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QUASI–CONFORMAL RIGIDITY OF NEGATIVELY CURVED THREE MANIFOLDS *

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Abstract. In this paper we study the rigidity of infinite volume 3-manifolds with sectional curvature $-b^2 \leq K \leq -1$ and finitely generated fundamental group. In-particular, we generalize the Sullivan's quasi-conformal rigidity for finitely generated fundamental group with empty dissipative set to negative variable curvature 3-manifolds. We also generalize the rigidity of Hamenstädt or more recently Besson-Courtois-Gallot, to 3-manifolds with infinite volume and geometrically infinite fundamental group.

1. Introduction. Let $\widetilde{M}$ be a simply connected complete Riemannian manifold with sectional curvature $-b^2 \leq K \leq -1$. Let $\mathrm{ISO}(\widetilde{M})$ denote the group of isometries of $\widetilde{M}$. Let Γ be a non-elementary, torsion-free, discrete subgroup of $\mathrm{ISO}(\widetilde{M})$, and set $M := \widetilde{M}/\Gamma$.

First we recall some terminologies that is required for the statement of the theorem. Let S_∞ denote the boundary of $\widetilde{M}$. On S_∞ one can define a metric in the following way. Let v be a vector in the unit tangent bundle $S\widetilde{M}$. The geodesic $v(t)$ defines two points on S_∞ given by $v(\infty)$ and $v(-\infty)$. Let π_t be the projection of $S_\infty \backslash v(-\infty)$ along the geodesics which are asymptotic to $v(-\infty)$ to the horosphere which is tangent to $v(-\infty)$ and passing through $v(t)$. Let $\mathrm{dist}_{v,t}$ be the distance on the horosphere induced by restriction of the Riemannian distance, dist. On $S_\infty \backslash v(-\infty) \times S_\infty \backslash v(-\infty)$ define a function η_v as $\eta_v(\xi, \zeta) := e^{-l_v(\xi,\zeta)}$ with $l_v(\xi, \zeta) := \sup\{t | \mathrm{dist}_{v,t}(\pi_t(\xi), \pi_t(\zeta)) \leq 1\}$. By our curvature assumption $-b^2 \leq K \leq -1$, the function η_v is a *distance* on $S_\infty \backslash v(-\infty)$, see [25].

Every element of $\gamma \in \Gamma$ has either exactly one or two fixed points in S_∞, and γ is called loxodromic if it has two fixed points [4]. The group Γ is called *purely loxodromic* if all $\gamma \in \Gamma$ are loxodromic. The limit set of Γ denoted by Λ_Γ is the unique minimal closed Γ-invariant subset of S_∞ [22]. If Γ is purely loxodromic and $\Lambda_\Gamma = S_\infty$, then it can be either cocompact or $\widetilde{M}/\Gamma$ is *geometrically infinite*, hence Γ has infinite co-volume. The *convex hull* CH_Γ is the smallest convex set in $\widetilde{M} \cup S_\infty$ containing Λ_Γ. The group Γ is called *convex-cocompact* if CH_Γ/Γ is compact.

The critical exponent of Γ is the unique positive number D_Γ such that the Poincaré series of Γ given by $\sum_{\gamma \in \Gamma} e^{-s\, \mathrm{dist}(x, \gamma x)}$ is divergent for $s < D_\Gamma$ and convergent for $s > D_\Gamma$. If the Poincaré series diverges at $s = D_\Gamma$ then Γ is called *divergent.*

Let $f : (X, \rho_X) \longrightarrow (Y, \rho_Y)$ be a embedding between two topological metric spaces. Then f is called *quasi-conformal* embedding [47] if there exists a constant $\kappa > 0$ such that, for any $x \in X$ and $r > 0$ there is $r_f(x, r) > 0$ with

$$f(X) \cap B'(f(x), r_f(x, r)) \subset f(B(x, r)) \subset B'(f(x), \kappa r_f(x, r)).$$

where B and B' denotes a ball in X and Y respectively. When $f(X) = Y$ then f is a quasi-conformal homeomorphism.

A torsion-free discrete subgroup Γ of $\mathrm{ISO}(\widetilde{M})$ is called *topologically tame* if $\widetilde{M}/\Gamma$ is homeomorphic to the interior of a compact manifold-with-boundary.

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THEOREM 1.1. *Let $\Gamma' \subset \mathrm{PSL}(2,\mathbb{C})$ be a topologically tame discrete group with $\Lambda_{\Gamma'} = S^2$, and isomorphic $\chi : \Gamma' \longrightarrow \Gamma$ to a convex-cocompact discrete subgroup Γ of $\mathrm{ISO}(\widetilde{M})$ (here $\widetilde{M}$ is n-dimensional). Let $f : S^2 \longrightarrow S_\infty$ be a quasi-conformal embedding which conjugate Γ' to Γ, i.e. $f \circ \gamma = \chi(\gamma) \circ f$, for $\gamma \in \Gamma'$. Then $D_\Gamma \geq D_{\Gamma'}$, and equality if and only if $\mathbb{H}^3$ embeds isometrically into $\widetilde{M}$ and the action of Γ stabilizes the image.*

To state our next theorem we need to introduce one additional terminology. We take $\widetilde{M}$ to be a 3-manifold in the following.

Let $\mathfrak{M}^\lambda_{\eta_v}$ denote the λ-dimensional hausdorff measure on $(S_\infty \backslash v(-\infty), \eta_v)$. We say Γ is *hausdorff-conservative* if there exists a constant $\alpha(v) > 0$ such that $\alpha^{-1} r^{D_\Gamma} \leq \mathfrak{M}^{D_\Gamma}_{\eta_v}(B(\xi, r) \cap \Lambda_\Gamma) \leq \alpha r^{D_\Gamma}$ for any ball $B(\xi, r)$ of radius r about $\xi \in \Lambda_\Gamma$ in $(S_\infty \backslash v(-\infty), \eta_v)$. From this definition, we note that if Γ is a finitely generated torsion-free discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$ with $D_\Gamma = 2$, then hausdorff-conservative implies *conservative* (classical definition, §5). Conversely, if Γ is a topologically tame, *conservative*, discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$, then Γ is hausdorff-conservative, see Proposition 5.2. We believe all finitely generated conservative discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$ are hausdorff-conservative, see Remark 5.3. For a convex-cocompact $\widetilde{M}/\Gamma$ with $-b^2 \leq K \leq -1$, it follows from [12], Γ is hausdorff-conservative. Now we are ready to state the theorem which generalizes Sullivan's quasi-conformal rigidity theorem.

THEOREM 1.2 (Main). *Let Γ be a topologically tame, purely loxodromic discrete subgroup of $\mathrm{ISO}(\widetilde{M})$ with $\Lambda_\Gamma = S_\infty$. Let Γ' be a topologically tame discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$. Suppose $f : S_\infty \longrightarrow S^2$ is a quasi-conformal homeomorphism conjugate Γ to Γ'. Then $D_\Gamma \geq D_{\Gamma'}$, and $\Gamma = \gamma\Gamma'\gamma^{-1}$ with $\gamma \in \mathrm{PSL}(2,\mathbb{C})$ if and only if $D_\Gamma = D_{\Gamma'}$ and Γ is hausdorff-conservative.*

COROLLARY 1.3. *Let $M = \widetilde{M}/\Gamma$ be a complete topologically tame 3-manifold with $-b^2 \leq K \leq -1$, Γ purely loxodromic, and $\Lambda_\Gamma = S_\infty$. Let $h : M \longrightarrow N$ be a quasi-isometric homeomorphism to a hyperbolic manifold N. Then M is isometric to N if and only if $D_\Gamma = 2$ and Γ is hausdorff-conservative.*

Let us point out that Theorem 1.2, generalizes known rigidity theorems in two directions for three dimensional manifolds.

First assume M is hyperbolic ($b = 1$) but not necessarily geometrically finite. Since M is topologically tame and $\Lambda_\Gamma = S^2$ we have $D_\Gamma = 2$ by analytical tameness (see Proposition 3.3). Hence by Theorem 1.2, M is quasi-conformal stable. This is a case of the Sullivan rigidity theorem for topologically tame Γ with empty dissipative set. Next let us assume M is compact with $-b^2 \leq K \leq -1$. Then the critical exponent D_Γ is equal to h_M the topological entropy of M, and by [16], any homotopy equivalence between M and a compact hyperbolic 3-manifold is induced by a homeomorphism. Therefore it follows from Corollary 1.3 we have: M is isometric to a compact hyperbolic 3-manifold if and only if they are homotopically equivalent and $h_M = 2$. This is the Hamenstädt's rigidity or more recently Besson-Courtois-Gallot theorem for 3-manifolds.

