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QRADRATIC SHEAVES AND SELF–LINKAGE*

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0. Introduction and first results. The present paper is devoted to the proof of a structure theorem for self–linked pure subschemes $C \subseteq \mathbb{P}^n_k$ of codimension 2 over a field k of characteristic $p \neq 2$.

We use the symbol C because the classical case to be studied was the one of reduced curves in $\mathbb{P}^3_k$. In this case, one can describe the notion of self–linkage in non–technical terms, saying that C is self–linked if and only if there are surfaces F and G such that their complete intersection is the curve C counted with multiplicity 2. This is a special case of the notion of linkage (C is linked to C' if $C \cup C'$ is the complete intersection of two surfaces F and G), classically introduced by R. Apery, F. Gaeta (see [Ap], [Gae]) and later deeply investigated by the algebraic point of view by Ch. Peskine and L. Szpiro and by P. Rao (see [P–S], [Rao1]). The special case of self–linkage was however studied before, in the work of E. Togliatti (see [To1], [To2]), and later D. Gallarati (see [Gal]), in the form of the theory of contact between surfaces.

In [Ca1] the theory of contact was related to a new theory, of the so called even sets of nodes, and later Rao used these ideas to obtain a structure theorem for projectively Cohen–Macaulay self–linked subschemes of codimension 2 in projective spaces (see [Rao2]).

Recently, Walter's structure theorem (see [Wa]) for subcanonical subschemes of codimension 3 opened the way to solving some old conjectures about even sets of nodes and contact of hypersurfaces (see [C–C]).

A basic ingredient was the algebraic concept of quadratic sheaves, generalizing to a greater extent the geometric notion of contact and even sets. This notion was applied in [C–C] to the classification of even sets of nodes on a surface $F \subseteq \mathbb{P}^3_k$ for low values of the degree d.

On the other hand, let $F \subseteq \mathbb{P}^3_k$ be a surface whose only singularities are an even set of nodes Δ. Then there is a curve C on F passing through the points of Δ, and a surface G such that $F \cap G = 2C$ as cycles (see [Ca1], [Gal]). With this in mind it is therefore only natural to apply the structure theorem for quadratic sheaves in order to obtain a structure theorem for self–linkage.

This is done in the present paper, where we generalize the previously cited result of P. Rao to arbitrary pure subschemes of $\mathbb{P}^n_k$ of codimension 2, (see also the survey [Ca2] of the second author for a preliminary version of these results).

MAIN THEOREM. *Let k be a field of characteristic $p \neq 2$ and $C \subseteq \mathbb{P}^n_k$ be a pure subscheme of codimension 2 which is self–linked through hypersurfaces $F := \{f = 0\}$ and $G := \{g = 0\}$ of respective degrees d, m. Let $\Im_C$ be its sheaf of ideals and set $\mathcal{F}_F := \Im_C / f\mathcal{O}_{\mathbb{P}^n_k}(-d)$, $\mathcal{F}_G := \Im_C / g\mathcal{O}_{\mathbb{P}^n_k}(-m)$.*

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Assume moreover that if $n \equiv 1 \mod 4$ *and* $d+m-n-1 = 2\varrho$, *then the following two equivalent congruences hold:*

$$\chi(\mathcal{O}_{\mathbb{P}^n_k}(\varrho)) - \chi(\mathcal{O}_{\mathbb{P}^n_k}(\varrho - d)) - \chi(\mathcal{O}_C(\varrho)) \equiv 0 \mod 2,$$
$$\chi(\mathcal{O}_{\mathbb{P}^n_k}(\varrho)) - \chi(\mathcal{O}_{\mathbb{P}^n_k}(\varrho - m)) - \chi(\mathcal{O}_C(\varrho)) \equiv 0 \mod 2.$$

Then there exist a locally free $\mathcal{O}_{\mathbb{P}^n_k}$*-sheaf* $\mathcal{E}$, *a symmetric map* $\alpha\colon \check{\mathcal{E}}(-d-m) \to \mathcal{E}$ *and a resolution*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-d) \oplus \check{\mathcal{E}}(-d-m) \xrightarrow{\begin{pmatrix} \gamma & {}^t\lambda \\ \lambda & \alpha \end{pmatrix}} \mathcal{O}_{\mathbb{P}^n_k}(-m) \oplus \mathcal{E} \longrightarrow \mathcal{F}_F \longrightarrow 0 \tag{1}$$

inducing exact sequences

$$0 \longrightarrow \check{\mathcal{E}}(-d-m) \xrightarrow{\begin{pmatrix} {}^t\lambda \\ \alpha \end{pmatrix}} \mathcal{O}_{\mathbb{P}^n_k}(-m) \oplus \mathcal{E} \longrightarrow \Im_C \longrightarrow 0, \tag{2}$$

$$0 \longrightarrow \check{\mathcal{E}}(-d-m) \xrightarrow{\alpha} \mathcal{E} \longrightarrow \mathcal{F}_G \longrightarrow 0. \tag{3}$$

Conversely, given a subscheme C *of codimension 2, assume that there does exist a sequence (2) with the above property of* α *being symmetric and* $\deg(\det(\alpha)) = m$. *Then* C *is self-linked through the hypersurfaces* F *and* G *of respective equations* $f := \det\begin{pmatrix} 0 & {}^t\lambda \\ \lambda & \alpha \end{pmatrix}$ *and* $g := \det(\alpha)$. □

It was conjectured by Ph. Ellia that if moreover C is subcanonical then C is the complete intersection of two hypersurfaces (see [E–B]). We have been trying to deduce this statement from our main theorem. In the meantime, Ellia's conjecture has been proven in [Ar] if $n \geq 4$, and in [F–L–K] when $n \geq 3$ and $k \cong \mathbb{C}$.

Moreover, in section 3 we discuss the problem whether the self-linkage ideal (f, g) is uniquely determined once the generator g of higher degree is fixed, and prove that the answer is positive under the condition that the subscheme be locally Gorenstein.

