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REFLECTION OF SHEAVES ON A CALABI-YAU VARIETY*

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Abstract. In this paper we study the reflection of stable sheaves on Calabi-Yau varieties and its effect on the moduli space. It is shown that the reflection defines isomorphisms between the Brill-Noether loci of moduli spaces.

Introduction. Let E be a torsion-free sheaf on a smooth projective K3 surface X and let $\varphi : H^0(X, E) \otimes \mathcal{O}_X \to E$ denote the natural evaluation map. If φ is either injective or surjective, then its cokernel or kernel is called the reflection of E. The reflection functor was first introduced by Mukai ([Mu]) and since then it has been exploited for the study of the moduli space of stable sheaves on K3 surfaces([Ma],[N],[Y]).

It seems significant to consider the reflection functor on higher dimensional Calabi-Yau variety X, in view of Kontsevich's homological mirror conjecture which predicts the existence of equivalence of the derived category $D^b(X)$ of coherent sheaves on X and the derived Fukaya category of its mirror. Inspired by the conjecture, Seidel and Thomas recently introduced an autoequivalence $T_{\mathcal{E}} : D^b(X) \to D^b(X)$ called the twist functor with respect to a spherical object $\mathcal{E}$([ST]). For $\mathcal{F} \in D^b(X)$, $T_{\mathcal{E}}(\mathcal{F})$ is defined to be the cone of the map $\mathrm{Hom}(\mathcal{E}, \mathcal{F}) \overset{\mathbb{L}}{\otimes} \mathcal{E} \to \mathcal{F}$, which coincides with Mukai's reflection in case $\mathcal{E} = \mathcal{O}_X$. However, the problem how this functor is related to the stability of sheaves has not been addressed.

In this paper we study the effect of the reflection functor on the moduli space of stable sheaves on higher dimensional Calabi-Yau varieties instead of the derived category. We shall show that under suitable minimality assumption on the first Chern classes, the reflection preserves the stability of sheaves on arbitrary smooth projective varieties. Further we define the Brill-Noether locus of the moduli space of sheaves on Calabi-Yau varieties and prove that the reflection induces isomorphisms between the Brill-Noether loci for different Mukai vectors. This is a higher dimensional generalization of the results in [Ma],[Y] obtained for K3 surfaces. We also consider examples of reflections on a certain Calabi-Yau threefold which appears in string theory([COFKM]).

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1. Reflection of sheaves. Let X be a smooth projective variety of dimension d defined over the complex number field $\mathbb{C}$ and let H be an ample line bundle on X. For a line bundle $L \in \mathrm{Pic}\, X$, let $\deg L = L \cdot H^{d-1}$ denote its degree. The *minimal H-degree* $d_{\min}(H)$ is defined to be the following positive integer

$$d_{\min}(H) = \min\{\deg M \mid \mathcal{M} \in \mathrm{Pic}(X),\, \deg M > 0\}.$$

A line bundle $\mathcal{L}$ on X is said to be *H-minimal* if $\deg \mathcal{L} = d_{\min}(H)$. For example, $\mathcal{L}$ is H-minimal in one of the following cases:
(1) $\mathrm{Pic}\, X \cong \mathbb{Z}[H]$ and $\mathcal{L} = H$;
(2) $\deg \mathcal{L} = 1$.

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When $d = 2$, the pair $(\mathcal{L}, H)$ has been said to be *of degree one* in [N].

For a torsion-free sheaf E on X, let $\mu(E) = \deg(E)/\operatorname{rk} E$ denote its slope. For torsion-free sheaves E, F on X, we set

$$\delta(E, F) = \operatorname{rk} F c_1(E) - \operatorname{rk} E c_1(F) \in \operatorname{Pic} X.$$

We have $\deg \delta(E, F) = \operatorname{rk} E \operatorname{rk} F(\mu(E) - \mu(F))$.

LEMMA 1.1. *Let E_i $(1 \leq i \leq 3)$ be torsion-free sheaves of rank r_i on X, such that $\mu(E_1) > \mu(E_2) > \mu(E_3)$. Then we have*

$$r_2 \geq \frac{d_{\min}(H)}{\deg \delta(E_1, E_3)}(r_1 + r_3).$$

Proof. By definition of $d_{\min}(H)$, we have $\deg \delta(E_1, E_3) \geq d_{\min}(H)$ and $\deg \delta(E_2, E_3) \geq d_{\min}(H)$. Hence $\mu(E_1) - \mu(E_2) \geq d_{\min}(H)/r_1 r_2$ and $\mu(E_2) - \mu(E_3) \geq d_{\min}(H)/r_2 r_3$. Adding these inequalities, the claim follows. □

LEMMA 1.2. *Let E_i $(1 \leq i \leq 3)$ be torsion-free sheaves of rank r_i and* $\deg E_i = d_i$ *on X which fit in an exact sequence*

$$0 \to E_1 \to E_2 \to E_3 \to 0.$$

Let E_0 be a torsion-free sheaf of rank r_0 and degree d_0.

(1) $\deg \delta(E_2, E_1) = \deg \delta(E_3, E_1)$.

