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INDEX OF DIRAC OPERATOR AND SCALAR CURVATURE ALMOST NON-NEGATIVE MANIFOLDS*

FUQUAN FANG†

Abstract. A manifold M is called *scalar curvature almost non-negative* if for any constant $\varepsilon > 0$, there is a Riemannian metric g on M such that $s_g \cdot \mathrm{diam}(M,g)^2 \geq -\varepsilon$ and the sectional curvature $\mathrm{Sec}_g \leq 1$, where s_g (resp. $\mathrm{diam}(M,g)$) is the scalar curvature (resp. diameter) of (M,g).

Among others we prove that for a scalar curvature almost non-negative manifold M with $\hat{A}(M) \neq 0$ (resp. $\mathcal{A}(M) \in \mathbb{Z}_2$ nonzero and $\chi(M) \neq 0$ if $n = 2(\mathrm{mod}\ 8)$), there is a constant $\varepsilon(n) > 0$ such that, if the scalar curvature $s_M \geq -\varepsilon(n)$, then
(i) the fundamental group $\pi_1(M)$ is finite;
(ii) M admits a real analytic Ricci flat metric g_0 such that its Riemannian universal covering $\tilde{M}$ is isometric to the product of Ricci flat Kähler-Einstein manifolds and/or Joyce manifolds of dimension 8 with special holonomy group $Spin(7)$.

0. Introduction. Let M be a closed Spin manifold. It is well-known that M admits a metric with positive scalar curvature only if the index of the Dirac operator vanishes, by the classical Lichnerowicz formula and the Atiyah-Singer index theorem (cf. [LM]). The Gromov-Lawson conjecture, confirmed by Stolz [St], asserts the converse for simply connected Spin manifolds of dimension at least 5. There are many manifolds with non-negative scalar curvature but do not accept metrics of positive scalar curvature, e.g. torus, K_3-surfaces, etc.

A basic theorem of Bourguignon shows that such a Spin manifold must be Ricci flat. Starting from [St], Futaki [Fu] characterized all simply connected Spin manifolds of dimension at least 5 with non-negative scalar curvature. It turns out that such a manifold either admits a metric with positive scalar curvature, or it is the product of Ricci flat Kähler-Einstein manifolds and/or Joyce manifold of dimension 8 with Spin(7)-holonomy (manifolds in the latter class are called *rigidly scalar flat*). A key-point involved is that, every harmonic spinor must be parallel and therefore, the holonomy group must be special (cf. [Fu] [Wa]).

Recall that a manifold is called *almost flat* (resp. *almost Ricci flat*) if for any positive constant $\varepsilon > 0$, there is a Riemannian metric g on M such that

$$|\mathrm{Sec}_g \cdot \mathrm{diam}(M,g)^2| \leq \varepsilon \text{ (resp. } |\mathrm{Ric}_g \cdot \mathrm{diam}\ (M,g)^2| \leq \varepsilon),$$

where Sec_g (resp. Ric_g) is the sectional (resp. Ricci) curvature.

The celebrated Gromov theorem asserts that an almost flat manifold must be an infra-nilmanifold, i.e. a finite regular cover must be a nilmanifold. By the Bochner technique it is also well-known that an almost Ricci flat manifold has the first Betti number at most n, where n is the dimension. In contrast, however, very recently Lohkamp [Lo] proved that for every compact manifold M of dimension at least 3 and for any given positive constant ε, there is a metric g so that its scalar curvature satisfies that $|s_g \cdot \mathrm{diam}(M,g)^2| \leq \varepsilon$.

We call a manifold M is of *scalar curvature almost non-negative* if for any constant $\varepsilon > 0$, there is a Riemannian metric g on M such that $s_g \cdot \mathrm{diam}(M,g)^2 \geq -\varepsilon$ and the sectional curvature $\mathrm{Sec}_g \leq 1$, where s_g (resp. $\mathrm{diam}(M,g)$) is the scalar curvature

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† Nankai Institute of Mathematics, Nankai University, Tianjin 300071, P.R.C. (ffang@nankai.edu.cn).
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(resp. diameter) of (M, g). Note that almost flat manifolds must be of scalar curvature non-negative.

The main purpose of the present paper is to characterize manifolds with almost non-negative scalar curvature. We will prove that there are indeed nontrivial topological obstructions for manifolds being of scalar curvature almost non-negative. For example, following from our theorem the connected sum, $K_3 \# S^1 \times S^3$, of a K_3-surface with $S^1 \times S^3$, does not admits metrics of almost non-negative scalar curvature.

