

NEARLY KÄHLER GEOMETRY AND RIEMANNIAN FOLIATIONS *

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Abstract. We consider strict and complete nearly Kähler manifolds with the canonical Hermitian connection. The holonomy representation of the canonical Hermitian connection is studied. We show that a strict and complete nearly Kähler is locally a Riemannian product of homogenous nearly Kähler spaces, twistor spaces over quaternionic Kähler manifolds and 6-dimensional nearly Kähler manifolds. As an application we obtain structure results for totally geodesic Riemannian foliations admitting a compatible Kähler structure. Finally, we obtain a classification result for the homogenous case, reducing a conjecture of Wolf and Gray to its 6-dimensional form.

1. Introduction. Given an oriented Riemannian manifold (M^n, g) a fundamental piece of data is encoded in its holonomy group, to be denoted by $Hol(M, g)$. Indeed, the condition that $Hol(M, g)$ be different from $SO(n)$ has strong geometric implications at the level of the geometric structure supported by M (see [31] for an account). The search for new geometric structures supporting (for example) Einstein metrics or satisfying usefull curvature identities motivated the introduction by A. Gray of the concept of weak holonomy (see [12] and [32]). However, in the case when the weak holonomy group of a manifold acts transitively on its unit bundle it was shown [12] that only three groups, namely

$$U(n), G_2 \text{ and } Spin(9)$$

can potentially produce new geometric structures (other than those coming from Riemannian holonomy). This is certainly the case for weak holonomy G_2 (see [8]), where many homogenous examples were constructed. Also, it is now known that every 7 dimensional 3-sasakian manifold carry a metric with weak holonomy G_2 and, furthermore, there is no scarcity of such manifolds (see [3]). Very recently, the case of $Spin(9)$ was undertaken by Friedrich [9].

In this paper we will be concerned with the study of manifolds with weak holonomy $U(n)$. These manifolds are called nearly Kähler and also appear as one of the sixteen classes of almost Hermitian manifolds [15]. Many properties of this class of manifolds are now known (see [11, 13, 14]). Note the 6-dimensional case stands out because of the existence of Killing spinors (fact that is in this dimension equivalent to being nearly Kähler [16]) and because of carrying Einstein metrics.

Nearly Kähler manifolds can also be characterized as almost Hermitian manifolds admitting a metric Hermitian connection whose torsion is parallel and totally skew. From this point of view they are interesting in theoretical physics [10]. Manifolds carrying a connection with totally skew-symmetric, parallel, torsion were used in [5], as a point of departure to propose a framework somewhat parallel to weak holonomy, formulated in terms of G -structures (see [35] for a brief account). The main result of our paper is a classification result of nearly Kähler manifolds, given below.

THEOREM 1.1. *Let (M^{2n}, g, J) be a strict and complete, simply connected, nearly Kähler manifold. Then M is a Riemannian product whose factors belong to one of the following three classes :*

- 6 dimensional nearly Kähler manifolds;

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- homogenous nearly Kähler spaces of type I,II,III or IV;
- twistor spaces over quaternionic Kähler manifolds with positive scalar curvature, endowed with the canonical nearly Kähler metric

Of course, the previous decomposition coincides with the deRham decomposition of the manifold (M, g) . The homogenous spaces of type I and II and described in sections 2 and 3 and those of types III and IV in sections 5 and 6. The latter are generalized "twistor spaces", that is Riemannian submersions whose (totally geodesic) fibers are compact and simply connected Hermitian symmetric spaces and whose base spaces are compact and simply connected symmetric spaces.

Hence, weak holonomy $U(n)$ does not produce any new geometric structure, except possibly in real dimension 6. But even in this case, the only known examples are homogenous.

The proof of our theorem goes as follows. In view of the results of [5] we are mainly concerned with the case when the holonomy of the canonical Hermitian connection is reducible. Then, our point of departure is to show that one can suppose that the holonomy representation is of special algebraic type, that is it has strong algebraic properties related to the torsion tensor of the Hermitian connection. This will be done in section 3. In the case of special algebraic torsion, we show in section 4 that the manifold M is the total space of a fibration whose (totally geodesic) fibers are compact and simply connected Hermitian symmetric spaces. It turns out that for such fibrations we can prove an analogue of the DeRham decomposition theorem, in the sense the one can always suppose that the fiber and the base space are irreducible, in the usual Riemannian sense. After developing some facts related to the Ricci tensor in section 5 we finally prove the classification result in section 6. It reposes on the observation that the torsion tensor points out in a nice way to the Riemannian holonomy group of the base manifold, hence allowing the use of the Berger classification theorem.

As a corollary of theorem 1.1 we show how one can use the classification of nearly Kähler manifolds in order to study geometric structures on Kähler manifolds. To cite some related results, recall that it was known for a long time that that a Kähler submersion has integrable and totally geodesic horizontal distribution, hence zero curvature [37, 18]. The same conclusion holds for nearly Kähler and almost Kähler submersions [7]. In twistor theory, one searches for a non-holomorphic fibration whose total space and fibers are complex manifolds. A reasonable way to see these objects arise is from Riemannian foliations. Unfortunately, the following result shows that it cannot happen in the presence of a Kähler metric, except for a few special cases.

THEOREM 1.2. *Let (M, g, J) be a simply connected Kähler manifold supporting a foliation \mathcal{F} such that (M, g, \mathcal{F}) becomes a totally geodesic Riemannian foliation. If the leaves of \mathcal{F} are complex manifolds then (M, g) is a Riemannian product whose factors belong to one the following three classes :*

- twistor spaces over quaternionic Kähler manifolds of positive scalar curvature ;
- compact homogenous Kähler manifolds corresponding to nearly Kähler manifolds of types III and IV;
- Kähler manifolds.

Furthermore, the foliation \mathcal{F} is obtained in a canonical way from the above decomposition.

This is not a very surprising result, as geometric structures on Kähler manifolds are rather rare. As an example which parallels theorem 1.2 we cite the result that a complex contact manifold with a Kähler-Einstein metric is the twistor space of a

quaternionic Kähler manifold of positive sectional curvature [23]. The main ingredient of the proof of the theorem 1.2, given in section 7, is the observation that Kähler manifolds admitting a Riemannian foliation as above have a canonical nearly Kähler metric. Next we use the results leading to the proof of theorem 1.1.

Finally, in section 8 we study homogenous nearly Kähler manifolds using theorem 1.1. A classification result involving the previously defined spaces of type *I, II, III* and *IV* is obtained. As a consequence, we recover by geometric means the classification of naturally reductive Riemannian 3-symmetric spaces, which was obtained in his general form in [13] using Lie theory. We are also able to reduce a conjecture of Wolf and Gray relating to nearly Kähler homogenous spaces to its 6-dimensional version.

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2. Preliminaries. A nearly Kähler manifold is an almost Hermitian manifold (M^{2n}, g, J) such that

$$(\nabla_X J)X = 0$$

for every vector field X on M (here ∇ denotes the Levi-Civita connection associated to metric g). It is called *strict* if $\nabla_X J \neq 0$ whenever $X \in TM, X \neq 0$.

Recall that the tensor ∇J has a number of important algebraic properties that can be summarized as follows : the tensors A and B defined for X, Y, Z in TM by $A(X, Y, Z) = \langle (\nabla_X J)Y, Z \rangle$ and $B(X, Y, Z) = \langle (\nabla_X J)Y, JZ \rangle$ are skew-symmetric and have type $(0, 3) + (3, 0)$ as real 3-forms. Another object of particular importance is the canonical Hermitian connection defined by

$$\bar{\nabla}_X Y = \nabla_X Y + \frac{1}{2}(\nabla_X J)JY.$$

It is easy to see that $\bar{\nabla}$ is the unique Hermitian connection on M with totally skew-symmetric torsion (see for example [10]). Note that the torsion of $\bar{\nabla}$ given by $T(X, Y) = (\nabla_X J)JY$ vanishes iff (M, g, J) is a Kähler manifold.