Note that it also follows from Theorem 1.2, the quasi-conformal version of the Hamenstädt's theorem for compact 3-manifold M can be stated as:

COROLLARY 1.4. *Let Γ be a cocompact discrete subgroup of $\mathrm{ISO}(\widetilde{M})$. Let $\Gamma' \subset \mathrm{PSL}(2,\mathbb{C})$ be a discrete group. Suppose $f : S_\infty \longrightarrow S^2$ is a quasi-conformal* *homeomorphism conjugate* Γ *to* Γ', *Then* $D_\Gamma \geq D_{\Gamma'}$, *and equality if and only if* $\widetilde{M}/\Gamma$ *is isometric to* $\mathbb{H}^3/\Gamma'$.

The proves of these theorems relies on our next result,

THEOREM 1.5. *Let* $M = \tilde{M}/\Gamma$ *be a topologically tame* 3*-manifold with* $-b^2 \leq \mathcal{K} \leq -1$. *Suppose that* Γ *is purely loxodromic and that* $\Lambda(\Gamma) = S_\infty$. *Then* $2 \leq D$ *and* Γ *is harmonically ergodic. If* $D = 2$ *then* Γ *is also divergent.*

In section 2, we state some of the topological properties of negatively pinched 3-manifolds. In particular, we define *geometrically infinite ends* for negatively pinched 3-manifolds, and then state our theorem which describe the geometrical properties of this type of end, it is a crucial step in the proof of Theorem 1.5. Section 3 discusses measures on S_∞ and the ergodicity of Γ with respect to these measures. In section 4, we give proofs of part I of the theorems. And section 5 is used to complete the proofs.

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2. Topological Ends. Every isometry of $\tilde{M}$ can be extend to a Lipschitz map on $S_\infty := \partial\tilde{M}$ [22]. For a torsion-free Γ, every element $\gamma \in \Gamma$ is one of the following types: (1) *parabolic* if it has exactly one fixed point in $\tilde{M} \cup S_\infty$ which lies in S_∞; (2) *loxodromic* if it has exactly two distinct fixed points in $\tilde{M} \cup S_\infty$, both lying in S_∞.

Denote by $\Lambda(\Gamma) \subset \partial\tilde{M}$ the limit set of Γ, which is the unique minimal closed $\Gamma-$invariant subset of S_∞. Most of the important properties of the limit set in the constant curvature space continue to hold in the variable curvature space [15]. In particular: (i) $\Lambda(\Gamma) = \overline{\Gamma x} \cap S_\infty$; (ii) $\Lambda(\Gamma)$is the closure of the set of fixed points of loxodromic elements of Γ; and (iii) $\Lambda(\Gamma)$ is a perfect subset of Γ. The set $\Omega(\Gamma) := S_\infty \backslash \Lambda(\Gamma)$ is the region of discontinuity. The action of Γ on $\tilde{M} \cup \Omega(\Gamma)$ is proper and discontinuous, see [15]. The manifold $M_\Gamma := \tilde{M} \cup \Omega(\Gamma)/\Gamma$ with possibly nonempty boundary is traditionally called the Kleinian manifold. We also let $\Lambda_c(\Gamma)$ denote the conical limit set of Γ, i.e. $\xi \in \Lambda_c(\Gamma)$ if for some $x \in \tilde{M}$ (and hence for every x) there exist a sequence (γ_n) of elements in Γ, a sequence (t_n) of real numbers, and a real number $C > 0$, such that $\gamma_n x \longrightarrow \xi$ and $\mathrm{dist}(c_x^\xi(t_n), \gamma_n x) < C$ where c_x^ξ is the geodesic ray connecting x and ξ. Equivalently, a point belongs to $\Lambda_c(\Gamma)$ if it belongs to infinitely many shadows cast by balls of some fixed radius centered at points of a fixed orbit of Γ. Note that $\Lambda_c(\Gamma)$ is a $\Gamma-$invariant subset of $\Lambda(\Gamma)$, hence a dense subset.

PROPOSITION 2.1 (Margulis Lemma). *There exists a number* ϵ_b *which only depend on the pinching constant* b *of* M, *such that the group* Γ_ϵ *generated by elements in* Γ *of length at most* ϵ_b *with respect to a fixed point in* M *is almost nilpotent of rank at most* 2. *Then the number,* $2\epsilon_b$ *is called the* Margulis constant.

Note that, if M is orientable and Γ is torsion-free, then Margulis Lemma implies Γ_{ϵ_b} is abelian.

Let $\epsilon \leq \epsilon_b$ be given. Then M may be written as the union of a *thin part* $M_{[0,\epsilon)}$ consisting of all points at which there is based a homotopically nontrivial loop of length $\leq \epsilon$ and a *thick part* $M_{[\epsilon,\infty)} = \overline{M - M_{[0,\epsilon)}}$. Note that $M_{[\epsilon,\infty)}$ is compact if

M is of finite volume. Also the thin part of M is completely classified by the next proposition.

PROPOSITION 2.2. *Each connected component of* $M_{[0,\epsilon)}$ *is diffeomorphic to one of the following :*
parabolic rank-1 cusp : $S^1 \times \mathbb{R} \times [0, \infty)$.
parabolic rank-2 cusp : $T^2 \times [0, \infty)$.
solid torus about the axis of a loxodromic γ : $D^2 \times S^1$.

For simplicity we restrict to the case where M has no cusps. It follows from the existence of a compact core $C(M)$ for M [14], that M has only finitely many ends [5]. In fact, each component of $\partial C(M)$ is the boundary of a neighborhood of an end of M, and this gives a bijective correspondence between ends of M and components of $\partial C(M)$.

We define the simplicial ruled surfaces as follows. Let S be a surface of positive genus and let T_P be a triangulation defined with respect to a finite collection P of points of S. This means that T_P is a maximal collection of nonisotopic essential arcs with end points in P; these arcs are the *edges* of the triangulation, and the components of the complement in S of the union of the edges are the *faces*. Let $f : S \longrightarrow M$ be a map which takes edges to geodesic arcs and faces to nondegenerate geodesic ruled triangles in M. The map f induces a singular metric on S. If the total angle about each vertex of S with respect to this metric is at least 2π, then the pair (S, f) is called a *simplicial ruled surface*. It follows from the definition of the induced metric on S that f preserves lengths of paths and is therefore distance non-increasing. Any geodesic ruled triangle in M has Gaussian curvature at most $-a^2$. This means that each 2-simplex of S inherits a Riemannian metric of curvature at most $-a^2$. Since we have required the the total angle at each vertex to be at least 2π, by Gauss-Bonnet theorem the curvature of S is negative in the induced metric.

DEFINITION 2.3. *An end E is said to be a* geometrically infinite *if there exists a divergent sequence of geodesics, i.e: there exists a sequence of closed geodesics* $\alpha_k \subset M^\circ_\epsilon$, *such that for any neighborhood U of E, there exists some positive integer N such that* $\alpha_k \subset U$ *for all* $k > N$. *If in addition for some surface S_E we have that U is homeomorphic to* $S_E \times [0, \infty)$, *and there exists a sequence of simplicial ruled surfaces* : $S_E \xrightarrow{f_l} U$ *such that* $f_l(S_E)$ *is homotopic to* $S_E \times 0$ *in U and leaves every compact subset of M, then E is said to be* simply degenerate. *The sequence* $(S_E \xrightarrow{f_l} U)$ *is called an* exiting sequence. *A end which is not geometrically infinite will be called* geometrically finite.

THEOREM 2.4 (Hou). *Let* $M = \tilde{M}/\Gamma$ *be a topologically tame negatively pinched 3-manifold with Γ purely loxodromic. Then all geometrically infinite ends of M are simply degenerate. And if* $\Lambda(\Gamma) = S_\infty$, *then there are no nonconstant positive superharmonic functions, or nonconstant subharmonic functions bounded above, on M.*

3. Γ-action. In this section we will study the action of Γ on S_∞ and prove ergodicity of Γ for topologically tame 3-manifolds with $\Lambda(\Gamma) = S_\infty$. We will prove that for such a manifold, the Green series is divergent, and that the Poincaré series is also divergent if $D = 2$. Theorem 1.5 will also be proved in this section.

In some situations we will take the dimension of M to be 3, otherwise we will assume M is n-dimensional in general.

Set the following notations throughout the paper. Let $\Gamma' \subset \mathrm{PSL}(2,\mathbb{C})$ be a discrete orsion-free subgroup. Denote $S^2 := \partial\mathbb{H}^3$, and $S_\infty := \partial\widetilde{M}$. There are many equivalent ways of equipping S_∞ with a metric which is compatible with Γ-action. Fix a point $x \in \widetilde{M}$. Let ξ, ζ in S_∞ be given. Set $c_y^\zeta(t)$ as the geodesic ray connecting y and ζ.

