different from $\mathfrak{c}_n = \mathfrak{sp}(\mathbb{C}^{2n})$. In all the cases the underlying conic affine special Kähler manifold is a Lagrangian cone $M \subset V$ generated by the G-orbit of a highest weight vector of a $G^{\mathbb{C}}$-module V of symplectic type. The $G^{\mathbb{C}}$-module V admits a G-invariant real structure τ compatible with the symplectic structure Ω, which defines a Hermitian metric $\gamma = \sqrt{-1}\Omega(\cdot, \tau\cdot)$. The affine special Kähler metric is the restriction of $g = \mathrm{Re}\,\gamma$ to M. The list is the following:

A) $\mathfrak{l} = \mathfrak{sl}_{n+3}(\mathbb{C})$, $\bar{M} = \mathbb{C}H^n = \mathrm{SU}(n,1)/\mathrm{S}(\mathrm{U}(n)\mathrm{U}(1))$, $V = \mathbb{C}^{n+1} \oplus (\mathbb{C}^{n+1})^*$

BD) $\mathfrak{l} = \mathfrak{so}_{n+5}(\mathbb{C})$, $\bar{M} = (\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)) \times (\mathrm{SO}(n-1,2)/\mathrm{SO}(n-1)\mathrm{SO}(2))$, $V = \mathbb{C}^2 \otimes \mathbb{C}^{n+1}$

E6) $\mathfrak{l} = \mathfrak{e}_6(\mathbb{C})$, $\bar{M} = \mathrm{SU}(3,3)/\mathrm{S}(\mathrm{U}(3)\mathrm{U}(3))$, $V = \bigwedge^3 \mathbb{C}^6$

E7) $\mathfrak{l} = \mathfrak{e}_7(\mathbb{C})$, $\bar{M} = \mathrm{SO}^*(12)/\mathrm{U}(6)$, $V^{(32)} = V(\pi_6)$ (semispinor)

E8) $\mathfrak{l} = \mathfrak{e}_8(\mathbb{C})$, $\bar{M} = \mathrm{E}_7^{(-25)}/\mathrm{E}_6\mathrm{SO}(2)$, $V^{(56)} = V(\pi_1)$

F) $\mathfrak{l} = \mathfrak{f}_4(\mathbb{C})$, $\bar{M} = \mathrm{Sp}(\mathbb{R}^6)/\mathrm{U}(3)$, $V^{(14)}(\pi_3) = \bigwedge_0^3 \mathbb{C}^6$

G) $\mathfrak{l} = \mathfrak{g}_2(\mathbb{C})$, $\bar{M} = \mathbb{C}H^1 = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$, $V = \bigvee^3 \mathbb{C}^2$.

Here $V(\lambda)$ denotes the irreducible module $G^{\mathbb{C}}$-module with highest weight $\lambda = \sum \lambda_i \pi_i$, where π_i are the fundamental weights. The notation $V^{(d)}$ indicates that the module has complex dimension d. Notice that in the cases A) and BD) $n = \dim_{\mathbb{C}} \bar{M}$. The only redundancy in this list occurs the case $n = 1$. In fact, the Dynkin diagrams $A_3 = B_3$ define the same projective special Kähler manifold $\mathbb{C}H^1 = \mathrm{SU}(1,1)/\mathrm{S}(\mathrm{U}(1)\mathrm{U}(1)) = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. In both cases the corresponding conic manifold M is a linear Lagrangian subspace in the vector space V.

Note that it may happen that projective special Kähler manifolds are isometric as Riemannian manifolds, but nevertheless their special geometry is different: The diagram G_2 defines $\bar{M} = \mathbb{C}H^1$ but in this case the underlying conic affine special Kähler manifolds $M \subset V$ is not a linear subspace, as for type A), n=1. In fact, $V = \bigvee^3 \mathbb{C}^2$ is the symmetric cube of the defining representation $\mathbb{C}^2$ of $G^{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C})$. The Zariski closure of $M \subset V$ is the nonlinear cone $M' = \{u^3 | u \in \mathbb{C}^2\} \subset V$ and $M \subset M'$ is open.

