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A GENERALIZATION OF THE YAMABE FLOW FOR MANIFOLDS WITH BOUNDARY *

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1. Introduction. Let M be a compact manifold of dimension $n \geq 3$ with boundary ∂M, and let g be a Riemannian metric on M. We denote by R the scalar curvature of M, and by H the mean curvature of ∂M. Moreover, we denote by

$$\overline{R} = \frac{\int_M R\, dV}{\int_M dV}$$

the mean value of the scalar curvature on M, and by

$$\overline{H} = \frac{\int_{\partial M} H\, dA}{\int_{\partial M} dA}$$

the mean value of the mean curvature on ∂M.

For a closed manifold M, the Yamabe conjecture [22] asserts that in each conformal class there is a metric of constant scalar curvature. In order to find such a metric, R. Hamilton introduced the Yamabe flow

$$\frac{\partial}{\partial t} g = -(R - \overline{R})g. \tag{1}$$

R. Ye [23] proved that the initial value problem (1) possesses a unique solution which exists for all $t \geq 0$. The Yamabe flow is also studied in a recent work of H. Schwetlick and M. Struwe [16]. To determine the asymptotic behavior of the solution, it is convenient to distinguish three cases:

(a) The conformal class contains a metric of negative scalar curvature
(b) The conformal class contains a metric of vanishing scalar curvature.
(c) The conformal class contains a metric of positive scalar curvature.

In cases (a) and (b), it follows from the maximum principle that every solution of (1) converges to a metric of constant curvature. The same holds in case (c) if M is locally conformally flat (see [23]).

In this paper, we study two generalizations of the Yamabe problem for manifolds with boundary. In the first case, we try to find a conformal metric which has constant scalar curvature in the interior and vanishing mean curvature at the boundary. In the second case, we look for a conformal metric with vanishing scalar curvature in the interior and constant mean curvature at the boundary. These generalizations of the Yamabe problem were studied by J.F. Escobar [4, 5]. Related questions were addressed by M. Ahmedou [1], V. Felli and M. Ahmedou [6], and by Z.C. Han and

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Y.Y. Li [9, 10].

To construct solutions to these elliptic problems, we study the corresponding parabolic problems. In the first case, we start with an initial metric g_0 such that $H_0 = 0$. Then we deform the metric by the Yamabe flow

$$\frac{\partial}{\partial t} g = -(R - \overline{R}) g \tag{2}$$

in M with the boundary condition

$$H = 0$$

on ∂M. In the second case, we assume that the initial metric satisfies $R_0 = 0$. In this case, we deform the metric by

$$\frac{\partial}{\partial t} g = -2(H - \overline{H}) g \tag{3}$$

on ∂M. Moreover, we require that

$$R = 0$$

in M. In order to analyze the long time behavior of the solutions of equation (2), we consider three cases:

(Ia) The conformal class contains a metric with negative scalar curvature and vanishing mean curvature.
(Ib) The conformal class contains a metric with vanishing scalar curvature and vanishing mean curvature.
(Ic) The conformal class contains a metric with positive scalar curvature and vanishing mean curvature.

THEOREM 1.1. *In case (Ic), we assume in addition that M is locally conformally flat and ∂M is umbilic. Then every solution to the initial boundary value problem (2) converges to a metric with constant scalar curvature in the interior and vanishing mean curvature at the boundary.*

To study the asymptotic behavior of the solutions of equation (3), we consider the following three cases:

(IIa) The conformal class contains a metric with negative mean curvature and vanishing scalar curvature.
(IIb) The conformal class contains a metric with vanishing mean curvature and vanishing scalar curvature.
(IIc) The conformal class contains a metric with positive mean curvature and vanishing scalar curvature.

THEOREM 1.2. *In case (IIc), we assume in addition that M is locally conformally flat and ∂M is umbilic. In this case, we also assume that the boundary of the universal cover of M is connected. Then every solution to the initial boundary value problem (3) converges to a metric with constant mean curvature at the boundary and vanishing scalar curvature in the interior.*

The evolution equation (2) equals the gradient flow to the functional

$$V^{-\frac{n-2}{n}} \int_M R\, dV$$

up to a constant factor. Similarly, the evolution equation (3) is the gradient flow to the functional

$$A^{-\frac{n-2}{n-1}} \int_{\partial M} H\, dA.$$

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2. The case of vanishing mean curvature. Let g_0 be a Riemannian metric on M such that $H_0 = 0$ on ∂M. We study the initial boundary value problem

$$\frac{\partial}{\partial t} g = -(R - \overline{R}) g$$

in M with the boundary condition

$$H = 0$$

on ∂M and the initial condition

$$g = g_0$$

for $t = 0$.

LEMMA 2.1. *For every metric g_0 on M, there exists a metric g conformal to g_0 which satisfies one of the conditions (Ia), (Ib), (Ic).*

Proof. Suppose that $u \in W^{1,2}(M)$ minimizes the functional

$$\int_M |\nabla_0 u|^2\, dV_0 + \frac{n-2}{4(n-1)} \int_M R_0\, u^2\, dV_0$$

with respect to the constraint

$$\int_M u^2\, dV_0 = 1.$$

By replacing u by $|u|$, it follows that u is of one sign. Hence, we may assume that $u > 0$. The function u satisfies

$$-\Delta_0 u + \frac{n-2}{4(n-1)} R_0\, u = \lambda\, u$$

in M and

$$\frac{\partial}{\partial \nu_0} u = 0$$

on ∂M. Therefore, the metric $g = u^{\frac{4}{n-2}}\, g_0$ satisfies

$$R = \frac{n-2}{4(n-1)}\, \lambda\, u^{-\frac{4}{n-2}}$$

in M and

$$H = 0$$

on ∂M. From this the assertion follows.

