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THE LIE TRIPLE SYSTEM OF THE SYMMETRIC SPACE F_4/SPIN(9) *

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Abstract. The Lie triple system of the rank-one compact symmetric space $F_4/Spin(9)$ (the Cayley projective plane) it is described in terms of an algebraic structure associated with the spinor representation of $Spin(9)$. We also discuss related questions in this 16-dimensional representation using the octonion algebra approach.

1. Introduction. Let G/K be the compact symmetric space $F_4/Spin(9)$. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of the Lie groups G and K respectively. Fix on $\mathfrak{g}$ the positive definite scalar product $\langle,\rangle = -\Phi/72$, where Φ is the Killing form of $\mathfrak{g}$. Denote by $\mathfrak{m}$ the orthogonal complement to $\mathfrak{k}$ in $\mathfrak{g}$. The isotropy representation of the Cayley plane G/K $\mathrm{Ad}_\mathfrak{m} : K \to \mathrm{End}(\mathfrak{m})$, $k \mapsto \mathrm{Ad}\, k|\mathfrak{m}$ is the (16-dimensional) spinor representation of $K \simeq Spin(9)$. There exist nine orthogonal endomorphisms $E_j : \mathfrak{m} \to \mathfrak{m}$ satisfying the relations

$$E_jE_p + E_pE_j = 0,\ j \neq p, \quad E_j^2 = Id,\ E_j^* = E_j^{-1} = E_j, \quad j,p = 0,\dots,8$$

and such that the linear 9-dimensional subspace U of $\mathrm{End}(\mathfrak{m})$ spanned by $E_j, j = 0,\dots,8$ is invariant under conjugation by elements of $\mathrm{Ad}_\mathfrak{m} K : (\mathrm{Ad}_\mathfrak{m} k)U(\mathrm{Ad}_\mathfrak{m} k^{-1}) = U$. In the present paper we prove that for any $w, \xi, \eta \in \mathfrak{m}$ the triple commutator $[w, [\xi, \eta]]$ is given by the following formula

$$[w,[\xi,\eta]] = 3\langle w,\eta\rangle\xi - 3\langle w,\xi\rangle\eta + \sum\nolimits_{j=0}^{8}\langle E_j\eta, w\rangle E_j\xi - \sum\nolimits_{j=0}^{8}\langle E_j\xi, w\rangle E_j\eta.$$

In [My] we found a complicated expression for this commutator using the Lie bracket identity $\left[\sqrt{-\mathrm{ad}_w^2}(\xi), \sqrt{-\mathrm{ad}_w^2}(\eta)\right] = -[w,[\xi,\eta]]$, where we consider the vector-functions $w \mapsto \sqrt{-\mathrm{ad}_w^2}(\zeta)$, $\zeta \in \mathfrak{m}$ as the vector fields on the set $\mathfrak{m}^0$ of all nonzero $w \in \mathfrak{m}$. To simplify this expression we prove here that the vector fields $\{\left(|w|^2 Id - |w|\sqrt{-\mathrm{ad}_w^2}\right)(\zeta),\ \zeta \in \mathfrak{m}\}$ commute on $\mathfrak{m}^0$ ($|w|^2 = \langle w, w\rangle$). To this end we use the realization of the spinor representation of the Lie group K in terms of the octonion algebra.

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2. The Lie triple system of F_4/Spin(9).

2.1. The spinor representation of Spin(9). Let G/K be the Riemannian symmetric space $F_4/Spin(9)$, a unique exceptional compact symmetric space of rank one. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of the groups G and K respectively. Let

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Φ be the Killing form of the Lie algebra $\mathfrak{g}$ and $\mathfrak{m}$ be the orthogonal complement to $\mathfrak{k}$ in $\mathfrak{g}$ with respect to Φ, i.e. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is the $\operatorname{Ad} K$-invariant direct sum decomposition of $\mathfrak{g}$.