(2) *If* $\deg \delta(E_0, E_2) \geq 0$, *then we have*

$$r_2(\deg \delta(E_3, E_1) - \deg \delta(E_3, E_0)) \geq (r_2 - r_0) \deg \delta(E_3, E_1), \tag{2a}$$

$$r_2(\deg \delta(E_3, E_1) - \deg \delta(E_1, E_0)) \geq (r_2 + r_0) \deg \delta(E_3, E_1). \tag{2b}$$

(3) *If* $\deg \delta(E_3, E_0) \geq 0$, *then we have*

$$r_3(\deg \delta(E_2, E_1) - \deg \delta(E_0, E_2)) \geq (r_3 - r_0) \deg \delta(E_2, E_1).$$

Proof. (1) is obvious. We shall prove (2*a*) only, since the other cases can be treated similarly. It suffices to show $-r_2 \deg \delta(E_3, E_0) \geq -r_0 \deg \delta(E_3, E_1)$. By the asusmption $\deg \delta(E_0, E_2) \geq 0$, we have $d_0 \geq \frac{r_0}{r_2} d_2$. Hence

$$\begin{aligned} -r_2 \deg \delta(E_3, E_0) &= -r_0 r_2 d_3 + r_2 r_3 d_0 \\ &\geq -r_0(r_2 d_3 - r_3 d_2) \\ &= -r_0\{(r_1 + r_3) d_3 - r_3(d_1 + d_3)\} \\ &= -r_0(r_1 d_3 - r_3 d_1) = -r_0 \deg \delta(E_3, E_1). \end{aligned}$$

□

For a coherent sheaf E we denote by $\operatorname{hd} E$ its homological dimension. We note that if $\operatorname{hd} E \leq q$, then $\mathcal{E}xt^i(E, \mathcal{O}_X) = 0$ for all $i > q$. We have $\operatorname{hd} E \leq d - 1$ (resp. $d - 2$) if E is torsion-free (resp. reflexive). In particular, if $d \leq 3$, then every reflexive sheaf E on X satisfies $\operatorname{hd} E \leq 1$.

LEMMA 1.3. *Let F be a vector bundle on X.*

(1) *If E is a coherent sheaf with* $\operatorname{codim}\operatorname{Supp} E \geq 2$, *then we have* $\operatorname{Ext}^1(E, F) = 0$.

(2) *Let E, Q be torsion-free sheaves which fit in the exact sequence*

$$0 \to E \to F \to Q \to 0.$$

Then E is locally free if and only if $\operatorname{hd} Q \leq 1$.

Proof. To prove (1), it suffices to show that $\mathcal{E}xt^i(E, F) = 0$ for $i \leq 1$, since the local-to-global spectral sequence

$$E^{p,q} = H^p(\mathcal{E}xt^q(E, F)) \Rightarrow E^{p+q} = \operatorname{Ext}^{p+q}(E, F)$$

would then yield $\operatorname{Ext}^1(E, F) \cong H^1(\mathcal{E}xt^0(E, F)) = 0$. Let $\mathcal{O}_X(q) = H^{\otimes q}$. We shall prove the claim by showing that $H^0(\mathcal{E}xt^i(E, F)(q)) = 0$ for $i \leq 1$ and sufficiently large q. We choose $q >> 0$ such that $H^0(\mathcal{E}xt^i(E, F)(q)) = \operatorname{Ext}^i(E, F(q))$. Since F is locally free, Serre duality yields $\operatorname{Ext}^i(E, F(q)) \cong \operatorname{Ext}^{d-i}(F(q), E \otimes \omega_X)^\vee \cong H^{d-i}(X, E \otimes \omega_X \otimes F^\vee(-q))^\vee$ where ω_X denotes the canonical bundle of X. Hence, by the assumption $\operatorname{codim}\operatorname{Supp} E \geq 2$, we obtain $\operatorname{Ext}^i(E, F(q)) = 0$ for $i \leq 1$. Thus (1) is proved. (2) follows from a general fact that for an exact sequence $0 \to E \to F \to Q \to 0$ with F locally free, we have $\operatorname{hd} E = \max\{0, \operatorname{hd} Q - 1\}$. □

In this paper we shall consider the following two types of reflections. Let E_1 and Q be torsion-free sheaves such that $\operatorname{Ext}^1(Q, E_1) \neq 0$. For a non-zero subspace $U \subset \operatorname{Ext}^1(Q, E_1)$, we have a natural isomorphism $\operatorname{Ext}^1(Q, U \otimes E_1) \cong \operatorname{Hom}(U^\vee, \operatorname{Ext}^1(Q, E_1))$. Let $\epsilon \in \operatorname{Ext}^1(Q, U^\vee \otimes E_1)$ denote the element corresponding to the inclusion $U \hookrightarrow \operatorname{Ext}^1(Q, E_1)$. The following extension defined by ϵ is called the *universal extension.*

$$0 \to U^\vee \otimes E_1 \to E \to Q \to 0.$$

Assume $\operatorname{Hom}(E_1, E) \neq 0$ and let $U \subset \operatorname{Hom}(E_1, E)$ be a non-zero subspace such that the evaluation map

$$\varphi : U \otimes E_1 \to E$$

is injective (resp.surjective). Then $\operatorname{Coker}\varphi$ (resp.$\operatorname{Ker}\varphi$) is called the *universal division* of E.