Let M be Spin manifold. Let S^+(resp. S^-) be the Spinor bundles on M. Let $\mathcal{D}^+ : \Gamma(S^+) \to \Gamma(S^-)$ denote the Dirac operator. The Atiyah-Singer index theorem asserts that ind $(\mathcal{D}^+) = \hat{A}(M)$ is a Spin cobordism invaraint and defines a ring homomorphism

$$\mathcal{A} : \Omega_*^{Spin} \to KO^{-*}(pt),$$

which is non-zero only if the dimension is 0(mod 4) or 1, 2(mod 8). In particular, if $n = 0(\text{mod } 4)$, $\mathcal{A}(M) = \hat{A}(M)$ is the $\hat{A}$-genus of M.

Let $\mathcal{M}(n, d)$ denote the set of all closed Riemannian n-manifolds such that the sectional $K_M \leq 1$, and diam$M \leq d$.

A manifold $M \in \mathcal{M}(n, d)$ is called *non-collapsing* if there is no a sequence of metrics of bounded curvature collapsing to a lower dimensional space in the Gromov-Hausdorff topology.

THEOREM A. *Let $M \in \mathcal{M}(n, d)$ be a closed non-collapsing Spin manifold such that $\mathcal{A}(M) \neq 0$. There is a constant $\varepsilon(n) > 0$ such that if the scalar curvature $s_M \geq -\varepsilon(n)$, then M admits a Ricci flat real analytic Riemannian metric g_0 with restricted holonomy group $Hol^0(M, g_0)$ a product whose irreducible components are $SU(m)$, $Sp(m)$, G_2 or $Spin(7)$.*

By the local formula for $\hat{A}$-genus (Euler characteristic), $M \in \mathcal{M}(n, d)$ is non-collapsing if $\hat{A}(M) \neq 0$ (resp. the Euler characteristic $\chi(M) \neq 0$). Therefore Theorem A together with the Cheeger-Gromoll splitting theorem [CGr] implies immediately

COROLLARY 0.1. *Let $M \in \mathcal{M}(n, d)$ be a closed Spin manifold. Suppose that $\hat{A}(M) \neq 0$ (resp. $\mathcal{A}(M) \neq 0$ and $\chi(M) \neq 0$ if $n = 2(\text{mod } 8)$). There is a constant $\varepsilon(n) > 0$ such that, if the scalar curvature $s_M \geq -\varepsilon(n)$, then*
(i) the fundamental group $\pi_1(M)$ is finite;
(ii) M admits a real analytic Ricci flat metric g_0 such that its Riemannian universal covering $\tilde{M}$ is isometric to the product of Ricci flat Kähler-Einstein manifolds and/or Joyce manifolds of dimension 8 with special holonomy group $Spin(7)$.

The above theorem together with the Stolz theorem [St] clearly implies

COROLLARY 0.2. *Let $M \in \mathcal{M}(n, d)$ be a closed non-collapsing Spin manifold. If M is simply connected and $n \geq 5$. There is a constant $\varepsilon(n) > 0$ such that, if the scalar curvature $s_M \geq -\varepsilon$, then either M admits a metric of positive scalar curvature or admits a Ricci flat metric.*

The proof of the Theorem A together with Corollaries 3.1 and 3.2 in [Wa] imply the following two corollaries

COROLLARY 0.3. *Let $M \in \mathcal{M}(4, D)$ be a closed Spin manifold so that $\hat{A}(M) \neq 0$. There is a constant $\varepsilon > 0$ such that, if the scalar curvature $s_M \geq -\varepsilon$, then M is bi-*

Lipschitz isometric to a Calabi-Yau K_3-surface.

COROLLARY 0.4. *Let $M \in \mathcal{M}(8, D)$ be a closed Spin manifold. Assume that $\hat{A}(M) \neq 0$. There is a constant $\varepsilon > 0$ such that, if the scalar curvature $s_M \geq -\varepsilon$, then M is simply connected.*

Observe that the $\hat{A}$-genus of a K_3-surface $\hat{A}(K_3) \neq 0$. Corollary 0.1 implies easily that the connected sum $K_3 \# S^1 \times S^3$ does not admit an almost non-negative scalar curvature metric. This also answers in negative the following question:

PROBLEM 0.5. *Let M (resp. N) be a manifold with positive (resp. non-negative) scalar curvature. Does the connected sum $M \# N$ admit a metric with almost non-negative scalar curvature?*

This is maybe interesting by comparing with the amazing discovery of Schoen-Yau [SY], that the connected sum of any two n-manifolds ($n \geq 3$) with positive scalar curvature admits a metric with positive scalar curvature.