Since the equation preserves the conformal structure, we may write $g = u^{\frac{4}{n-2}}\, g_0$ for a positive function u on M. Then the scalar curvature of g is given by

$$R = u^{-\frac{n+2}{n-2}} \Big(- \frac{4(n-1)}{n-2}\, \Delta_0 u + R_0 u \Big).$$

This implies

$$\frac{\partial}{\partial t} u = (n-1)\, u^{-\frac{4}{n-2}}\, \Delta_0 u - \frac{n-2}{4}\, (R_0\, u^{-\frac{4}{n-2}} - \overline{R})\, u$$

in M. Moreover, the boundary condition $H = 0$ is equivalent to

$$\frac{\partial}{\partial \nu_0} u = 0$$

on ∂M. The standard regularity theory for quasilinear parabolic equations guarantees that this equation has a solution on a small time interval.

PROPOSITION 2.2. *The conformal factor is uniformly bounded for all $t \geq 0$.*

Proof. There are three possibilities:

(a) Suppose that there exists a metric g_0 in the conformal class such that $R_0 < 0$ and $H_0 = 0$. If we write $g = u^{\frac{4}{n-2}} g_0$, then we obtain

$$\frac{\partial}{\partial t} u = (n-1)\, u^{-\frac{4}{n-2}}\, \Delta_0 u - \frac{n-2}{4}\, (R_0 u^{-\frac{4}{n-2}} - \overline{R}) u$$

in M and

$$\frac{\partial}{\partial \nu_0} u = 0$$

on ∂M. Since $R_0 < 0$, it follows from the maximum principle that the quotient

$$\max_{x \in M} u \big/ \min_{x \in M} u$$

is bounded. Since the volume of M is constant, it follows that u is bounded.

(b) Suppose that there exists a metric g_0 in the conformal class such that $R_0 = 0$ and $H_0 = 0$. As above, we write $g = u^{\frac{4}{n-2}} g_0$. This yields

$$\frac{\partial}{\partial t} u = (n-1)\, u^{-\frac{4}{n-2}}\, \Delta_0 u + \frac{n-2}{4}\, \overline{R} u$$

in M and

$$\frac{\partial}{\partial \nu_0} u = 0$$

on ∂M. Again, the maximum principle implies that

$$\max_{x\in M} u / \min_{x\in M} u$$

is bounded. Since the volume of M is constant, it follows that u is bounded.

(c) We now assume that there exists a metric g_0 in the conformal class such that $R_0 > 0$ and $H_0 = 0$. By assumption, M is locally conformally flat and ∂M is umbilic. Since ∂M has zero mean curvature, it follows that ∂M is totally geodesic. Hence, we may double M to obtain a compact manifold without boundary. The result follows now from a result of R. Ye [23].

As a consequence, we obtain the following result.

COROLLARY 2.3. *The evolution equation (2) has a unique solution for all* $t \geq 0$.
We now want to determine the asymptotic behavior of the solution.

THEOREM 2.4. *The solution converges to a metric of constant scalar curvature as* $t \to \infty$.

Proof. We define a functional $E[u]$ as

$$E[u] = V^{-\frac{n-2}{n}} \int_M R\,dV,$$

where $g = u^{\frac{4}{n-2}} g_0$. Then we obtain

$$\delta E[u] = 2\,V^{-\frac{n-2}{n}} \int_M (R - \overline{R})\,u^{-1}\,\delta u\,dV.$$

Hence, under the evolution equation (2) the functional $E[u]$ changes according to

$$\frac{d}{dt} E[u] = -\frac{n-2}{2}\,V^{-\frac{n-2}{n}} \int_M (R - \overline{R})^2\,dV.$$

We now define

$$\lim_{t\to\infty} E[u] = \alpha.$$

Since the functional $E[u]$ is real analytic, we can apply the Lojasiewicz-Simon inequality (see [17], Theorem 3 or [12], Proposition 3.3). Therefore, there exists a real number θ such that $0 < \theta < \frac{1}{2}$ and

$$(E[u] - \alpha)^{2(1-\theta)} \leq C \int_M (R - \overline{R})^2\,dV$$

for all $t \geq t_0$. Letting

$$z^2 = \int_M (R - \overline{R})^2\,dV$$

we obtain

$$\left(\int_T^\infty z(t)^2\,dt \right)^{2(1-\theta)} \leq C z(T)^2$$

for all $T \geq t_0$. By Lemma 4.1 in [12], we conclude that

$$\int_0^\infty z(t)\,dt < \infty.$$

Therefore, the limit

$$\lim_{t \to \infty} u(t) = v$$

exists in $L^2(M)$. This proves the assertion.

3. The case of vanishing scalar curvature. We now consider a Riemannian metric g_0 such that $R_0 = 0$ in M. We study the initial boundary value problem

$$\frac{\partial}{\partial t} g = -(R - \overline{R})g$$

in M with the boundary condition

$$H = 0$$

on ∂M and the initial condition

$$g = g_0$$

for $t = 0$.