LEMMA 1.4. *Let E_1 be a μ-stable vector bundle and let Q be a torsion-free sheaf on X such that $\delta(Q, E_1)$ is H-minimal. Let U be a non-zero vector space and let E be a torsion-free sheaf given by the following non-split extension*

$$(*) \qquad 0 \to U \otimes E_1 \to E \to Q \to 0.$$

Then E is μ-stable if Q is μ-stable and the coboundary map $\delta : U^\vee \to \operatorname{Ext}^1(Q, E_1)$ *induced from $(*)$ is injective. In particular, every universal extension is μ-stable.*

Proof. We proceed by induction on $s := \dim U$. Assume $s = 1$. If E were not μ-stable, then there would exist a μ-stable subsheaf $F \subset E$ with $\mu(F) \geq \mu(E)$. Let $\overline{F} \subset Q$ denote its image by the composite map $F \hookrightarrow E \to Q$. Assume that f is not trivial. Then we have $\mu(F) \leq \mu(\overline{F}) \leq \mu(Q)$ by stability of F and Q. If either $\mu(F) < \mu(\overline{F})$ or $\mu(\overline{F}) < \mu(Q)$, then $\deg \delta(Q, F) > 0$. So the minimality assumption yields $\delta(Q, F) \geq \deg \delta(Q, E_1)$. On the other hand, by Lemma 1.2 (2a) we must have $\operatorname{rk} E(\deg \delta(Q, E_1) - \deg \delta(Q, F)) \geq (\operatorname{rk} E - \operatorname{rk} F) \deg \delta(Q, E_1) > 0$, a contradiction. Thus we obtain $\mu(F) = \mu(\overline{F}) = \mu(Q)$, which implies that f is an isomorphism between F and Q. This again contradicts the assumption that the original sequence is not split. Therefore we obtain $f = 0$. It follows that there exists a non-trivial map $g : F \to E_1$, whose image is denoted by $\overline{F}_1$. By stability, we have $\mu(F) \leq \mu(\overline{F}_1) \leq \mu(E_1)$. As before, if at least one of the inequalities is strict, then the minimality yields $\deg \delta(E_1, F) \geq \deg \delta(Q, E_1)$. This is impossible, since Lemma 1.2 (2b) yields $\operatorname{rk} E(\deg \delta(Q, E_1) - \deg \delta(E_1, F)) \geq (\operatorname{rk} E + \operatorname{rk} F) \deg \delta(Q, E_1) > 0$. Hence it follows that $\mu(F) = \mu(\overline{F}_1) = \mu(E_1)$, hence g is an isomorphism between F and E_1. Then we obtain $\deg \delta(E, E_1) = \deg \delta(E, F) > 0$, which contradicts the assumption $\mu(F) \geq \mu(E)$. Therefore we have shown that E is μ-stable in case $s = 1$.

If $s > 1$, we choose a one dimensional subspace $U_1 \subset U$ and let $\overline{E} = E/U_1 \otimes E_1$ and $\overline{U} = U/U_1$. In view of the following exact sequences

$$0 \to \overline{U} \otimes E_1 \to \overline{E} \to Q \to 0,$$

and

$$0 \to U_1 \otimes E_1 \to E \to \overline{E} \to 0,$$

the assumption that Q is μ-stable implies that $\delta : U^\vee \to \operatorname{Ext}^1(Q, E_1)$ is injective if and only if the two maps $\delta_1 : U_1^\vee \to \operatorname{Ext}^1(\overline{E}, E_1)$, $\bar{\delta} : \overline{U}^\vee \to \operatorname{Ext}^1(Q, E_1)$ are injective. So $\overline{E}$ is μ-stable by the inductive assumption and E is μ-stable by the case $s = 1$. Thus the proof is complete. □

LEMMA 1.5. *Let E_1 be a μ-stable vector bundle of rank r_1 and and E a μ-stable torsion-free sheaf of rank r. Assume* $\operatorname{Hom}(E_1, E) \neq 0$ *and let $U \subset$* $\operatorname{Hom}(E_1, E)$ *be a subspace of dimension $s \neq 0$ with $sr_1 < r$. If $\delta(E, E_1)$ is H-minimal, then the natural evaluation map $\varphi : U \otimes E_1 \to E$ is injective and its cokernel is a μ-stable torsion-free sheaf.*

Proof. We shall prove the claim by induction on s. Assume $s = 1$ and let $E' = \operatorname{Im} \varphi$. If $\operatorname{rk} E' < r_1$, then we would have $\mu(E_1) < \mu(E') < \mu(E)$ by stability. Thus Lemma 1.1 yields $\operatorname{rk} E' \geq r + r_1$, which is a contradiction. Therefore we have $\operatorname{rk} E' = r_1$, hence φ is injective as desired.

Next we show that $Q := \operatorname{Coker} \varphi$ is a μ-stable torsion-free sheaf. Assume that it has a nontrivial torsion subsheaf $\overline{T}$ and let T denote the inverse image of $\overline{T}$ in E. If we had $\operatorname{codim} \operatorname{Supp} \overline{T} \geq 2$, then $\operatorname{Ext}^1(\overline{T}, E_1) = 0$ by Lemma 1.3. So the exact sequence

$$0 \to E_1 \to T \to \overline{T} \to 0$$

splits. Then E must have a torsion subsheaf $\overline{T}$, which is impossible. Thus we see that $\operatorname{codim} \operatorname{Supp} \overline{T} = 1$. Then, since $c_1(\operatorname{Supp} \overline{T}) = c_1(T) - c_1(E_1)$ is strictly effective, we have $\mu(E) > \mu(T) > \mu(E_1)$. Then Lemma 1.1 implies $\operatorname{rk} T \geq r + r_1$ which is a contradiction. Hence Q is torsion-free. If Q were not μ-stable, then there would exist a μ-stable quotient sheaf $Q \to G$ with $\mu(E) < \mu(G) \leq \mu(Q)$. Hence we have $\deg \delta(G, E) \geq \deg \delta(E, E_1)$ by minimality of $\delta(E, E_1)$. However, this is impossible since Lemma 1.2 (3) gives $\operatorname{rk} Q(\deg \delta(E, E_1) - \deg \delta(G, E)) \geq (\operatorname{rk} Q - \operatorname{rk} G) \deg \delta(E, E_1) > 0$. Therefore Q is μ-stable.