Homogeneous affine Sasakian spheres. An affine hypersphere $(S, g, \hat{\nabla})$ is called *homogeneous* if $\mathrm{Aut}(S) = \mathrm{Aut}(S, g, \hat{\nabla})$ acts transitively on S. Note that in general $\mathrm{Aut}(S)$ is a proper subgroup of $\mathrm{Isom}(S) = \mathrm{Aut}(S, g)$. If S has an affine Sasakian structure σ then let $\mathrm{Aut}_\sigma(S)$ be the subgroup of those automorphisms in $\mathrm{Aut}(S)$ which commute with the flow of the vector field σ. We call $\mathrm{Aut}_\sigma(S)$ the group of automorphisms of the affine Sasakian sphere S. Clearly, any affine Sasakian hypersphere S with a transitive action of $\mathrm{Aut}_\sigma(S)$ is a circle bundle over a homogeneous projective special Kähler manifold. If $\bar{M}$ is homogeneous then $\mathrm{Isom}(S)$ acts transitively on S. But note that, in general, the canonical isometric S^1-action on S does not preserve the connection $\hat{\nabla}$. The following theorem is a consequence of the above classification.

THEOREM 18. *Let S be the affine Sasakian hypersphere over a homogeneous projective special Kähler manifold $\bar{M} = G/K$ of a real semisimple Lie group G. If the special Kähler metric of $\bar{M}$ is negative definite then $\bar{M}$ belongs to the above list A)-G) and G acts transitively by automorphisms of the Lorentzian affine Sasakian hypersphere S.*

Proof. By construction, the G-action on $\bar{M}$ is induced by a G-action on the symplectic vector space V which preserves the geometric data on V. Hence G acts also on the canonical circle bundle S over M preserving the affine Sasakian geometry on S. Note now that in all the cases the centre $Z(K) \cong \mathrm{U}(1)$ of K acts non-trivially, and hence transitively, on the fibre of $S \to \bar{M}$ over the canonical base point $o = eK$

in $\bar{M} = G/K$. This follows, for example, from the fact that K is the stabilizer of the line $l = \mathbb{C}v \subset V$ generated by a highest weight vector $v \in V^\tau$ of the $G^{\mathbb{C}}$-module V. In fact, K contains a (compact) Cartan subgroup of G, which cannot act trivially on l. (Notice, that the semisimple part of K, however, acts trivially on l.) □

Clearly, the $\mathrm{Aut}(S)$ action on the affine sphere S extends to a linear (with respect to the flat connection ∇) action on the parabolic cone which contains S. Hence, if $G \subset \mathrm{Aut}(S)$ acts transitively, the affine sphere S arises as a generic G-orbit in a real vector space W. If G is semisimple then S must be the level-set of a homogeneous G-invariant polynomial on W.

THEOREM 19. *Let S be an affine Sasakian hypersphere with Lorentzian metric. If $\mathrm{Aut}_\sigma(S)$ contains a semisimple transitive group G then the affine sphere $(S, g, \hat{\nabla})$ arises as a hypersurface which is defined by a G-invariant homogeneous quartic polynomial on a real vector space W.*

Proof. S identifies with the canonical circle bundle in the parabolic cone $M = C(S)$ which is special Kähler. The action of $\mathrm{Aut}_\sigma(S)$ on S extends to an action on M which preserves the special Kähler data on M. Using a compatible Lagrangian immersion we may therefore as well assume that the action of $G = \mathrm{Aut}_\sigma(S)$ on S is induced by an action of G on a Hermitian symplectic vector space (V, γ, Ω). In fact, we identify S as an affine sphere in the real vector space $W = V^\tau$, and S is a level set of the Kähler potential k, which is, as a function on V^τ, homogeneous of degree 2 and invariant by G. We claim that k^2 is a quartic polynomial. Since G acts with cohomogeneity one, it is sufficient to show that V^τ admits a homogeneous G-invariant quartic polynomial, which is then necessarily proportional to k^2. To show this it is clearly enough to construct a (complex) homogeneous $G^{\mathbb{C}}$-invariant quartic polynomial on V. The existence of such a polynomial on V follows by the following general argument.