In [22], Gromov defined a metric on S_∞ as follows. For $y, z \in \widetilde{M}$, let us consider arbitrary continuous curve $c(t)$ in $\widetilde{M}$ with initial point and end point denoted by $c(t_0) = y$ and $c(t_1) = z$ respectively. Define a nonnegative real-valued function $\mathcal{G}_x$ on $\widetilde{M} \times \widetilde{M}$ by

$$\mathcal{G}_x(y,z) := \inf_{\text{all } c} \left(\int_{[t_0,t_1]} e^{-\operatorname{dist}(x,c(t))} \mathrm{d}t \right).$$

In particular, Gromov showed the function $\mathcal{G}_x$ extends continuously to $S_\infty \times S_\infty$. Every element of Γ extends to S_∞ as a Lipschitz map with respect to $\mathcal{G}_x$.

In [31] the following metrics are shown to be equivalent to the Gromov's metric.

K_x **metric** : Let B_ζ denote the Busemann function based at x_0. Set $B_\zeta(x,y) = B_\zeta(x) - B_\zeta(y)$, for $x, y \in \widetilde{M}$,
the function $B_\zeta(x,y)$ is called the Busemann cocycle. Define $\beta_x : S_\infty \times S_\infty \longrightarrow \mathbb{R}$ by $\beta_x(\xi,\zeta) := B_\xi(x,y) + B_\zeta(x,y)$ where y is a point on the geodesic connecting ξ and ζ. The K_x metric is then defined by

$$K_x(\xi,\zeta) := e^{-\frac{1}{2}\beta_x(\xi,\zeta)}.$$

L_x **metric:** Let $\alpha_x(\xi,\zeta)$ denote the distance between x and the geodesic connecting ξ and ζ. The function $L_x : S_\infty \times S_\infty \longrightarrow \mathbb{R}$ is then defined by

$$L_x(\xi,\zeta) := e^{-\alpha_x(\xi,\zeta)}.$$

d_x **metric:** Define a function $l_x : S_\infty \times S_\infty \longrightarrow \mathbb{R}$ by $l_x(\xi,\zeta) := \sup\{\tau | \operatorname{dist}(c_\xi^x(\tau), c_\zeta^x(\tau)) = 1\}$. Geometrically, a neighborhood about ξ in S_∞ with respect to the topology induced by l_x is the shadow cast by the intersection of 1-ball about $c_x^\xi(\tau)$ and τ-sphere about x. The d_x metric is then defined by

$$d_x(\xi,\zeta) := e^{-l_x(\xi,\zeta)}.$$

It was originally observed for symmetric spaces by Mostow [36] that the boundary map is quasi-conformal. This property continue to hold in negatively curved spaces, see [25] and [40]. Here we give a proof of this fact with respect to the above metrics.

PROPOSITION 3.1. *Let h be a quasiisometry between two negatively pinched curved spaces. The boundary extension map $\bar{h}$ is quasi-conformal on the boundary with respect to d_x, L_x, K_x, η_v-metrics.*

Proof. For the proof of η_v-metric See Proposition 3.1 in [25]. Fix $x \in \tilde{M}$. Let us take d_x-metric. Set $\lambda \geq L$. Denote by $S(x;y,R)$ the shadow cased from x of the metric sphere $S(y,R)$ with center located at y and radius R, i.e. $S(x;y,R) = \{\xi \in S_\infty | c_x^\xi \cap S(y,R) \neq \emptyset\}$. Let $B(\xi,r)$ be a ball of radius r in S_∞. Using triangle comparison we can show there exists a constant $\alpha_b \geq 1$ depends on pinching constant b such that

$$S(x; c_x^\xi(t_r), \lambda) \subset B(\xi, r) \subset S(x; c_x^\xi(t_r), \alpha_b \lambda)$$

for some $t_r > 0$ which depends only on r. The images $\bar{\phi}(S(x; c_x^\xi(t_r), \lambda))$ and $\bar{\phi}(S(x; c_x^\xi(t_r), \alpha_b\lambda))$ are quasi-spheres, i.e. there exists a constant $\beta_\phi > 0$ depends on ϕ such that $S(\bar{\phi}(x); \bar{\phi}(c_x^\xi(t_r)), \beta_\phi^{-1}\lambda) \subset \bar{\phi}(S(x; c_x^\xi(t_r), \lambda))$ and $\bar{\phi}(S(x; c_x^\xi(t_r), \alpha_b\lambda) \subset S(\bar{\phi}(x); \bar{\phi}(c_x^\xi(t_r)), \beta_\phi \alpha_b \lambda)$. On the other hand, by estimates in [11] there exists positive numbers $A_1(\beta_\phi, \lambda)$ and $A_2(\alpha_b, \beta_\phi, \lambda)$ such that

$$B(\bar{\phi}(\xi), A_1 e^{-R}) \subset S(\bar{\phi}(x); \bar{\phi}(c_x^\xi(t_r)), \beta_\phi^{-1}\lambda),$$

$$S(\bar{\phi}(x); \bar{\phi}(c_x^\xi(t_r)), \beta_\phi \alpha_b \lambda) \subset B(\bar{\phi}(\xi), A_2 e^{-R})$$

where $R = \text{dist}(\bar{\phi}(x), \bar{\phi}(c_x^\xi(t_r)))$. Hence the result follows by setting $r_\phi(\xi, r) = A_1 e^{-R}$ and $\kappa = A_2/A_1$. □

PROPOSITION 3.2. *Let $f : \partial\widetilde{N} \longrightarrow S_\infty$ be a embedding conjugate Γ_1 to Γ_2 under isomorphism $\chi : \Gamma_1 \longrightarrow \Gamma_2$ ($f \circ \gamma = \chi(\gamma) \circ f$). Then $f(\Lambda_{\Gamma_1}) = \Lambda_{\Gamma_2}$.*

Proof. Let $\gamma \in \Gamma_1$. Since $\gamma f^{-1}(\Lambda_{\Gamma_2}) = f^{-1}(\chi(\gamma)\Lambda_{\Gamma_2})$, and by Γ_2-invariance of Λ_{Γ_2}, we have $f^{-1}(\Lambda_{\Gamma_2})$ is Γ_1-invariant closed set. Note that $f^{-1}(\Lambda_2)$ is nonempty, since fixed points of elements of Γ_1 are also fixed points of elements of Γ_2, hence $f(\Lambda_{\Gamma_1}) \subseteq \Lambda_{\Gamma_2}$. Similarly we also have $f(\Lambda_{\Gamma_1}) \supseteq \Lambda_{\Gamma_2}$, and result follows. □

PROPOSITION 3.3. *Let Γ be a topologically tame, torsion-free, discrete subgroup of* $\text{ISO}(\widetilde{M})$ *with $\Lambda_\Gamma = S_\infty$. Let Γ' be a topologically tame, discrete subgroup of* $\text{PSL}(2, \mathbb{C})$. *Suppose $f : S_\infty \longrightarrow S^2$ is a homeomorphism conjugate Γ to Γ'. Then $D_{\Gamma'} = 2$ and Γ' is divergent.*

Proof. By Proposition 3.2 the hyperbolic manifold $N = \mathbb{H}^3/\Gamma'$ is topologically tame and $\Lambda_{\Gamma'} = S^2$. It follows from analytical tameness and Theorem 9.1 of [10], there exists no non-trivial positive superharmonic function on N with respect to the hyperbolic Laplacian Δ. Let $P(y, \xi)$ denote the Poisson kernel on $\mathbb{H}^3$. The $D_{\Gamma'}$-dimensional conformal measure (Patterson-Sullivan measure, see end of §3) σ_y has Radon-Nikodym derivative of $P(y, \xi)^{D_{\Gamma'}}$, i.e. $\frac{d\sigma_y}{d\gamma^*\sigma_y}(\xi) = P(\gamma^{-1}y, \xi)^{D_{\Gamma'}}$. The Γ'-invariant function $h(y) := \sigma_y(S^2)$ satisfies $\Delta h = D_{\Gamma'}(D_{\Gamma'} - 2)h$, which implies h is non-trivial superharmonic if $D_{\Gamma'} \neq 2$. And it follows that Γ' must also be divergent. □

Let C be a subset of S_∞. Let λ-dimensional Hausdorff measure of C on the metric space(S_∞, ρ_x) be denoted by $\mathfrak{M}^\lambda_{\rho_x}(C)$. Observe that for any $x \in \widetilde{M}$ and any $\gamma \in \Gamma$, we have $\gamma^*\mathfrak{M}^\lambda_{K_x} = \mathfrak{M}^\lambda_{K_{\gamma^{-1}x}}$; this follows from the straightforward identity.