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First results. For the reader's benefit we recall the following definitions and results proved from [C–C]. From now on we always assume that k is a field of characteristic $p \neq 2$.

DEFINITION 0.1. Let X be a projective, locally Cohen–Macaulay scheme. We say that a coherent, locally Cohen–Macaulay sheaf of $\mathcal{O}_X$–modules $\mathcal{F}$ is a $\delta/2$–quadratic sheaf on X, $\delta = 0, 1$, if there exists a symmetric bilinear map

$$\mathcal{F} \times \mathcal{F} \to \mathcal{O}_X(-\delta)$$

inducing an isomorphism $\sigma\colon \mathcal{F}(\delta) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

REMARK 0.2. If $\mathcal{F}$ is $\delta/2$–quadratic, then it is reflexive since the natural map $\mathcal{F} \to \mathcal{F}^{\check{}\check{}}$ equals $\check{\sigma}^{-1} \circ \sigma(-\delta)$.

Assume now that $F \subseteq \mathbb{P}^n_k$ is a hypersurface of degree d and that $\mathcal{F}$ is a $\delta/2$–quadratic sheaf on F.

The main result of sections 1 and 2 of [C–C] concerns a characterization of quadratic sheaves on hypersurfaces in $\mathbb{P}^n_k$ (including the needed parity condition as pointed out in theorem 9.1 of [E–P–W]).

THEOREM 0.3. *Let $F \subseteq \mathbb{P}^n_k$ be a hypersurface of degree d and let $\mathcal{F}$ be a $\delta/2$–quadratic sheaf on F. Then $\mathcal{F}$ fits into an exact sequence of the form*

$$0 \longrightarrow \check{\mathcal{E}}(-d-\delta) \xrightarrow{\varphi} \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0 \tag{0.3.1}$$

where $\mathcal{E}$ is a locally free $\mathcal{O}_{\mathbb{P}^n_k}$–sheaf and φ is a symmetric map if and only if the following parity condition holds: if $n \equiv 1 \mod 4$ and $n+1-d-\delta = 2r$, then also $\chi(\mathcal{F}(-r))$ is even.

Moreover we can choose $\mathcal{E}$ such that $H^j_*(\mathbb{P}^n_k, \mathcal{E}) = 0$ for $n > j > (n-1)/2$. □

REMARK 0.4. Following [Ca1] we say that $\mathcal{F}$ is split symmetric if one can choose $\mathcal{E}$ to be a direct sum of line bundles. This is possible if and only if $\mathcal{F}$ is arithmetically Cohen-Macaulay i.e., if and only if $H^i_*(\mathbb{P}^n_k, \mathcal{F}) = 0$ for each $i = 1, \dots, n-2$.

1. From self–linkage to quadratic sheaves. Our first main theorem is an application of the theory of quadratic sheaves. For another application see [C–C].

DEFINITION 1.1. Let $C \subseteq \mathbb{P}^n_k$ be a pure subscheme of codimension 2 and let $\Im_C \subseteq \mathcal{O}_{\mathbb{P}^n_k}$ be its sheaf of ideals: C is said to be self-linked with respect to the complete intersection $X := F \cap G$ of the two hypersurfaces F, G of respective degrees d, m, if $C \subseteq X$ and one of the following equivalent conditions holds

i) $\Im_X : \Im_C = \Im_C$;

ii) $\Im_C/\Im_X = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{O}_C, \mathcal{O}_X)$.

For the above well-known equivalence see e.g. theorem 21.23 of [Ei] (see also [P–S]).

REMARK 1.2. Indeed, since $\Im_C$ and $\Im_X$ coincide with $\mathcal{O}_{\mathbb{P}^n_k}$ in codimension 1 and are torsion free, every $\phi \in \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n_k}}(\Im_C, \Im_X)$ is given by a rational function, which is in turn regular by the normality of $\mathcal{O}_{\mathbb{P}^n_k}$.

Condition i) of definition 1.1 can be thus rewritten as (see [Ca2], proposition 2.6)

$$\Im_C = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n_k}}(\Im_C, \Im_X). \tag{1.2.1}$$

By duality for finite maps (see [Ha], exercise II.6.10),

$$\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{O}_C, \mathcal{O}_X) = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{O}_C, \omega_X(n+1-m-d)) = \omega_C(n+1-m-d). \tag{1.2.2}$$

Finally, let C, $F := \{f = 0\}$, $G := \{g = 0\}$ be as above, and assume $m = \deg(G) \geq d = \deg(F)$. Then we can replace g by $g+af$ (a is a homogeneous polynomial of degree $m-d$), and obtain by Bertini's theorem that G is smooth outside C. Indeed, if C is reduced we can even have that G is smooth at the generic points of C, so that G is a normal, whence irreducible, hypersurface. Moreover in this case (cfr. [Ca1], proposition 2.6) condition i) is equivalent to $cycle(X) = 2C$.

DEFINITION 1.3. Let $\mathcal{F}_G := \Im_C/g\mathcal{O}_{\mathbb{P}^n_k}(-m)$, $\mathcal{F}_F := \Im_C/f\mathcal{O}_{\mathbb{P}^n_k}(-d)$.

By the above definition it follows that both $\mathcal{F}_G$ and $\mathcal{F}_F$ are obviously locally Cohen–Macaulay. By remark 1.2, since

$$\Im_X/g\mathcal{O}_{\mathbb{P}^n_k}(-m) \cong (f\mathcal{O}_{\mathbb{P}^n_k}(-d) + g\mathcal{O}_{\mathbb{P}^n_k}(-m))/g\mathcal{O}_{\mathbb{P}^n_k}(-m) \cong f\mathcal{O}_G(-d),$$

we have pairings

$$\mathcal{F}_G \times \mathcal{F}_G \longrightarrow \Im_X/g\mathcal{O}_{\mathbb{P}^n_k}(-m) \cong f\mathcal{O}_G(-d),$$

and analogously

$$\mathcal{F}_F \times \mathcal{F}_F \longrightarrow g\mathcal{O}_F(-m).$$

To verify that a twist of $\mathcal{F}_G$ yields a quadratic sheaf on G (similarly for $\mathcal{F}_F$ on F) we have to show that the pairing is perfect, i.e. $\mathcal{F}_G \cong \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{F}_G, \mathcal{O}_G(-d))$.