Assume that the claim has been proved up to $s - 1$. Let $U \subset \operatorname{Hom}(E_1, E)$ be a subspace of $\dim U = s$ and let $Q = \operatorname{Coker}[\varphi : U \otimes E_1 \to E]$. If U_1 is one dimensional subspace of U, then $\overline{E} = E/U_1 \otimes E_1$ is μ-stable. Since $\overline{U} = U/U_1$ is of dimension

$s-1$, from the exact sequence

$$0 \to \overline{U} \otimes E_1 \to \overline{E} \to Q \to 0$$

and the inductive assumption it follows that Q is μ-stable. Hence we are done. □

LEMMA 1.6. *Let E, E_1, U be as in Lemma 1.5. If $sr_1 > r$, then the evaluation map $\varphi : U \otimes E_1 \to E$ is surjective in codimension 1 and $\operatorname{Ker}\varphi$ is μ-stable.*

Proof. Let $\overline{E} = \operatorname{Im}\varphi$. We have the inequalities $\mu(E_1) \le \mu(\overline{E}) \le \mu(E)$ by stability and at least one of them is not strict by Lemma 1.1. Since $sr_1 > r$, we cannot have $\mu(E_1) = \mu(\overline{E})$, hence $\mu(\overline{E}) = \mu(E)$. It follows that φ is generically surjective. Its cokernel must have support of codimension ≥ 2 since otherwise we would have $\mu(E) > \mu(\overline{E})$, a contradiction.

Let $E' = \operatorname{Ker}\varphi$. By a similar argument as in th proof of Lemma 1.4, if E' were not μ-stable, then there would exist a μ-stable subsheaf $F \subset E'$ with $\mu(F) > \mu(E')$. Then, since $U \otimes E_1$ is semistable, we have $\mu(E_1) \ge \mu(F) > \mu(E')$. If we had $\mu(E_1) > \mu(F)$, then Lemma 1.1 would yield $\operatorname{rk} F \ge (s+1)r_1 - r > \operatorname{rk} E'$, a contradiction. Hence we have $\mu(E_1) = \mu(F)$. Then, since there exists a projection $U \otimes E_1 \to E_1$ such that the composite map $F \hookrightarrow U \otimes E_1 \to E_1$ is not zero, F must be isomorphic to E_1. However this implies $\operatorname{Hom}(E_1, E') \ne 0$, which is not possible. Hence E' is μ-stable. □

Let E_i $(1 \le i \le n)$ be n distinct μ-stable vector bundles of rank r_i. The resuts in this section can be generalized in the following way. Since this will not be needed in the rest of this paper, we leave its proof to the reader, which is an easy induction on n.

PROPOSITION 1.7.

(1) *If Q is a torsion-free sheaf such that $\delta(Q, E_i)$ are H-minimal for all i. Let U_i be vector spaces and let E be a sheaf given by the extension*

$$0 \to \bigoplus_{i=1}^{n} U_i \otimes E_i \to E \to Q \to 0.$$

Then E is μ-stable if and only if Q is μ-stable and the coboundary maps $\delta_i : U_i^\vee \to \operatorname{Ext}^1(Q, E_i)$ are injective for all i.

(2) *Let E be a μ-stable sheaf of rank r. Let $U_i \subset \operatorname{Hom}(E_i, E)$ be subspaces of dimension s_i and let $\varphi : \bigoplus_{i=1}^n U_i \otimes E_i \to E$ be the evaluation map. If $\sum_{i=1}^n s_i r_i < r$, then $\operatorname{Coker}\varphi$ is a μ-stable torsion-free sheaf. If $\sum_{i=1}^n s_i r_i > r$, then φ is generically surjective and $\operatorname{Ker}\varphi$ is μ-stable.*

So far we have been concerned with the construction of μ-stable torsion-free sheaves from the given ones. We may obtain locally free sheaves or reflexive sheaves from codimension two subschemes by means of a generalization of the Serre correspondence, as follows. Let Z be a Cohen Macaulay closed subscheme of codimension two on a polarized smooth projective variety (X, H). Let ω_X denote the canonical bundle of X and let $\omega_Z = \mathcal{E}xt^2(\mathcal{O}_Z, \omega_X)$ denote the dualizing sheaf.

PROPOSITION 1.8. *Let (X, H) and Z be as above. Let $\mathcal{L}$ be a line bundle and E_1 a μ-stable vector bundle on X. Assume that $\delta(\mathcal{L}, E_1)$ is H-minimal and $H^i(E_1 \otimes \mathcal{L}^\vee) = 0$ for $i = 1, 2$. Let $U \subset H^0(E_1 \otimes \omega_Z \otimes \omega_X^\vee \otimes \mathcal{L}^\vee)$ be a non-zero subspace such that the evaluation map*

$$\varphi : U \otimes E_1^\vee \to \omega_Z \otimes \omega_X^\vee \otimes \mathcal{L}^\vee$$

is surjective(resp. surjective in codimension two). Then there exists an extension

$$0 \to U^\vee \otimes E_1 \to E \to \mathcal{I}_Z \otimes \mathcal{L} \to 0.$$

which defines a μ-stable vector bundle(resp. reflexive sheaf)E.