A family of finite Borel measures $[\nu_y]_{y \in \widetilde{M}}$. will be called a *$\lambda$-conformal density under the action of Γ* if for every $x \in \widetilde{M}$ and every $\gamma \in \Gamma$ we have $\gamma^*\nu_y = \nu_{\gamma^* y}$, and the Radon-Nikodym derivative $\frac{d\nu_y}{d\gamma^*\nu_y}(\zeta)$ at any point $\zeta \in S_\infty$ is equal to $e^{-\lambda B_\zeta(\gamma^{-1}y, y)}$. (This is to be interpreted as being vacuously true if, for example, the measures in the family are all identically zero). Although there can not be any Γ-invariant non-trivial finite Borel measure on Λ_Γ for non-elementary Γ, we can always define a Γ-invariant non-trivial locally finite measure Π_{ν_x} on $\Lambda_\Gamma \times \Lambda_\Gamma$ by setting $\mathrm{d}\Pi_{\nu_x}(\xi,\zeta) = e^{\lambda\beta_x(\xi,\zeta)}\mathrm{d}\nu_x(\xi)\mathrm{d}\nu_x(\zeta)$. The measure Π_{ν_x} corresponds to the Bowen-Margulis measure on the unit tangent bundle $S\widetilde{M}$ see [31].

Let us recall a fundamental fact about conformal density, which was originally proved by Sullivan for $\Gamma \subset \mathrm{SO}(n,1)$ and generalized to the pinched negatively curved spaces in [46]. It relates the divergence of Γ at the critical exponent D_Γ with ergodicity of the D_Γ-conformal density under the action of Γ.

We will say that two Borel measures on S_∞ are in the same Γ-class if the Radon-Nikodym derivative of $\gamma^*\nu_1$ with respect to ν_1 is equal to the Radon-Nikodym derivative of $\gamma^*\nu_2$ with respect to ν_2.

PROPOSITION 3.4 (see [46]). *Let Γ be a nonelementary, discrete, torsion-free and divergent at D_Γ. Suppose $[\nu]$ is a D_Γ-conformal density under the action of Γ, then Γ act ergodically on Λ_Γ and $\Lambda_\Gamma \times \Lambda_\Gamma$ with respect to $[\nu_x]$ and $[\Pi_{\nu_x}]$ respectively.*

PROPOSITION 3.5 (see [37]). *Let Γ be nonelementary and discrete. Suppose that Γ acts ergodically on S_∞ with respect to a measure ν defined on S_∞. Then every measure of S_∞ in the same measure class as ν is a constant multiple of ν.*

PROPOSITION 3.6. *Let Γ be a non-elementary discrete subgroup of the isometry group of $\tilde{M}$. If $[\nu_y]^D_{y\in\tilde{M}}$ is a non-trivial Γ-invariant D-conformal density, then $D \neq 0$.*

Proof. Suppose $D = 0$. Then ν_y is a Γ-invariant non-trivial finite Borel measure. Since Γ is non-elementary, there exists a loxodromic element γ in Γ. Let $\xi,\zeta \in S_\infty$ be the two distinct fixed points of γ. Let $<\gamma>$ be the group generated by γ. Then ν_y is clearly $<\gamma>$-invariant. But γ is loxodromic, so we must have $\mathrm{supp}(\nu_y) \subset \{\xi,\zeta\}$. Then, by the fact that $\Lambda(\Gamma)$ is infinite, we have ν_y is an infinite measure, which is a contradiction. □

PROPOSITION 3.7. *Let Γ be a discrete subgroup of* $\mathrm{ISO}(\widetilde{M})$. *Suppose $\mathfrak{M}^\lambda_{K_x}$ is a finite measure. Then $\mathfrak{M}^\lambda_{K_x}$ is a λ-conformal density under the action of Γ.*

There is a canonical way of constructing D_Γ-dimensional conformal density which is due to Patterson-Sullivan as follows; By applying a adjusting function we can always assume the Poincaré series diverges at D_Γ. The measures

$$\mu_{x,s} := \frac{\sum_{\gamma\in\Gamma} e^{-s\,\mathrm{dist}(z,\gamma x)}\delta_{\gamma x}}{\sum_{\gamma\in\Gamma} e^{-s\,\mathrm{dist}(z,\gamma z)}} \quad ; s > D_\Gamma$$

converges weakly to a limiting measure μ_x as $s_n \to D_\Gamma$ through a subsequence. It is trivial to see that μ_x is supported on Λ_Γ. The measure $[\mu_x]$ is called Patterson-Sullivan measure which is D_Γ-conformal under Γ see [39], [42].

From now on for $\Gamma' \subset \mathrm{PSL}(2,\mathbb{C})$, we will denote the Patterson-Sullivan measure on $\Lambda_{\Gamma'}$ by σ_y.

Let Λ^c_Γ denote the set of conical limit points in Λ_Γ. Recall a point $\xi \in \Lambda_\Gamma$ is in Λ^c_Γ if and only if there exists $\{\gamma_n\} \subset \Gamma$ such that $\mathrm{dist}(\gamma_n c^\xi_x(t_n), x) < c$ for some $c > 0$ and sequence of t_n. Obviously Λ^c_Γ is Γ-invariant, and non-empty (a loxodromic fixed points are in Λ_Γ^c), hence it is a dense Γ-invariant subset of Λ_Γ. A equivalent definition for the conical limit point ξ is that it must be contained in infinitely many shadows $S(x;\gamma_n x, c)$. Hence $\Lambda_\Gamma^c = \cup_{\lambda>0}\cap_{m\geq 1}\cup_{n>m} S(x,\gamma_n x,\lambda)$. It is a easy fact from the construction of μ_x, no points in Λ_Γ^c can be a atom for μ_x, and if $\mathrm{supp}(\mu_x)\subseteq\Lambda_\Gamma^c$ then Γ is divergent. In fact it is a deep result of Sullivan that Γ is divergent if and only if $\mathrm{supp}(\mu_x)\subseteq\Lambda_\Gamma^c$.

LEMMA 3.8 (see [12]; Sullivan's Shadow Lemma). *Let μ_x be a D_Γ-conformal density with respect to Γ, which is not a single atom. Then there exists constants $\alpha>0$ and $\lambda_o\geq 0$, such that,*

$$\alpha^{-1}e^{-D_\Gamma \operatorname{dist}(x,\gamma^{-1}x)} \leq \mu_x(S(x;\gamma x,\lambda)) \leq \alpha e^{-D_\Gamma \operatorname{dist}(x,\gamma^{-1}x)+2D\lambda},$$

for all $\gamma\in\Gamma$ and $\lambda\geq\lambda_o$.

PROPOSITION 3.9. *Let $\Gamma\subset \mathrm{ISO}(\widetilde{M})$ be a discrete subgroup. Suppose either $\Lambda_\Gamma=\Lambda_\Gamma^c$ or $\Lambda_\Gamma=S_\infty$ and Γ is divergent. Then μ_x is positive on all non-empty relative open subsets of Λ_Γ.*

Proof. Suppose $\Lambda_\Gamma=S_\infty$. It suffices to show μ_x is positive for any non-empty open ball $B(\xi,r)$ with respect to the d_x-metric. Fix $\lambda>\lambda_o$. Let $\zeta\in\Lambda_\Gamma^c\cap B(\xi,r)$ (note that Λ_Γ^c is dense in Λ_Γ so the intersection is nonempty). Then we can choose $\gamma\in\Gamma$ such that $S(x;\gamma x,\lambda)\subset B(\xi,r)$. By assumption Γ is divergent, we have $\mathrm{supp}(\mu_x)\subseteq\Lambda_\Gamma^c$. Since no points of Λ_Γ^c can be a atom for conformal density, the result follows from Lemma 3.8. Same argument works if $\Lambda_\Gamma=\Lambda_\Gamma^c$ □

Let us define a function $\Theta:\tilde{M}\times\tilde{M}\times S_\infty\longrightarrow\mathbb{R}^+$ by $\Theta(x,y,\xi):=\exp(-B_\xi(x,y))$.

Harmonic Density

Let λ_1 and $\tilde{\lambda}_1$ denote the first of the spectrum of Δ on $M=\tilde{M}/\Gamma$, and of $\tilde{\Delta}$ on $\tilde{M}$, respectively. Recall that for a noncompact open manifold, the first of the spectrum is defined as

$$\lambda_1 := \inf_{f\in C_o^\infty, f\neq 0}\left(\frac{\int|\nabla f|^2}{\int f^2}\right)$$

where C_o^∞ is the space of smooth functions on M with compact support. Note that we always have $\lambda_1\leq\tilde{\lambda}_1$.

The λ_1-harmonic functions has been studied by Ancona in [2] and [3].