Let $\psi \in \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{F}_G, \mathcal{O}_G(-d))$, then ψ induces $\psi' : \Im_C \to \mathcal{O}_G(-d) \cong \Im_X / g\mathcal{O}_{\mathbb{P}^n_k}(-m)$. If ψ' is in the image of a $\phi \in \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n_k}}(\Im_C, \Im_X)$, then, by remark 1.2, $\phi \in \Im_C$, and clearly ϕ induces a zero ψ if and only if $\phi \in g\mathcal{O}_{\mathbb{P}^n_k}(-m)$.

Thus, we are left with the verification that

$$\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n_k}}(\Im_C, \Im_X) \to \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n_k}}(\Im_C, \Im_X / g\mathcal{O}_{\mathbb{P}^n_k}(-m))$$

is surjective. By the $\mathcal{E}xt^i_{\mathcal{O}_{\mathbb{P}^n_k}}$–exact sequence, the above surjectivity is equivalent to the injectivity of $j: \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n_k}}(\Im_C, g\mathcal{O}_{\mathbb{P}^n_k}(-m)) \to \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n_k}}(\Im_C, \Im_X)$.

To this purpose, consider

$$0 \longrightarrow \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{O}_C, \Im_X) \xrightarrow{i} \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{O}_C, \mathcal{O}_G(-d)) \longrightarrow$$
$$\longrightarrow \mathcal{E}xt^2_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{O}_C, g\mathcal{O}_{\mathbb{P}^n_k}(-m)) \xrightarrow{j'} \mathcal{E}xt^2_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{O}_C, \Im_X)$$

where j' corresponds to j via the natural equivalence $\mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n_k}}(\Im_C, \cdot) \cong \mathcal{E}xt^2_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{O}_C, \cdot)$. It follows that j is injective if and only if i is an isomorphism. We have the exact sequence

$$\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{O}_{\mathbb{P}^n_k}, \Im_X) \longrightarrow \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n_k}}(\Im_C, \Im_X) \longrightarrow$$
$$\longrightarrow \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{O}_C, \Im_X) \longrightarrow \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{O}_{\mathbb{P}^n_k}, \Im_X) \cong 0.$$

Since $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{O}_{\mathbb{P}^n_k}, \Im_X) \cong \Im_X$ and $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n_k}}(\Im_C, \Im_X) \cong \Im_C$ then

$$\mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{O}_C, \Im_X) \cong \Im_C / \Im_X \cong \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{O}_C, \mathcal{O}_X) \cong \omega_C(n+1-m-d).$$

We also have

$$\mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{O}_C, \mathcal{O}_{\mathbb{P}^n_k}(-d)) \longrightarrow \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{O}_C, \mathcal{O}_G(-d)) \longrightarrow$$
$$\longrightarrow \mathcal{E}xt^2_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{O}_C, g\mathcal{O}_{\mathbb{P}^n_k}(-m-d)) \xrightarrow{g} \mathcal{E}xt^2_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{O}_C, \mathcal{O}_{\mathbb{P}^n_k}(-d)).$$

Since $C \subseteq G$, then the multiplication by g is zero. Moreover, $\mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{O}_C, \mathcal{O}_{\mathbb{P}^n_k}(-d)) = 0$ since C is locally Cohen–Macaulay. We obtain that

$$\mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{O}_C, \mathcal{O}_G(-d)) \cong \mathcal{E}xt^2_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{O}_C, g\mathcal{O}_{\mathbb{P}^n_k}(-m-d)) \cong \omega_C(n+1-m-d).$$

The following easy lemma thus concludes the proof that $i: \omega_C(m+d-n-1) \to \omega_C(m+d-n-1)$ is an isomorphism.

LEMMA 1.4. *If $\mathcal{H}$ is a coherent $\mathcal{O}_{\mathbb{P}^n_k}$-sheaf and $\varrho: \mathcal{H} \to \mathcal{H}$ is either injective or surjective, then it is an isomorphism.*

Proof. Let ϱ be injective (resp. surjective) and $\mathcal{K} := \operatorname{coker}(\varrho)$ (resp. $\mathcal{K} := \ker(\varrho)$). By Serre'stheorem B we get $h^1(\mathbb{P}^n_k, \mathcal{H}(t)) = 0$ (resp. $h^1(\mathbb{P}^n_k, \mathcal{K}(t)) = 0$) for each t large enough, hence $h^0(\mathbb{P}^n_k, \mathcal{K}(t)) = 0$ for t large enough, whence $\mathcal{K} = 0$ by Serre's theorem A. □

Set $d = 2d' + \delta$, $m = 2m' + \mu$, $\delta, \mu \in \{0, 1\}$. Then $\mathcal{F}_G(d')$ is a $\delta/2$–quadratic sheaf on G, and similarly $\mathcal{F}_F(m')$ is a $\mu/2$–quadratic sheaf on F. According to theorem 0.3, we obtain two locally free $\mathcal{O}_{\mathbb{P}^n_k}$–sheaves $\mathcal{E}'_G$ and $\mathcal{E}'_F$ provided that the respective parity conditions hold.