LEMMA 3.1. *For every metric g_0 on M, there exists a metric g conformal to g_0 which satisfies one of the conditions (IIa), (IIb), (IIc).*

Proof. Suppose that $u \in W^{1,2}(M)$ minimizes the functional

$$\int_M |\nabla_0 u|^2\,dV_0 + \frac{n-2}{2(n-1)} \int_{\partial M} H_0\, u^2\,dV_0$$

with respect to the constraint

$$\int_{\partial M} u^2\,dA_0 = 1.$$

By replacing u by $|u|$, it follows that u is of one sign. Hence, we may assume that $u > 0$. The function u satisfies

$$\frac{\partial}{\partial \nu_0} u + \frac{n-2}{2(n-1)} H_0\, u = \lambda\, u$$

on ∂M and

$$\Delta_0 u = 0$$

in M. Therefore, the metric $g = u^{\frac{4}{n-2}}\, g_0$ satisfies

$$H = \frac{n-2}{2(n-1)} \lambda\, u^{-\frac{2}{n-2}}$$

on ∂M and

$$R = 0$$

in M. From this the assertion follows.

As in the previous section, we may write $g = u^{\frac{4}{n-2}} g_0$ for a positive function u on M. Then the mean curvature of g is given by

$$H = u^{-\frac{n}{n-2}} \left(\frac{2(n-1)}{n-2} \frac{\partial}{\partial \nu_0} u + H_0 u \right).$$

This implies

$$\frac{\partial}{\partial t} u = -(n-1)\, u^{-\frac{2}{n-2}} \frac{\partial}{\partial \nu_0} u - \frac{n-2}{2} (H_0\, u^{-\frac{2}{n-2}} - \overline{H})\, u$$

on ∂M. Moreover, the condition $R = 0$ is equivalent to

$$\Delta_0 u = 0$$

in M. In the first step, we show that this equation has a solution on a small time interval. To this end, we need some estimates for the linearized equation. We begin with an estimate for the elliptic problem.

LEMMA 3.2. *Let ϕ be an harmonic function M with respect to the a metric g_0. Then we have the estimates*

$$\|\nabla \phi\|_{L^p(\partial M)} \leq C\, \|\phi\|_{W^{1,p}(\partial M)}$$

and

$$\|\nabla \phi\|_{L^p(\partial M)} \leq C\, \|\frac{\partial}{\partial \nu_0} \phi\|_{L^p(\partial M)}.$$

Proof. The first estimate is a consequence of [20], equation (4.71) on p. 168 and equation (4.82) on p. 170. The second estimate follows from [20], equation (4.103) on p. 172.

Since this estimate plays a key role in our subsequent arguments, we give a proof for the model problem on the half-space $\{x_n \geq 0\}$. For abbreviation, we denote by f the restriction of ϕ to the boundary, and by g the normal derivative of f at the boundary. Since ϕ is harmonic, we have

$$\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \phi = 0.$$

Taking the Fourier transform in the first $n-1$ variables, we obtain

$$\left(\frac{\partial^2}{\partial x_n^2} - |\xi|^2 \right) \hat{\phi}(\xi, x_n) = 0.$$

Moreover, we have

$$\hat{\phi}(\xi, 0) = \hat{f}(\xi)$$

and

$$\frac{\partial}{\partial x_n}\hat{\phi}(\xi,0) = -\hat{g}(\xi).$$

This implies

$$\hat{\phi}(\xi,x_n) = e^{-|\xi|\,x_n}\,\hat{f}(\xi) = \frac{1}{|\xi|}\,e^{-|\xi|\,x_n}\,\hat{g}(\xi).$$

From this it follows that

$$\widehat{\nabla\phi}(\xi,0) = -(i\xi,|\xi|)\,\hat{f}(\xi) = \frac{1}{|\xi|}\,(i\xi,|\xi|)\,\hat{g}(\xi).$$

Therefore, the assertion follows from Mikhlin's theorem (see [18], p. 109).

Given a function ϕ on ∂M, we can extend ϕ to M such that

$$\Delta_0\phi = 0.$$

We now define

$$B_0\phi = -(n-1)\frac{\partial}{\partial\nu_0}\phi$$

and

$$B\phi = -(n-1)\,u^{-\frac{2}{n-2}}\,\frac{\partial}{\partial\nu_0}\phi$$

on ∂M. Note that

$$B = u^{-\frac{2}{n-2}}\,B_0,$$

hence

$$\frac{\partial}{\partial t}B = -\frac{2}{n-2}\,u^{-1}\,\frac{\partial}{\partial t}u\,B.$$

LEMMA 3.3. *Let ϕ be a solution of the linear initial boundary value problem*

$$\frac{\partial}{\partial t}\phi = -(n-2)\,u^{-\frac{2}{n-2}}\,\frac{\partial}{\partial\nu_0}\phi + f$$

on ∂M,

$$\Delta_0\phi = 0$$

on ∂M, and

$$\phi = 0$$

for $t = 0$. We assume that u is uniformly bounded above and below. Then we have the estimate

$$\|\phi\|_{W^{1,2}(\partial M\times[0,T])} \leq C\,\|f\|_{L^2(\partial M\times[0,T])}.$$

Proof. Integration by parts gives

$$\begin{aligned}\frac{d}{dt}\int_M |\nabla_0\phi|^2\,dV_0 &= 2\int_M \nabla_0\frac{\partial}{\partial t}\phi\cdot\nabla_0\phi\,dV_0\\ &= 2\int_{\partial M}\frac{\partial}{\partial t}\phi\,\frac{\partial}{\partial\nu_0}\phi\,dA_0\\ &= -2(n-1)\int_{\partial M} u^{-\frac{2}{n-2}}\Big(\frac{\partial}{\partial\nu_0}\phi\Big)^2\,dA_0 + 2\int_{\partial M} f\,\frac{\partial}{\partial\nu_0}\phi\,dA_0.\end{aligned}$$