Proof. We have the local-to-global spectral sequence

$$E^{p,q} = H^p(\mathcal{E}xt^q(\mathcal{I}_Z \otimes \mathcal{L}, E_1)) \Rightarrow E^{p+q} = \mathrm{Ext}^{p+q}(\mathcal{I}_Z \otimes \mathcal{L}, E_1).$$

Since $H^i(\mathcal{H}om(\mathcal{I}_Z \otimes \mathcal{L}^\vee, E_1)) \cong H^i(E_1 \otimes \mathcal{L}^\vee) = 0$ for $i = 1, 2$ by assumption, we obtain an isomorphism

$$\mathrm{Ext}^1(\mathcal{I}_Z \otimes \mathcal{L}, E_1) \cong H^0(\mathcal{E}xt^1(\mathcal{I}_Z \otimes \mathcal{L}, E_1)) \cong H^0(E_1 \otimes \omega_Z \otimes \omega_X^\vee \otimes \mathcal{L}^\vee).$$

Since we may consider U as a subspace of $\mathrm{Ext}^1(\mathcal{I}_Z \otimes \mathcal{L}, E_1)$, by Lemma 1.4 the universal extension

$$0 \to U^\vee \otimes E_1 \to E \to \mathcal{I}_Z \otimes \mathcal{L} \to 0.$$

defines a μ-stable torsion-free sheaf E. Consider the induced exact sequence

$$0 \to \mathcal{L}^\vee \to E^\vee \to U \otimes E_1^\vee \to \omega_Z \otimes \omega_X^\vee \otimes \mathcal{L}^\vee \to \mathcal{E}xt^1(E, \mathcal{O}_X) \to 0.$$

The map $U \otimes E_1^\vee \to \omega_Z \otimes \omega_X^\vee \otimes \mathcal{L}^\vee$ is identified with φ, so we obtain $\mathcal{E}xt^1(E, \mathcal{O}_X) = 0$ in case φ is surjective. Since Z is Cohen-Macaulay, we have $\mathrm{hd}\, E \leq 1$([O,Proposition 1.4]), hence $\mathcal{E}xt^q(E, \mathcal{O}_X) = 0$ for all $q > 1$. We note that the singularity set $S(E)$ of E is equal to $\cup_{q=1}^{\dim X} \mathrm{Supp}\, \mathcal{E}xt^q(E, \mathcal{O}_X)$, hence E is locally free if φ is surjective. Similarly, if φ is surjective in codimension two, then we have $\mathrm{codim}\, S(E) \geq 3$. This implies that E is reflexive by [O, Proposition 1.2]. Hence we are done. □

2. Brill-Noether loci on Calabi-Yau varieties. Let (X, H) be a polarized Calabi-Yau variety of dimension d. In this section we shall introduce the Brill-Noether locus of the moduli space of μ-stable sheaves on X. For this purpose, we define the notion of the Mukai vector of coherent sheaves, following [Ty]. Let

$$\widetilde{H}(X) = \oplus_{i=0}^{d} H^{2i}(X, \mathbb{Z})$$

There exists an involution $* : \widetilde{H}(X) \to \widetilde{H}(X)$ which is defined by

$$*_{|H^{4i}(X,\mathbb{Z})} = \mathrm{id}, \quad *_{|H^{4i+2}(X,\mathbb{Z})} = -\,\mathrm{id}.$$

For $u \in \widetilde{H}(X)$, let $[u]_i$ denote the i-th component of u. We can define a symmetric bilinear form $(\, , \,)$ on $\widetilde{H}(X)$ as follows.

$$(u, v) = -[u^* \cdot v]_{2d} = (-1)^d [v^* \cdot u]_{2d}.$$

We define the *Mukai vector* $v(E) \in \widetilde{H}(X)$ of E by

$$v(E) = ch(E) \cdot \sqrt{Td(X)}$$

where $\mathrm{Td}(X)$ denotes the Todd class of X. Then, for coherent sheaves E, F on X, Riemann-Roch yields

$$\begin{aligned} \chi(E, F) : &= \sum_{i=0}^{3} (-1)^i \dim \mathrm{Ext}^i(E, F) \\ &= (v(E), v(F)). \end{aligned}$$

For example, if $d = 3$, we have

$$Td(X) = (1, 0, \frac{c_2(X)}{12}, 0), \quad \sqrt{Td(X)} = (1, 0, \frac{c_2(X)}{24}, 0).$$

Hence for a coherent sheaf E on Calabi-Yau threefold X of rank r, Chern classes $c_i(E) = c_i$, $v(E)$ is given by

$$v(E) = (r, ch_1(E), ch_2(E) + \frac{r}{24}c_2(X), ch_3(E) + \frac{1}{24}ch_1(E) \cdot c_2(X)).$$

For $u = (u_0, u_1, u_2, u_3)$, $v = (v_0, v_1, v_2, v_3) \in \widetilde{H}(X)$, we define an element $\delta(u, v) \in H^2(X, \mathbb{Z})$ by

$$\delta(u, v) = v_0 u_1 - u_0 v_1.$$

For fixed $v \in \widetilde{H}(X)$, let $\mathcal{M}(v)$ denote the moduli space of torsion-free sheaves E with $v(E) = v$ on X, which are μ-stable with respect to H. For a fixed vector bundle E_1 and positive integers i, j, we define *the Brill-Noether locus of type* (i, j) as the locally closed subset of $\mathcal{M}(v)$ defined by

$$\mathcal{M}(v)_{i,j} = \{E \in \mathcal{M}(v) | \dim \mathrm{Hom}(E_1, E) = i, \dim \mathrm{Ext}^1(E, E_1) = j\}.$$