PROPOSITION 3.10 (Ancona). *For each $s<\lambda_1$, the elliptic operator $\tilde{\Delta}+sI$ has a Green function $G_s(x,y)$, and there exists a function $f:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ such that $\sum_{\gamma\in\Gamma}\hat{G}_s(x,\gamma y)$ converges for $s<\lambda_1$ and diverges for $s\geq\lambda_1$, where $\hat{G}_s(x,\gamma y):=\exp(f(\operatorname{dist}(y,\gamma y)))G_s(x,\gamma y)$. Furthermore, $\mathfrak{P}_s(x,y,\zeta):=\lim_{z\to\zeta}\frac{G_s(x,z)}{G_s(y,z)}$ defines the Poisson kernel of $\tilde{\Delta}+sI$ at $\zeta\in S_\infty$.*

Similarly to the construction of μ_x from Z_Γ, one can also construct a family of Borel measures from $\sum_{\gamma\in\Gamma}\hat{G}_s(x,\gamma y)$.

PROPOSITION 3.11. *Let x be any point of $\tilde{M}$. There exists a family of Borel measures $[\omega_y^1]_{y\in\tilde{M}}$ on S_∞ such that (i) for all $x,y\in\tilde{M}$, Radon-Nikodym derivative $d\omega_y^1/d\omega_x^1$ at any point $\zeta\in S_\infty$ is equal to $\mathfrak{P}_{\lambda_1}(x,y,\zeta)$ and (ii) ω_x^1 is of mass 1.*

Let us denote the harmonic density of $\tilde{\Delta}$ by $[\omega_y]_{y\in\tilde{M}}$ with ω_x normalized of mass 1. By definition this means that every harmonic function f on M with boundary values f_∞ is given by

$$f(x) = \int_{S_\infty} f_\infty(\xi)\mathrm{d}\omega_x(\xi).$$

The existence and uniqueness of harmonic density follows from the solvability of the Dirichlet problem on $\tilde{M} \cup S_\infty$ see [1] and the Riesz Representation Theorem. The Radon-Nikodym derivative of ω_x at $\xi \in S_\infty$ is given by the Poisson kernel $\mathfrak{P}(x, y, \xi)$ of $\tilde{\Delta}$, i.e. $\frac{\mathrm{d}\omega_y}{\mathrm{d}\omega_x}(\xi) = \mathfrak{P}(y, x, \xi)$. For any Γ-invariant subset $C \subset S_\infty$, the function h_C on $\tilde{M}$ defined by $h_C(y) := \int_{S_\infty} \chi_C \mathfrak{P}(y, x, \xi)\mathrm{d}\omega_x(\xi)$ is Γ-invariant, hence defines a harmonic function on M.

PROPOSITION 3.12. *Let $M = \tilde{M}/\Gamma$ be a negatively pinched topologically tame 3-manifold with $\Lambda(\Gamma) = S_\infty$. Then Γ is ergodic with respect to harmonic density $[\omega_y]_{y\in\tilde{M}}$.*

Proof. Suppose not, and let $C \subset S_\infty$ be a Γ-invariant subset with $\omega_x(C) > 0$ and $\omega_x(C^c) > 0$. By Fatou's conical convergence theorem, we have $\chi_C(\xi) = \lim_{t\to\infty} h_C(c_y^\xi(t))$ for $\xi \in S_\infty$. Hence, h_C defines a positive nonconstant Γ-invariant harmonic function, which contradicts Theorem 2.4. Therefore Γ must be ergodic. □

PROPOSITION 3.13. *Let M be noncompact and satisfy the hypothesis of Proposition 3.12. Then $\omega_x^1 = \omega_x$.*

Proof. Let us note that $\lambda_1 = 0$. This follows from the fact that for a noncompact, complete Riemannian manifold M, if $\lambda_1(M) > 0$ then there exists a positive Green's function G on M. If such a G exists, then $1 - \exp(-G)$ defines a positive superharmonic function, which is a contradiction to Theorem 2.4. Hence we must have $\lambda_1 = 0$. Therefore $\mathfrak{P}_{\lambda_1} = \mathfrak{P}$, i.e $\frac{\mathrm{d}\omega_y^1}{\mathrm{d}\omega_x^1} = \frac{\mathrm{d}\omega_y}{\mathrm{d}\omega_x}$. Hence, by Proposition 3.12 and uniqueness, we have the desired result. □

Superharmonic Functions

Let $\xi \in S_\infty$ be given. Let $\mathcal{E}$ be a continuous unit vector field on $\tilde{M}$ with $\mathcal{E}(x) = \Phi_x(\xi)$. Then by using the first length variation formula, one can show that B_ξ is C^1 and that $-\operatorname{grad} B_\xi = \mathcal{E}$. In fact, the Busemann function is $C^{2,\alpha}$, see [27].

Let $[\mu_y]^D$ denote D-conformal density. Let us define a nonnegative function u on $\tilde{M}$ by

$$u(y) := \int_{S_\infty} \Theta^D(y, x, \xi)\mathrm{d}\mu_x(\xi).$$

PROPOSITION 3.14. *The function u is a Γ-invariant and positive. It is superharmonic if $D \leq (n-1)a$, and subharmonic if $(n-1)a \leq D \leq (n-1)b$.*

Proof. We can write $u(y)$ as $\mu_y(S_\infty)$. Since $u(\gamma y) = \gamma^*\mu_y(S_\infty) = \mu_y(S_\infty) = u(y)$ for $\gamma \in \Gamma$, we have that u is Γ-invariant.

Let $x \in \tilde{M}$ be fixed. It follows from, $|\nabla B_\xi(y,x)| = |\mathcal{E}| = 1$ and Rauch'stheorem that we have $\exp(-DB_\xi(y,x))D(D-(n-1)b) \leq \Delta\Theta^D \leq \exp(-DB_\xi(y,x))D(D-(n-1)a)$. This implies the result. □

PROPOSITION 3.15. *Suppose Γ is nonelementary, i.e. has no abelian subgroup of finite index. Suppose that there are no nontrivial Γ-invariant positive-valued superharmonic function on $\tilde{M}$. Then $(n-1)a \leq D \leq (n-1)b$.*

Proof. Let Γx be the orbit of x under Γ. Then the growth rate of the number of points of Γx in $\text{ball}(x,r)$ as r increases is bounded by $\text{vol}(\text{ball}(x,r))$. By the volume comparison theorem we have $C_n \exp((n-1)br) \geq \text{vol}(\text{ball}(x,r))$, for some constant C_n which depends only on dimension n. Therefore, when $s > (n-1)b$ we get $Z_\Gamma(x,s) < \infty$, which implies that $D \leq (n-1)b$.

Next suppose that we have $D \leq (n-1)a$. Then by Proposition 3.14, $u(x)$ is a Γ-invariant positive superharmonic and $\Delta u \leq D(D-(n-1)a)u$. It now follows from the hypothesis that u is constant and that either $D = 0$ or $D = (n-1)a$. However, since Γ is nonelementary and $[\mu_y]$ is Γ-invariant, Proposition 3.6 implies that $D \neq 0$. Hence, $D = (n-1)a$, and the result follows. □

The next proposition was originally proved by Sullivan [41] using a Borel-Cantelli type of argument. The proof is purely measure theoretic (see [37], [46]). The proposition relates the ergodicity of Γ with the divergence of the Poincaré series at D.

PROPOSITION 3.16 (Sullivan). *Suppose that Γ is nonelementary, discrete and torsion-free, and is divergent at D. Then Γ is ergodic with respect to $[\mu]^D$.*

PROPOSITION 3.17. *Suppose $D = (n-1)a$ and there are no nontrivial positive superharmonic functions on M. Then Γ is divergent.*

Proof. Fix a point $y \in \tilde{M}$. Let us assume the Poincaré series converges at D (i.e. $\sum_{\gamma\in\Gamma} \exp(-(n-1)a\,\text{dist}(x,\gamma y)) < \infty$). Then this series defines a nontrivial Γ-invariant function on $\tilde{M}$. Let us denote this function by $h(x)$. Since $\exp(-D\,\text{dist}(z,\gamma y)) \leq \exp(D\,\text{dist}(x,z))\exp(-D\,\text{dist}(x,\gamma y))$ for $z \in \tilde{M}$, it follows that for any given number $N > 0$ there is a constant $C > 0$ such that $\sum_{\gamma\in\Gamma}\exp(-D\,\text{dist}(z,\gamma y)) \leq C\sum_{\gamma\in\Gamma}\exp(D\,\text{dist}(x,\gamma y))$ for $\text{dist}(z,x) \leq N$. Hence the series converges uniformly on compact subsets of $\tilde{M}$. We will show that the convergence of the Poincaré series at $(n-1)a$ implies existence of nontrivial positive superharmonic function on M.