Fix a base $\{W_b\}$ in $\mathfrak{m}$. Let $\{w_b\}$ be the coordinates in $\mathfrak{m}$ with respect to this basis. With any vector-function $\tau : \mathfrak{m} \to \mathfrak{m}$, $\tau = \sum_b \tau_b(w) W_b$ we can associate the vector field $\sum_b \tau_b(w) \frac{\partial}{\partial w_b}$. By [My, Corollary 3.10.1] for any vectors $\xi, \eta \in \mathfrak{m}$ the following Lie bracket identity holds on the set $\mathfrak{m}^0$ of all nonzero points w from $\mathfrak{m}$

$$\left[\sum\nolimits_b \left(\sqrt{-\operatorname{ad}_w^2}(\xi)\right)_b \frac{\partial}{\partial w_b}, \sum\nolimits_b \left(\sqrt{-\operatorname{ad}_w^2}(\eta)\right)_b \frac{\partial}{\partial w_b}\right] = -\sum\nolimits_b [w, [\xi, \eta]]_b \frac{\partial}{\partial w_b}. \quad (1)$$

But for the symmetric space $F_4/Spin(9)$ (of rank one) the positive restricted root system Σ^+ has only two elements σ and 2σ with multiplicities $m_\sigma = 8$ and $m_{2\sigma} = 7$ [He, Table VI]. Therefore for any $w \in \mathfrak{m}^0$ there exists a restricted root decomposition $\mathfrak{m} = V_8(w) \oplus V_7(w) \oplus V_1(w)$ of the space $\mathfrak{m}$, where the subspaces $V_8(w), V_7(w), V_1(w)$ are eigenspaces of ad_w^2 of dimension $8, 7, 1$ respectively and the space $V_1(w) = \langle w \rangle$ (the Cartan subspace) is generated by the vector w. Let $\mathbf{\Pi}_w^{(8)}, \mathbf{\Pi}_w^{(7)}, \mathbf{\Pi}_w^{(1)}$ be the orthogonal projectors from $\mathfrak{m}$ onto the spaces $V_8(w), V_7(w), V_1(w)$ respectively. Then there exists a unique positive definite scalar product $\langle , \rangle = c\Phi$ on the compact Lie algebra $\mathfrak{g}$ such that $\sqrt{-\operatorname{ad}_w^2}|\mathfrak{m} = |w|(\mathbf{\Pi}_w^{(8)} + 2\mathbf{\Pi}_w^{(7)})$, $|w|^2 \stackrel{\text{def}}{=} \langle w, w \rangle$. Since $\Phi(w,w) \stackrel{\text{def}}{=} \operatorname{Tr} \operatorname{ad}_w^2 = 2\operatorname{Tr}(\operatorname{ad}_w^2|\mathfrak{m})$ and $-\operatorname{ad}_w^2|\mathfrak{m} = \langle w, w \rangle(\mathbf{\Pi}_w^{(8)} + 4\mathbf{\Pi}_w^{(7)})$, the constant $c = -1/72$.

The Ad-representation $R_\mathfrak{m}$ of $K \simeq Spin(9)$ in $\mathfrak{m}$ is a faithful real representation in a 16-dimensional space [He, GG] and is a unique irreducible representation of the group $Spin(9)$ in dimension 16 [On]. Moreover, $\operatorname{Ad}_\mathfrak{m} K \simeq Spin(9)$ acts transitively on the 15-dimensional sphere of all vectors from $\mathfrak{m}$ of constant length; the isotropy group $K_w \stackrel{\text{def}}{=} \{k \in K : \operatorname{Ad} k(w) = w\}$ of any nonzero $w \in \mathfrak{m}$ is isomorphic to $Spin(7)$ [On, Ch.I, §5, Example 5]. The spaces $V_8(w), V_7(w), V_1(w)$ are $\operatorname{Ad} K_w$-invariant. It is clear that $V_8(w)$ and $V_7(w)$ are simple K_w-moduli (see [On, Ch.I, §5, Examples 4,5]). To describe the Lie triple system of $F_4/Spin(9)$ in terms associated with the Lie group $\operatorname{Ad}_\mathfrak{m} K \simeq Spin(9)$ we consider the construction of the spinor representation of $Spin(9)$ in more detail.