This implies

$$(n-1)\int_0^T\int_{\partial M} u^{-\frac{2}{n-2}}\Big(\frac{\partial}{\partial\nu_0}\phi\Big)^2\,dA_0\,dt \le \int_0^T\int_{\partial M} f\,\frac{\partial}{\partial\nu_0}\phi\,dA_0\,dt.$$

Since u is uniformly bounded above and below, it follows that

$$\int_0^T\int_{\partial M}\Big(\frac{\partial}{\partial\nu_0}\phi\Big)^2\,dA_0\,dt \le C\int_0^T\int_{\partial M} f^2\,dA_0\,dt.$$

Using the estimate

$$\|\phi\|_{W^{1,2}(\partial M)} \le C\,\|\frac{\partial}{\partial\nu_0}\phi\|_{L^2(\partial M)}$$

from the previous lemma, we obtain

$$\|\phi\|_{W^{1,2}(\partial M\times[0,T])} \le C\,\|f\|_{L^2(\partial M\times[0,T])}.$$

This proves the assertion.

LEMMA 3.4. *Let m be sufficiently large, and let ϕ be the solution of the linear initial boundary value problem*

$$\frac{\partial}{\partial t}\phi = -(n-2)\,u^{-\frac{2}{n-2}}\,\frac{\partial}{\partial\nu_0}\phi + f$$

on ∂M,

$$\Delta_0\phi = 0$$

in M and

$$\phi = 0$$

for $t = 0$. We assume that the function u is uniformly bounded above and below. Moreover, we assume that $\|u\|_{W^{m,2}(\partial M\times[0,T])} \le C$. Then we obtain the estimate

$$\|\phi\|_{W^{m+1,2}(\partial M\times[0,T])} \le C\,\|f\|_{W^{m,2}(\partial M\times[0,T])}.$$

Proof. We claim that

$$\|\phi\|_{W^{k+1,2}(\partial M\times[0,T])} \le C\,\|f\|_{W^{k,2}(\partial M\times[0,T])}$$

for $k \leq m$. To prove this, we proceed by induction on k. For $k = 0$, the assertion follows from the previous lemma. We now assume that the assertion holds for $k-1$. The function ϕ satisfies the evolution equation

$$\frac{\partial}{\partial t}\phi = B\phi + f$$

with the initial condition $\phi = 0$ for $t = 0$. From this it follows that

$$\frac{\partial}{\partial t}B\phi = BB\phi - \frac{n-2}{2}\,u^{-1}\,\frac{\partial}{\partial t}u\,B\phi + Bf$$

and $B\phi = 0$ for $t = 0$. The induction hypothesis implies that

$$\begin{aligned}
&\|\phi\|_{W^{k+1,2}(\partial M\times[0,T])} \\
&\leq C\,\|B_0\phi\|_{W^{k,2}(\partial M\times[0,T])} \\
&= C\,\|u^{\frac{2}{n-2}}\,B\phi\|_{W^{k,2}(\partial M\times[0,T])} \\
&\leq C\,\|B\phi\|_{W^{k,2}(\partial M\times[0,T])} \\
&\leq C\,\|u^{-1}\,\frac{\partial}{\partial t}u\,B\phi\|_{W^{k-1,2}(\partial M\times[0,T])} + C\,\|Bf\|_{W^{k-1,2}(\partial M\times[0,T])} \\
&= C\,\|u^{-\frac{n}{n-2}}\,\frac{\partial}{\partial t}u\,B_0\phi\|_{W^{k-1,2}(\partial M\times[0,T])} + C\,\|u^{-\frac{2}{n-2}}\,B_0 f\|_{W^{k-1,2}(\partial M\times[0,T])} \\
&\leq C\,\|B_0\phi\|_{W^{k-1,2}(\partial M\times[0,T])} + C\,\|B_0 f\|_{W^{k-1,2}(\partial M\times[0,T])} \\
&\leq C\,\|\phi\|_{W^{k,2}(\partial M\times[0,T])} + C\,\|f\|_{W^{k,2}(\partial M\times[0,T])} \\
&\leq C\,\|f\|_{W^{k,2}(\partial M\times[0,T])}.
\end{aligned}$$

This completes the proof.

PROPOSITION 3.5. *The initial boundary value problem*

$$\frac{\partial}{\partial t}g = -2Hg$$

on ∂M and

$$R = 0$$

in M has a unique solution on a small time interval.

Proof. Letting $g = u^{\frac{4}{n-2}}\,g_0$, we obtain the initial boundary value problem

$$\frac{\partial}{\partial t}u = -(n-1)\,u^{-\frac{2}{n-2}}\,\frac{\partial}{\partial \nu_0}u - \frac{n-2}{2}\,u^{\frac{n-4}{n-2}}\,H_0$$

on ∂M,

$$\Delta_0 u = 0$$

in M and

$$u = 1$$

for $t = 0$. We define a map $\mathcal{F} : W^{m,2}(\partial M \times [0,T]) \to W^{m+1,2}(\partial M \times [0,T]) \subset W^{m,2}(\partial M \times [0,T])$ in the following way: Given a function $u \in W^{m,2}(\partial M \times [0,T])$ satisfying $u = 1$ for $t = 0$, let $\phi = \mathcal{F}(u)$ be the unique solution of the linear initial boundary value problem