Let us define $\Phi \in \Lambda^3(M)$ by $\Phi(X, Y, Z) = \langle (\nabla_X J)Y, Z \rangle$ whenever X, Y, Z are in TM . A fact of crucial importance for us will be that the form Φ is parallel with respect to the canonical Hermitian connection, that is :

$$(2.1) \quad \bar{\nabla}\Phi = 0.$$

This can be deduced from the $\bar{\nabla}$ parallelism of the tensor ∇J which was proven in [1]. This particularly implies that the holonomy group of $\bar{\nabla}$ is contained at each point in the $U(n)$ -stabiliser of the three form Φ . The following decomposition result shows that one can restrict attention to strict nearly Kähler manifolds.

PROPOSITION 2.1. [19, 24] *Let (M, g, J) be a complete and simply connected nearly Kähler manifold. Then M is a Riemannian product $M_1 \times M_2$ where M_1 and M_2 are Kähler respectively strict nearly Kähler manifolds.*

We discuss now briefly some properties of the Ricci tensor of nearly Kähler manifolds. Recall that it was shown in [24] that if (M, g, J) is strict then it has positive

Ricci curvature, hence (M, g, J) is compact as soon as it is complete. Moreover, under these conditions the fundamental group is finite hence the requirement of simple connectivity (to be done in most cases in this work) is not too restrictive. Now, the Ricci- \star curvature is defined by :

$$Ric^*(X, Y) = \frac{1}{2} \sum_{i=1}^{2n} R(X, JY, e_i, J e_i)$$

where R is the curvature tensor of (M, g) and $\{e_1, \dots, e_{2n}\}$ a local frame field. The difference of the Ricci and Ricci- \star curvature tensors, to be denoted by r , is given by the formula (see [14]) :

$$\langle rX, Y \rangle = \sum_{i=1}^{2n} \langle (\nabla_{e_i} J)X, (\nabla_{e_i} J)Y \rangle .$$

Then r is symmetric, positive and commutes with J ; furthermore we have $\bar{\nabla}r = 0$ (see [24]). It is well known [14] that if the tensor r has a single eigenvalue then (M, g) is an Einstein manifold and furthermore the first Chern class of (M, J) vanishes.

From now on we will work with a strict and complete nearly Kähler manifold (M^{2n}, g, J) . The torsion tensor of the canonical Hermitian connection can be used to define a vector product on TM by setting

$$X \bullet Y = T(X, Y)$$

whenever X, Y belong to TM . The algebraic properties of the torsion tensor T translate into analogous ones for the previous defined vector product. If V, W are distributions of TM we will usually denote by $V \bullet W$ the linear span of $\{x \bullet y : x \in V, y \in W\}$.

Let us recall that the first Chern class of (M, J) is represented by the closed 2-form γ_1 defined by $8\pi\gamma_1(X, Y) = \langle CX, JY \rangle$ for all X, Y in TM . Here C denotes the symmetric endomorphism $Ric - 5Ric^*$. It has been proven in [24] that the tensor C is $\bar{\nabla}$ -parallel. As γ_1 is a closed form we obtain easily

$$(2.2) \quad -C(X \bullet Y) = X \bullet CY + CX \bullet Y$$

for all X and Y in TM .

REMARK 2.1. *A basic fact about C is that if E_i and E_j are eigenspaces of C with corresponding eigenvalues λ_i and λ_j then $-(\lambda_i + \lambda_j)$ is an eigenvalue of C if $E_i \bullet E_j \neq 0$.*

PROPOSITION 2.2. *Suppose that M is simply connected. If C has a single eigenvalue then it splits as a Riemannian product whose factors are strict nearly Kähler manifolds such that their corresponding tensors C and r have exactly one eigenvalue.*

Proof. For the inverse implication see [14]. Let us suppose now that C has a single eigenvalue. Then $C = 0$ by (2.2) and more, $Ric = \frac{5}{4}r$. We recall now the expression of the Ricci curvature of (M, g) we obtained in [24]. If $V_i, 1 \leq i \leq p$ are the eigenspaces of r corresponding to the eigenvalue λ_i then

$$(2.3) \quad Ric(X, Y) = \frac{\lambda_i}{4} \langle X, Y \rangle + \frac{1}{\lambda_i} \sum_{s=1}^p \lambda_s \langle r^s(X), Y \rangle$$

for all X and Y in V_i , where the tensors r^s , $1 \leq s \leq p$ are defined by $\langle r^s(X), Y \rangle = -\text{Tr}_{V_s}(\nabla_X J)(\nabla_Y J)$. Let us now suppose that the eigenvalues of r are ordered as follows : $0 < \lambda_1 < \dots < \lambda_p$. Then, on V_p we must have $\lambda_p = \frac{1}{\lambda_p} \sum_{s=1}^p \lambda_s r^s$. As $r = \sum_{s=1}^p r^s$ this implies easily that $r^s = 0$, $1 \leq s \leq p-1$ on V_p . In other words, $V_p \bullet H = 0$ where $H = \bigoplus_{s=1}^{p-1} V_s$ and it follows that $V_p \bullet V_p \subseteq V_p$ and $H \bullet H \subseteq H$. Hence, the distributions V_p and H are ∇ -parallel, thus -by DeRham's decomposition theorem- M is a Riemannian product $M_1 \times M_2$. It is easy to show that M_1, M_2 are strict nearly Kähler manifolds and M_1 has the property that the corresponding tensors r and C have exactly one eigenvalue. Now we apply the same procedure to the manifold M_2 and we conclude by induction. \square

REMARK 2.2. (i) Note that a result analogous to proposition 2.1 was already proven by different means in [36] under the assumptions of Riemannian irreducibility and constant scalar curvature. Note that the latter is now known to hold for all strict nearly Kähler manifolds. By the work in [33], proposition 2.1 remains true if we replace the vanishing of the Chern form with the vanishing of the Chern class.

(ii) Manifolds with $C = 0$ and r with a single eigenvalue are automatically Einstein.

In the rest of this section we will set up some facts concerning the holonomy representation of the canonical Hermitian connection. We begin with the following elementary result in representation theory. We also include its proof as we were unable to find it in the literature.

LEMMA 2.1. Let (V^{2n}, g, J) a real vector space equipped with a scalar product g and a compatible complex structure J . Let $V^{\mathbb{C}}$ the complex vector space obtained from V by defining the complex multiplication to be $i.v = Jv$. Let π be a real representation of a group G on V respecting the inner product and the complex structure. Let $\pi^{\mathbb{C}}$ be the complex representation of G on $V^{\mathbb{C}}$ induced by π . If $\pi^{\mathbb{C}}$ is irreducible then the following possibilities may occur :

(i) π is irreducible;

(ii) there exists an irreducible G -submodule of V , to be denoted by \mathcal{V} , such that $V = \mathcal{V} \oplus J\mathcal{V}$, an orthogonal direct sum.

Proof. If π is irreducible then we are in the case of (i). Suppose now that the representation of G on V is real reducible. Let \mathcal{V} be a proper invariant subspace. Then $\mathcal{V} \cap J\mathcal{V}$ is preserved by J and G -invariant so it follows that $\mathcal{V} \cap J\mathcal{V} = 0$. The same argument implies that we have the direct sum decomposition $V = \mathcal{V} \oplus J\mathcal{V}$. We will show now that the representation of G on \mathcal{V} is irreducible. Indeed, if $E \subseteq \mathcal{V}$ is non-zero G module then as before $E \cap JE = 0$ and $E \oplus JE = \mathcal{V}$, hence E and \mathcal{V} have the same dimension and thus $E = \mathcal{V}$. It remains to show that the sum $\mathcal{V} \oplus J\mathcal{V}$ is orthogonal. Using Riesz's representation theorem we obtain the existence of a skew-symmetric endomorphism of \mathcal{V}_m , to be denoted by F , such that :

$$\langle v, Jw \rangle = \langle Fv, w \rangle \quad \text{for all } v, w \text{ in } \mathcal{V}.$$

Of course, F must be G -invariant. We extend F to an endomorphism f of V , by setting $f(Jv) = J(Fv)$ for all v in \mathcal{V} . Obviously this is complex linear and G -invariant, so using the irreducibility of $V^{\mathbb{C}}$ and Schur's lemma we obtain that $f = z1_{V^{\mathbb{C}}}$ for some complex number z . The definition of f and the skew-symmetry of F imply now the vanishing of F . \square

Before applying this to our geometric context let us recall the following important result.