Some remarks on the above results are in order.

REMARK 0.6. The Seiberg-Witten theory [Wi] implies that 4-manifolds with non-trivial Seiberg-Witten invariant do not accept any metric with positive scalar curvature. Furthermore, Witten ([Wi]) proved that a scalar curvature non-negative 4-manifold with non-trivial Seiberg-Witten invariant is hyper-Kähler, if $b_2^+ \geq 2$. Seiberg-Witten theory may be used to study scalar curvature almost non-negative 4-manifolds in the spirit of the presented paper.

REMARK 0.7. Theorem A should be compared with a recent result in [Lot] where sectional curvature is assumed being almost non-negative.

REMARK 0.8. Theorem A holds identically, if the upper bound for the sectional curvature is replaced by a upper bound for the Ricci curvature together with a positive lower bound for the conjugate radii.

Now let us start to sketch the idea in the proof of Theorem A.

Suppose Theorem A is false, then there is a sequence of n-dimensional Spin manifolds (M_i, g_i) so that $\mathcal{A}(M_i) \neq 0$, the scalar curvatures $s_{g_i} \geq -1/i$, the diameter diam $M_i \leq D$ and $\mathrm{Sec}_{g_i} \leq 1$ for some positive constant D. By the Gromov precompactness theorem and the Cheeger-Gromov theorem, passing to a subsequence if necessary, we may assume that (M_i, g_i) converges in $L^{2,p}$-class to a manifold (X, g_∞) (since M_i are non-collapsing.) Let $\phi_i \in \Gamma(S_i^+)$ be a harmonic spinor with unit L^2-norm. Using the Lichnerowicz formula and the scalar curvature bound we prove that ϕ_i converges to a parallel harmonic spinor ϕ_∞ with nontrivial norm. Therefore ϕ_i is almost parallel for i large, which implies that the Ricci curvature of (M_i, g_i) converges in L^p-class to zero. By regularity of Einstein equation we know that there is a real analytic Ricci flat metric on X. A contradiction.

1. Preliminaries. In this section we give some necessary preliminary results needed in next sections.

a). Harmonic coordiantes

A local coordinate $(h^1, ..., h^n)$ is *harmonic* if each component is a harmonic function, i.e., $\triangle h^i = 0$ for $i = 1, ..., n$, where $\triangle$ is the Laplacian opertator. In a harmonic coordinate, the Ricci curvature of the metric tensor g satisfies the equation

$$(\mathrm{Ric}_g)_{ij} = -\frac{1}{2} \triangle g_{ij} + Q(g, \partial g).$$

here $g_{ij} = g(\frac{\partial}{\partial h^i}, \frac{\partial}{\partial h^j})$ and $Q(.,.)$ is a quadratic form of its variables (c.f. [Pe]).

The existence of harmonic coordinate ball of uniform radius is studied in [JK] [An], in terms of various curvature bound. We quote the following

THEOREM 1.1 [AN]. *Let (M, g) be a Riemannian n-manifold (not necessarily complete) such that*

$$|Ric_M| \leq \lambda, \ inj\, B(x, \frac{1}{2} \cdot dist(x, \partial M)) \geq i_0(x) > 0 \tag{1.4.1}$$

Then, given any $C > 1$ and $\alpha \in (0,1)$, there is an $\varepsilon_0 = \varepsilon_0(\lambda, C, n, \alpha)$ with the following property: given any $x \in M$, there is a harmonic coordinate system $U = \{u_i\}_1^m$ defined on $B(x, \varepsilon(x)) \subset M$ such that if $g_{ij} = g(\nabla u_i, \nabla u_j)$, then $g_{ij}(x) = \delta_{ij}$ and

$$C^{-1} \cdot I \leq g(y) \leq C \cdot I, \ (\text{as bilinear forms}) \tag{1.4.2}$$

