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ON THE AMPLE CONE OF AN AMPLE HYPERSURFACE *

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For a smooth ample hypersurface D of dimension at least 3 in a smooth projective variety X, the restriction map $r : \mathrm{Pic}(X) \to \mathrm{Pic}(D)$ is an isomorphism by weak Lefschetz. In a recent paper [3], Hassett, Lin and Wang gave an example of a pair $D \subset X$ with $\dim D = 3$ such that the cone of ample divisor classes of D is strictly larger than the image under r of the cone of ample divisor classes of X; in other words, the weak Lefschetz principle can fail for ample cones.

The purpose of this note is to point out that not only is the ample cone of a hypersurface D often larger than the ample cone of the ambient space X, in general it cannot even be computed from birational models of X. This contrasts with the conjecture of [2] (already disproved for semi-ample hypersurfaces in [5]), which predicted that such a description is possible. Note that in the special case studied in [3], the ample cone is the union of ample cones of birational models of X (Remark 2.3). However, a slight generalization of the example gives hypersurfaces with ample cone larger than the union of ample cones of all birational models of the ambient space (Corollary 3.1).

For a projective variety Y, denote by $\mathcal{N}(Y) \subset \mathrm{Pic}_{\mathbb{R}}(Y)$ the cone of nef divisor classes; this is the closure of the ample cone and also the dual of the Mori cone by Kleiman's criterion. Facets (codimension-one faces) of $\mathcal{N}(Y)$ correspond in favourable cases to primitive (non-factorizable) contraction morphisms on Y.

1. The ambient space. I use the notation of [3]. Let $\varphi : X \to \mathbb{P}^4$ be the blowup of two distinct points $p_1, p_2 \in \mathbb{P}^4$. The line $p_1p_2 = l' \subset \mathbb{P}^4$ has proper transform $l \subset X$. The Picard group $\mathrm{Pic}(X) = \langle H, E_1, E_2 \rangle$ is generated by the classes of exceptional divisors E_1, E_2 of φ and the proper transform H of a general hyperplane section of $\mathbb{P}^4$ through p_1, p_2. Let $\mathcal{B}$, $\mathcal{C}$ be the cones in $\mathrm{Pic}_{\mathbb{R}}(X)$ spanned by the rays $H + E_1$, $H + E_2$, $H + E_1 + E_2$ and H, $H + E_1$, $H + E_2$ respectively.

PROPOSITION 1.1.

(i) The nef cone of X is $\mathcal{N}(X) = \mathcal{B}$; the three facets of the cone correspond to the contractions of E_i, and to a small contraction $\psi : X \to \bar{X}$ contracting the single extremal curve $l \cong \mathbb{P}^1$ with $K_X \cdot l = 1$.

(ii) The antiflip $\psi^+ : X^+ \to \bar{X}$ of ψ exists. The proper transform E_i^+ of E_i is isomorphic to $\mathbb{P}^3$ blown up in a point; $E_1^+ \cap E_2^+ \cong \mathbb{P}^2$ is the exceptional divisor of both blowups and also the small exceptional locus of ψ^+.

(iii) Let $\alpha : X^+ \dashrightarrow X$ be the flip and identify the Picard groups of X and X^+ by the map $\alpha_* \colon \mathrm{Pic}(X^+) \xrightarrow{\sim} \mathrm{Pic}(X)$ induced by proper transform on divisors composed with the isomorphisms between the Picard groups of X, X^+ and their class groups. Then $\mathcal{N}(X^+) = \mathcal{C}$. The three facets of the cone $\mathcal{C}$ correspond to the contraction ψ^+ and two different $\mathbb{P}^1$-bundle structures $\pi_i : X^+ \to Z$. The image Z is the blowup of $\mathbb{P}^3$ in one point. The fibres l_i of π_i are generically the lines of the original $\mathbb{P}^4$ through the point p_i. On E_i^+, π_i is the identity; on E_{3-i}^+ it is the unique projection to $\mathbb{P}^2$.

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Proof. Once the statement is formulated, the proof becomes an elementary exercise; my personal favourite is toric geometry, which provides a clean and uniform proof. In the lattice $N = \mathbb{Z}^4$, the fan spanned by

$$v_0 = (-1,-1,-1,-1),\ v_3 = (1,0,0,0), \ldots, v_6 = (0,0,0,1)$$

describes $\mathbb{P}^4$. Barycentric subdivision of the cones $v_0v_3v_4v_5$ and $v_0v_3v_4v_6$ by the vertices

$$v_1 = (0,0,0,-1),\ v_2 = (0,0,-1,0)$$

leads to X. In this language, E_i are the torus-invariant divisors corresponding to the rays v_i and l is the torus orbit corresponding to the facet $v_0v_3v_4$. A short check using toric Mori theory [4] shows that there are three primitive contractions on X: the contractions of E_i and a contraction ψ with one-dimensional exceptional locus l. The extremal curves define three inequalities on ample classes which gives $\mathcal{N}(X) = \mathcal{B}$.