Let V be a real vector space of dimension 9 endowed with a positive definite bilinear form Q. Let $e_0, .., e_8$ be an orthonormal basis of V. The Clifford algebra $\mathbb{C}l_+(9)$ in terms of this basis is defined as the real associative algebra with unit 1, generators $e_0, .., e_8$ and defining relations

$$e_j \cdot e_p + e_p \cdot e_j = 0, \quad j \neq p, \qquad e_j^2 = 1, \qquad j, p = 0, \dots, 8.$$

Let $pin_+(9)$ be the subgroup of the multiplicative group of all invertible elements of $\mathbb{C}l_+(9)$ generated by vectors of length one in V. If $Q(v,v) = 1$ then $v \cdot v = 1$, so that $v \in pin_+(9)$. The Lie group $Spin_+(9) \simeq Spin(9)$ is the subgroup of $pin_+(9)$ consisting of even elements, i.e.

$$Spin(9) = \{v_1 \cdot v_2 \cdot \cdots \cdot v_{2p},\ Q(v_j, v_j) = 1,\ j = 1, .., 2p,\ p \in \mathbb{N}\}.$$

Moreover, the group $Spin(9)$ preserves under conjugation the space V: $gVg^{-1} = V$ [Po, Lecture 13].

There exists a faithful 16-dimensional representation ρ_{16} of $pin_+(9)$ by orthogonal matrices [Po, Lecture 15]. In other words, $\operatorname{Ad}_\mathfrak{m} K \subset \rho_{16}(pin_+(9)) \subset O(16)$. Therefore, there exist nine orthogonal (with respect to the form $\langle , \rangle$ on $\mathfrak{m}$) linear transformations $E_p : \mathfrak{m} \to \mathfrak{m}$ satisfying the relations

$$E_jE_p + E_pE_j = 0,\ j \neq p,\ E_j^2 = Id,\ E_j^* = E_j^{-1} = E_j,\ j,p = 0,..,8. \tag{2}$$

Before going into further details we describe these transformations in terms of the octonion multiplication. To this end consider the algebra $\mathbb{C}\mathrm{a}$ of octonions with the standard basis $\mathbf{e}_1 = 1, \mathbf{e}_2, \dots, \mathbf{e}_8$, $\mathbf{e}_p^2 = -1, p = 2, \dots, 8$. To simplify notation we shall often write $ab \cdot c$ and $a \cdot bc$ instead of $(ab)c$ and $a(bc)$ for any elements a, b, c of the nonassociative algebra $\mathbb{C}\mathrm{a}$. Denote by aba the common element $a(ba) = (ab)a$ (the flexible law). We can identify $\mathfrak{m}$ with $\mathbb{C}\mathrm{a}^2$ such that for each $w = (w_1, w_2) \in \mathfrak{m} = \mathbb{C}\mathrm{a}^2$ (see [Po, Lecture 15]):

$$E_0(w_1, w_2) = (-w_1, w_2), \tag{3}$$
$$E_p(w_1, w_2) = (\mathbf{e}_p\overline{w}_2, \overline{w}_1\mathbf{e}_p),\ p = 1, \dots, 8. \tag{4}$$

(the operators $\{\varepsilon_jE_j, \varepsilon_j = \pm 1, j = 0,..,8\}$ also satisfy conditions (2)). Then for any vectors $\xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2) \in \mathfrak{m}$ we have $\langle \xi, \eta \rangle = \langle \xi_1, \eta_1 \rangle + \langle \xi_2, \eta_2 \rangle$, where $\langle \xi_i, \eta_i \rangle = \frac{1}{2}(\xi_i\overline{\eta}_i + \eta_i\overline{\xi}_i)$, $i = 1, 2$, and $|w|^2 = |w_1|^2 + |w_2|^2$, where $|w_i|^2 = w_i\overline{w}_i = \overline{w}_iw_i$.