$$\frac{\partial}{\partial t}\phi = -(n-1)\, u^{-\frac{2}{n-2}} \frac{\partial}{\partial \nu_0}\phi - \frac{n-2}{2}\, u^{\frac{n-4}{n-2}}\, H_0$$

on ∂M,

$$\Delta_0 \phi = 0$$

in M and

$$\phi = 1$$

for $t = 0$. We claim that the map $\mathcal{F}$ has a fixed point. To prove this, we consider a function $\tilde{\phi}$ such that

$$\frac{\partial}{\partial t}\tilde{\phi} = -(n-1)\, \tilde{u}^{-\frac{2}{n-2}} \frac{\partial}{\partial \nu_0}\tilde{\phi} - \frac{n-2}{2}\, \tilde{u}^{\frac{n-4}{n-2}}\, H_0$$

on ∂M,

$$\Delta_0 \tilde{\phi} = 0$$

in M and

$$\tilde{\phi} = 1$$

for $t = 0$. Then the difference $\phi - \tilde{\phi}$ satisfies

$$\begin{aligned}\frac{\partial}{\partial t}(\phi - \tilde{\phi}) &= -(n-1)\, u^{-\frac{2}{n-2}} \frac{\partial}{\partial \nu_0}(\phi - \tilde{\phi}) \\ &\quad - (n-1)\, (\phi^{-\frac{2}{n-2}} - \tilde{\phi}^{-\frac{2}{n-2}})\, \frac{\partial}{\partial \nu_0}\tilde{\phi} \\ &\quad - \frac{n-2}{2}\, (u^{\frac{n-4}{n-2}} - \tilde{u}^{\frac{n-4}{n-2}})\, H_0.\end{aligned}$$

on ∂M,

$$\Delta_0(\phi - \tilde{\phi}) = 0$$

in M and

$$\phi - \tilde{\phi} = 0$$

for $t = 0$. Since $\tilde{\phi}$ is bounded in $W^{m+1,2}(\partial M \times [0,T])$, we conclude that

$$\begin{aligned}\|\phi - \tilde{\phi}\|_{W^{m+1,2}(\partial M \times [0,T])} &\leq C\, \|(u^{-\frac{2}{n-2}} - \tilde{u}^{-\frac{2}{n-2}})\, \frac{\partial}{\partial \nu_0}\tilde{\phi}\|_{W^{m,2}(\partial M \times [0,T])} \\ &\quad + C\, \|u^{\frac{n-4}{n-2}} - \tilde{u}^{\frac{n-4}{n-2}}\|_{W^{m,2}(\partial M \times [0,T])} \\ &\leq C\, \|u - \tilde{u}\|_{W^{m,2}(\partial M \times [0,T])}.\end{aligned}$$

Hence, if T is sufficiently small, then we obtain

$$\|\phi - \tilde{\phi}\|_{W^{m,2}(\partial M \times [0,T])} \leq \frac{1}{2}\, \|u - \tilde{u}\|_{W^{m,2}(\partial M \times [0,T])}.$$

By the contraction mapping principle, the map $\mathcal{F}$ has a fixed point. From this the assertion follows.

To derive uniform estimates for u, we employ the following variant of a result of R. Schoen and S.T. Yau [15].

PROPOSITION 3.6. *Let M be a compact Riemannian manifold with positive mean curvature on the boundary and vanishing scalar curvature in the interior. We assume that M is locally conformally flat and ∂M is umbilic. Moreover, we assume that the boundary of the universal cover of M is connected. Then the universal cover of M is conformally equivalent to a dense open subset Ω of B^n.*

Proof. After a conformal change of the metric, we may assume that M has positive scalar curvature in the interior and vanishing mean curvature on the boundary. To see this, we consider the first eigenfunction of the Laplace operator with the Neumann boundary condition

$$\frac{\partial}{\partial \nu_0} u + \frac{n-2}{2(n-1)}\, H_0\, u = 0$$

on ∂M. Then the function u is positive, and satisfies

$$-\Delta_0 u = \lambda\, u$$

in M. This implies

$$\frac{n-2}{2(n-1)} \int_{\partial M} H_0\, u\, dA_0 = -\int_{\partial M} \frac{\partial}{\partial \nu_0} u\, dA_0 = -\int_M \Delta_0 u\, dV_0 = \lambda \int_M u\, dV_0.$$

Since H_0 is positive, it follows that $\lambda > 0$. Hence, the conformally modified metric $g = u^{\frac{4}{n-2}}\, g_0$ has positive scalar curvature in the interior and vanishing mean curvature at the boundary. Since ∂M is umbilic, we conclude that ∂M is totally geodesic. Passing to the universal cover, we obtain a new manifold $\tilde{M}$ such that $\tilde{M}$ is simply connected and $\partial \tilde{M}$ is connected. Since the boundary of $\tilde{M}$ is totally geodesic, we can double $\tilde{M}$. The resulting manifold $\hat{M}$ will be complete and simply connected. Since $\hat{M}$ has positive scalar curvature, it follows from a theorem of R. Schoen and S.T. Yau [15] that $\hat{M}$ is conformally equivalent to an open subset of S^n whose complement has Hausdorff dimension at most $\frac{n-2}{2}$. Consequently, $\tilde{M}$ must be conformally equivalent to a dense open subset of S^n_+.