As we know, the $G^{\mathbb{C}}$-module V is associated to a Dynkin diagramm Δ of the type A, B, D, E, F or G. We give some more detail how this correspondence works. (See [AC] for a complete account.) Let $N = N(\Delta) = L/L_o$ be the compact symmetric quaternionic Kähler manifold which is associated to the Dynkin diagramm Δ. (See [Wo].) L is the compact simple Lie group with trivial centre associated to Δ and $L_o = \mathrm{Sp}(1)H$ is the stabilizer of a point $o \in N$. The complexified isotropy representation is a product $T_oN \otimes \mathbb{C} = \mathbb{C}^2 \otimes_{\mathbb{C}} V$. The group $\mathrm{Sp}(1)$ acts by the standard representation on $\mathbb{C}^2$, and V is a complex module for H which admits a skew symmetric bilinear invariant. It follows that the maximal semisimple subgroup $H' \subset H$ is a compact form of a complex semisimple group $G^{\mathbb{C}}$ which acts on V. In this way, we have associated a $G^{\mathbb{C}}$-module V to the Dynkin diagramm Δ. Now the quaternionic Weyl tensor, see [Sa], of the quaternionic Kähler manifold $N = N(\Delta)$ at the point $o \in N$ gives rise to a nonzero $G^{\mathbb{C}}$-invariant element $Q \in S^4V^*$. This shows the existence of a nontrivial $G^{\mathbb{C}}$-invariant homogeneous quartic polynomial Q on V. □

In examples it is not difficult to guess the quartic invariant Q directly from the G-module V^τ. This gives an explicit description of the corresponding affine hyperspheres.

EXAMPLES.

A) $G = \mathrm{SU}(n,1)$, $V^\tau = \mathbb{C}^{n,1}$, $Q(v) = g(v,v)^2$, where g is the $\mathrm{SU}(n,1)$-invariant Hermitian product.

BD) $G = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(n-1,2)$, $V^\tau = \mathbb{R}^2 \otimes \mathbb{R}^{n-1,2} \cong \mathrm{Hom}((\mathbb{R}^2)^*, \mathbb{R}^{n-1,2})$. Let ω be a $\mathrm{SL}(2,\mathbb{R})$-invariant symplectic form on $\mathbb{R}^2$ and g the $\mathrm{SO}(n-1,2)$-invariant scalar product, defining identifications $\Phi_\omega : \mathbb{R}^2 \cong (\mathbb{R}^2)^*$, $\Phi_g : \mathbb{R}^{n-1,2} \cong (\mathbb{R}^{n-1,2})^*$. For $A \in \mathrm{Hom}((\mathbb{R}^2)^*, \mathbb{R}^{n-1,2})$ let $A^* \in \mathrm{Hom}((\mathbb{R}^{n-1,2})^*, \mathbb{R}^2)$ be the dual morphism. Then $Q(A) = \det(A^*\Phi_g A \Phi_\omega)$.

E6) $G^{\mathbb{C}} = \mathrm{SL}(6,\mathbb{C})$, $V = (\wedge^3\mathbb{C}^6)^*$. To any 3-form α we can associate the operator

$$A_\alpha : \mathbb{C}^6 \longrightarrow \left(\bigwedge^5 \mathbb{C}^6\right)^* = \mathbb{C}^6, \quad v \mapsto \alpha \wedge \iota_v \alpha .$$

Then $Q(\alpha) = \mathrm{trace}(A_\alpha^2)$. It is easy to check that $Q \neq 0$ by evaluating Q on $dz^1 \wedge dz^2 \wedge dz^3 + dz^4 \wedge dz^5 \wedge dz^6$. This example is discussed in detail in [H] and the corresponding real symplectic $\mathrm{SL}(6,\mathbb{R})$-module is also considered. Here we are interested in the real structure τ invariant under the real form $G = \mathrm{SU}(3,3)$ of $\mathrm{SL}(6,\mathbb{C})$. It is induced by the $\mathrm{SU}(3,3)$-invariant pseudo-Hermitian form on $\mathbb{C}^6 = \mathbb{C}^{3,3}$. In fact, this form induces a G-invariant pseudo-Hermitian form γ on $V = (\wedge^3\mathbb{C}^6)^*$. This determines a G-invariant real structure τ on V such that $-i\gamma(\cdot, \tau\cdot) = \Omega$ is the $G^{\mathbb{C}}$-invariant symplectic form of V: $\Omega(\alpha,\beta) dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4 \wedge dz^5 \wedge dz^6 = \alpha \wedge \beta$.