We equip $\mathcal{M}(v)_{i,j}$ with the reduced induced scheme structure. For a positive integer s, let

$$\mathcal{N}_0(v)^s : Sch/\mathbb{C} \to Sets$$

denote the functor which associates to a scheme Y over $\mathbb{C}$ the set $\mathcal{N}_0(v)^s(Y)$ consisting of equivalence classes of the pairs $(\tau, \mathcal{E})$ such that $\mathcal{E}$ is a flat family of μ-stable sheaves with Mukai vector v and τ is a locally free sheaf of $\mathcal{O}_Y$-modules of rank s with an injection

$$\tau \hookrightarrow q_*(\mathcal{E} \otimes p^* E_1^\vee).$$

were $p : X \times Y \to X$, $q : X \times Y \to Y$ denote the natural projections. We note that such τ corresponds to an injection on $X \times Y$:

$$q^*\tau \otimes p^* E_1 \to \mathcal{E}.$$

Then two pairs $(\tau, \mathcal{E})$, $(\tau', \mathcal{E}')$ are said to be equivalent if there exist a line bundle L on Y and an isomorphism $\varphi : \mathcal{E} \cong \mathcal{E}' \otimes q^* L$ such that $\varphi(q^*\tau \otimes p^* E_1) = q^*\tau' \otimes p^* E_1 \otimes q^* L$.

When $E_1 = \mathcal{O}_X$, the above functor coincides with the functor of coherent systems defined in [He]. By the same argument as given there, this functor has a coarse moduli scheme $\mathcal{N}_0(v)^s$ whose $\mathbb{C}$-valued points is the following set

$$\mathcal{N}_0(v)^s = \{(E, U) | E \in \mathcal{M}(v), U \subset \mathrm{Hom}(E_1, E), \dim U = s\}.$$

Let p be the natural projection

$$p : \mathcal{N}_0(v)^s \to \mathcal{M}(v)$$

which sends (E, U) to E. Let

$$\mathcal{N}_0(v)^s_{i,j} = p^{-1}(\mathcal{M}(v)_{i,j}).$$

Then $p : \mathcal{N}(v)^s_{i,j} \to \mathcal{M}(v)_{i,j}$ is a $\mathrm{Gr}(s, i)$-bundle over $\mathcal{M}(v)_{i,j}$.

Similarly for $w \in \widetilde{H}(X)$ and a positive integer t, we define the functor

$$\mathcal{N}_1(w)^t : Sch/\mathbb{C} \to Sets$$

by defining $\mathcal{N}_1(w)^t(Y)$ to be the set of isomorphism classes of pairs $(\eta, \mathcal{Q})$ such that $\mathcal{Q}$ is a flat family of μ-stable sheaves on X with Mukai vector w and η is a locally free sheaf of $\mathcal{O}_Y$-modules of rank t with an injection

$$\eta \hookrightarrow \mathcal{E}xt^1_q(\mathcal{Q}, p^*E_1 \otimes \omega_q)$$

where $\mathcal{E}xt^1_q(\mathcal{Q}, p^*E_1)$ denotes the relative extension sheaf with respect to q and ω_q is the relative dulaizing sheaf. There exists a moduli space $q : \mathcal{N}_1(w)^t \to \mathcal{M}(w)$ whose set of $\mathbb{C}$-valued points is given by

$$\mathcal{N}_1(w)^t = \{(Q, V) \mid Q \in \mathcal{M}(w),\ V \subset \operatorname{Ext}^1(Q, E_1),\ \dim V = t\}.$$

As before, if we define

$$\mathcal{N}_1(w)^t_{i,j} = q^{-1}(\mathcal{M}(w)_{i,j}),$$

then $q : \mathcal{N}_1(w)^t_{i,j} \to \mathcal{M}(w)_{i,j}$ is a $\mathrm{Gr}(t, j)$-bundle over $\mathcal{M}(w)_{i,j}$.

Assume that E_1 is μ-stable and *rigid*, that is, $\operatorname{Ext}^1(E_1, E_1) = 0$. We assume further that its Mukai vector $v(E_1)$ satisfies the condition that $\delta(v, v(E_1))$ is H-minimal. For a positive integer s satisfying $sv(E_1)_0 < v_0$, we let

$$w = v - sv(E_1).$$

We notice that for every $E \in \mathcal{M}(v)_{i,j}$, we have $\operatorname{Hom}(E, E_1) = 0$, since otherwise E must be isomorphic to E_1 and hence $j = 0$, which is a contradiction. For an element $(E, U) \in \mathcal{N}_0(v)^s_{i,j}$, let Q denote the universal division defined by the sequence

$$0 \to U \otimes E_1 \to E \to Q \to 0.$$

Since E_1 is μ-stable and rigid, this induces the exact sequences

$$0 \to U \to \operatorname{Ext}^1(Q, E_1) \to \operatorname{Ext}^1(E, E_1) \to 0$$

and

$$0 \to U \to \operatorname{Hom}(E_1, E) \to \operatorname{Hom}(E_1, Q) \to 0.$$

Hence $\dim \operatorname{Hom}(E_1, Q) = i - s$ and $\dim \operatorname{Ext}^1(Q, E_1) = j + s$.