It is convenient to rewrite such conditions. If $n \equiv 1 \mod 4$ and $d+m-n-1 = 2\varrho$, then we want that $\chi(\mathcal{F}_F(\varrho)) \equiv \chi(\mathcal{F}_G(\varrho)) \equiv 0 \mod 2$. By two obvious exact sequences the above conditions are equivalent to the two congruences

$$\begin{aligned} &\chi(\mathcal{O}_{\mathbb{P}^n_k}(\varrho)) - \chi(\mathcal{O}_{\mathbb{P}^n_k}(\varrho - d)) - \chi(\mathcal{O}_C(\varrho)) \equiv 0 \mod 2, \\ &\chi(\mathcal{O}_{\mathbb{P}^n_k}(\varrho)) - \chi(\mathcal{O}_{\mathbb{P}^n_k}(\varrho - m)) - \chi(\mathcal{O}_C(\varrho)) \equiv 0 \mod 2. \end{aligned} \tag{1.5}$$

In turn the two congruences above are equivalent each other. In fact, it suffices to show that

$$\chi(\mathcal{O}_{\mathbb{P}^n_k}(\varrho - d)) + \chi(\mathcal{O}_{\mathbb{P}^n_k}(\varrho - m)) = 0.$$

To this purpose recall that

$$\chi(\mathcal{O}_{\mathbb{P}^n_k}(h)) = \frac{(h+n)\dots(h+1)}{n!}.$$

In our case $n = 4a + 1$ and ϱ is even. Since, by the linkage condition, dm is even too, the parity of $d + m - n - 1$ yields $d = 2d'$, $m = 2m'$, hence

$$\chi(\mathcal{O}_{\mathbb{P}^n_k}(\varrho - d)) = \frac{(2a+i)\dots(-2a+i)}{(4a+1)!} = -\frac{(2a-i)\dots(-2a-i)}{(4a+1)!} = \chi(\mathcal{O}_{\mathbb{P}^n_k}(\varrho - m)),$$

where $i = m' - d'$.

We define $\mathcal{E}_G := \mathcal{E}'_G(-d')$ and $\mathcal{E}_F := \mathcal{E}'_F(-m')$. Then theorem 0.3 rewrites as follows.

PROPOSITION 1.6. *Assume that if $n \equiv 1 \mod 4$ and $n + 1 - d - m = 2\varrho$ then the two equivalent congruences (1.5) above hold.*

Then there exist two locally free $\mathcal{O}_{\mathbb{P}^n_k}$–sheaves $\mathcal{E}_G$ and $\mathcal{E}_F$, such that $H^i_*(\mathbb{P}^n_k, \mathcal{E}_G) = H^i_*(\mathbb{P}^n_k, \mathcal{E}_F) = 0$ for $n > i > (n-1)/2$, fitting into exact sequences

$$\begin{aligned} &0 \longrightarrow \check{\mathcal{E}}_F(-d-m) \xrightarrow{\alpha_F} \mathcal{E}_F \longrightarrow \mathcal{F}_F \longrightarrow 0, \\ &0 \longrightarrow \check{\mathcal{E}}_G(-d-m) \xrightarrow{\alpha_G} \mathcal{E}_G \longrightarrow \mathcal{F}_G \longrightarrow 0, \end{aligned} \tag{1.6.1}$$

where ${}^t\alpha_F = \alpha_F$, ${}^t\alpha_G = \alpha_G$. □

In the above proposition and in what follows, the superscript t denotes the dual morphism twisted by $-d-m$.

REMARK 1.7. Let $\ell \subseteq \mathbb{P}^n_k$ be a line disjoint from C. Then the restrictions of the sequences (1.6.1) to ℓ are still exact. On the other hand $\mathcal{F}_{F|\ell} \cong \mathcal{O}_{F\cap\ell}$ and $\mathcal{F}_{G|\ell} \cong \mathcal{O}_{G\cap\ell}$, hence

$$c_1(\mathcal{E}_F) = -\frac{1}{2}\,\mathrm{rk}(\mathcal{E}_F)(d+m) + \frac{d}{2}, \qquad c_1(\mathcal{E}_G) = -\frac{1}{2}\,\mathrm{rk}(\mathcal{E}_G)(d+m) + \frac{m}{2}.$$

2. From quadratic sheaves to self–linkage. From now on we will assume that the parity condition holds.

Let $\mathcal{O}_{\mathbb{P}^n_k}(-d)\xrightarrow{f}\mathcal{O}_{\mathbb{P}^n_k}$ be the multiplication by f. Then it induces a map $\vartheta\colon \mathcal{O}_{\mathbb{P}^n_k}(-d)\to \mathcal{F}_G$. Since we have a chain of homomorphisms

(2.1)

$$\begin{aligned}\operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{O}_{\mathbb{P}^n_k}(-d),\mathcal{E}_G)\longrightarrow \operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{O}_{\mathbb{P}^n_k}(-d),\mathcal{F}_G)\longrightarrow\\ \xrightarrow{\partial}\operatorname{Ext}^1_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{O}_{\mathbb{P}^n_k}(-d),\check{\mathcal{E}}_G(-d-m))\cong\\ \cong H^1(\mathbb{P}^n_k,\check{\mathcal{E}}_G(-m))\cong H^{n-1}(\mathbb{P}^n_k,\mathcal{E}_G(m+d-n-1))\check{}=0\end{aligned}$$

we can lift ϑ to a map $\lambda\colon \mathcal{O}_{\mathbb{P}^n_k}(-d)\to\mathcal{E}_G$.

Argueing as in (2.1) we obtain that the natural map

$$\operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{E}_G,\Im_C)\to\operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{E}_G,\mathcal{F}_G)$$

is surjective whence we infer that the surjection $\mathcal{E}_G\to\mathcal{F}_G$ can be lifted to $\nu\colon\mathcal{E}_G\to\Im_C$. Notice that the map $\nu\circ\lambda$ is given by $\widetilde{f}\in H^0(\mathbb{P}^n_k,\mathcal{O}_{\mathbb{P}^n_k}(d))$ congruent $f \mod g$.

Let $r:=\operatorname{rk}(\mathcal{E}_G)$. We have a map

$$\Lambda^{r-1}(\alpha_G)\colon\Lambda^{r-1}(\check{\mathcal{E}}_G(-d-m))\to\Lambda^{r-1}\mathcal{E}_G.$$

Since $\Lambda^{r-1}\check{\mathcal{E}}_G\cong\mathcal{E}_G\otimes\det(\mathcal{E}_G)^{-1}$ and $\Lambda^{r-1}\mathcal{E}_G\cong\check{\mathcal{E}}_G\otimes\det(\mathcal{E}_G)$, twisting by $\det(\mathcal{E}_G)^{-1}(-d)$ and taking remark 1.7 into account, we obtain a map $\alpha_G^{adj}\colon\mathcal{E}_G\to\check{\mathcal{E}}_G(-d)$ such that $\alpha_G\circ\alpha_G^{adj}(-m)\colon\mathcal{E}_G(-m)\to\mathcal{E}_G$ is the multiplication by g and $\alpha_G^{adj}(d)=\check{\alpha}_G^{adj}$ since α_G, hence α_G^{adj}, is symmetric.