PROPOSITION 3.7. *The conformal factor is uniformly bounded for all $t \geq 0$.*

Proof. There are three possibilities:

(a) Suppose that there exists a metric g_0 in the conformal class such that $H_0 < 0$ and $R_0 = 0$. If we write $g = u^{\frac{4}{n-2}} g_0$, then we obtain

$$\frac{\partial}{\partial t} u = -(n-1)\, u^{-\frac{2}{n-2}}\, \frac{\partial}{\partial \nu_0} u - \frac{n-2}{2}\, (H_0 u^{-\frac{2}{n-2}} - \overline{H}) u$$

on ∂M and

$$\Delta_0 u = 0$$

in M. Since $H_0 < 0$, it follows from the maximum principle that the quotient

$$\max_{x \in \partial M} u / \min_{x \in \partial M} u$$

is bounded on ∂M. Since the area of ∂M is constant, it follows that u is bounded on ∂M.

(b) Suppose that there exists a metric g_0 in the conformal class such that $H_0 = 0$ and $R_0 = 0$. As above, we write $g = u^{\frac{4}{n-2}} g_0$. This yields

$$\frac{\partial}{\partial t} u = -(n-1)\, u^{-\frac{2}{n-2}} \frac{\partial}{\partial \nu_0} u + \frac{n-2}{2} \overline{H} u$$

on ∂M and

$$\Delta_0 u = 0$$

in M. Again, the maximum principle implies that

$$\max_{x \in \partial M} u / \min_{x \in \partial M} u$$

is bounded on ∂M. Since the area of M is constant, it follows that u is bounded on ∂M.

(c) We now assume that there exists a metric g_0 in the conformal class such that $H_0 > 0$ and $R_0 = 0$. By assumption, M is locally conformally flat and ∂M is umbilic. By Proposition 3.6, the universal cover of M is conformally equivalent to a dense open subset Ω of B^n.

We write $g = u^{\frac{4}{n-2}} g_0$, where g_0 denotes the standard metric on B^n. Then the function u satisfies $u \to \infty$ for $x \to B^n \setminus \Omega$. We will use the Alexandrov reflection principle (see [7]) to show that u is bounded. To this end, we consider an arbitrary point on the boundary of B^n. We choose a conformal mapping from B^n to the half-space $H^n = \{x_n \geq 0\}$ which maps this point to infinity. The function u admits an asymptotic expansion of the form

$$u = a + \sum_i \frac{b_i x_i}{|x|^2} + \ldots$$

for $|x| \to \infty$ and some $a > 0$. We now write $g_{ij} = w^{\frac{4}{n-2}} \delta_{ij}$. The function w satisfies

$$w = \frac{2^{\frac{n-2}{2}} a}{|x|^{n-2}} + \sum_i \frac{2^{\frac{n-2}{2}} b_i x_i}{|x|^n} + \ldots$$

for $|x| \to \infty$. We reflect the function w at the hyperplane $\{x_1 = \lambda\}$, and denote the resulting function by w_λ. The function w satisfies the evolution equation

$$\frac{\partial}{\partial t} w = (n-1)\, w^{-\frac{2}{n-2}} \frac{\partial}{\partial x_n} w + \frac{n-2}{2} \overline{H} w$$

on ∂H^n and

$$\sum_i \frac{\partial^2}{\partial x_i^2} w = 0$$

in H^n. Similarly, for the function w_λ we have

$$\frac{\partial}{\partial t} w_\lambda = (n-1)\, w_\lambda^{-\frac{2}{n-2}} \frac{\partial}{\partial x_n} w_\lambda + \frac{n-2}{2} \overline{H} w_\lambda$$

on ∂H^n and

$$\sum_i \frac{\partial^2}{\partial x_i^2} w_\lambda = 0$$

in H^n. If we choose λ large enough, then

$$w \geq w_\lambda$$

for $x_1 \leq \lambda$ and $t = 0$. In view of the asymptotic expansion for w, it is clear that this is possible. By virtue of the maximum principle, this inequality remains valid for all $t \geq 0$. Therefore, we obtain

$$w \geq w_\lambda$$

for all $x_1 \leq \lambda$ and $t \geq 0$. In particular, we deduce

$$\frac{\partial}{\partial x_1} w \leq 0$$

for $x_1 = \lambda$ and $t \geq 0$. Using the asymptotic expansion for w, we obtain

$$\frac{\partial}{\partial x_1} w = -\frac{(n-2)\, 2^{\frac{n-2}{2}}\, a x_1}{|x|^n} + \frac{2^{\frac{n-2}{2}}\, b_1}{|x|^n} - \sum_i \frac{n\, 2^{\frac{n-2}{2}}\, b_i x_i x_1}{|x|^{n+2}} + \ldots$$

for $|x| \to \infty$. Hence, for $x_1 = \lambda$ we have

$$\frac{\partial}{\partial x_1} w == -\frac{(n-2)\, 2^{\frac{n-2}{2}}\, a \lambda}{|x|^n} + \frac{2^{\frac{n-2}{2}}\, b_1}{|x|^n} - \sum_i \frac{n\, 2^{\frac{n-2}{2}}\, b_i x_i \lambda}{|x|^{n+2}} + \ldots$$

for $|x| \to \infty$. Thus, we conclude

$$-(n-2) a \lambda + b_1 \leq 0,$$

hence

$$\frac{b_1}{a} \leq (n-2)\lambda.$$

Therefore, the gradient of u at infinity can be estimated as

$$\frac{(\nabla_0)_1 u}{u} \leq (n-2)\lambda.$$

Similarly, we find

$$\frac{|(\nabla_0)_i u|}{u} \leq C$$

for $i = 1, \ldots, n-1$. Hence, the tangential part of the gradient of $\log u$ is bounded on every compact subset of ∂M. Therefore, the function $\log u$ is bounded on every compact subset of ∂M.