By (2) for any nonzero $w \in \mathfrak{m}$ the vectors $E_jw, j = 0, \dots, 8$ form an orthogonal basis of some subspace $V_9(w) \subset \mathfrak{m}$. Moreover, since $gVg^{-1} = V$, where $g \in Spin(9)$, for each $k \in K, j = 0, \dots, 8$ the endomorphism $(\mathrm{Ad}_\mathfrak{m}\, k)E_j(\mathrm{Ad}_\mathfrak{m}\, k^{-1})$ is a linear combination of endomorphisms $E_p, p = 0, \dots, 8$. Hence $\mathrm{Ad}\, K_w(V_9(w)) = V_9(w)$. Using the dimension arguments we obtain the splitting $V_9(w) = V_8(w) \oplus V_1(w)$ and the identity

$$\sum_{j=0}^{8} \langle E_jw, \xi \rangle E_jw = |w|^2(\mathbf{\Pi}_w^{(8)} + \mathbf{\Pi}_w^{(1)})(\xi) \stackrel{\mathrm{def}}{=} |w|^2\mathbf{\Pi}_w^{(9)}(\xi), \quad w, \xi \in \mathfrak{m}^0. \tag{5}$$

2.2. Two operator-functions. We continue with the previous notations. Our interest in this subsection focuses on what will be shown to be the most important ingredients of the calculation of the Lie triple system. Consider on $\mathfrak{m}^0$ two operator–functions: $\mathbf{A} : w \mapsto \mathbf{A}_w$,

$$\mathbf{A}_w \stackrel{\mathrm{def}}{=} |w|^2(-\mathbf{\Pi}_w^{(7)} + \mathbf{\Pi}_w^{(1)}) = |w|^2(-Id + \mathbf{\Pi}_w^{(9)} + \mathbf{\Pi}_w^{(1)}) \tag{6}$$

and $\mathbf{B} : w \mapsto \mathbf{B}_w$,

$$\mathbf{B}_w \stackrel{\mathrm{def}}{=} \sum\nolimits_{j=0}^{8} \langle E_jw, w \rangle E_j. \tag{7}$$

Proposition 1. *For any* $w = (w_1, w_2), \xi = (\xi_1, \xi_2) \in \mathfrak{m}^0$

$$\mathbf{A}_w(\xi_1, \xi_2) = (w_1\overline{\xi}_1w_1 + w_1\xi_2 \cdot \overline{w}_2,\ w_2\overline{\xi}_2w_2 + \overline{w}_1 \cdot \xi_1w_2). \tag{8}$$

Proof. By (5) and definition (6)

$$\mathbf{A}_w(\xi) = -\langle w, w \rangle \xi + \sum\nolimits_{j=0}^{8} \langle E_jw, \xi \rangle E_jw + \langle w, \xi \rangle w. \tag{9}$$

Since $E_0(w_1,w_2)=(-w_1,w_2)$, we first readily verify that

$$\begin{aligned}&-(|w_1|^2+|w_2|^2)(\xi_1,\xi_2)+\langle(-w_1,w_2),(\xi_1,\xi_2)\rangle(-w_1,w_2)\\&+\langle(w_1,w_2),(\xi_1,\xi_2)\rangle(w_1,w_2)\\&=(w_1\overline{\xi}_1w_1-|w_2|^2\xi_1,\ w_2\overline{\xi}_2w_2-|w_1|^2\xi_2),\end{aligned}\tag{10}$$

using the identities $(ab)\overline{b}=a(b\overline{b})$ and $\overline{b}(ba)=(\overline{b}b)a$, $a,b\in\mathbb{C}\mathrm{a}$. By relations (4) $\sum\limits_{j=1}^{8}\langle E_jw,\xi\rangle E_jw=((\mu+\nu)\overline{w}_2,\ \overline{w}_1(\mu+\nu))$, where $\mu=\sum\limits_{j=1}^{8}\langle \mathbf{e}_j\overline{w}_2,\xi_1\rangle\mathbf{e}_j$ and $\nu=\sum\limits_{j=1}^{8}\langle\overline{w}_1\mathbf{e}_j,\xi_2\rangle\mathbf{e}_j$. But for any octonions $a,b,c:\langle ca,b\rangle=\langle c,b\overline{a}\rangle$ [Ok, (1.36),(3.8)] and since $\langle ca,b\rangle=\langle\overline{ca},\overline{b}\rangle$ we have also the identity $\langle ac,b\rangle=\langle c,\overline{a}b\rangle$. Therefore $\mu=\sum\limits_{j=1}^{8}\langle\mathbf{e}_j,\xi_1w_2\rangle\mathbf{e}_j=\xi_1w_2$, $\nu=\sum\limits_{j=1}^{8}\langle\mathbf{e}_j,w_1\xi_2\rangle\mathbf{e}_j=w_1\xi_2$ and

$$\sum_{j=1}^{8}\langle E_jw,\xi\rangle E_jw=(w_1\xi_2\cdot\overline{w}_2+|w_2|^2\xi_1,\ \overline{w}_1\cdot\xi_1w_2+|w_1|^2\xi_2)\tag{11}$$