We can then define

$$\mu:=\check{\lambda}(-d)\circ\alpha_G^{adj}\colon\mathcal{E}_G\to\mathcal{O}_{\mathbb{P}^n_k}.$$

PROPOSITION 2.2. *$\Im_C$ coincides with the sheaf of ideals of $\mathcal{O}_{\mathbb{P}^n_k}$ generated by g and* $\operatorname{im}(\mu)$.

Proof. Let us consider the second sequence in (1.6.1). We have a chain map

(2.2.1)
$$\begin{array}{ccccccccc}0&\longrightarrow&\check{\mathcal{E}}_G(-d-m)&\xrightarrow{\alpha_G}&\mathcal{E}_G&\longrightarrow&\mathcal{F}_G&\longrightarrow&0\\ &&\downarrow\beta&&\downarrow\nu&&\downarrow id&&\\ 0&\longrightarrow&\mathcal{O}_{\mathbb{P}^n_k}(-m)&\xrightarrow{g}&\Im_C&\longrightarrow&\mathcal{F}_G&\longrightarrow&0\end{array}$$

where β is induced by the restriction of ν.

By the mapping cone construction we obtain a resolution

$$0\longrightarrow\check{\mathcal{E}}_G(-d-m)\xrightarrow{s}\mathcal{O}_{\mathbb{P}^n_k}(-m)\oplus\mathcal{E}_G\longrightarrow\Im_C\longrightarrow 0$$

where s has components β, α_G. Recall that $\alpha_G(m)\circ\alpha_G^{adj}$ is the multiplication by g whence $(\nu\circ\alpha_G)(m)\circ\alpha_G^{adj}=g\nu$, since $(\nu\circ g)(m)=g\nu$.

Diagram (2.2.1) yields $g\beta=\nu\circ\alpha_G$. Composing on the right with $\alpha_G^{adj}\circ\lambda$, we obtain

$$g\beta(m)\circ\alpha_G^{adj}\circ\lambda=g\nu\circ\lambda\colon\mathcal{O}_{\mathbb{P}^n_k}(-d)\to\Im_C(m)\subseteq\mathcal{O}_{\mathbb{P}^n_k}(m).$$

Since g is obviously a non–zero divisor in $\Im_C$, then $\nu\circ\lambda=\beta(m)\circ\alpha_G^{adj}\circ\lambda\colon\mathcal{O}_{\mathbb{P}^n_k}(-d)\to\mathcal{O}_{\mathbb{P}^n_k}$ and since $\nu\circ\lambda$ is given by $\widetilde{f}\equiv f \mod g$ the same is true for $\beta(m)\circ\alpha_G^{adj}\circ\lambda$.

It follows from the above identities that $s\circ{}^t\mu = (\nu\circ\lambda(-m), g\lambda(-m))$ is represented by the product matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \tilde{f} \\ g \end{pmatrix}.$$

Thus we obtain the following commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathcal{O}_{\mathbb{P}^n_k}(-d-m) & \xrightarrow{\binom{\tilde f}{g}} & \mathcal{O}_{\mathbb{P}^n_k}(-m)\oplus\mathcal{O}_{\mathbb{P}^n_k}(-d) & \longrightarrow & \Im_X & \longrightarrow 0 \\ & \downarrow{}^t\mu & & \downarrow\left(\begin{smallmatrix}1&0\\0&\lambda\end{smallmatrix}\right) & & \downarrow & \\ 0 \longrightarrow & \check{\mathcal{E}}_G(-d-m) & \xrightarrow{s} & \mathcal{O}_{\mathbb{P}^n_k}(-m)\oplus\mathcal{E}_G & \longrightarrow & \Im_C & \longrightarrow 0 \end{array} \tag{2.2.2}$$

whose right column is the inclusion.

The mapping cone of (2.2.2) is a resolution of $\Im_C/\Im_X \cong \omega_C(n+1-d-m)$ (see (1.2.2)). Therefore the dual of the mapping cone of diagram (2.2.2) yields a resolution of $\Im_C$ (see proposition 2.5 of [P–S]) and $\Im_C$ coincides with the sheaf of ideals locally generated by the maximal minors of

$$\begin{pmatrix} \beta & 1 & 0 \\ \alpha_G & 0 & \lambda \end{pmatrix}$$

which is the ideal locally generated by the maximal minors of (α_G, λ). Our statement follows then from the very definition of μ and the identity $\det(\alpha_G) = g$. □

Since $\mu\circ\alpha_G = g({}^t\lambda)$, then μ induces an endomorphism ψ of $\mathcal{F}_G$ fitting into the following commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \check{\mathcal{E}}_G(-d-m) & \xrightarrow{\alpha_G} & \mathcal{E}_G & \longrightarrow & \mathcal{F}_G & \longrightarrow 0 \\ & \downarrow{}^t\lambda & & \downarrow\mu & & \downarrow\psi & \\ 0 \longrightarrow & \mathcal{O}_{\mathbb{P}^n_k}(-m) & \xrightarrow{g} & \Im_C & \longrightarrow & \mathcal{F}_G & \longrightarrow 0. \end{array}$$

Since $\mathcal{F}_G := \Im_C/g\mathcal{O}_{\mathbb{P}^n_k}(-m)$, proposition 2.2 implies the surjectivity of ψ. Moreover ψ is also injective by lemma 1.4 above. We can replace the given surjection $\pi\colon\mathcal{E}_G \to \mathcal{F}_G$ with $\psi\circ\pi$, whence we may also replace ν by μ in the arguments of proposition 2.2, thus obtaining the following