F) $G = \mathrm{Sp}(\mathbb{R}^6)$, $V^\tau = \bigwedge_0^3 \mathbb{R}^6$ is the kernel of the map $\bigwedge^3 \mathbb{R}^6 \ni \alpha \mapsto \omega \wedge \alpha \in \bigwedge^5 \mathbb{R}^6$, where ω is the symplectic form on $\mathbb{R}^6$. The G-invariant quartic polynomial Q is just the restriction of the $\mathrm{SL}(6,\mathbb{C})$-invariant quartic polynomial on $\bigwedge^3 \mathbb{C}^6$, see previous example, to the subspace $\bigwedge_0^3 \mathbb{R}^6$.

G) $G = \mathrm{SL}(2,\mathbb{R})$, $V^\tau = \bigvee^3 \mathbb{R}^2$. The elements of V^τ can be considered as homogeneous cubic polynomials p on $\mathbb{R}^2$. Let $q(p) = \det(\partial^2 p) \in \bigvee^2 \mathbb{R}^2$ be the determinant of the Hessian of p. Then $Q = D(q(p))$ is the discriminant of $q(p)$.

REMARKS:

1) In all the above examples (A-E) the group $R^* \cdot G$ acts with an open orbit on V^τ, in other words V^τ with the action of $R^* \cdot G$ is a real *prehomogeneous vector space.* Complex irreducible prehomogeneous vector spaces were classified in [SK]. Another description of the quartic invariant for the complex prehomogeneous vector spaces associated to some of the complex simple Lie algebras was recently given in [Cl].
2) The Sasaki field σ of the affine hypersphere $S \subset V^\tau$ can be easily computed from the real quartic invariant Q. From Lemma 2 ii) it follows that $\sigma = J\xi$ is precisely the Hamilton vector field X_k associated to the Kähler potential k. We can normalize the G-invariant real symplectic structure on V^τ (or the invariant Q) such that Q is related to the Kähler potential k by the formula $Q = k^2$. Then we have $X_Q = 2kX_k$ and therefore, since $k = 1/2$ on S, we have $\sigma = X_Q$ on S.

Compact quotients. Let G be one of the real semi-simple Lie groups from the list A)-G). By Theorem 18, G acts transitively and properly on a Lorentzian affine hypersphere $S = G/\tilde{K}$ which fibers over a Hermitian symmetric space G/K of non-compact type, $K = \tilde{K}Z(K)$. By a result of Borel [Bo], G admits cocompact lattices $\Gamma \leq G$. This allows to construct compact Clifford-Klein forms

$$S_\Gamma = \Gamma \backslash G/\tilde{K}$$

for the Lorentzian homogeneous spaces $S = G/\tilde{K}$. The spaces S_Γ admit an isometric S^1-action (induced from the affine Sasakian structure) with finite stabilizers, the orbit space being a Hermitian locally symmetric space

$$\bar{M}_\Gamma = {}_\Gamma\backslash G/K .$$

In his influential paper [Kul], Kulkarni observed the existence of non-trivial circle bundles over compact locally complex hyperbolic spaces, carrying a Lorentzian metric of constant curvature 1. This corresponds to the complex hyperbolic case $\bar{M} = \mathbb{C}\mathrm{H}^n = \mathrm{SU}(n,1)/S(\mathrm{U}(n)\mathrm{U}(1))$, i.e. case A) in our list. In this sense, our construction generalizes Kulkarni's construction of *compact* Lorentzian space-forms. It seems worthwile to further study the particular Lorentzian geometry of the homogeneous spaces S occuring in examples B) to G), and their compact Clifford-Klein forms. However, in this paper we content ourselves with summarizing what was just explained:

COROLLARY 20. *Let $\bar{M}$ be one of the Hermitian symmetric spaces appearing in the list A)-G), and $\bar{M}_\Gamma$ a compact Clifford-Klein form for $\bar{M}$. Then $\bar{M}_\Gamma$ is the orbit space of an isometric S^1-action on a compact Clifford-Klein S_Γ for the Lorentzian homogeneous space S associated to $\bar{M}$.*

Acknowledgements. We thank Philipp Lohrmann and the anonymous referee for helpful remarks.