Adding equations (10) and (11) we obtain the assertion of Proposition 1. □

We wish to describe some defining relations between the operator-functions $\mathbf{A}$ and $\mathbf{B}$.

PROPOSITION 2. *For any nonzero* $w=(w_1,w_2),\xi=(\xi_1,\xi_2)\in\mathfrak{m}=\mathbb{C}\mathrm{a}^2$

$$\mathbf{B}_w(\xi)=2\Big(w_1w_2\cdot\overline{\xi}_2,\ \overline{\xi}_1\cdot w_1w_2\Big)+(|w_2|^2-|w_1|^2)(-\xi_1,\ \xi_2)\tag{12}$$

and

$$\mathbf{B}_w=4|w|^2\mathbf{\Pi}_w^{(1)}-|w|^2Id-2\mathbf{A}_w=|w|^2(-\mathbf{\Pi}_w^{(8)}+\mathbf{\Pi}_w^{(7)}+\mathbf{\Pi}_w^{(1)}).\tag{13}$$

Proof. To prove (12) it is sufficient to see that $\langle E_0w,w\rangle E_0\xi=\langle(-w_1,w_2),(w_1,w_2)\rangle(-\xi_1,\xi_2)$ and $\sum_{j=1}^{8}\langle E_jw,w\rangle E_j\xi=((\mu+\nu)\overline{\xi}_2,\ \overline{\xi}_1(\mu+\nu))$, where the octonion $\mu+\nu$ is given by:

$$\begin{aligned}\mu+\nu&=\sum\nolimits_{j=1}^{8}(\langle\mathbf{e}_j\overline{w}_2,w_1\rangle+\langle\overline{w}_1\mathbf{e}_j,w_2\rangle)\mathbf{e}_j\\&=\sum\nolimits_{j=1}^{8}(\langle\mathbf{e}_j,w_1w_2\rangle+\langle\mathbf{e}_j,w_1w_2\rangle)\mathbf{e}_j=2w_1w_2\end{aligned}$$

(see the proof of Proposition 1). To check the second assertion of the proposition, we can use the already proved relations (8) and (12). Indeed, since $ab\cdot\overline{c}+ac\cdot\overline{b}=a(b\overline{c}+c\overline{b})$, $b\cdot\overline{c}a+c\cdot\overline{b}a=(b\overline{c}+c\overline{b})a$ [Po, Lecture 15,(1)] and $\langle a,b\rangle=\langle\overline{a},\overline{b}\rangle$, we have

$$4\langle w,\xi\rangle w-\langle w,w\rangle\xi-2\mathbf{A}_w(\xi)=$$

$$2\Big((w_1\overline{\xi}_1+\xi_1\overline{w}_1)w_1+w_1(w_2\overline{\xi}_2+\xi_2\overline{w}_2),(\overline{w}_1\xi_1+\overline{\xi}_1w_1)w_2+w_2(\overline{w}_2\xi_2+\overline{\xi}_2w_2)\Big)$$

$$-(|w_1|^2+|w_2|^2)(\xi_1,\xi_2)-2(w_1\overline{\xi}_1w_1+w_1\xi_2\cdot\overline{w}_2,\ w_2\overline{\xi}_2w_2+\overline{w}_1\cdot\xi_1w_2)$$

$$= 2\Big(w_1w_2 \cdot \bar{\xi}_2, \bar{\xi}_1 \cdot w_1w_2\Big) + \Big((|w_1|^2 - |w_2|^2)\xi_1, (|w_2|^2 - |w_1|^2)\xi_2\Big) = \mathbf{B}_w.$$