THEOREM 2.1. [5] *Let (M^n, g) be a simply connected Riemannian manifold equipped with a metric connexion ∇' whose torsion tensor, denoted by T , is ∇' -parallel and totally skew. If the Lie algebra of the stabiliser of T in $\mathfrak{so}(n)$ acts irreducibly on some tangent space, then M is a homogenous space with irreducible isotropy unless $\text{Stab}_{\mathfrak{so}(n)}T$ equals G_2 or $SU(3)$ acting as usually on \mathbb{R}^7 and \mathbb{R}^6 respectively.*

REMARK 2.3. *The two exceptions in theorem 2.1 are weak holonomy structures, precisely nearly parallel G_2 and six dimensional nearly Kähler manifolds. The basic ingredient for the proof of the previous result is a structure theorem for the space of algebraic curvature tensors of a Lie algebra preserving a three form. The homogenous structure on M is obtained by proving that ∇' is an Ambrose-Singer connexion, that is $\nabla'R' = 0$. Also, it would be interesting to have a differential geometric proof of this result.*

This remark motivates the setting, in the nearly Kähler case, of the following definition.

DEFINITION 2.1. *A strict nearly Kähler manifold (M^{2n}, g, J) with $n \geq 4$ is said homogenous of type I if the holonomy representation of the canonical Hermitian connection is real irreducible.*

By the previous discussion, in this case $\bar{\nabla}$ is a Ambrose-Singer connexion and moreover, M is isotropy irreducible. We can now use the previous preliminaries in order to distinguish nearly Kähler manifolds by holonomic means, as follows.

PROPOSITION 2.3. *Let (M^{2n}, g, J) a strict, complete and simply connected nearly Kähler manifold. Then the following possibilities may occur :*

- (i) *M is homogenous of type I, unless M is of dimension 6 ;*
- (ii) *we have a $\bar{\nabla}$ -parallel, orthogonal, decomposition $TM = \mathcal{V} \oplus H$ with $H = J\mathcal{V}$;*
- (iii) *the holonomy representation of the canonical Hermitian connection is complex reducible.*

Proof. We use that the torsion of the canonical Hermitian connection is parallel and totally skew. In particular, this implies that if the holonomy representation of $\bar{\nabla}$ is real irreducible then the same is true for representation of the $\mathfrak{so}(2n)$ -stabilizer of the three form Φ . We conclude by lemma 2.1 and theorem 2.1. \square

Notice the difference with the classical Riemannian case, where there is no scarcity of holonomy irreducible structures. We mention apart the coresponding version of proposition 2.3 in six dimensions. We first set first the following

DEFINITION 2.2. *Let (M, g, J) be a 6-dimensional nearly Kähler manifold. It is called Hermitian irreducible if the holonomy representation of the canonical Hermitian connexion is irreducible in the real sense (and hence the Hermitian holonomy equals $SU(3)$).*

COROLLARY 2.1. *Let (M, g, J) be a complete and simply connected nearly Kähler 6-fold. The following cases may occur :*

- (i) *(M, g, J) is Hermitian irreducible;*
- (ii) *we have a $\bar{\nabla}$ -parallel, orthogonal, decomposition $TM = \mathcal{V} \oplus H$ with $H = J\mathcal{V}$;*
- (iii) *(M, g, J) is one of the spaces PC^3, \mathbb{F}^3 equipped with its canonical nearly Kähler metric.*

Proof. Follows immediately from proposition 2.3 and [1] (see also [24]). \square

Thus, in order to achieve the classification of strict nearly Kähler manifolds it remains to treat the cases when the holonomy representation of the canonical Hermitian connection is reducible, first in the real sense precised at the point (ii) of the proposition 2.3 and next in the general complex sense. We leave open the problem of classifying Hermitian irreducible nearly Kähler 6-folds.

3. Curvature properties and reducibility. In this section we will consider a strict nearly Kähler manifold (M^{2n}, g, J) which is reducible in the real Hermitian sense, that is the tangent bundle of M admits a $\bar{\nabla}$ -parallel, orthogonal, decomposition:

$$TM = \mathcal{V} \oplus H.$$

In the first part of this section we will show that it is always possible to suppose that the previous decomposition is of special type, that is $\mathcal{V} \bullet \mathcal{V} \subseteq \mathcal{V}$. This will be done by a carefull examination of the curvature tensor of the canonical Hermitian connection, to be denoted by \bar{R} . Some preliminaries are required.

LEMMA 3.1. *Let X be in H, Y in TM and V, W in \mathcal{V} . We have :*

$$\bar{R}(X, Y, V, W) = \langle [\nabla_V J, \nabla_W J]X, Y \rangle - \langle (\nabla_X J)Y, (\nabla_V J)W \rangle .$$

Proof. Let us recall the relation (see [14] page 237):

$$(3.1) \quad \begin{aligned} \bar{R}(X, Y, Z, T) = R(X, Y, Z, T) - \frac{1}{2} \langle (\nabla_X J)Y, (\nabla_Z J)T \rangle + \\ \frac{1}{4} \left[\langle (\nabla_X J)Z, (\nabla_Y J)T \rangle - \langle (\nabla_X J)T, (\nabla_Y J)Z \rangle \right] \end{aligned}$$

where R denotes the curvature tensor of the Levi-Civita connection of g . It follows by the first Bianchi identity that

$$R(X, Y, V_1, V_2) + R(Y, V_1, X, V_2) + R(V_1, X, Y, V_2) = 0.$$

Or $\bar{R}(Y, V_1, X, V_2) = 0$ so by (3.1) we get

$$\begin{aligned} R(Y, V_1, X, V_2) = \frac{1}{2} \langle (\nabla_Y J)V_1, (\nabla_X J)V_2 \rangle \\ - \frac{1}{4} \left[\langle (\nabla_Y J)X, (\nabla_{V_1} J)V_2 \rangle - \langle (\nabla_Y J)V_2, (\nabla_{V_1} J)X \rangle \right]. \end{aligned}$$

In the same manner we obtain an analogous formula for $R(V_1, X, Y, V_2)$ and the result follows easily. \square

COROLLARY 3.1. *Let (M^{2n}, g, J) be complete, strict and simply connected nearly Kähler manifold satisfying the conditions of (ii) in proposition 2.3. Then (M, g) is a homogenous space.*

Proof. Let $V_i, 1 \leq i \leq 4$ be in \mathcal{V} . Then using lemma 3.1 we get

$$(3.2) \quad \bar{R}(V_1, V_2, JV_3, JV_4) = \langle [\nabla_{V_1} J, \nabla_{V_2} J]V_3, V_4 \rangle + \langle (\nabla_{V_1} J)V_2, (\nabla_{V_3} J)V_4 \rangle .$$

Using now that $\bar{R}(V_1, V_2, JV_3, JV_4) = \bar{R}(V_1, V_2, V_3, V_4) = \bar{R}(JV_1, JV_2, JV_3, JV_4)$ it follows that (3.2) completely determines the curvature \bar{R} on TM . We recall now that $\bar{\nabla}(\nabla J) = 0$, a fact proven in [1], which is equivalent to

$$(3.3) \quad \bar{\nabla}_U((\nabla_X J)Y) = (\nabla_X J)\bar{\nabla}_U X + (\nabla_{\bar{\nabla}_U X} J)Y$$

whenever U, X, Y are in TM . Thus, derivating formulas of type (3.2) we get easily that $\bar{\nabla} \bar{R} = 0$, hence $\bar{\nabla}$ is an Ambrose-Singer connection. As M is simply connected, we have that (M, g, J) is a homogenous space (see [28]). \square

It is convenient to distinguish now a second class of homogenous nearly Kähler manifolds as follows.