Set $\text{dist}_{\gamma y}(x) := \text{dist}(x,\gamma y)$. First, we have

$$\Delta h(x) = \sum_{\gamma\in\Gamma} \exp(-D\,\text{dist}_{\gamma y}(x))D(D|\nabla\,\text{dist}_{\gamma y}(x)|^2 - \Delta\,\text{dist}_{\gamma y}(x)).$$

By Rauch'stheorem and $|\nabla\,\text{dist}_{\gamma y}(x)|^2 = 1$ we get

$$\Delta h(x) \leq \sum_{\gamma\in\Gamma} \exp(-D\,\text{dist}(x,\gamma y))D(D-(n-1)a),$$

which implies $\Delta h \leq 0$. We consider the series $\sum_{\gamma\in\Gamma}\log\tanh(\frac{(n-1)a\,\text{dist}_{\gamma y}(x)}{2})$. It is easy to see that the convergence of this series on the set of points bounded away from

Γy follows from the convergence of the Poincaré series at $D = (n-1)a$. Denote this series by $-f$. Then by direct computation and Rauch's theorem we have $\Delta f(x) \leq 0$ for $x \in \tilde{M}\backslash\Gamma y$. Hence $1 - \exp(-f(x))$ defines a nontrivial positive Γ-invariant superharmonic function on $\tilde{M}$.

Therefore, the convergence of the Poincaré series at $D = (n-1)a$ give raise to contradictions to our hypothesis, and the result follows. □

COROLLARY 3.18. *Suppose $D = (n-1)a$ and there are no nontrivial positive superharmonic functions on M. Then, Γ is ergodic with respect to $[\mu]^D$.*

Proof. The corollary follows from Proposition 3.16 and Proposition 3.17. □

COROLLARY 3.19. *Let $M = \tilde{M}/\Gamma$ be a topologically tame 3-manifold with $-b^2 \leq \mathcal{K} \leq -1$ and $\Lambda(\Gamma) = S_\infty$. If $D = 2$, then Γ is divergent, hence ergodic with respect to $[\mu]^D$.*

Proof. The corollary follows from Theorem 2.4, Proposition 3.17 and Corollary 3.18. □

Proof. [Proof of Theorem 1.5] Under the hypothesis of Theorem 1.5, it follows from Proposition 3.15 that $D \in [2, 2b]$. That Γ is harmonically ergodic follows from Proposition 3.12. If $D = 2$, then by Corollary 3.19 we have Γ is divergent. □

4. Part I of Theorems 1.1 and 1.2. Let Γ be a torsion-free discrete subgroup of $\mathrm{ISO}(\widetilde{M})$ with $D_\Gamma = 2$. We assume Γ is either convex-cocompact or $\Lambda_\Gamma = S_\infty$, hausdorff-conservative and divergent.

PROPOSITION 4.1. *The measure $\mathfrak{M}^2_{K_x}$ is finite and positive on all non-empty relative open subsets of Λ_Γ, and $\mathfrak{M}^2_{K_x}(A) = 0$ if and only if $\mathfrak{M}^2_{\eta_v}(A) = 0$ for $A \subset \Lambda_\Gamma\backslash v(-\infty)$.*

Proof. First note that if we replace $\mathrm{dist}_{v,t}$ with dist in the definition of η_v we get a equivalent metric by Lemma 4 in [26].

Let $x \in \widetilde{M}$ be any point. Denote $H_{v,x}$ the horosphere tangent to $v(\infty)$ and passing through x. Take two vectors U^ζ, U^ξ in $S\widetilde{M}$ that are asymptotic to $v(-\infty)$ and passing through $H_{v,x}$ with $U^\zeta(\infty) = \zeta$ and $U^\xi(\infty) = \xi$. Then there exists a positive constant α such that for any unite tangent vectors v^ζ, v^ξ at x which are asymptotic to ζ and ξ respectively, we have $\mathrm{dist}(g_t U^\zeta, g_t v^\zeta) \leq \alpha e^{-t}$ and $\mathrm{dist}(g_t U^\xi, g_t v^\xi) \leq \alpha e^{-t}$, where g_t is the flow. This gives $\mathrm{dist}(g_\tau U^\zeta, g_\tau U^\xi) \leq 2\alpha e^{-\tau} + 1$ with $\tau = l_x(\zeta, \xi)$. On the other hand we also have $\beta^{-1}e^t \leq \mathrm{dist}(g_t U^\zeta, g_t U^\xi) \leq \beta e^{bt}$ for some positive constant β, which gives $e^{-s} \geq \beta^{-1}$ and $e^{-s} \leq \beta^{1/b}$ when $\mathrm{dist}(g_s U^\zeta, g_s U^\xi) = 1$. Hence $\frac{\beta^{-1}\beta^{1/b}}{2\alpha+1}e^{-s} \leq e^{-\tau} \leq \beta e^{-s}$. Therefore η_v and K_x are equivalent on all points in $S_{v,x}$, where $S_{v,x}$ is the shadow of $H_{v,x}$ cased from $v(-\infty)$. By compactness of S_∞ there are $\{v_1, \ldots, v_n\} \subset S\widetilde{M}$ such that $\cup_1^n S_{v_i,x} = S_\infty$. Since $0 < \mathfrak{M}^2_{\eta_{v_i}}(S_{v_i,x} \cap \Lambda_\Gamma) < \infty$, we have $\mathfrak{M}^2_{K_x}$ is positive and finite on Λ_Γ. It follows from Propositions 3.7, 3.5 and 3.9 the measure $\mathfrak{M}^2_{K_x}$ is positive on all relative open subsets. Let $A \subset S_\infty\backslash v(-\infty)$ be a $\mathfrak{M}^2_{\eta_v}$-null set. Let $\delta > 0$. Note that $\cup_{x\in\widetilde{M}} S_{v,x} = S_\infty\backslash v(-\infty)$. Hence there is $B \subset A$ with $B \subset S_{v,z}$ such that $\mathfrak{M}^2_{K_z}(A\backslash B) < \delta$. But $\mathfrak{M}^2_{K_z}(B) \leq c\mathfrak{M}^2_{\eta_v}(B)$ for some $c > 0$. By finiteness we have $\mathfrak{M}^2_{K_z}(A) < \delta$. Same argument holds for the rest of the proposition. □

COROLLARY 4.2. *The measures μ_x and $\mathfrak{M}^2_{\eta_v}$ are absolutely continuous with respect to each other. In-particular $\mathfrak{M}^2_{\eta_v}$ is supported on Λ_Γ.*

Proof. The result follows from Proposition 3.7, 3.5 and Proposition 4.1. □

We use Mostow and Gehring's original idea to show the regularity of quasiconformal map [36], [20]. This method was extended in [25]. We will follow their presentations, but with necessary generalizations that will allow us to prove our theorems using results from previous sections.

Take the unite ball model of $\mathbb{H}^3$. Let u be a unit tangent vector at the origin. Let $\mathcal{O}_u$ be the unit circle on $\partial\mathbb{H}^3 = S^2$ which is contained in the unique totally geodesic plane perpendicular to u and passing through the origin. Also denote the point $u(\infty)$ on S^2 by ς. Then for any pair $(p, \varsigma) \in \mathbb{B}_u := \mathcal{O}_u \times \varsigma$ there is a unique semi-circle connecting them. The bundle of all these semi-circles is the upper hemisphere Ω_u of S^2. We denote this bundle space by $(\Omega_u, \pi_u, \mathbb{B}_u)$
where π_u is the projection.

Let $\phi : S^2 \longrightarrow S_\infty$ be a quasi-conformal embedding conjugate Γ' to Γ under isomorphism $\chi : \Gamma' \longrightarrow \Gamma$, here Γ' is a topologically tame, torsion-free, discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$ with $\Lambda_{\Gamma'} = S^2$. And let ψ be the inverse of ϕ when it is a quasi-conformal homeomorphism.

Let ρ_u be the metric on $S^2\backslash u(-\infty)$ which is defined same as η_v with $v(-\infty) = \phi(u(-\infty))$. The hausdorff measure $\mathfrak{M}^2_{\rho_u}$ on $S^2\backslash u(-\infty)$ with respect to ρ_u-metric is the usual Lebesgue measure. Hence there exists a constant $\omega > 0$ such that for all $\theta \in S^2\backslash u(-\infty)$, we have $\mathfrak{M}^2_{\rho_u}(B_{\rho_u}(\theta, r)) = \omega r^2$.