Taking into account definition (6) we complete the proof. □

We can now supplement Proposition 2 with

COROLLARY 2.1. *For any $w \in \mathfrak{m}^0$, $p = 0, \dots, 8$:*
1) $\mathbf{B}_w E_p + E_p \mathbf{B}_w = 2\langle E_p w, w\rangle Id$;
2) $\mathbf{\Pi}_w^{(1)} E_p \mathbf{\Pi}_w^{(7)} = 0$;
3) $\mathbf{\Pi}_w^{(7)} E_p \mathbf{\Pi}_w^{(7)} = \langle E_p w, w\rangle |w|^{-2} \mathbf{\Pi}_w^{(7)}$.

Proof. The first assertion is an consequence of (7) and defining relations (2). The second holds because $V_7(w) \perp E_p w$ and $E_p^* = E_p$. To prove 3) it is sufficient to see that $2|w|^2\mathbf{\Pi}_w^{(7)} = \mathbf{B}_w + |w|^2 Id - 2|w|^2\mathbf{\Pi}_w^{(1)}$, $\mathbf{B}_w\mathbf{\Pi}_w^{(7)} = |w|^2\mathbf{\Pi}_w^{(7)}$ and use 1), 2). Indeed, $(\mathbf{B}_w + |w|^2 Id - 2|w|^2\mathbf{\Pi}_w^{(1)})E_p\mathbf{\Pi}_w^{(7)} = (-E_p\mathbf{B}_w + 2\langle E_p w, w\rangle Id)\mathbf{\Pi}_w^{(7)} + |w|^2 E_p \mathbf{\Pi}_w^{(7)}$. □

The following property of the mapping $\mathbf{A}$ is needed for the proof of Proposition 4.

PROPOSITION 3. *For any vectors $\xi, \eta \in \mathfrak{m}$ the vector fields $\sum_b (\mathbf{A}_w\xi)_b \frac{\partial}{\partial w_b}$ and $\sum_b (\mathbf{A}_w\eta)_b \frac{\partial}{\partial w_b}$ commute on $\mathfrak{m}^0$.*

Proof. It is sufficient to show that the vector-function $Y_w(\xi,\eta) \stackrel{\text{def}}{=} (d/dt)_0 \mathbf{A}_{(w+t\mathbf{A}_w\xi)}(\eta)$ is symmetric for exchanges of two variables ξ and η. Since $2\mathbf{A}_w = 4|w|^2\mathbf{\Pi}_w^{(1)} - |w|^2 Id - \mathbf{B}_w$,

$$2\mathbf{A}_w\xi = 4\langle w, \xi\rangle w - \langle w, w\rangle\xi - \sum_{p=0}^{8}\langle E_p w, w\rangle E_p\xi. \tag{14}$$

Therefore

$$4Y_w(\xi,\eta) = 16\langle w,\eta\rangle\langle w,\xi\rangle w - 4\langle w,w\rangle\langle\xi,\eta\rangle w - 4\sum\nolimits_{p=0}^{8}\langle E_p w, w\rangle\langle E_p\xi,\eta\rangle w$$

$$\begin{aligned}
&+ 16\langle w,\eta\rangle\langle w,\xi\rangle w - 4\langle w,\eta\rangle\langle w,w\rangle\xi - 4\langle w,\eta\rangle\sum\nolimits_{p=0}^{8}\langle E_p w, w\rangle E_p\xi \\
&- 8\langle w,w\rangle\langle w,\xi\rangle\eta + 2\langle w,w\rangle\langle w,\xi\rangle\eta + 2\sum\nolimits_{p=0}^{8}\langle E_p w, w\rangle\langle E_p\xi, w\rangle\eta \\
&- 8\langle w,\xi\rangle\sum\nolimits_{j=0}^{8}\langle E_j w, w\rangle E_j\eta + 2\langle w,w\rangle\sum\nolimits_{j=0}^{8}\langle E_j w,\xi\rangle E_j\eta \\
&+ 2\sum\nolimits_{j,p=0}^{8}\langle E_p w, w\rangle\langle E_j w, E_p\xi\rangle E_j\eta.
\end{aligned}$$