THEOREM 2.1. *There exists an isomorphism between Brill-Noether loci*

$$f : \mathcal{N}_0(v)^s_{i,j} \to \mathcal{N}_1(w)^s_{i-s,j+s}.$$

Proof. We shall show that for every $\mathbb{C}$-scheme Y, there exists a natural bijection between the two sets $\mathcal{N}_0(v)^s_{i,j}(Y)$ and $\mathcal{N}_1(w)^s_{i-s,j+s}(Y)$. A pair $(\mathcal{E}, \tau) \in \mathcal{N}_0(v)^s_{i,j}(Y)$ with $\tau \hookrightarrow q_*(\mathcal{E} \otimes p^*E_1^\vee)$ yields an exact sequence on $X \times Y$

$$0 \to q^*\tau \otimes p^*E_1 \to \mathcal{E} \to \mathcal{Q} \to 0$$

By Lemma 1.5, $\mathcal{Q}$ is a family of sheaves in $\mathcal{N}_1(w)^s_{i-s,j+s}$, whcih is flat over Y. Since we have $q_*\mathcal{H}om(\mathcal{E}, p^*E_1) = 0$ by stability, taking $\mathcal{H}om_q(\ \ , p^*E_1)$ we obtain the injection

$$0 \to \tau^\vee \otimes p^*E_1 \to \mathcal{E}xt^1_q(\mathcal{Q}, p^*E_1 \otimes \omega q).$$

So we can define the map f by $f((\mathcal{E},\tau)) = (\mathcal{Q},\tau) \in \mathcal{N}_1(w)^s_{i-s,j+s}(Y)$. Now we will give the inverse map g of f. For $(\mathcal{Q},\eta) \in \mathcal{N}_1(w)^s_{i-s,j+s}(Y)$, we have $q_*\mathcal{H}om(\mathcal{E}, q^*\tau \otimes p^*E_1 \otimes \omega_q) = 0$ since $\mu(\mathcal{E}_t) > \mu(E_1)$ for all $t \in Y$ by assumption. Hence the Leray spectral sequence yields the isomorphism

$$\operatorname{Ext}^1(\mathcal{Q}, q^*\eta^\vee \otimes p^*E_1) \cong H^0(Y, \mathcal{E}xt^1_q(\mathcal{Q}, q^*\eta^\vee \otimes p^*E_1 \otimes \omega_q)).$$

We may identify the injection $\eta \hookrightarrow \mathcal{E}xt^1(\mathcal{Q}, p^*E_1 \otimes \eta^\vee)$ with an element of $H^0(X \times Y, \mathcal{E}xt^1_q(\mathcal{Q}, q^*\eta^\vee \otimes p^*E_1 \otimes \omega_q))$, so this gives the following extension on $X \times Y$

$$0 \to \eta^\vee \otimes p^*E_1 \to \mathcal{E} \to \mathcal{Q} \to 0.$$

Let $q_*(\eta^\vee) \hookrightarrow q_*(\mathcal{E} \otimes p^*E_1^\vee)$ be the push-down of this map by q. If we define $g((\mathcal{Q},\eta)) = (\mathcal{E}, q_*\eta^\vee)$, then clearly this is the inverse of f. Thus the theorem is proved. □

Assume that $\delta(v, v(E_1))$ is H-minimal. For a positive integer s with $sv(E_1)_0 > v_0$, let

$$w^* = sv(E_1) - v.$$

For positive integers i, j, we define another Brill-Noether loci whose $\mathbb{C}$-valued points are given by

$$\mathcal{M}(v)^*_{i,j} = \{E \in \mathcal{M}(v) \mid E \text{ is locally free}, \dim \operatorname{Hom}(E, E_1) = i, \dim \operatorname{Ext}^1(E_1, E) = j\},$$

and

$$\mathcal{N}_0(v)^{s*}_{i,j} = \{(E,U) \mid E \in \mathcal{M}(v)^*_{i,j},\ U \subset \operatorname{Hom}(E, E_1),\ \dim U = s\},$$

$$\mathcal{N}_0(v)^{s**}_{i,j} = \{(E,U) \mid E \in \mathcal{M}(v)_{i,j},\ \operatorname{hd} E \le 1,\ U \subset \operatorname{Hom}(E_1, E),\ \dim U = s,\ U \otimes E_1 \to E \text{ is surjective}\}.$$

Then we have

THEOREM 2.2. *There exists an isomorphism between Brill-Noether loci*

$$f^* : \mathcal{N}_0(v)^{s*}_{i,j} \to \mathcal{N}_0(w^*)^{s**}_{j+s,i-s}.$$

Proof. We consider only $\mathbb{C}$-valued points. Since the argument works for famillies, the claim follows as in Theorem 2.1. For $(E,U) \in \mathcal{N}_0(v)^{s*}_{i,j}(\mathbb{C})$, under the isomorphism $\operatorname{Hom}(E, E_1) \cong \operatorname{Hom}(E_1^\vee, E^\vee)$, the natural map

$$\varphi : E \to U^\vee \otimes E_1$$

is identified with the dual of the evaluation map

$$U^\vee \otimes E_1^\vee \to E^\vee.$$

Hence $Q = \operatorname{Coker} \varphi$ is a μ-stable torsion-free sheaf with $\operatorname{hd} Q \le 1$ by Lemma 1.3 and Lemma 1.5. So $f^*((E,U)) = (Q, U^\vee)$ belongs to $\mathcal{N}_0(v)^{s**}_{j+s,i-s}$. For $(Q,V) \in \mathcal{N}_0(v)^{s**}_{i,j}$, let E denote the kernel of the evaluation map $V \otimes E_1 \to Q$. E is a μ-stable vector bundle by Lemma 1.3 and Lemma 1.6. If we let $g^*((Q,V)) = (E, V^\vee)$, g^* is the inverse of f^*. □