PROPOSITION 2.3. *There exists $\gamma \in H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d-m))$ such that*

$$f = \det\begin{pmatrix} \gamma & {}^t\lambda \\ \lambda & \alpha_G \end{pmatrix}.$$

Proof. Recall that we are now assuming $\nu = \mu$, whence by the very definition of μ, $\mu\circ\lambda \equiv f \mod g$. Since

$$\mu\circ\lambda = \det\begin{pmatrix} 0 & {}^t\lambda \\ \lambda & \alpha_G \end{pmatrix},$$

then we obtain the existence of a γ such that

$$f = \det\begin{pmatrix} 0 & {}^t\lambda \\ \lambda & \alpha_G \end{pmatrix} + \gamma g = \det\begin{pmatrix} 0 & {}^t\lambda \\ \lambda & \alpha_G \end{pmatrix} + \det\begin{pmatrix} \gamma & {}^t\lambda \\ 0 & \alpha_G \end{pmatrix} = \det\begin{pmatrix} \gamma & {}^t\lambda \\ \lambda & \alpha_G \end{pmatrix}.$$ □

Since $f \neq 0$ we get a coherent sheaf $\mathcal{F}$ supported on F and an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-d) \oplus \check{\mathcal{E}}_G(-d-m) \xrightarrow{\left(\begin{smallmatrix}\gamma & {}^t\lambda \\ \lambda & \alpha_G\end{smallmatrix}\right)} \mathcal{O}_{\mathbb{P}^n_k}(-m) \oplus \mathcal{E}_G \longrightarrow \mathcal{F} \longrightarrow 0.$$

It follows that $\mathcal{F}(m')$ is $\mu/2$–quadratic and we can easily construct the following exact sequence of the vertical complexes C_i's

$$\begin{array}{ccccccccc} & & C_1 & & C_2 & & C_3 & & \\ 0 & \longrightarrow & \check{\mathcal{E}}_G(-d-m) & \longrightarrow & \mathcal{O}_{\mathbb{P}^n_k}(-m) \oplus \mathcal{E}_G & \longrightarrow & \Im_C & \longrightarrow & 0 \\ & & \downarrow \left(\begin{smallmatrix}0 \\ id\end{smallmatrix}\right) & & \downarrow id & & \downarrow \eta & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^n_k}(-d) \oplus \check{\mathcal{E}}_G(-d-m) & \longrightarrow & \mathcal{O}_{\mathbb{P}^n_k}(-m) \oplus \mathcal{E}_G & \longrightarrow & \mathcal{F} & \longrightarrow & 0. \end{array}$$

The associated long exact sequence gives $\mathcal{O}_{\mathbb{P}^n_k}(-d) \cong H_1(C_1) \cong H_0(C_3) \cong \ker(\eta)$. On the other hand $f \in \ker(\eta)$, thus $\mathcal{F} \cong \mathcal{F}_F$.

If we set $\mathcal{E} := \mathcal{E}_G$ and $\alpha := \alpha_G$, the above discussion proves the "only if" part of the statement of the following main theorem.

THEOREM 2.4. *Let k be a field of characteristic $p \neq 2$ and $C \subseteq \mathbb{P}^n_k$ be a pure subscheme of codimension 2 which is self-linked through hypersurfaces $F := \{f = 0\}$ and $G := \{g = 0\}$ of respective degrees d, m. Let $\Im_C$ be its sheaf of ideals and set $\mathcal{F}_F := \Im_C / f\mathcal{O}_{\mathbb{P}^n_k}(-d)$, $\mathcal{F}_G := \Im_C / g\mathcal{O}_{\mathbb{P}^n_k}(-m)$.*

Assume that if $n \equiv 1 \mod 4$ and $d + m - n - 1 = 2\varrho$, then the following two equivalent congruences hold:

$$\chi(\mathcal{O}_{\mathbb{P}^n_k}(\varrho)) - \chi(\mathcal{O}_{\mathbb{P}^n_k}(\varrho - d)) - \chi(\mathcal{O}_C(\varrho)) \equiv 0 \mod 2,$$
$$\chi(\mathcal{O}_{\mathbb{P}^n_k}(\varrho)) - \chi(\mathcal{O}_{\mathbb{P}^n_k}(\varrho - m)) - \chi(\mathcal{O}_C(\varrho)) \equiv 0 \mod 2.$$

Then there exist a locally free $\mathcal{O}_{\mathbb{P}^n_k}$–sheaf $\mathcal{E}$, a symmetric map $\alpha: \check{\mathcal{E}}(-d-m) \to \mathcal{E}$ and a resolution

$$(2.4.1) \qquad 0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-d) \oplus \check{\mathcal{E}}(-d-m) \xrightarrow{\left(\begin{smallmatrix}\gamma & {}^t\lambda \\ \lambda & \alpha\end{smallmatrix}\right)} \mathcal{O}_{\mathbb{P}^n_k}(-m) \oplus \mathcal{E} \longrightarrow \mathcal{F}_F \longrightarrow 0$$

inducing exact sequences

$$(2.4.2) \qquad 0 \longrightarrow \check{\mathcal{E}}(-d-m) \xrightarrow{\left(\begin{smallmatrix}{}^t\lambda \\ \alpha\end{smallmatrix}\right)} \mathcal{O}_{\mathbb{P}^n_k}(-m) \oplus \mathcal{E} \longrightarrow \Im_C \longrightarrow 0,$$
$$(2.4.3) \qquad 0 \longrightarrow \check{\mathcal{E}}(-d-m) \xrightarrow{\alpha} \mathcal{E} \longrightarrow \mathcal{F}_G \longrightarrow 0.$$

Conversely, given a subscheme C of codimension 2, assume that there does exist a sequence (2.4.2) with the above property of α being symmetric and $\deg(\det(\alpha)) = m$. *Then C is self-linked through the hypersurfaces F and G of respective equations $f := \det \begin{pmatrix} 0 & {}^t\lambda \\ \lambda & \alpha \end{pmatrix}$ and $g := \det(\alpha)$.*