DEFINITION 3.1. *A simply connected, strict, nearly Kähler manifold is homogeneous of type II iff the holonomy representation of $\bar{\nabla}$ on $T_m^{\mathbb{C}}(M)$ is complex irreducible and satisfies the conditions of (ii) in proposition 2.3.*

REMARK 3.1. (i) *In section 8 we will study in more detail the class of nearly Kähler manifolds having the property that $\bar{\nabla}$ is an Ambrose-Singer connection.*

(ii) *A typical example of nearly Kähler manifold of type II is provided by products $G \times G$ where G is a compact Lie group without center (see [13]). However, it is not straightforward to see that these examples exhaust our class of spaces. This problem is studied in [25].*

Hence, when the holonomy representation of the canonical Hermitian connexion is complex irreducible, (M, g, J) is homogeneous of type I or II. Therefore, it remains to understand the complex reducible case. In this situation, using parallel transport we get a $\bar{\nabla}$ -parallel decomposition $TM = \mathcal{V} \oplus H$ which is stable by J . For this reason, all the distributions we will consider from now on will be preserved by J .

Now, let us state and prove the following important consequence of lemma 3.1.

COROLLARY 3.2. *Let X, Y be in H and V, W be in \mathcal{V} respectively. Then :*

- (i) $(\nabla_X J)(\nabla_V J)W = 0$;
- (ii) $(\nabla_X J)(\nabla_Y J)Z$ belongs to H whenever Z belongs to H ;
- (iii) $(\nabla_V J)(\nabla_W J)X$ belongs to H ;
- (iv) $(\nabla_X J)(\nabla_Y J)V$ belongs to \mathcal{V} .

Proof. (i) We use that $\bar{R}(JX, JZ, V, W) = \bar{R}(X, Z, V, W)$ for all Z in TM , lemma 3.1 and the algebraic properties of the tensor ∇J (see Section 2).

(ii) Reversing the roles of \mathcal{V} and H we obtain that $(\nabla_V J)(\nabla_X J)Y = 0$ for all X, Y in H and V in \mathcal{V} . Taking the scalar product with Z in H gives now the result.

(iii) We choose $Y = V_1$ in lemma 3.1, where V_1 belongs to \mathcal{V} . Since $\bar{R}(X, V_1, V, W) = 0$ we get by (i) that $[\nabla_V J, \nabla_W J]X$ belongs to H . But $[\nabla_V J, \nabla_{JW} J]JX = -\{\nabla_V J, \nabla_W J\}X$ is equally in H and the result follows.

(iv) It suffices to interchange the roles of the distributions \mathcal{V} and H . \square

We can now show the following result which mainly asserts that the tangent bundle of a reducible nearly Kähler manifold contains an integrable distribution.

PROPOSITION 3.1. *Let (M^{2n}, g, J) be a complete, strict nearly Kähler manifold and let us suppose that M is complex reducible and simply connected. Then M splits as a Riemannian product $Z \times N$ where Z and N are complete and strict nearly Kähler manifolds and the tangent bundle of N contains a proper $\bar{\nabla}$ -parallel distribution with \mathcal{V} with $\mathcal{V} \bullet \mathcal{V} = 0$.*

Proof. We will show first that one can find a $\bar{\nabla}$ -parallel distribution \mathcal{V} in TM such that $\mathcal{V} \bullet \mathcal{V} \subseteq \mathcal{V}$. Indeed, the reducibility of M implies the existence of a $\bar{\nabla}$ -parallel decomposition $TM = E \oplus F$. Let F_0 be the distribution generated by elements of the form $(v \bullet w)_F$ where v, w belong to E and the subscript denotes orthogonal projection on F . Using corollary 3.2, (i) we get

$$(\nabla_X J)(\nabla_Y J)(v \bullet w)_E + (\nabla_X J)(\nabla_Y J)(v \bullet w)_F = 0$$

whenever X, Y are in F . But the first term of this sum belongs to E by corollary 3.2, (iv) while the second is in F by corollary 3.2, (ii). It follows that each term of the previous sum vanishes and this implies easily that $F \bullet F_0 = 0$. As F_0 is contained in F we get that $F_0 \bullet F_0 = 0$. Obviously, F_0 is $\bar{\nabla}$ -parallel (one uses (3.3)) hence we may take $\mathcal{V} = F_0$ when $F_0 \neq 0$. If $F_0 = 0$ then $E \bullet E \subseteq E$ and we set $\mathcal{V} = E$.

Therefore, let us consider a $\bar{\nabla}$ -parallel decomposition $TM = \mathcal{V} \oplus H$, such that $\mathcal{V} \bullet \mathcal{V} \subseteq \mathcal{V}$. Let $r_1 : V \rightarrow V$ be the tensor defined by $\langle r_1 V, W \rangle = -Tr_{\mathcal{V}}(\nabla_V J)(\nabla_W J)$ for all V, W in \mathcal{V} . Then r_1 is parallel with respect to the unitary connection induced by $\bar{\nabla}$ in the bundle \mathcal{V} thus we have the decomposition $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$ with $\mathcal{V}_0 = Ker(r_1)$ and \mathcal{V}_1 the orthogonal complement of \mathcal{V}_0 in \mathcal{V} . Since the corollary 3.2, (i) implies that $(\nabla_X J)r_1 W = 0$ for all X in H and W in \mathcal{V} it follows that $H \bullet \mathcal{V}_1 = 0$. Thus, we have the $\bar{\nabla}$ parallel decomposition

$$TM = \mathcal{V}_1 \oplus H_1$$

where $H_1 = \mathcal{V}_0 \oplus H$. Since the very definition of \mathcal{V}_0 implies that $\mathcal{V} \bullet \mathcal{V}_0 = 0$ it is now easy to establish that $H_1 \bullet H_1 \subseteq H_1, \mathcal{V}_1 \bullet \mathcal{V}_1 \subseteq \mathcal{V}_1$ and $\mathcal{V}_1 \bullet H_1 = 0$. It follows that the distributions \mathcal{V}_1 and H_1 are in fact $\bar{\nabla}$ -parallel and we conclude by the decomposition theorem of de Rham. \square

REMARK 3.2. *In the case when $C \neq 0$ (see section 2 for definitions) proposition 3.1 can be given a simple algebraic proof. Let $\lambda_i, 1 \leq i \leq p$ be the (pairwise distinct) eigenvalues of C ordered such that $|\lambda_1| \leq \dots \leq |\lambda_p|$. If $E_i, 1 \leq i \leq p$ are the corresponding eigenspaces using the remark 2.1 it follows that $E_p \bullet E_p = 0$. Nevertheless, in the case when $C = 0$ (which implies that M is Einstein) the geometric arguments of the proof of the proposition 3.1 are needed.*

We will show now that the class of nearly Kähler manifolds appearing in proposition 3.1 leads to another class introduced by the following definition.

DEFINITION 3.2. *Let (M^{2n}, g, J) be a strict and complete nearly Kähler manifold. It is said to have special algebraic torsion if there exists a $\bar{\nabla}$ -parallel decomposition $TM = \mathcal{V} \oplus H$ with $\mathcal{V} \bullet \mathcal{V} = 0$ and $H \bullet H = \mathcal{V}$.*

REMARK 3.3. *A strict nearly Kähler manifold (M, g, J) admitting a $\bar{\nabla}$ -parallel decomposition $TM = \mathcal{V} \oplus H$ where $\mathcal{V} \bullet \mathcal{V} = 0$ and $H \bullet H \subseteq \mathcal{V}$ is of special algebraic type. To see this, let us set $\mathcal{V}_0 = H \bullet H$ and let \mathcal{V}_1 be the orthogonal complement of \mathcal{V}_0 in \mathcal{V} . By the definition of \mathcal{V}_1 we must have $\mathcal{V}_1 \bullet H = 0$ hence $\mathcal{V}_1 \bullet TM = 0$ as $\mathcal{V} \bullet \mathcal{V} = 0$. The fact that M is strict implies now that $\mathcal{V}_1 = 0$, thus $H \bullet H = \mathcal{V}$. Using a similar argument one can also prove that $\mathcal{V} \bullet H = H$.*