PROPOSITION 4.3 (see [36], [25]). *The measure $\phi^*\mathfrak{M}^1_{\eta_v}$ is absolutely continuous with respect to measure $\mathfrak{M}^1_{\rho_u}$ on semi-circles. Here $\mathfrak{M}^1_{\eta_v}$ and $\mathfrak{M}^1_{\rho_u}$ are 1-dimensional hausdorff measures with respect to the η_v-metric and ρ_u-metric respectively.*

Proof. Let $\mathfrak{L}$ be the Lebesgue measure on $\mathbb{B}_u$. Then for all $P \in \mathbb{B}_u$ we have the following derivative

$$\lambda(P) := \lim_{r\to 0} \frac{\mathfrak{M}^2_{\eta_v}(\bar{\phi} \circ \pi_u^{-1}(B_{\rho_u}(P, r) \cap \mathbb{B}_u))}{\mathfrak{L}(B_{\rho_u}(P, r) \cap \mathbb{B}_u)}$$

exists and finite for $\mathfrak{L}$-almost everywhere, see [18].

Choose $P \in \mathbb{B}_u$ with $\lambda(P) < \infty$. For a semi-circle $l := \pi_u^{-1}(P)$, let $U_r(l)$ denote the r-neighborhood of l, then $\limsup_{r\to 0} \mathfrak{M}^2_{\eta_v}(\bar{\phi}(U_r(l)))/r < \infty$. For any compact $K \subset l$ with $\mathfrak{M}^1_{\rho_u}(K) = 0$, choose a number $C > 0$ with $\mathfrak{M}^2_{\eta_v}(\bar{\phi}(U_r(l)))/r < C$. Let $\epsilon > 0$ be given, by Besicovic's covering theorem there exists $\{\theta_1, \ldots, \theta_k\} \subset K$ such that $kr < \epsilon$, $K \subset \cup_1^k B_\rho(\theta_i, r)$ and any three of the balls $B_\rho(\theta_i, r)$ with distinct centers are disjoint.

Let $s_i := \inf\{s > 0 | \bar{\phi}(B_\rho(\theta_i, r)) \subset B_\eta(\bar{\phi}(\theta_i), s)\}$ and $\kappa > 0$ (conformal constant) provided by Proposition 3.1. Then we have $\bar{\phi}(K) \subset \cup_1^k B_\eta(\bar{\phi}(\theta_i), s_i)$, $\bar{\phi}(S^2) \cap B_\eta(\bar{\phi}(\theta_i), s_i/\kappa) \subset \bar{\phi}(B_\rho(\theta_i, s_i))$. Since Γ is hausdorff-conservative and by Proposition 3.2, Corollary 4.2, there exists $\alpha > 0$ such that

$$\begin{aligned}\left(\sum_1^k s_i\right)^2 &\le k\sum_1^k s_i^2 \le k\kappa^2\alpha \sum_1^k \mathfrak{M}^2_{\eta_v}(\bar{\phi}(B_\rho(\theta_i, r))) \\ &\le 2\kappa^2\alpha k \mathfrak{M}^2_{\eta_v}(\bar{\phi}(U_r(K))) \le 2\kappa^2\alpha k \mathfrak{M}^2_{\eta_v}(\bar{\phi}(U_r(l))) \\ &\le 2\kappa^2 C\alpha(kr) \le \text{const} \quad \epsilon.\end{aligned}$$

Note the fact that any three of $\bar{\phi}(B_\rho(\theta_i, r))$ do not intersect is used to bound $\sum_1^k \mathfrak{M}^2_{\eta_v}(\bar{\phi}(B_\rho(\theta_i, r)))$ by $2\mathfrak{M}^2_{\eta_v}(\bar{\phi}(U_r(K)))$.

Therefore the result follows from the last inequality. □

The balls $B_\rho(\theta, r)$, $\theta \in S^2 \backslash u(-\infty)$, $r > 0$ form a Vitali relation for the Lebesgue measure $\mathfrak{M}^2_{\rho_u}$. The following derivative

$$J(\theta) := \lim_{r \to 0} \frac{\mathfrak{M}^2_{\eta_v}(\phi(B_\rho(\theta, r)))}{\mathfrak{M}^2_\rho(B_{\rho_u}(\theta, r))}$$

exists and finite for $\mathfrak{M}^2_\rho$-almost every $\theta \in S^2 \backslash u(-\infty)$.

PROPOSITION 4.4. *Let* Lip_ϕ *be defined by* $\mathrm{Lip}_\phi : \theta \longrightarrow \limsup_{r \to 0} r_\phi(\theta, r)/r$. *Then* $\mathrm{Lip}_\phi \in L^2_{\mathrm{loc}}(S^2 \backslash u(-\infty), \mathfrak{M}^2_{\rho_u})$. *In-fact there exists a constant* $k > 0$ *such that*

$$\sqrt{J(\theta)}/k \le \liminf_{r \to 0} r_\phi(\theta, r)/r \le \limsup_{r \to 0} r_\phi(\theta, r)/r \le k\sqrt{J(\theta)}.$$

Proof. Let $\epsilon > 0$. There is $r_\epsilon > 0$ such that for any $r < r_\epsilon$ we have

$$\omega f(\theta) r^2/2 \le \mathfrak{M}^2_{\eta_v}(\bar{\phi}(B_\rho(\theta, r))) \le (2\omega f(\theta) + \epsilon) r^2$$

where the fact that $\mathfrak{M}^2_{\rho_u}$ is Lebesgue measure, i.e. $\mathfrak{M}^2_{\rho_u}(B_{\rho_u}(\theta, r)) = \omega r^2$ for some constant $\omega > 0$ has been used. Since Γ is hausdorff-conservative and by Proposition 3.2, Corollary 4.2, there exists some constant $\alpha > 0$ such that

$$(r_\phi(\theta, r)/\beta)^2/\alpha \le \mathfrak{M}^2_\eta(\bar{\phi}(B_\rho(\theta, r))) \le \alpha(r_\phi(\theta, r))^2.$$

Hence we have

$$\sqrt{(\omega/2\alpha f(\theta))}r \le r_\phi(\theta, r) \le \sqrt{\alpha(2f(\theta)\omega + \epsilon))}\beta r$$

and the result follows by letting $\epsilon \to 0$. □

LEMMA 4.5. *The image under* ϕ *of almost every semi-circle has locally finite* $\mathfrak{M}^1_{\eta_v}$*-measure.*

Proof. Let $f : \Omega_u \longrightarrow \mathbb{B}_u \times [0,1]$ be a diffeomorphism which maps $\pi_u^{-1}(P)$ over P onto $P \times [0,1]$. For every compact subset $C \subset \Omega_u$ we can find a positive number α such that

- For all $x \in f(C)$ the Jacobian of f^{-1} at x are $< \alpha$,
- For all $P \in \mathbb{B}_u$ and $y \in \pi_u^{-1}(P) \cap C$ the local dilations at y of $f|_{\pi_u^{-1}(P)}$ are $< \alpha$.

Since $\bar{\phi}$ is a embedding, $\bar{\phi}(\Omega_u)$ is relative compact subset of $S_\infty \backslash v(-\infty)$ and we have by Proposition 4.4, $\int_{\Omega_u} \mathrm{Lip}^2_\phi \, d\mathfrak{M}^2_{\rho_u} \le k^2 \mathfrak{M}^2_{\eta_v}(\bar{\phi}(\Omega_u)) < \infty$, and Hölder inequality gives $\int_{\Omega_u} \mathrm{Lip}_\phi \, d\mathfrak{M}^2_{\rho_u} < \infty$. Hence

$$\begin{aligned} \int_{\mathbb{B}_u} \left(\int_{\pi_u^{-1}(P) \cap C} \mathrm{Lip}_\phi \, d\mathfrak{M}^1_{\eta_v} \right) d\mathfrak{L} &\le \alpha \int_{f(C)} \mathrm{Lip}_\phi \circ f^{-1} d\mathfrak{L} dt \\ &\le \alpha^2 \int_C \mathrm{Lip}_\phi \, d\mathfrak{M}^2_{\rho_u} < \infty \end{aligned}$$

where $\mathrm{d}t$ is Lebesgue measure on $[0,1]$. Now by Proposition 4.3, $\bar{\phi}$ is absolutely continuous on $\pi_u^{-1}(P)$ therefore

$$\mathfrak{M}^1_{\eta_v}(\bar{\phi}(\pi_u^{-1}(P)\cap C)) \leq \int_{\pi_u^{-1}(P)\cap C} \mathrm{Lip}_\phi \, \mathrm{d}\mathfrak{M}^1_{\rho_u} < \infty.$$

□

Next we adapt the idea in [25] to prove the inequality part of Theorems 1.1.