Proof. There remains to prove the converse assertion. Let $P \in \mathbb{P}^n_k$ and consider in $\mathcal{O}_{\mathbb{P}^n_k,P}$ the maximal minors f_i of the matrix $(\lambda\ \alpha)$ obtained by deleting the i^{th} column: in particular $g = f_1$. Recall that $\Im_{C,P} = (f_1, \dots, f_{r+1})$ in $\mathcal{O}_{\mathbb{P}^n_k,P}$, hence for each pair of indices $i, j = 2, \dots r+1$ we have

$$u_{i,j} f + v_{i,j} g = f_i f_j$$

for suitable $u_{i,j}, v_{i,j} \in \mathcal{O}_{\mathbb{P}^n_k,P}$ (the determinantal identity (1.2) of [Ca1] with $k = j = 1$).

It follows that neither f nor g are identically zero, else C would have a codimension one component. The same identity shows that $\Im_C^2 \subseteq \Im_X$. Moreover it is always true that $\Im_X \subseteq \Im_C$, thus F and G have no common components.

We now prove that $\Im_X : \Im_C = \Im_C$. To this purpose we first check that $\mathcal{F} := \operatorname{coker}(\alpha) \cong \Im_C / g\mathcal{O}_{\mathbb{P}^n_k}(-m)$. Indeed the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \check{\mathcal{E}}_G(-d-m) & \longrightarrow & \mathcal{O}_{\mathbb{P}^n_k}(-m) \oplus \mathcal{E}_G & \longrightarrow & \Im_C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle (0\ id)} & & & & \\ 0 & \longrightarrow & \check{\mathcal{E}}_G(-d-m) & \longrightarrow & \mathcal{E}_G & \longrightarrow & \mathcal{F} & \longrightarrow & 0. \end{array}$$

induces a surjection $\Im_C \to \mathcal{F}$. Argueing as above we obtain that its kernel is $g\mathcal{O}_{\mathbb{P}^n_k}(-m)$.

The map α induces an isomorphism $\mathcal{F} \cong \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{F}, \mathcal{O}_G(-d))$, since $\deg(f) = d$. We have $h \in \Im_{X,P} : \Im_{C,P}$ if and only if $h\Im_{C,P} \subseteq \Im_{X,P}$.

Taking residue classes $\mod g$, this is equivalent to $h\mathcal{F}_P \subseteq \Im_{X,P}/g\mathcal{O}_{\mathbb{P}^n_k,P}(-m) \cong \mathcal{O}_{G,P}(-d)$, since $g \in \Im_X$, hence to the fact that $h \mod g$ is in $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n_k,P}}(\mathcal{F}_P, \mathcal{O}_{G,P}(-d)) \cong \mathcal{F}_P$, i.e. $h \in \Im_{C,P}$. □

3. Self–linkage of generically Gorenstein subschemes. In this section we shall inspect more deeply the case when C is a generically Gorenstein pure subscheme of codimension 2 of $\mathbb{P}^n_k$ which is self–linked through two hypersurfaces $F := \{f = 0\}$ and $G := \{g = 0\}$ of degrees $\deg(F) =: d \leq m := \deg(G)$.

The first hypothesys implies that C is also generically locally complete intersection since its codimension is 2. Owing to the isomorphism (1.2.2) we have

$$\Im_C/\Im_X \cong \omega_C(n+1-m-d)$$

whence at each generic point $P \in C$ the sheaf $\omega_{C,P}$ is invertible and a lift of a generator yields y such that $\Im_{C,P} = (f,g,y)\mathcal{O}_{\mathbb{P}^n_k,P}$. Since C is generically complete intersection and $\Im_X = (f,g)$ then $\Im_C$ is either (f,y) or (g,y) at P. In the first case we have locally $g = af + by$. By changing globally g with $g + cf$ for a suitable c we can assume that a is invertible at each generic point of C, hence at each generic point $P \in C$ we may assume that we are in the case $\Im_C = (g,y)$ holds generically.

Let now $H := \{h = 0\}$ be another hypersurface such that C is also self–linked through H and G. Recall that we defined $X := F \cap G$. If we set $\overline{X} := H \cap G$, then the following proposition holds.

PROPOSITION 3.1. *Let C be a generically Gorenstein pure subscheme of codimension 2 of $\mathbb{P}^n_k$ which is self–linked through the two complete intersections $X = F \cap G$ and $\overline{X} = H \cap G$. If $\deg(F) \leq \deg(G)$ then $X = \overline{X}$*

Proof. At each generic point $P \in C$ we have $\Im_C = (y,g)$ and $\Im_X = (f,g)$. By the factoriality of $\mathcal{O}_{\mathbb{P}^n_k,P}$ and since (y,g) is a system of parameters for the regular local ring $\mathcal{O}_{\mathbb{P}^n_k,P}$, the condition $\Im_X : \Im_C = \Im_C$ amounts to the identity $f = gz - y^2$ up to units.

Moreover C is self–linked with respect to both X and $\overline{X}$, whence in $\mathcal{O}_{\mathbb{P}^n_k,P}$

$$(f,g):(y,g) = (y,g) = (h,g):(y,g).$$

It follows that $y^2 \in (h,g)$, hence $(f,g) = (y^2,g) \subseteq (h,g)$ in $\mathcal{O}_{\mathbb{P}^n_k,P}$. Since such an inclusion holds at each generic point $P \in C$ then $(f,g) \subseteq (h,g)$. Changing the roles of f and h we obtain $(h,g) \subseteq (f,g)$, hence equality must hold. □

REMARK 3.2. The condition $\deg(F) \leq \deg(G)$ is necessary as shows the following easy example. Let C be the origin in the affine plane with coordinates h, g. Let $f = h^2 - g^2$. Then $(f,g) \neq (f,h)$.