REFERENCES

[AC] D. V. ALEKSEEVSKY, V. CORTÉS, *Classification of stationary compact homogeneous special pseudo- Kähler manifolds of semisimple groups*, Proc. London Math. Soc. (3) 81:1 (2000), pp. 211–230.

[ACD] D. V. ALEKSEEVSKY, V. CORTÉS, C. DEVCHAND, *Special complex manifolds*, J. Geom. Phys. 42 (2002), pp. 85–105.

[BC] O. BAUES, V. CORTÉS, *Realisation of special Kähler manifolds as parabolic spheres*, Proc. Am. Math. Soc. 129:8 (2001), pp. 2403–2407.

[BG] C. BOYER, K. GALICKI, *3-Sasakian manifolds*, Surveys in differential geometry: essays on Einstein manifolds, pp. 123–184, Surv. Differ. Geom. VI, Int. Press, Boston, MA, 1999.

[Bo] A. BOREL, *Compact Clifford-Klein forms of symmetric spaces*, Topology, 2 (1963), pp. 111–122.

[Ca1] E. CALABI, *Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens*, Michigan Math. J. 5 (1958), pp. 105–126.

[Ca2] E. CALABI, *Complete affine hyperspheres I*, Symposia Mathematica, Vol. X (Convegno di Geometria Differenziale, INDAM, Rome, 1971), pp. 19–38, Academic Press, London, 1972.

[Cl] J.-L. CLERC, *Special prehomogeneous vector spaces associated to F_4, E_6, E_7, E_8 and simple Jordan algebras of rank 3*, preprint June 28, 2002.

[C1] V. CORTÉS, *On hyper-Kähler manifolds associated to Lagrangian Kähler submanifolds of T^*C^n*, Trans. Amer. Math. Soc. 350:8 (1998), pp. 3193–3205.

[C2] V. CORTÉS, *A holomorphic representation formula for parabolic hyperspheres*, Proceedings of the international conference "PDEs, Submanifolds and Affine Differential Geometry" (Warsaw 2000), eds. B. Opozda, U. Simon and M. Wiehe, Banach Center Publications (Polish Academy of Sciences, Institute of Mathematics) Vol. 57, pp. 11–16, available as math.DG/0107037.

[F] D. S. FREED, *Special Kähler manifolds*, Comm. Math. Phys., 203:1 (1999), pp. 31–52.

[H] N. HITCHIN, *The geometry of three-forms in six dimensions*, J. Differential Geom. 55:3 (2000), pp. 547–576.

[Kul] R. S. KULKARNI, *Proper actions and pseudo-Riemannian space forms*, Adv. in Math. 40:1 (1981), pp. 10–51.

[Kur] M. KURANISHI, *On the locally complete families of complex analytic structures*, Ann. of Math. (2) 75 (1962), pp. 536–577.

[L] Lu, Zhiqin, *A note on special Kähler manifolds*, Math. Ann. 313:4 (1999), pp. 711–713.

[NS] K. Nomizu, T. Sasaki, *Affine differential geometry. Geometry of affine immersions*, Cambridge Tracts in Mathematics, 111, Cambridge University Press, Cambridge, 1994.

[Po] A. V. Pogorelov, *On the improper convex affine hyperspheres*, Geometriae Dedicata, 1:1 (1972), pp. 33–46.

[Sa] S. Salamon, *Quaternionic Kähler manifolds*, Invent. Math. 67:1 (1982), pp. 143–171.

[SK] M. Sato, T. Kimura, *A classification of irreducible prehomogeneous vector spaces and their relative invariants*, Nagoya Math. J. 65 (1977), pp. 1–155.

[Wo] J. A. Wolf, *Complex homogeneous contact manifolds and quaternionic symmetric spaces*, J. Math. Mech. 14 (1965), pp. 1033–1047.