PROPOSITION 3.2. *Let (M^{2n}, g, J) be a nearly Kähler manifold equipped with a $\bar{\nabla}$ -parallel distribution \mathcal{V} satisfying $\mathcal{V} \bullet \mathcal{V} = 0$. If M is simply connected, it splits as a Riemannian product $Z \times N$ where both factors are strict nearly Kähler manifolds and N has special algebraic torsion.*

Proof. Set $H_0 = \mathcal{V} \bullet H$ which is obviously contained in H . If H_1 is the orthogonal complement of H_0 in H we get a J invariant, $\bar{\nabla}$ -parallel decomposition $H = H_0 \oplus H_1$. Then $\mathcal{V} \bullet H_0 \subseteq H_0$ and $\mathcal{V} \bullet H_1 \subseteq H_0$ by the definition of H_0 . But then $\mathcal{V} \bullet H_1$ is orthogonal to H_0 hence $\mathcal{V} \bullet H_1 = 0$. Using corollary 3.2, (iii) we obtain that $H \bullet H_0 \subseteq \mathcal{V}$ and since $\mathcal{V} \bullet H_1 = 0$ we must have $H_0 \bullet H_1 = 0$. So $H_0 \bullet H_0 \subseteq \mathcal{V}$ and it is now easy to see that $H_1 \bullet H_1 \subseteq H_1$. Hence the decomposition $TM = E_1 \oplus E_2$ is $\bar{\nabla}$ -parallel with $E_i \bullet E_i \subseteq E_i, i = 1, 2, E_1 \bullet E_2 = 0$ and we conclude using the de Rham decomposition theorem and the previous remark. \square

The results of this section lead easily to the following theorem which roughly classifies strict nearly Kähler manifolds up to those having special algebraic torsion.

THEOREM 3.1. *Let (M^{2n}, g, J) be a complete, simply connected and strict nearly Kähler manifold. Then M decomposes as Riemannian product whose factors belong to one of the following three classes of strict nearly Kähler manifolds :*

- homogenous spaces of type I and II;
- Hermitian irreducible 6-dimensional manifolds;
- manifolds with special algebraic torsion.

4. Special algebraic torsion. Let (M, g, J) be nearly Kähler with special algebraic torsion. Thus, we have a $\bar{\nabla}$ -parallel decomposition $TM = \mathcal{V} \oplus H$ with $\mathcal{V} \bullet \mathcal{V} = 0$ and $H \bullet H = \mathcal{V}$. Our first observation is that \mathcal{V} being $\bar{\nabla}$ -parallel with $\mathcal{V} \bullet \mathcal{V} = 0$ it is integrable. We would like to establish that \mathcal{V} induces on M the structure of a fibration, with smooth base space. We first prove the following result, showing that the torsion tensor of the canonical Hermitian connection completely determines the curvature in the \mathcal{V} -direction.

PROPOSITION 4.1. *Let X, Y be in H and V_1, V_2, V_3 in \mathcal{V} . Then :*

$$(4.1) \quad \bar{R}((\nabla_X J)JY, V_1, V_2, V_3) = \langle JY, [\nabla_{V_1} J, [\nabla_{V_2} J, \nabla_{V_3} J]]X \rangle .$$

Moreover, we have $\bar{\nabla}_U \bar{R}(V_1, V_2, V_3, V_4) = 0$.

Proof. Using the second Bianchi identity for the Hermitian connection $\bar{\nabla}$ (see [21]) we obtain

$$\begin{aligned} &(\bar{\nabla}_X \bar{R})(Y, V_1, V_2, V_3) + (\bar{\nabla}_Y \bar{R})(V_1, X, V_2, V_3) + (\bar{\nabla}_{V_1} \bar{R})(X, Y, V_2, V_3) + \\ &\bar{R}((\nabla_X J)JY, V_1, V_2, V_3) + \bar{R}((\nabla_Y J)JV_1, X, V_2, V_3) + \bar{R}((\nabla_{V_1} J)JX, Y, V_2, V_3) = 0. \end{aligned}$$

Since the distributions \mathcal{V} and H are $\bar{\nabla}$ -parallel we get $(\bar{\nabla}_X \bar{R})(Y, V_1, V_2, V_3) = (\bar{\nabla}_Y \bar{R})(V_1, X, V_2, V_3) = 0$ and using lemma 3.1 and (3.3) it can be seen that the term $(\bar{\nabla}_{V_1} \bar{R})(X, Y, V_2, V_3)$ also vanishes. Now, we compute the last two curvature terms in the Bianchi identity using lemma 3.1 and the corollary 3.1, (i) and the result follows by calculus. Let us prove now the second assertion. We first remark that formula (4.1) can be given the form :

$$(4.2) \quad (\nabla_X J)\bar{R}(V_1, V_2)V_3 = [\nabla_{V_3} J, [\nabla_{V_1} J, \nabla_{V_2} J]]X$$

Derivating this in the direction U in TM , we get by using (3.3) that

$$\langle (\nabla_X J) \left[\bar{\nabla}_U \bar{R}(V_1, V_2, V_3) \right], Y \rangle = 0$$

and we conclude using the special algebraic torsion assumptions, and the strictness of M . \square

As $H \bullet H = \mathcal{V}$ it follows that the restriction of the tensor \bar{R} to the distribution \mathcal{V} is completely determined by the formula (4.1). We can now prove the following.

PROPOSITION 4.2. *There exists a Riemannian manifold (N, h) and a Riemannian submersion with totally geodesic fibers $\pi : M \rightarrow N$ whose vertical distribution equals \mathcal{V} . With respect to the induced metric and almost complex structures, each fiber is a simply connected, compact, Hermitian symmetric space.*

Proof. Consider the foliation induced by the distribution \mathcal{V} . One can easily show that \mathcal{V} is totally geodesic hence we obtain by the second part of the proposition 4.1 that each leaf is a Hermitian symmetric space. We look now at the Ricci curvature of the leaves. Let $\widehat{Ric} : \mathcal{V} \rightarrow \mathcal{V}$ be the Ricci curvature in the vertical direction. We consider V be in \mathcal{V} with $V \neq 0$ and let $\{e_i\}$ be a local orthonormal basis in \mathcal{V} . Taking $V_1 = V_3 = e_i, V_2 = V$ in (4.2), we obtain after taking the trace over H that :

$$\langle \widehat{Ric}(V), r(V) \rangle = 2 \sum_{e_i \in \mathcal{V}} \|(\nabla_V J)(\nabla_{e_i} J)\|^2 > 0$$

(here, we considered that the norm of the linear operator A on TM is given by $\|A\|^2 = Tr(AA^*)$). Now the tensor r preserves \mathcal{V} and its restriction to \mathcal{V} is ∇ -parallel with strictly positive eigenvalues. It follows then easily that each leaf has positive Ricci curvature and we use a result of [20] to obtain that the leaves are compact and simply connected. The simple connectivity of the leaves implies that the foliation has trivial leaf holonomy, so using the criterion from [34], page 90, we get a smooth fibration $\pi : M \rightarrow N$. The metric part of our statement is a standard one, which is left to the reader. \square

We will refer to the fibration $\pi : M \rightarrow N$ as the *canonical* fibration of the nearly Kähler manifold with special algebraic torsion M . Its generic fiber will be denoted by F . Before going further, let us examine a few geometric properties related to special algebraic torsion.