The fan of X^+ is obtained by replacing the cones $v_0v_1v_3v_4$ and $v_0v_2v_3v_4$ by the cones $v_0v_1v_2v_3$, $v_0v_1v_2v_4$ and $v_1v_2v_3v_4$. The new codimension-two face v_1v_2 represents the small exceptional locus $E_1^+ \cap E_2^+ \cong \mathbb{P}^2$. There are three primitive contractions again; the two $\mathbb{P}^1$-bundle structures π_i come from projecting the whole fan along the vector v_i for $i = 1, 2$. The tasks of drawing the pictures and filling in the details are left to the reader. □

2. The hypersurface. Let now $D' \subset \mathbb{P}^4$ be a smooth hypersurface of degree $d > 2$ containing p_1 and p_2; assume for simplicity that the line l' is either contained in D' or is at most simply tangent to D' at p_i for $i = 1, 2$ while meeting it transversally elsewhere. Let $D \subset X$ be the proper transform of D'. A simple computation shows that D is ample in X; hence by the Lefschetz principle, the Picard groups of X and D can be identified by the restriction isomorphism $r : \mathrm{Pic}(X) \to \mathrm{Pic}(D)$.

PROPOSITION 2.1.
(i) If $l' \subset D'$, then

$$\mathcal{N}(D) = \mathcal{B}.$$

(ii) Assume that D' does not contain l', but for both indices $i = 1, 2$ one of the following holds:
a. D' contains a line through p_i, or
b. the line l' is tangent to D' at p_{3-i}.
Then

$$\mathcal{N}(D) = \mathcal{B} \cup \mathcal{C}.$$

(iii) In all other cases, there is a strict inclusion

$$\mathcal{N}(D) \supset \mathcal{B} \cup \mathcal{C}.$$

Proof. $\mathcal{N}(D) \supset \mathcal{B}$ is clear as a restriction of an ample class from X to D is also ample. Part (i) follows immediately from Proposition 1.1(i), since every face of the nef cone of X corresponds to a contraction with D containing at least one fibre.

To prove (ii), let $\bar{D} \subset \bar{X}$, $D^+ \subset X^+$ be the proper transforms of D in $\bar{X}$ and X^+. As D meets l in a finite number of points, the map $\psi|_D : D \to \bar{D}$ is finite and

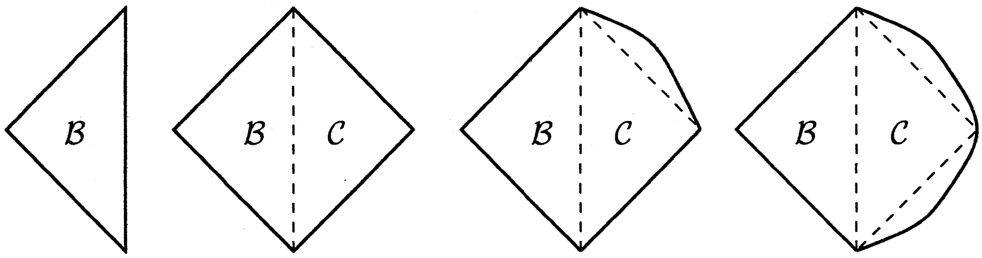

FIG. 2.1. *Possibilities for the shape of $\mathcal{N}(D)$*

birational and there is a Cartesian diagram

$$\begin{array}{ccc} & \widetilde{D} & \\ \swarrow & & \searrow \\ D & & D^+ \\ \searrow & & \swarrow \\ & \bar{D}. & \end{array}$$

By the conditions put on D', the map $\sigma : \widetilde{D} \to D$ is the blowup of the intersection points of l and D; the map $\tau : \widetilde{D} \to D^+$ is again finite and birational. Write $\beta : D^+ \dashrightarrow D$ for the composition $\sigma \circ \tau^{-1}$.

Take any ample divisor class A^+ on X^+. Part (ii) will follow if I can show that the divisor class $r(\alpha_* A^+)$ is ample on D. However, using the fact that τ is finite birational,

$$r(\alpha_* A^+) = \beta_* r^+(A^+) = \sigma_* \tau^* r^+(A^+),$$

where $r^+ : \operatorname{Pic}(Y^+) \to \operatorname{Pic}(X^+)$ is the restriction map. As A^+ is ample on Y^+, $r^+(A^+)$ is an ample Cartier divisor class on X^+ and hence $\tau^* r^+(A^+)$ is ample on $\widetilde{D}$. Now the ampleness of $r(\alpha_* A^+)$ on D follows by applying Lemma 2.2 to the map $\sigma : \widetilde{D} \to D$.