Since all generators E_p are symmetric operators and $w \in V_9(w)$, $\sum_{p=0}^{8}\langle E_p w, w\rangle\langle E_p\xi, w\rangle\eta = \langle w,w\rangle\langle w,\xi\rangle\eta$. Taking into account relations (2) we obtain

$$\begin{aligned}
&\sum\nolimits_{j,p=0}^{8}\langle E_p w, w\rangle\langle E_j w, E_p\xi\rangle E_j\eta \\
&= -\sum\nolimits_{j,p=0}^{8}\langle E_p w, w\rangle\langle E_p w, E_j\xi\rangle E_j\eta + 2\sum\nolimits_{j=0}^{8}\langle E_j w, w\rangle\langle w,\xi\rangle E_j\eta \\
&= -\langle w,w\rangle\sum\nolimits_{j=0}^{8}\langle E_j w,\xi\rangle E_j\eta + 2\langle w,\xi\rangle\sum\nolimits_{j=0}^{8}\langle E_j w, w\rangle E_j\eta.
\end{aligned}$$

Hence

$$4Y_w(\xi,\eta) = S(w,\xi,\eta)w - 4\sum_{j=-1}^{8}\langle E_j w, w\rangle(\langle w,\xi\rangle E_j\eta + \langle w,\eta\rangle E_j\xi),$$

where $S(w,\xi,\eta) = S(w,\eta,\xi) \in \mathbb{R}$ and we put $E_{-1} \stackrel{\text{def}}{=} Id$. Thus $Y_w(\xi,\eta) = Y_w(\eta,\xi)$. □

But for $w \in \mathfrak{m}^0$ $\sqrt{-\operatorname{ad}_w^2}|\mathfrak{m} = |w|(\mathbf{\Pi}_w^{(8)} + 2\mathbf{\Pi}_w^{(7)}) = |w|Id - |w|^{-1}\mathbf{A}_w$ and the function $w \mapsto |w|$ is constant along any vector field $\sqrt{-\operatorname{ad}_w^2}(\xi)$, $\xi \in \mathfrak{m}$, so that as an immediate consequence of (1) and Proposition 3 we have the equality $[w,[\xi,\eta]] = (d/dt)_0\mathbf{A}_{(w+t\xi)}(\eta) - (d/dt)_0\mathbf{A}_{(w+t\eta)}(\xi)$. Now, using (9) or (14) and Proposition 1, we obtain

PROPOSITION 4. *For any* $w,\xi,\eta \in \mathfrak{m}$

$$[w,[\xi,\eta]] = 3\langle w,\eta\rangle\xi - 3\langle w,\xi\rangle\eta + \sum_{j=0}^{8}\langle E_j\eta, w\rangle E_j\xi - \sum_{j=0}^{8}\langle E_j\xi, w\rangle E_j\eta,$$

or in terms of the octonion algebra ($\mathfrak{m} = \mathbb{C}\mathrm{a}^2$)

$$\begin{aligned} &[(w_1,w_2),[(\xi_1,\xi_2),(\eta_1,\eta_2)]] \\ &= \Big((\xi_1\overline{\eta}_1\cdot w_1 + \xi_1\eta_2\cdot\overline{w}_2 + w_1\overline{\eta}_1\cdot\xi_1 + w_1\eta_2\cdot\overline{\xi}_2) \\ &- (\eta_1\overline{\xi}_1\cdot w_1 + \eta_1\xi_2\cdot\overline{w}_2 + w_1\overline{\xi}_1\cdot\eta_1 + w_1\xi_2\cdot\overline{\eta}_2), \\ &\quad(\xi_2\cdot\overline{\eta}_2 w_2 + \overline{\xi}_1\cdot\eta_1 w_2 + w_2\cdot\overline{\eta}_2\xi_2 + \overline{w}_1\cdot\eta_1\xi_2) \\ &- (\eta_2\cdot\overline{\xi}_2 w_2 + \overline{\eta}_1\cdot\xi_1 w_2 + w_2\cdot\overline{\xi}_2\eta_2 + \overline{w}_1\cdot\xi_1\eta_2)\Big). \end{aligned} \tag{15}$$

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