PROPOSITION 4.3. *Let (M, g, J) be a nearly Kähler with special algebraic torsion and corresponding decomposition $TM = \mathcal{V} \oplus H$. Define a Riemannian metric \bar{g} on M by $\bar{g} = 2g_{\mathcal{V}} \oplus g_H$ where $g_{\mathcal{V}}$ and g_H are the restrictions of the metric g to the distributions \mathcal{V} and H respectively. We also define a new almost complex structure \bar{J} by setting $\bar{J}_{\mathcal{V}} = -J$ and $\bar{J}|_H = J$. Then (M, \bar{g}, \bar{J}) is a Kähler manifold of positive Ricci curvature, and hence simply connected.*

Proof. The proof of the fact that (M, \bar{g}, \bar{J}) is Kähler is essentially the same with the one given in the 6-dimensional case in [1]. Let us denote by $\tilde{\nabla}$ the Levi-Civita connection of the metric \bar{g} . In the standard way we get

$$\begin{aligned} \tilde{\nabla}_X Y &= \nabla_X Y, \quad \tilde{\nabla}_X V = \bar{\nabla}_X V - \frac{1}{2}(\nabla_X J)JV \\ \tilde{\nabla}_V X &= \bar{\nabla}_V X, \quad \tilde{\nabla}_V W = \nabla_V W \end{aligned}$$

whenever X, Y are in H and V, W in \mathcal{V} . After a routine computation we obtain

$$(4.3) \quad \begin{aligned} Ric_{\bar{g}} &= Ric_g - \frac{r}{4} \text{ on } H \\ Ric_{\bar{g}} &= Ric_g + \frac{3r}{4} \text{ on } \mathcal{V} \end{aligned}$$

and of course $Ric_{\bar{g}}(V, X) = 0$ for V, X belonging to \mathcal{V} and H respectively. Using (2.3) we see that the Ricci tensor of \tilde{g} is strictly positive and by a theorem of Kobayashi [20] M is simply connected. \square

We are now going to prove that the standard de Rham decomposition of the fiber induces a splitting at the level of the canonical fibration.

THEOREM 4.1. *Let (M, g, J) be a nearly Kähler manifold with special algebraic torsion. Then M is a Riemannian product*

$$M_1 \times M_2 \times \dots \times M_q$$

where each $M_i, 1 \leq i \leq q$, is a nearly Kähler manifold with special algebraic torsion and such that the canonical fibration has irreducible fiber.

Proof. If the fiber F is irreducible there is nothing to prove. Otherwise it decomposes as $F_1 \times F_2$ and using parallel transport we obtain a ∇ -parallel decomposition $\mathcal{V} = \mathcal{V}^1 \oplus \mathcal{V}^2$. Hence $R((\nabla_X J)Y, V, V_1, V_2) = 0$ for all X, Y in H, V in \mathcal{V} and V_i in $\mathcal{V}^i, i = 1, 2$. Using (4.1) we obtain

$$(4.4) \quad [\nabla_V J, [\nabla_{V_1} J, \nabla_{V_2} J]] = 0.$$

Taking $V = V_1$ we get :

$$(\nabla_{V_1} J)^2(\nabla_{V_2} J) + (\nabla_{V_2} J)(\nabla_{V_1} J)^2 = 2(\nabla_{V_1} J)(\nabla_{V_2} J)(\nabla_{V_1} J).$$

We change V_2 in JV_2 in the previous relation and we use that $(\nabla_{JV_2} J) = (\nabla_{V_2} J)J$. We get

$$(\nabla_{V_1} J)^2(\nabla_{V_2} J) + (\nabla_{V_2} J)(\nabla_{V_1} J)^2 = -2(\nabla_{V_1} J)\nabla_{V_2} J(\nabla_{V_1} J)$$

hence we must have

$$(\nabla_{V_1} J)^2(\nabla_{V_2} J) + (\nabla_{V_2} J)(\nabla_{V_1} J)^2 = (\nabla_{V_1} J)(\nabla_{V_2} J)(\nabla_{V_1} J) = 0.$$

This implies that $(\nabla_{V_1} J)^3(\nabla_{V_2} J) = 0$ and since $-(\nabla_{V_1} J)^2$ is a nonnegative operator we have that $(\nabla_{V_1} J)^2(\nabla_{V_2} J) = 0$. We equally have $(\nabla_{V_2} J)(\nabla_{V_1} J)^2 = 0$. It follows that

$$\|((\nabla_{V_2} J)(\nabla_{V_1} J))X\|^2 = \|((\nabla_{V_1} J)(\nabla_{V_2} J))X\|^2 = 0$$

for all X in H , that is

$$(4.5) \quad (\nabla_{V_2} J)(\nabla_{V_1} J) = (\nabla_{V_1} J)(\nabla_{V_2} J) = 0$$

whenever V_1 and V_2 belong to \mathcal{V}_1 and \mathcal{V}_2 respectively. We define now the subspaces H^1 and H^2 of H by

$$H^i = \mathcal{V}^i \bullet H, \quad i = 1, 2.$$

Using (4.4) and the fact that $\mathcal{V} \bullet H = H$ (see remark 3.3) we easily obtain that $H = H^1 \oplus H^2$. After standard manipulations it follows that $\mathcal{V}^1 \bullet H^2 = \mathcal{V}^2 \bullet H^1 = H^1 \bullet H^2 = 0$ and further that $\mathcal{V}^i \bullet H^i = H^i, H^i \bullet H^i = \mathcal{V}^i$ for $i = 1, 2$. We are going to show now that the four distributions $\mathcal{V}^1, \mathcal{V}^2, H^1, H^2$ are $\overline{\nabla}$ -parallel. Using that $\mathcal{V}^1 \bullet H^1 = H^1$ and (3.3) we establish that H^1 is $\overline{\nabla}$ -parallel. For the parallelism of \mathcal{V}^1 we use that $H^1 \bullet H^1 = \mathcal{V}^1$ and the remaining cases are analogous. Now, the distributions $E_i = \mathcal{V}^i \oplus H^i, i = 1, 2$ are $\overline{\nabla}$ -parallel. Since $E_i \bullet E_i \subseteq E_i, i = 1, 2$ and $E_1 \bullet E_2 = 0$ they are in fact ∇ -parallel and we conclude by the de Rham's theorem, as M is simply connected. \square

We can therefore admit without loss of generality that the fiber F of the canonical fibration is an irreducible Riemannian manifold. This assumption is to be made in the rest of the present section.

Let us define a tensor $\mathcal{R} : H \times H \rightarrow \text{End}_-(H)$ by

$$\mathcal{R}(X, Y) = \overline{R}(X, Y) + \nabla_{(\nabla_X J)Y}$$

for all X, Y in H . Now, if X^* denotes the horizontal lift of the vector field X of N , using standard computations from [2], page 241, we get

$$(4.6) \quad \mathcal{R}(X^*, Y^*)Z^* = (R^h(X, Y)Z)^*$$

whenever X, Y, Z are vector fields on N (here R^h denotes the curvature tensor of N).

LEMMA 4.1. *We have*

$$\sum_{e_i \in H} \mathcal{R}(e_i, (\nabla_V J)e_i) = \nabla_{r(V)} J$$

for all V in \mathcal{V} and all orthonormal basis $\{e_i\}$ of H .

Proof. We compute the given sum using the base $\{J e_i\}$ and we take into account that $\overline{R}(JX, JY) = \overline{R}(X, Y)$ whenever X, Y belong to H . \square

THEOREM 4.2. *The Riemannian manifold (N, h) is irreducible.*

Proof. If (M, h) wasn't irreducible its tangent bundle admitted a decomposition $TN = E_1 \oplus E_2$ parallel with respect of the Levi-Civita connection of h , to be denoted by ∇^h . We obviously have $H = H_1 \oplus H_2$ where

$$H_i = \{v \in H : (d\pi)v \in E_i\}$$

for $i = 1, 2$. Now, the operator $R^h(X, Y)$ preserves E_1 and E_2 for all X, Y in TN . It follows by (4.5) that $\mathcal{R}(X, Y)$ preserves H_1 and H_2 whenever X and Y are in H . Hence lemma 4.1 easily implies that $\mathcal{V} \bullet H_i \subseteq H_i$ for $i = 1, 2$. It follows that $H_1 \bullet H_2$ and \mathcal{V} are orthogonal but the fact that $H_1 \bullet H_2 \subseteq \mathcal{V}$ gives that $H_1 \bullet H_2 = 0$. We get a decomposition

$$TM = \mathcal{V}' \oplus H_2$$

where $\mathcal{V}' = \mathcal{V} \oplus H_1$. This decomposition satisfies $\mathcal{V}' \bullet \mathcal{V}' \subseteq \mathcal{V}'$ and $H_2 \bullet H_2 \subseteq \mathcal{V}'$. Let us show now that it is also $\overline{\nabla}$ -parallel.