Proof. [Proof. Part I of Theorems 1.1 and 1.2] For Theorem 1.2, the inequality follows from Theorem 1.5 and Proposition 3.3. Let Γ and Γ' be as in Theorem 1.1 and satisfies those conditions. Note that by Proposition 3.3, $D_{\Gamma'} = 2$. Let g be the Riemannian metric of $\widetilde{M}$. Set $h = (D_\Gamma/2)g$ as the new metric of $\widetilde{M}$. The boundary space of $(\widetilde{M}, g)$ and $(\widetilde{M}, h)$ can be trivially identified, and $\eta_{(2/D_\Gamma)v} = \eta_v^{D_\Gamma/2}$. The critical exponent of Γ with respect to h is 2, hence by Lemma 4.5 there is a non-trivial curve in $S_\infty\backslash v(-\infty)$ with finite $D_\Gamma/2$-dimensional hausdorff measure with respect to η_v. However as noted before the curvature assumption $-b^2 \leq K \leq -1$ of g implies the η_v-metric is a distance on $S_\infty\backslash v(-\infty)$, but the distance-hausdorff dimension is ≥ 1 for any non-trivial curves. Therefore we have $D_\Gamma/2 \geq 1$. □

LEMMA 4.6. *Let Γ' be a divergent, torsion-free discrete subgroup of* $\mathrm{PSL}(2,\mathbb{C})$ *with $\Lambda_{\Gamma'} = S^2$ and $D_{\Gamma'} = 2$. Then the maps ϕ and ψ are absolutely continuous with respect to σ_y and μ_x.*

Proof. By ergodicity of Γ, Γ' and equivariance of $\bar{\phi}$, $\bar{\psi}$ and also Proposition 4.2, its suffices to show there exists a $A \subset S^2\backslash u(-\infty)$ with $\mathfrak{M}^2_{\rho_u}(A) > 0$ such that the Radon-Nikodym derivative of $\bar{\phi}$ at every $x \in A$ with respect to $\mathfrak{M}^2_{\rho_u}$ and $\mathfrak{M}^2_{\eta_v}$ is non-zero. Using the fact that η_v is a distance function, it follows from Proposition 4.3, for $\mathfrak{L}$-almost all $P \in \mathbb{B}_u$ the length of $\bar{\phi}(\pi_u^{-1}(P)) > 0$ is bounded by $\int_{\pi_u^{-1}(P)} \mathrm{Lip}_\phi \, \mathrm{d}\mathfrak{M}^1_{\rho_u}$. Hence if we set $A := \{x \in \Omega_u | \mathrm{Lip}_\phi(x) > 0\}$, then for $\mathfrak{L}$-almost all $P \in \mathbb{B}_u$, $\mathfrak{M}^1_{\rho_u}(\pi_u^{-1}(P) \cap A) > 0$ which implies $\mathfrak{M}^2_{\rho_u}(A) > 0$. Therefore the result follows from Proposition 4.4. □

5. Part II of Theorems 1.1 and 1.2. Let $\xi_1, \xi_2, \xi_3, \xi_4 \in S_\infty$. The *cross-ratio* $|\xi_1, \xi_2, \xi_3, \xi_4|$ of these four points is defined as

$$|\xi_1, \xi_2, \xi_3, \xi_4| := \frac{e^{-\beta_x(\xi_1,\xi_2)} e^{-\beta_x(\xi_3,\xi_4)}}{e^{-\beta_x(\xi_1,\xi_3)} e^{-\beta_x(\xi_2,\xi_4)}}.$$

This definition is consistent with the hyperbolic space cross-ratio.

If Γ_1, Γ_2 are discrete subgroups of $\widetilde{M}$ such that both Γ_1, Γ_2 are divergent, and there exists a equivariant (under some group morphism χ), nonsingular (with respect to μ_1, μ_2 Patterson-Sullivan measures on Λ_{Γ_1} and Λ_{Γ_2} respectively), measurable map $f : \Lambda_{\Gamma_1} \longrightarrow \Lambda_{\Gamma_2}$. Then

$$\mathrm{d}(f \times f)^*\Pi_2(\xi, \zeta) = e^{-D_{\Gamma_2}\beta_y(f\xi, f\zeta)} g(\xi) g(\zeta) \mathrm{d}\mu_1(\xi) \mathrm{d}\mu_2(\zeta)$$

where $g := \frac{\mathrm{d}f^*(\mu_2)}{\mathrm{d}(\mu_1)}$, and Π_i is the measure defined in §3 through μ_i. From the properties of f, $(f \times f)^*\Pi_2$ is a constant $a > 0$ multiple of Π_1. Hence $e^{D_{\Gamma_2}\beta_y(f\xi, f\zeta)} g(\xi) g(\zeta) = a e^{D_{\Gamma_1}\beta_x(\xi,\zeta)}$. Therefore for μ_1-almost everywhere we have

$$|f(\xi_1), f(\xi_2), f(\xi_3), f(\xi_4)| = |\xi_1, \xi_2, \xi_3, \xi_4|^{D_{\Gamma_1}/D_{\Gamma_2}}.$$

This was the idea of Sullivan for the following lemma:

LEMMA 5.1. *Let* Γ_1, Γ_2 *be discrete subgroups of* $\mathrm{ISO}(\widetilde{M})$ *with* $D_{\Gamma_1} = D_{\Gamma_2}$ *and* Γ_1, Γ_2 *are divergent. Suppose there exists a equivariant nonsingular measurable map* $f : \Lambda_{\Gamma_1} \longrightarrow \Lambda_{\Gamma_2}$ *with respect to Patterson-Sullivan measures space* $(\Lambda_{\Gamma_1}, \mu_1)$ *and* $(\Lambda_{\Gamma_2}, \mu_2)$. *Then* f *preserves cross-ratio* μ_1*-almost everywhere.*

For a finitely generated discrete subgroup Γ of $\mathrm{PSL}(2,\mathbb{C})$. The *conservative set* of Γ on S^2 coincides with Λ_Γ up-to Lebesgue measure zero. The group Γ is called *conservative* if and only if Λ_Γ has full Lebesgue measure. Since for a topologically tame Γ, the hausdorff dimension of Λ_Γ is equal to D_Γ, therefore we have the following:

PROPOSITION 5.2. *Let* Γ *be a topologically tame, torsion-free discrete subgroup of* $\mathrm{PSL}(2,\mathbb{C})$ *with conservative* Γ, *then* Γ *is hausdorff-conservative.*

REMARK 5.3. *It is a conjecture that all finitely generated discrete subgroup* Γ *of* $\mathrm{PSL}(2,\mathbb{C})$ *are topologically tame.*

Next we recall the statement of Sullivan's quasi-conformal stability for discrete subgroups of $\mathrm{PSL}(2,\mathbb{C})$.

THEOREM 5.4 (Sullivan [43]). *Let* Γ *be a discrete subgroup of* $\mathrm{PSL}(2,\mathbb{C})$. *Then* Γ *is quasi-conformally stable (i.e. if* f *is a quasi-conformal automorphism of* S^2 *with* $f\Gamma f^{-1} \subset \mathrm{PSL}(2,\mathbb{C})$, *then* f *is a Möbius transformation) if and only if* Γ *is conservative.*

COROLLARY 5.5. *Let* $N = \mathbb{H}^3/\Gamma$ *be a complete hyperbolic* 3*-manifold for a conservative* Γ. *Then* N *is quasi-isometrically stable, i.e. If there is a quasi-isometric homeomorphism* $h : N \longrightarrow M$ *to a hyperbolic manifold* M, *then* N *is isometric to* M.

Proof. [Proof. Theorem 1.2 part II] By Theorem 1.5, Γ is divergent for $D_\Gamma = 2$. From Proposition 3.3, Γ' is also divergent and $D_{\Gamma'} = 2$. Lemma 4.6 then implies f is absolutely continuous with respect to σ_y and μ_x. Hence by Lemma 5.1, f preserves cross ratio σ_y-everywhere. By Proposition 3.2, $\Lambda_{\Gamma'} = S^2$ and since σ_y is non-zero constant multiple of Lebesgue measure, we can modify f on the Lebesgue measure null subset of S^2 to a map which is cross ration preserving on S^2. We denote the new map also by f. By Bourdon's theorem [9], f extends into the space as a isometry, i.e. $\mathbb{H}^3$ and $\widetilde{M}$ are isometric. Hence the result follows from Theorem 5.4. □

Proof. [Proof. Theorems 1.1 part II] Here f embeds S^2 into S_∞. If we suppose $D_\Gamma = D_{\Gamma'} = 2$, then by using same argument as the proof of Theorem 1.2, f extends to a isometric embedding of $\mathbb{H}^3$ into $\widetilde{M}$ by [9]. Since $f(S^2)$ is a Λ_Γ-invariant closed subset of S_∞, by Proposition 3.2, $f(S^2) = \Lambda_\Gamma$. Hence the boundary space of the isometric embedded image of $\mathbb{H}^3$ coincides with Λ_Γ, therefore the result follows. □

Proof. [Proof. Corollary 1.3] This follows from Propositions 3.1, 3.3, and Theorem 1.2. □

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