$$\varepsilon(x)^{1+\alpha} \|g_{ij}(y)\|_{C^{1,\alpha}} \leq C \tag{1.4.3}$$

for all $y \in B(x, \varepsilon(x))$ (resp. $\frac{\varepsilon(x)^{2-\frac{n}{p}}}{\sqrt{\mathrm{vol}\ B(x,\varepsilon(x))}} \|g_{ij}\|_{L^{2,p}(B(x,\varepsilon(x)))} \leq C$), where

$$\frac{\varepsilon(x)}{i_0(x)} \geq \varepsilon_0 \cdot \frac{dist(x, \partial M)}{diamM} > 0 \tag{1.4.4}$$

where $B(x, r)$ is the metric ball with radius r, I the identity matrix and inj is the injectivity radius.

b). Compactness theorems

THEOREM 1.2 [CG]. *Let (M_i, g_i) be a sequence of compact Riemannian manifolds whose sectional curvature, diameter, and injectivity radius satisfy*

$$\lambda \leq Sec._{g_i} \leq \Lambda, \ \ diam \leq d, \ \ i_M > i_0,$$

where the constants are independent of i. Then, replacing M_i by a subsequence if necessary, M_i converges to a metric space X, such that
(i) X is a differentiable manifold;
(ii) there is a diffeomorphsim $f_i : X \to M_i$ for all sufficiently large i;
(iii) the pullback metrics $f_i^*(g_i)$ converges in $C^{1,\alpha}$-class to a $C^{1,\alpha}$ (resp. $L^{2,p}$) Riemannian metric g_∞ in X, for any prescribed real number $\alpha \in (0,1)$ (resp. positive integer $p > 1$).

For the sake of simplicity, in the rest of the paper we fix the real number $\alpha \in (0,1)$ (resp. the integer $p > n$).

For a sequence (M_i, g_i) as above, the limit metric g_∞ is not necessarily C^2. However, if the L^p-Ricci curvature (for p large) of g_∞ vanishes identically, then by the elliptic regularity of the Einstein equation one may conclude the smoothness of g_∞ (indeed real analytic). That is

THEOREM 1.3 [AN]. *Let (M_i, g_i) be a sequence as in Theorem 1.2. If the L^p-Ricci curvature (for some large p) of g_∞ vanishes identically. Then (M_i, g_i) has a subsequence converging to a C^∞ Riemannian manifold X, such that*
(i) there is a diffeomorphsim $f_i : X \to M_i$ for all sufficiently large i;
(ii) the pullback metrics $f_i^*(g_i)$ converges in C^∞-class to a real analytic Riemannian metric g_∞.

2. Convergence of harmonic spinors. Let M be a closed Spin Riemannian manifold. Let S^+ (resp. S^-) be the positive (resp. negative) spinor bundle. Let $D : \Gamma(S^+) \to \Gamma(S^-)$ be the Dirac operator. The classical Lichnerowicz formula reads

$$D^2 = \nabla^*\nabla + \frac{1}{4}s \tag{2.1}$$

where s is the scalar curvature of M.

Let M_i be a sequence of closed Spin Riemannian n-manifolds such that

$$\operatorname{diam}(M_i)^2 \cdot s_i \geq -\varepsilon_i \ ; \ \operatorname{Sec}_{g_i} \leq 1$$

where s_{g_i} is the scalar curvature of M_i. By rescaling we may assume that $\operatorname{diam}(M_i) \leq 1$ and $s_{g_i} \geq -\varepsilon_i$. Note that the sectional curvature $\operatorname{Sec}_{g_i} \geq -\lambda(n)$.

Let $\phi_i \in \Gamma(S_i^+)$ denote the harmonic spinors (i.e. $D_i\phi_i = 0$.) By normalizing we may assume that

$$\frac{1}{\operatorname{vol}(M_i)} \int_{M_i} |\phi_i|^4 = 1 \tag{2.2}$$

LEMMA 2.3. *Let ϕ_i be as above. Then*

$$|\phi_i|_{L^\infty} \leq C$$

where C is a constant depending only on n, ε_i and $\lambda(n)$.