In order to clarify the role of the hypothesys that C is generically Gorenstein in proposition 3.1, in the remaining part of this section we shall give an example where:

a) C is a pure subscheme of codimension 2 of $\mathbb{P}^n_k$, which is not generically Gorenstein;

b) C is self–linked through $X := F \cap G$, with $F := \{f = 0\}$ and $G := \{g = 0\}$ of degrees $\deg(F) =: d \leq m := \deg(G)$;

c) for each choice of the second generator $\widehat{g} := af + g$ of $\mathfrak{I}_X$, there exists h with $\deg(h) = \deg(f)$ such that C is self–linked through $\overline{X} := H \cap \widehat{G}$, where $\widehat{G} := \{\widehat{g} = 0\}$ and $H := \{h = 0\}$,

d) but $\overline{X} \neq X$.

EXAMPLE 3.3. Let $n \geq 2$, x and y independent linear forms in $\mathbb{P}^n_k$ and $C \subseteq \mathbb{P}^n_k$ the subscheme associated to the ideal (x^2, xy, y^2). Notice that C is not generically Gorenstein.

CLAIM 3.3.1. *Let C be as above. Assume that C is self–linked through two hypersurfaces $F := \{f = 0\}$ and $G := \{g = 0\}$, with $\deg(F) =: d \leq m := \deg(G)$. Then $d = 2$, $m = 3$ and there exist two other linear forms x', y' on $\mathbb{P}^n_k$ such that $(x,y) = (x',y')$ and*

i) *either $f = x'^2$ and $g = y'^3$ mod f,*

ii) *or $f = x'y'$ and $g = x'^3 - y'^3$ mod f.*

Proof. Since $\deg(C) = 3$ then $\deg(F \cap G) = 6$ hence $d = 2$ and $m = 3$. There exists a Hilbert–Burch resolution of $\mathfrak{I}_C$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-3)^2 \xrightarrow{A} \mathcal{O}_{\mathbb{P}^n_k}(-2)^3 \xrightarrow{B} \mathfrak{I}_C \longrightarrow 0$$

where

$$A := \begin{pmatrix} y & 0 \\ -x & y \\ 0 & -x \end{pmatrix}, \qquad B := (x^2, xy, y^2).$$

Then the module $\omega_C(n-4) := \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n_k}}\big(\mathfrak{I}_C, \mathcal{O}_{\mathbb{P}^n_k}(-5)\big)$ has a dual resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-5) \xrightarrow{{}^tB} \mathcal{O}_{\mathbb{P}^n_k}(-3)^3 \xrightarrow{{}^tA} \mathcal{O}_{\mathbb{P}^n_k}(-2)^2 \longrightarrow \omega_C(n-4) \longrightarrow 0.$$

In particular $\omega_C(n-4)$ has two generators e_1, e_2 subject to the relations $ye_1 = xe_2 = xe_1 - ye_2 = 0$. On the other hand we have an isomorphism $\mathfrak{I}_C/\mathfrak{I}_X \cong \omega_C(n-4)$.

Therefore $\mathfrak{I}_C/\mathfrak{I}_X$ has two generators, which lie in degree 2, satisfying the above relations.

Up to a linear change $(x,y) \to (x',y')$ of generators we can assume $f = x'^2$ or $f = x'(x' + cy)$.

We consider then the two possible cases.

$c = 0$, whence $f = x'^2$. Hence, modulo f, we can find $y' = ay + bx'$ such that either $g = x'y'^2$ or $g = y'^3$. The first case is impossible since (f, g) must be a regular sequence. In the second case let $e_1 = y'^2$, $e_2 = x'y' \bmod \Im_X$. We obtain therefore a self–linkage of C.

$c \neq 0$ whence we may set $y' = x' + cy$ and therefore $f = x'y'$. Hence, modulo f, $g = ax'^3 + by'^3$ where $ab \neq 0$, thus we can assume $g = x'^3 - y'^3$. In this case $e_1 = x'^2$, $e_2 = y'^2 \bmod \Im_X$. □

CLAIM 3.3.2. *For each linear form $L \in k[x, y]$ there exist linear forms $\xi, \eta, M \in k[x, y]$ such that*

$$y^3 + Lx^2 = \eta^3 + M\xi^2$$

where x and ξ (resp. ξ and η) are linearly independent.

Proof. Let $L := L_0 x + L_1 y$.

Assume that $L_0 \neq 0$. Then we may set

$$\eta := R'\left(x + \frac{L_1}{3L_0}y\right), \qquad \xi = y, \qquad M := -\frac{L_1^2}{3L_0}x + \left(1 - \frac{L_1^3}{27L_0^2}\right)y,$$

where R' is a fixed cube root of L_0.

Next consider the case $L_0 = L_1 = 0$. Then we may take $\eta = y$, ξ arbitrary and $M = 0$.

Finally let $L_0 = 0$ and $L_1 \neq 0$. Let a be such that

$$(3a^2 - L_1)^2 - 12a^4 = 0. \tag{3.3.2.1}$$

Notice that $a \neq 0$. It follows that $3ay^2 + 3a^2xy + a^3x^2 - L_1xy$ is the square of a linear form $\xi := R''(6ay + (3a^2 - L_1)x)$, where R'' is a fixed square root of $(12a)^{-1}$. Setting $\eta := y + ax$ and $M := -x$, then $y^3 + Lx^2 = \eta^3 + M\xi^2$. Notice that x and ξ are independent, else $a = 0$ which is not a root of equation (3.3.2.1). On the other hand also ξ and η are independent for $L_1 \neq -3a^2$, in which case substituting in (3.3.2.1) we would obtain $24a^4 = 0$, a contradiction. □

Set now $f := x^2$, $g := y^3$, $\widehat{g} := y^3 + Lx^2$ and $h := \xi^2$. Since ξ and η are independent $(h, \widehat{g}) = (\eta^3, \xi^2)$ so that C is self–linked through $H := \{h = 0\}$ and $\widehat{G} := \{\widehat{g} = 0\}$, but $f \notin (h, g)$, since x and ξ are independent.

Thus we have checked that our example satisfies conditions a), b), c) and d) above.

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