As $\overline{\nabla}_{X^*} Y^* = (\nabla_X Y)^*$ for all X, Y in TN it follows that $\overline{\nabla}_X Y$ belongs to H_i whenever X is in H and Y is in $H_i, i = 1, 2$. It remains to see that $\overline{\nabla}_V X$ belongs to H_i whenever X is in H_i, V is in \mathcal{V} and $i = 1, 2$. Indeed if X is in E_1 we know that $[V, X^*]$ is in \mathcal{V} so by the parallelism of H it follows that $\overline{\nabla}_V X^* = -(\nabla_V J)JX^*$ which belongs to H_1 as we already shown that $\mathcal{V} \bullet H_1 \subseteq H_1$. Thus, H_1 is $\overline{\nabla}$ -parallel and of course the same holds for H_2 .

Hence we may apply corollary 3.2, (i) for the distribution \mathcal{V}' and we get

$$(\nabla_X J)(\nabla_V J)W = 0$$

for V, W in \mathcal{V}' and X in H_2 . If we set $\mathcal{V}^i = H_i \bullet H_i, i = 1, 2$ it follows that $\mathcal{V}^1 \bullet H_2 = 0$. Applying the same reasoning to the decomposition $TM = \mathcal{V}'' \oplus H_1$ where $\mathcal{V}'' = \mathcal{V} \oplus H_1$ we get $\mathcal{V}^2 \bullet H_1 = 0$. It is now routine to verify that \mathcal{V}^1 and \mathcal{V}^2 are orthogonal thus $\mathcal{V} = \mathcal{V}^1 \oplus \mathcal{V}^2$. Since $\mathcal{V}^i, i = 1, 2$ are ∇ -parallel (one uses the formula 3.3) by the irreducibility of \mathcal{V} we must have, say $\mathcal{V}^1 = 0$. But this would imply $H_1 \bullet \mathcal{V} = 0$ and further $H_1 \bullet TM = 0$ a contradiction, since (M, g, J) is strict. \square

To summarize the results of this section, nearly Kähler manifolds with special algebraic torsion have the property that up to Riemannian products of the total space, the fiber (a compact, simply connected, Hermitian symmetric space) and the base space of the canonical fibration are irreducible Riemannian manifolds.

5. Metric properties. In this section we will consider (M^{2n}, g, J) , a nearly Kähler manifold with special algebraic torsion and decomposition $TM = \mathcal{V} \oplus H$. We also suppose that the fiber of the canonical fibration is irreducible and recall that the base space has to be irreducible, too. Let us define a tensor $F : H \rightarrow H$ by

$$F = - \sum_{e_i \in \mathcal{V}} (\nabla_{e_i} J)^2$$

where $\{e_i\}$ is a local orthonormal frame in \mathcal{V} . Our main objective will be to determine the maximal number of eigenvalues of the tensors F and C . This will be used for the final classification result in the next section. It will also separate the nearly Kähler manifolds with special algebraic torsion into two classes.

The fact that C is $\bar{\nabla}$ -parallel and symmetric, together with the irreducibility of the fiber imply that there exists a real constant λ such that $C = \lambda 1_{\mathcal{V}}$ on \mathcal{V} . This leads to the following:

PROPOSITION 5.1. *The tensor C has at most three eigenvalues.*

Proof. We know that $-C(V \bullet X) = V \bullet CX + CV \bullet X$ for all V in \mathcal{V} and X in H (see section 2). As \mathcal{V} is an eigenspace of C with eigenvalue λ this implies that $S(V \bullet X) + V \bullet SX = 0$ whenever V, X are in \mathcal{V} and H respectively, where $S = C + \frac{\lambda}{2}$ on H . Let us denote by \mathcal{L}_V^H the projection of the Lie derivative \mathcal{L}_V on H . Taking into account that S^2 is $\bar{\nabla}$ -parallel it follows using the above algebraic property of S that $\mathcal{L}_V^H S^2 = 0$ for all V in \mathcal{V} . It is now an elementary exercise to see that S^2 projects on a symmetric, ∇^h -parallel endomorphism of N which has to be a multiple of identity by the irreducibility of N , hence proving our assertion. \square

Using the fact that $H \bullet H = \mathcal{V}$ it is straightforward to show that $r|_H = 2F$. Hence when C has a single eigenvalue, that is $C = 0$ it follows also that F has a single eigenvalue. We concentrate now on the cases when C has two or three eigenvalues. We will need several preliminary results.

LEMMA 5.1. *We have :*

(i)

$$(5.1) \quad Ric((\nabla_X J)Y, V) = \frac{1}{4}r(V, (\nabla_X J)Y) + \langle Y, F(V \bullet X) + V \bullet FX \rangle$$

whenever X, Y are in H and V is in \mathcal{V} .

(ii) *The base manifold (N, h) is Einstein with Einstein constant $\mu > 0$ and furthermore*

$$(5.2) \quad Ric + \frac{r}{4} = \mu \cdot 1_H.$$

on H .

Proof. (i) is an immediate consequence of (4.2). To prove (ii), let X, Y be vector fields on N . Then $Ric^h(X, Y) \circ \pi = \sum_{e_i \in H} \mathcal{R}(X^*, e_i, Y^*, e_i) = \frac{1}{2} \langle r(X^*), Y^* \rangle + Ric_{\bar{\nabla}}(X^*, Y^*)$ where $\{e_i\}$ is a local orthonormal frame in H . But $Ric_{\bar{\nabla}} = \frac{1}{4}(3Ric + Ric^*)$ (see [14]) thus

$$Ric^h(X, Y) \circ \pi = \langle (Ric + \frac{r}{4})X^*, Y^* \rangle.$$

Recall that r is a $\bar{\nabla}$ -parallel tensor, as well as Ric (see [24]). Hence, it is straightforward to see that Ric^h is ∇^h -parallel and since (N, h) is irreducible, we must have

$Ric^h = \mu \cdot h$ for some constant μ . That $\mu > 0$ follows by the fact that the Ricci curvature of g is strictly positive (see [24]). \square

We mention now without proof the following elementary result.

LEMMA 5.2. *Let Q be a $\bar{\nabla}$ -parallel subbundle of H such that $\mathcal{V} \bullet Q = 0$. Then there exists a ∇^h -parallel subbundle E of TN such that $Q = \pi^*E$.*

We are now able to treat the cases when the tensor C admits exactly two or three eigenvalues.

PROPOSITION 5.2. (i) *If the tensor C has exactly two eigenvalues then $F = k \cdot 1_H$ with $k > 0$ constant. Furthermore, the eigenvalues and eigenbundles of each of the tensors r, C, Ric are given in the following table.*

(5.3)

Eigenvalue	r	Ric	C	Eigenbundle
λ_1	$\frac{n-d}{d}k$	$\frac{n+7d}{4d}k$	$\frac{4(n-3d)}{d}k$	\mathcal{V}
λ_2	$2k$	$\frac{n+2d}{2d}k$	$-\frac{2(n-3d)}{d}k$	H

Here d is the half dimension of the distribution \mathcal{V} .

(ii) *Suppose that the tensor C has exactly three eigenvalues. Then the tangent bundle of M admits a $\bar{\nabla}$ -parallel decomposition*

$$TM = \mathcal{V} \oplus H_1 \oplus H_2$$

with $H_1 \bullet H_1 = H_2 \bullet H_2 = 0$ and $H_1 \bullet H_2 = \mathcal{V}$. Moreover, (M, g) is a homogenous space and the base space of the canonical fibration is a simply connected, compact and irreducible symmetric space.

Proof. (i) Using the (2.2) it easy to see that the eigenvalues of C are of the form λ and $-\frac{\lambda}{2}$. Since by definition $C = Ric - 5Ric^*$ we get by (5.2) that $r = \frac{-\lambda+8\mu}{12}$ on H hence F has a single eigenvalue.