Proof. Note that

$$\Delta|\phi_i|^2 = 2\,\langle \nabla_i^*\nabla_i\phi_i, \phi_i\rangle - 2|\nabla_i\phi_i|^2$$

By the Lichnerowicz formula we get that

$$0 = \nabla_i^*\nabla_i\phi_i + \frac{1}{4}s_{g_i}\phi_i$$

Therefore

$$\Delta|\phi_i|^2 \leq -\frac{1}{2}s_{g_i}|\phi_i|^2 \leq \frac{1}{2}\varepsilon_i|\phi_i|^2$$

By [Ga] Proposition 3.2 it follows that

$$|\phi_i|^2_{L^\infty} \leq C_1 \frac{(\int_{M_i}|\phi_i|^4)^{\frac{1}{2}}}{\sqrt{vol(M_i)}} = C_1$$

The desired result follows. □

Let $2r$ denote a uniform lower bound for the conjugate radii of M_i. By Theorem 1.1, there are radii r balls, $B_i^1(r), \cdots, B_i^m(r)$, in the tangent spaces $T_{p_k^i}M_i$, where $p_1^i, \cdots, p_m^i \in M_i$ are m points so that the exponential maps $\exp_{p_k^i} : B_i^k(r) \subset T_{p_k^i}M_i \to M_i$ are embeddings and $\exp_{p_1^i}(B_i^1(r)), \cdots, \exp_{p_m^i}(B_i^m(r))$ is a harmonic coordinate covering of M_i. Fixing k, consider the pullback metric $\exp^*_{p^i_k}(g_i)$ on $B^k_i(r)$. By Theorem 1.2 there is an integer N such that for all $i > N$, there is a diffeomorphism $f^k_i : B^k(r) := B^k_N(r) \to B^k_i(r)$ so that the pullback metric tensors $(f^k_i)^*(\exp^*_{p^i_k}(g_i))$ converge in $L^{2,p}$-class (cf. [Ya])

Let $\hat{\phi}_i = (f^k_i)^*(\exp^*_{p^i_k}(\phi_i)) \in \Gamma(\hat{S}^+)$ be the pullback spinors, where $\hat{S}^+$ is a spinor bundle of $B^k(r)$, which is isomorphic to the pullback spinor bundles $\hat{S}^\pm_i$ on $B^k(r)$ for all i sufficiently large (passing to a subsequence if necessary.)

LEMMA 2.4. *Let M_i be as above. If $\lim_i \varepsilon_i = 0$, then a subsequence of $\hat{\phi}_i$ above converges in $L^{2,p}$-class (for any integer $p > 1$) to a non-trivial parallel harmonic spinor $\hat{\phi}$ with respect to the limit metric g_∞.*

Proof. Recall that all $\hat{\phi}_i$ satisfy

$$\nabla^*_i \nabla_i \hat{\phi}_i + \frac{1}{4} s_{g_i} \hat{\phi}_i = 0 \tag{2.4.1}$$

By Theorem 1.2 it follows that in the harmonic coordinate ball $B^k(r)$ and a local frame of sections of $\hat{S}^+$ (resp. $\hat{S}^-$), (2.4.1) gives a second order uniformly elliptic equation, such that

(2.4.2) the coefficients of the second order term are uniformly $C^{1,\alpha}$- (resp. $L^{2,p}$-) bounded;

(2.4.3) the coefficients of the first order terms (also the quadratic terms) are uniformly $C^{0,\alpha}$- (resp. $L^{1,p}$-) bounded;

(2.4.4) the coefficients of the zero order terms are uniformly L^p-bounded;

By [ADN] we know that $\|\hat{\phi}_i\|_{L^{2,p}} \leq C\|\hat{\phi}_i\|_{L^\infty}$, where C is a universal constant. Therefore, by Lemma 2.3 $\hat{\phi}_i$ contains a convergence subsequence in $L^{2,p'}$-(resp. $C^{1,\alpha}$-) topology for any $p' < p$, noting that there is a Sobolev embedding $L^{2,p} \subset C^{1,1-\frac{n}{p}}$.

Next we prove that the limit $\hat{\phi}$ is parallel with respect to the metric g_∞. By (2.2) $\hat{\phi}$ is not zero and the desired result follows.