3. Examples of reflection on a certain Calabi-Yau threefold. In this section we shall consider the reflection of stable sheaves on the "stringy" Calabi-Yau threefold $\mathbb{P}^{(1,1,2,2,2)}[8]$, which has been extensively studied in the context of mirror symmetry([COFKM]). Let $\widehat{X}$ denote a hypersurface of degree 8 in the weighted projective space $\mathbb{P}^{(1,1,2,2,2)}$. A typical example is given by the equation

$$x_1^8 + x_2^8 + x_3^4 + x_4^4 + x_5^4 = 0.$$

Let

$$p : \mathbb{P} \longrightarrow \mathbb{P}^{(1,1,2,2,2)}$$

be the blowup of the singular locus C of $\mathbb{P}^{(1,1,2,2,2)}$ which is a smooth curve of genus three. This induces a resolution $p : X \longrightarrow \widehat{X}$ where X is a smooth Calabi-Yau threefold contained in $\mathbb{P}$ as an anti-canonical divisor. $\mathbb{P}$ is isomorphic to the $\mathbb{P}^3$-bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ over $\mathbb{P}^1$. Let $M = \mathcal{O}_{\mathbb{P}}(1)$ be the tautological line bundle and L the fiber of the projection $\pi : \mathbb{P} \longrightarrow \mathbb{P}^1$. Then X is linearly equivalent to $-K_{\mathbb{P}} = 4M$ and the induced morphism $\pi : X \longrightarrow \mathbb{P}^1$ is a K3 fibration whose general fiber is a quartic surface in $\mathbb{P}^3$. Let $H = M_{|X}$ and $F = L_{|X}$. $\mathrm{Pic}(X)$ is generated by H, F and the intersection numbers are given by

$$H^3 = 8, \quad H^2 \cdot F = 4, \quad F^2 = 0.$$

Let E denote the exceptional divisor over C and let $l \in A^2(X)$ denote the class of a fiber of the ruling $E \to C$. Then we have $H = 2F + E$ and

$$4l = H \cdot E = H \cdot (H - 2F).$$

We define $h \in A^2(X)$ as the class defined by

$$4h = H \cdot F$$

which is represented by a smooth rational curve. We can calculate the intersection numbers of them as follows.

$$F \cdot l = 1, \quad F \cdot h = 0, \quad H \cdot l = 0, \quad H \cdot h = 1.$$

The second Chern class of X is $c_2(X) = 56h + 24l$.

We will consider an ample divisor $H_q = H + qF$ on X for fixed $q > 0$. For integers α, β, we let

$$\mathcal{L}_{\alpha,\beta} = \alpha H + \beta F.$$

Then we have

$$\mathcal{L}_{\alpha,\beta} \cdot H_q^2 = 4(2(q+1)\alpha + \beta) \equiv 0 \pmod 4.$$

For every integer α, the line bundle

$$\mathcal{L}_\alpha := \mathcal{L}_{\alpha,-2(q+1)\alpha+1}$$

satisfies $\mathcal{L}_\alpha \cdot H_q^2 = 4$, hence $d_{\min}(H_q) = 4$ and $\mathcal{L}_\alpha$ is H_q-minimal. By Lemma 1.3, every subspace $U \subset \mathrm{Ext}^1(\mathcal{L}_\alpha, \mathcal{O}_X)$ yields an H_q-stable bundle E which fits in the exact sequence

$$0 \to U \otimes \mathcal{O}_X \to E \to \mathcal{L}_\alpha \to 0.$$

Next we will give an example of E_1 which is not isomorphic to $\mathcal{O}_X$. For given Mukai vector v, the ample line bundle H_q is *suitable* for sufficiently large q. This means that a sheaf E of Mukai vector v is H_q-stable if and only if its restriction to general fiber X_t of π is H_t-stable. Let T_π denote the relative tangent bundle of the ambient $\mathbb{P}^3$-bundle $\pi : \mathbb{P} \to \mathbb{P}^1$ and let E_1 denote its restriction to X. Since E_1 is H_t-stable and rigid on general K3-fiber, E_1 is rigid and H_q-stable for sufficiently large q by [Th]. It has Mukai vector $v(E_1) = (3, 4H - 2F, 24(h + l), 8)$ and sits in the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(H)^{\oplus 3} \oplus \mathcal{O}_X(H - 2F) \to E_1 \to 0.$$

Let α be an integer and let $\beta = -(2q+2)\alpha + 24q + 17$. Then we have $\delta(\mathcal{L}_{\alpha,\beta}, E_1) = \mathcal{L}_\alpha$. So a subspace $U \subset \operatorname{Ext}^1(E_1, E)$ yields an H_q-stable bundle E by the extension

$$0 \to U \otimes E_1 \to E \to \mathcal{L}_{\alpha,\beta} \to 0.$$

By Theorem 2.1, from the two examples considered above we obtain Brill-Noether loci $\mathcal{N}(v)^s_{i,j}$ which are isomorphic to the Grassmann varieties.

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