LEMMA 3.8. *The mean curvature satisfies the evolution equation*

$$\frac{\partial}{\partial t} H = -(n-1)\, \frac{\partial}{\partial \nu} H + H(H - \overline{H})$$

on ∂M. Here, the function H is extended such that

$$\Delta H = 0$$

in M.

Proof. Let g_0 be a fixed metric with $R_0 = 0$. We write $g = u^{\frac{4}{n-2}} g_0$. We then have

$$H = u^{-\frac{2}{n-2}} \left(\frac{2(n-1)}{n-2}\, u^{-1}\, \frac{\partial}{\partial \nu_0} u + H_0 \right)$$

on ∂M and

$$\Delta_0 u = 0$$

in M. Differentiating both equations with respect to t, we obtain

$$\frac{\partial}{\partial t} H = -(n-1)\, u^{-\frac{2}{n-2}}\, \frac{\partial}{\partial \nu_0} H + H(H - \overline{H})$$

in M and

$$\Delta_0 H + 2u^{-1} \langle \nabla_0 u, \nabla_0 H \rangle = 0$$

on ∂M. If we set $g_0 = g$, we get

$$\frac{\partial}{\partial t} H = -(n-1)\, \frac{\partial}{\partial \nu} H + H(H - \overline{H})$$

on ∂M and

$$\Delta H = 0$$

in M.

LEMMA 3.9. *The mean curvature is uniformly bounded in $L^{n-1}(\partial M)$.*

Proof. Using the evolution equation for the mean curvature we obtain

$$\begin{aligned}
\frac{d}{dt}\left(\int_{\partial M} |H|^{n-1}\, dA \right) &= -(n-1)^2 \int_{\partial M} \operatorname{sign}(H)\, |H|^{n-2}\, \frac{\partial}{\partial \nu} H\, dA \\
&= -(n-1)^2 (n-2) \int_M |H|^{n-3}\, |\nabla H|^2\, dV \\
&= -4(n-2) \int_M \left|\nabla |H|^{\frac{n-1}{2}}\right|^2 dV.
\end{aligned}$$

Therefore, the expression

$$\int_{\partial M} |H|^{n-1}\, dA$$

is decreasing.

LEMMA 3.10. *The mean curvature is uniformly bounded in* $L^{\frac{(n-1)^2}{n-2}}(\partial M)$.

Proof. For abbreviation, let $\|\cdot\|_p = \|\cdot\|_{L^p(\partial M)}$ and $\|\cdot\|_{r,p} = \|\cdot\|_{W^{r,p}(\partial M)}$. From the proof of Lemma 3.9 we know that

$$\int_0^T \int_M |\nabla |H|^{\frac{n-1}{2}}|^2\, dV\, dt \leq C.$$

This implies

$$\int_0^T \||H|^{\frac{n-1}{2}}\|_{\frac{1}{2},2}^2\, dt \leq \int_0^T \||H|^{\frac{n-1}{2}}\|_2^2\, dt + C.$$

Since H is bounded in $L^{n-1}(\partial M)$, we obtain

$$\int_0^T \||H|^{\frac{n-1}{2}}\|_{\frac{1}{2},2}^2\, dt \leq C.$$

The Sobolev inequality yields

$$\int_0^T \||H|^{\frac{n-1}{2}}\|_{\frac{2(n-1)}{n-2}}^2\, dt \leq C,$$

hence

$$\int_0^T \|H\|_{\frac{(n-1)^2}{n-2}}^{n-1}\, dt \leq C.$$

Moreover, we have

$$\begin{aligned}
\frac{d}{dt}\bigg(\int_{\partial M} |H|^p\, dA\bigg) &= -(n-1)\, p \int_{\partial M} \operatorname{sign}(H)\, |H|^{p-1}\, \frac{\partial}{\partial \nu} H\, dA \\
&\quad + (p-n+1) \int_{\partial M} |H|^p\, (H - \overline{H})\, dA \\
&= -(n-1)\, p(p-1) \int_M |H|^{p-2}\, |\nabla H|^2\, dV \\
&\quad + (p-n+1) \int_{\partial M} |H|^p\, (H - \overline{H})\, dA \\
&= -(n-1)\, \frac{4(p-1)}{p} \int_M |\nabla |H|^{\frac{p}{2}}|^2\, dV \\
&\quad + (p-n+1) \int_{\partial M} |H|^p\, (H - \overline{H})\, dA
\end{aligned}$$

for all $p \geq 2$. At this point we use the estimate

$$\||H|^{\frac{p}{2}}\|_{\frac{1}{2},2}^2 \leq C \int_M |\nabla |H|^{\frac{p}{2}}|^2\, dA + \||H|^{\frac{p}{2}}\|_2^2$$