Integrating (2.4.1) we get

$$0 = \frac{1}{\mathrm{vol}\, M_i} \int_{M_i} |\nabla_i \phi_i|^2 + \frac{1}{4\mathrm{vol}\, M_i} \int_{M_i} s_{g_i} |\phi_i|^2$$

Since $\varepsilon_i \to 0$, the second term

$$\liminf_{i\to\infty} \frac{1}{4\mathrm{vol}\, M_i} \int_{M_i} s_{g_i} |\phi_i|^2 = 0$$

Hence, passing to a subsequence we may assume that

$$\lim_{i\to\infty} \frac{1}{\mathrm{vol}\, M_i} \int_{M_i} |\nabla_i \phi_i|^2 = 0 \tag{*}$$

We now use a trick in [Ya]. Suppose that $\hat{\phi}$ is not parallel with respect to g_∞. Set $\hat{a}_i = |\hat{\nabla}_i \hat{\phi}_i|^2$, $\hat{a} = |\hat{\nabla}_\infty \hat{\phi}|^2$. Then there is a point $x \in B^k(r)$ for some k such that $\hat{a} > b > 0$ on the radius δ ball $B_\delta(x, (B^k(r), \hat{g}_\infty))$ for some positive constants b and δ. Thus $\hat{a}_i > \frac{b}{2}$ on $B_\delta(x, (B^k(r), \hat{g}_i))$ for i large. Set $a_i = |\nabla_i \phi_i|^2$. By the Bishop-Gromov volume comparison theorem we get

$$1 - \frac{\text{vol }\{a_i \leq \frac{b}{2}\}}{\text{vol }(M_i)} \geq \frac{\text{vol } B_\delta(q_i, M_i)}{\text{vol }(M_i)} \geq \frac{\alpha(\delta)}{\alpha(D)}$$

where $q_i = \exp_{p_k^i} \circ f_i^k(x)$ and $\alpha(r)$ is the volume function of radius r ball in the hyperbolic space of curvature $-\lambda(n)$. Therefore

$$\frac{\text{vol }\{a_i < \frac{b}{2}\}}{\text{vol }(M_i)} \leq 1 - \frac{\alpha(\delta)}{\alpha(D)}$$

for sufficiently large i. On the other hand, by (*) above, for any fixed $b > 0$, it holds that

$$\lim_{i\to\infty} \frac{\text{vol }\{a_i < \frac{b}{2}\}}{\text{vol }(M_i)} = 1$$

A contradiction. □

3. Proof of Theorem A. Let M be a Spin manifold, and $\phi \in \Gamma(S^+)$ be a spinor. Using the first Bianchi identity we get

(3.1)
$$\sum_j e_j(\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i} - \nabla_{[e_i,e_j]})\phi = -\frac{1}{4}\sum_{j,k,l} e_j R_{ijkl} e_k e_l \phi = -\frac{1}{2}\sum_l \text{Ric}(e_i, e_l) e_l \phi$$

for any i, where $e_1, \cdots, e_n$ is an orthonormal basis.

Proof of Theorem A. Suppose not. Then there is a sequence of Riemannian manifolds (M_i, g_i) with $\mathcal{A}(M_i) \neq 0$ so that

$$\text{diam}(M_i) \leq 1,\ s_{g_i} \geq -i^{-1},\ \text{Sec}_{g_i} \leq 1$$

but no one of M_i admits the desired metrics.

By the Atiyah-Singer index theorem, the index $\text{Ind}(D_i) \neq 0$. Therefore there is a nonzero harmonic spinor $\phi_i \in \ker D_i$, where D_i is the Dirac operator. By normalizing we may assume (2.2) for all ϕ_i.

Applying (3.1) to ϕ_k and integrating both sides we get

$$\lim_{k\to\infty} \frac{1}{\text{vol } M_k} \int_{M_k} |\sum_l \text{Ric}(e_i, e_l) e_l \phi_k|^p = 0 \tag{3.2}$$

by Lemmas 2.3 and 2.4.

Since $\hat{\phi}$ is parallel with respect to g_∞ and

$$\frac{1}{\text{vol } M_k} \int_{M_k} |\phi_k|^4 = 1,$$

the C^0-norm

$$|\phi_k|_{C^0} \geq \frac{1}{2}$$

for k large. Therefore (3.2) implies that

$$\lim_{k\to\infty} \frac{1}{\text{vol } M_k} \int_{M_k} |\sum_l \text{Ric}(e_i, e_l) e_l|^p = 0$$

for any i and any prescribed integer $p > 1$. This proves that the limit metric g_∞ on X is L^p-Ricci flat. By Theorem 1.3 this implies that g_∞ is a smooth Ricci flat metric.

By Lemma 2.4 the limit of ϕ_i is a parallel spinor ϕ_∞ with respect to g_∞. By [Fu] we know that the holonomy of g_∞ must be one of the desired types. □

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