To prove (iii), take $A = \pi_i^* B$ where B is ample on Z; A is a nef but non-ample divisor class in the boundary of the cone $\mathcal{N}(X^+)$. If none of the special circumstances of (i)-(ii) happen, then D^+ does not contain any fibre of π_i, so π_i restricts to D^+ as a finite morphism. Hence the restriction of A to D^+ is ample. The argument of (ii) now shows that $r(\alpha_* A)$ is ample on D and as ampleness is an open condition, (iii) follows. □

LEMMA 2.2. *Let Y be a smooth threefold, $f \colon \widetilde{Y} \to Y$ the blowup of a closed point. If A is an ample divisor class on $\widetilde{Y}$, then its proper transform $B = f_* A$ is ample on Y.*

Proof. Let E be the exceptional divisor; then $f^* B = A + \lambda E$ with a non-negative integer λ. If C is a curve in Y with proper transform $\widetilde{C}$ in $\widetilde{Y}$, then

$$B \cdot C = f^* B \cdot \widetilde{C} = A \cdot \widetilde{C} + \lambda E \cdot \widetilde{C} > 0$$

as $A \cdot \widetilde{C} > 0$ and $E \cdot \widetilde{C} \geq 0$ since $\widetilde{C}$ is not contained in E. If S is any surface in X with proper transform $\widetilde{S}$ then

$$B^2 \cdot S = (A + \lambda E)^2 \widetilde{S} = A \cdot (A + \lambda E) \cdot \widetilde{S} \geq A^2 \cdot \widetilde{S} > 0.$$

The second equality holds since $(A + \lambda E) \cdot E \cdot \widetilde{S} = f^* B \cdot \widetilde{S}|_E = 0$ by the projection formula, as E and $\widetilde{S}$ intersect in a curve contracted by f; the first inequality uses $A \cdot E \cdot \widetilde{S} = A \cdot \widetilde{S}|_E \geq 0$. Finally in same way

$$B^3 = (A + \lambda E)^3 \geq (A + \lambda E)^2 \cdot A = (A + \lambda E) \cdot A^2 \geq A^3 > 0.$$

Hence by the Nakai–Moishezon criterion, B is ample. □

REMARK 2.3. Note that in the special case of smooth cubics studied in [3], the first condition of Proposition 2.1(ii) is always satisfied: a smooth cubic threefold is covered by lines. Hence either $l' \subset D'$ and $\mathcal{N}(D) = \mathcal{B}$, or $l' \not\subset D'$ and $\mathcal{N}(D) = \mathcal{B} \cup \mathcal{C}$. [3, Assumptions 2.5] fixes the type of the primitive contractions on D induced by π_i: if D' contains finitely many lines through p_i and l' is not tangent to D' at p_i, then (the Stein factorization of) $\pi_i|_D$ is a small flopping contraction ($K_D \cdot l_i = 0$); otherwise it is divisorial.

3. Good hypersurfaces. In the original example of [3], as well as in all cases where the conditions of Proposition 2.1(i)-(ii) are satisfied, the ample cone of the hypersurface D has an explicit description in terms of nef cones of models of its ambient space; D is good in the following sense:

Good hypersurface: Call a hypersurface $D \subset X$ *good*, if the nef cone $\mathcal{N}(D)$ is a union of cones $r(\alpha_{i*}\mathcal{N}(X_i))$, where $\alpha_i : X_i \dashrightarrow X$ are certain birational models of X isomorphic to X in codimension one.

Anticanonical hypersurfaces in certain toric varieties were conjectured to be good by Cox and Katz [2, Conjecture 6.2.8]. By [5], this conjecture can fail if D is only semi-ample. Alas, an ample but non-anticanonical $D \subset X$ can also fail to be good.

COROLLARY 3.1. *Take any smooth degree $d > 3$ hypersurface D' in $\mathbb{P}^4$ with no lines through p_i and with l' not tangent at p_i for both $i = 1, 2$ and let D be its proper transform in X as before. Then $D \subset X$ is not good.*

Proof. By Proposition 2.1(iii), $\mathcal{N}(D)$ is strictly larger than $\mathcal{B} \cup \mathcal{C}$. On the other hand, X has no further birational models which are isomorphic in codimension one. For if $\alpha_i : X_i \dashrightarrow X$ is any such model, then $\alpha_{i*}A$ is a moving divisor class for A ample on X_i. However, from the explicit description of Proposition 1.1 it is clear that the moving cone of X is the interior of $\mathcal{B} \cup \mathcal{C}$. So X_i is isomorphic to X or X^+. □

Note that a 3-dimensional ample and anticanonical hypersurface is always good: if $D \in |-K_X|$ is ample, $r(\mathcal{N}(X)) = \mathcal{N}(D)$ by Kollár's result [1]. To the best of my knowledge, the analogous result is not known if $\deg(D) > 3$.

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