(see [19], p. 27). Using the Sobolev inequality, we obtain

$$\begin{aligned}\frac{d}{dt}\|H\|_p^p &\leq -\frac{1}{C}\||H|^{\frac{p}{2}}\|_{\frac{1}{2},2}^2 + C\|H\|_{p+1}^{p+1} + C\\ &\leq -\frac{1}{C}\||H|^{\frac{p}{2}}\|_{\frac{2(n-1)}{n-2}}^2 + C\|H\|_{p+1}^{p+1} + C\\ &\leq -\frac{1}{C}\|H\|_{\frac{p(n-1)}{n-2}}^p + C\|H\|_{p+1}^{p+1} + C\\ &\leq -\frac{1}{C}\|H\|_{\frac{p(n-1)}{n-2}}^p + C\|H\|_q^{\theta(p+1)}\|H\|_{\frac{p(n-1)}{n-2}}^{(1-\theta)(p+1)} + C,\end{aligned}$$

hence

$$\frac{d}{dt}\|H\|_p^p \leq C\|H\|_q^{\frac{\theta(p+1)p}{\theta(p+1)-1}} + C$$

for $q \geq 2$. Here, we have to guarantee that $\theta(p+1) > 1$. The number θ is given by

$$\theta\left(\frac{p}{q} - \frac{n-2}{n-1}\right) = \frac{p}{p+1} - \frac{n-2}{n-1}.$$

Using this relation, we obtain

$$\theta(p+1) > 1 \iff q > n-1.$$

We now take $p = q = \frac{(n-1)^2}{n-2}$. In this case, we obtain

$$\frac{d}{dt}\|H\|_{\frac{(n-1)^2}{n-2}}^{\frac{(n-1)^2}{n-2}} \leq C\|H\|_{\frac{(n-1)^2}{n-2}}^{\frac{(n-1)(2n-3)}{n-2}} + C.$$

If we put

$$y = \|H\|_{\frac{(n-1)^2}{n-2}}^{\frac{(n-1)^2}{n-2}},$$

then we obtain

$$\frac{d}{dt}y \leq Cy^{\frac{2n-3}{n-1}} + C.$$

From this it follows that

$$\frac{d}{dt}\log(y+1) \leq Cy^{\frac{n-2}{n-1}} + C.$$

On the other hand, we have shown above that

$$\int_0^T y^{\frac{n-2}{n-1}}\,dt \leq C.$$

Thus, we conclude

$$y \leq C$$

at time T. Since T is arbitrary, this proves the assertion.

LEMMA 3.11. *The mean curvature is uniformly bounded in $L^p(\partial M)$ for all $p \geq 2$.*

Proof. From the proof of the previous lemma we know that

$$\frac{d}{dt}\|H\|_p^p \leq -\frac{1}{C}\|H\|^p_{\frac{p(n-1)}{n-2}} + C\|H\|_q^{\theta(p+1)}\|H\|^{(1-\theta)(p+1)}_{\frac{p(n-1)}{n-2}} + C$$

for all $p, q \geq 2$. For $q = \frac{(n-1)^2}{n-2}$ we have by Lemma 3.10

$$\|H\|_q \leq C.$$

From this it follows that

$$\frac{d}{dt}\|H\|_p^p \leq -\frac{1}{C}\|H\|^p_{\frac{p(n-1)}{n-2}} + C\|H\|^{(1-\theta)(p+1)}_{\frac{p(n-1)}{n-2}} + C.$$

Since $q > n-1$, we have $\theta(p+1) > 1$, hence $(1-\theta)(p+1) < p$. Thus, we conclude that $\|H\|_p$ is bounded.

PROPOSITION 3.12. *The function u is uniformly bounded in $W^{1,p}(\partial M)$ for all $t \geq 0$.*

Proof. Using Lemma 3.2, we obtain

$$\|u\|_{W^{1,p}(\partial M)} \leq C\,\|\frac{\partial}{\partial\nu_0}u\|_{L^p(\partial M)} \leq C\,\|u^{\frac{2}{n-2}}\,H - H_0\|_{L^p(\partial M)} \leq C$$

for all $p \geq 2$.

As a consequence, the function u is uniformly bounded in C^α. Therefore, we obtain the following result.

COROLLARY 3.13. *The evolution equation (2) has a unique solution for all $t \geq 0$.*

We now determine the asymptotic behavior of the solution.

THEOREM 3.14. *The solution converges to a metric of constant mean curvature as $t \longrightarrow \infty$.*

Proof. We define a functional $E[u]$ on the space of harmonic functions on M by

$$E[u] = A^{-\frac{n-2}{n-1}}\int_{\partial M} H\,dA,$$

where $g = u^{\frac{4}{n-2}}g_0$. Then we obtain

$$\delta E[u] = 2\,A^{-\frac{n-2}{n-1}}\int_{\partial M}(H - \overline{H})\,u^{-1}\,\delta u\,dA.$$

Hence, under the evolution equation (3) the functional $E[u]$ changes according to

$$\frac{d}{dt}E[u] = -(n-2)\,A^{-\frac{n-2}{n-1}}\int_{\partial M}(H - \overline{H})^2\,dA.$$

We now define

$$\lim_{t\longrightarrow\infty} E[u] = \alpha.$$

Since the functional $E[u]$ is real analytic, we can apply the Lojasiewicz-Simon inequality (see [17], Theorem 3 or [12], Proposition 3.3). Therefore, there exists a real number θ such that $0 < \theta < \frac{1}{2}$ and

$$(E[u] - \alpha)^{2(1-\theta)} \leq C \int_{\partial M} (H - \overline{H})^2 \, dA$$

for all $t \geq t_0$. Letting

$$z^2 = \int_{\partial M} (H - \overline{H})^2 \, dA$$

we obtain

$$\left(\int_T^\infty z(t)^2 \, dt \right)^{2(1-\theta)} \leq C z(T)^2$$

for all $T \geq t_0$. By Lemma 4.1 in [12], we conclude that

$$\int_0^\infty z(t) \, dt < \infty.$$

Therefore, the limit

$$\lim_{t \to \infty} u(t) = v$$

exists in $L^2(\partial M)$. This proves the assertion.

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