(ii) Let λ_1 and λ_2 be the eigenvalues of C on H , and denote the corresponding eigenbundles by H_1 and H_2 . Using lemma 5.2 we have that $\mathcal{V} \bullet H_1 \neq 0$, hence it follows by (2.2) that $-(\lambda + \lambda_1)$ is an eigenvalue of C . It is different from λ since $\mathcal{V} \bullet H_1$ is orthogonal to \mathcal{V} . Let us suppose that $-(\lambda + \lambda_1) = \lambda_1$. It follows that $\mathcal{V} \bullet H_1 \subseteq H_1$ and since M has special algebraic torsion we get that $H_2 \bullet \mathcal{V}$ is orthogonal ot $\mathcal{V} \oplus H_1$, that is $H_2 \bullet \mathcal{V}$ is contained in H_2 . Again by lemma 5.2 we have $H_2 \bullet \mathcal{V} \neq 0$ and using (2.2) we find that $\lambda_2 = -\frac{\lambda}{2} = \lambda_1$ an absurdity.

Hence $-(\lambda + \lambda_1) = \lambda_2$ a fact which obviously implies by (2.2) that $\mathcal{V} \bullet H_1 \subseteq H_2, \mathcal{V} \bullet H_2 \subseteq H_1$ and $H_1 \bullet H_2 \subseteq \mathcal{V}$. Let us show now that $H_1 \bullet H_1 = 0$. If $H_1 \bullet H_1 \neq 0$ it produces the eigenvalue $-2\lambda_2$ for C . Or $H_1 \bullet H_1$ is orthogonal to H_2 and $-2\lambda_1 \neq \lambda$ (since $\lambda_1 \neq \lambda_2$). We get $-2\lambda_1 = \lambda_1$ hence $\lambda_1 = 0$ and $H_1 \bullet H_1 \subseteq H_1$. We set $\mathcal{V}' = H_1$ and $H' = \mathcal{V} \oplus H_2$. Then using corollary 3.2, (ii) we obtain that $(\nabla_X J)(\nabla_Y J)Z$ belongs to H' whenever X, Y are in \mathcal{V} and Z is in $H_1 \bullet H_1$. As $\mathcal{V} \bullet (\mathcal{V} \bullet H_1) \subseteq \mathcal{V} \bullet H_2 \subseteq H_1$ we obtain easily that $\mathcal{V} \bullet (H_1 \bullet H_1) = 0$ and we conclude by lemma 5.2. In the same way it can be proven that $H_2 \bullet H_2 = 0$. Hence, we have three different splittings of special algebraic type $TM = \mathcal{V} \oplus H, TM = H_1 \oplus (\mathcal{V} \oplus H_2), TM = H_2 \oplus (\mathcal{V} \oplus H_1)$ each of which being $\bar{\nabla}$ -parallel. It follows that the restriction of the tensor $\bar{\nabla}_U \bar{R}$ to either \mathcal{V}, H_1 or H_2 vanishes. Or this implies easily that $\bar{\nabla} \bar{R} = 0$, so $\bar{\nabla}$ is an Ambrose-Singer connection hence (M, g, J) is a homogenous space [28]. That the base space of the canonical fibration is symmetric follows by the usual comparison between curvature tensors of the total and the base space in [2]. \square

Hence we obtained a new class of nearly Kähler manifolds motivating the following definition.

DEFINITION 5.1. *A nearly Kähler manifold is homogenous of type III if it has special algebraic torsion and the tensor C admits exactly three eigenvalues.*

REMARK 5.1. *In the case when C has exactly three eigenvalues, let d_1 and d_2 be the half dimensions of H_1 and H_2 respectively. Then the eigenvalues of the tensors r, Ric, C with their corresponding eigenbundles are listed in the following table.*

(5.4)

Eigenvalue	r	Ric	C	Eigenbundle
λ_1	$\frac{2d_1}{d}k$	$(\frac{d_1}{2d} + \frac{d_1}{d_2} + 1)k$	$4(\frac{2d_1}{d} - \frac{d_1}{d_2} - 1)$	\mathcal{V}
λ_2	k	$(\frac{d_1}{d} + \frac{d_1}{d_2} + \frac{1}{2})k$	$-4(\frac{d_1}{d} + \frac{d_1}{d_2} - 2)$	H_1
λ_3	$\frac{d_1}{d_2}k$	$(\frac{d_1}{d} + \frac{d_1}{2d_2} + 1)k$	$-4(\frac{d_1}{d} - \frac{2d_1}{d_2} + 1)$	H_2

Here, k is a positive constant and, moreover, the eigenvalues of the tensor F are k and $\frac{d_1}{d_2}k$ with eigenbundles H_1 and H_2 respectively. The proof will be omitted, as being a simple calculation, based on (2.3) and of the relations between eigenvalues of the tensors C, F and r developed in the proof of the proposition 5.2, (ii).

To summarize the results obtained so far in this section, one can reduce the study of special algebraic torsion to the case when the tensor F has a single eigenvalue.

6. The holonomy of the base manifold. We consider in this section a nearly Kähler manifold with special algebraic torsion and let $TM = \mathcal{V} \oplus H$ be the corresponding decomposition. Furthermore, we suppose that the fiber of the canonical fibration is irreducible and that the tensor F has a single eigenvalue, to be denoted by k .

Let m be a point of M . We define a vector subspace of $\mathfrak{so}(H_m)$ by

$$\mathfrak{p}_m = \{ \nabla_v J : v \text{ in } \mathcal{V}_m \}.$$

The dimension of \mathfrak{p}_m equals that of \mathcal{V}_m since (M, g, J) is strict. Let \mathfrak{k}_m be the vector subspace of $\mathfrak{so}(H_m)$ generated by

$$\{ [\nabla_v J, \nabla_w J] : v, w \in \mathcal{V}_m \}.$$

PROPOSITION 6.1. *We have $\mathfrak{k}_m \cap \mathfrak{p}_m = 0$. Then $\mathfrak{h}_m = \mathfrak{k}_m \oplus \mathfrak{p}_m$ is a Lie subalgebra of $\mathfrak{so}(H_m)$.*

Proof. Let us first prove that $[\mathfrak{p}_m, \mathfrak{k}_m] \subseteq \mathfrak{p}_m$. Indeed, if V_1, V_2, V_3 belong to \mathcal{V} we obtain using (4.2) that

$$[\nabla_{V_1} J, [\nabla_{V_2} J, \nabla_{V_3} J]] = -\nabla_{R(V_2, V_3)V_1} J$$

and the assertion is proven. Consider now z in $\mathfrak{p}_m \cap \mathfrak{k}_m$. Then $z = \nabla_v J$ with v in \mathcal{V} and $[\nabla_v J, x]$ belongs to \mathfrak{p}_m for all x in \mathfrak{p} . In particular, there exists w in \mathcal{V} such that $[\nabla_v J, \nabla_{Jv} J] = \nabla_w J$, which implies $2(\nabla_v J)^2 = \nabla_{Jv} J$. The left side of this equality is symmetric whilst the right is skew-symmetric. It follows that $z = 0$ and we proved that $\mathfrak{k}_m \cap \mathfrak{p}_m = 0$. Now, $[\mathfrak{p}_m, \mathfrak{p}_m] \subseteq \mathfrak{k}_m$ by definition and using the Jacobi identity and (4.2) we obtain

$$[[\nabla_{V_1} J, \nabla_{V_2} J], [\nabla_{V_3} J, \nabla_{V_4} J]] = [\nabla_{V_2} J, \nabla_{R(V_3, V_4)V_1} J] + [\nabla_{R(V_3, V_4)V_2} J, \nabla_{V_1} J]$$

for all V_i in $\mathcal{V}, 1 \leq i \leq 4$. It follows that \mathfrak{k}_m and \mathfrak{h}_m are Lie algebras. \square

REMARK 6.1. (i) We saw along the proof of the previous proposition that $[\mathfrak{p}_m, \mathfrak{p}_m] \subseteq \mathfrak{k}_m$ and $[\mathfrak{p}_m, \mathfrak{k}_m] \subseteq \mathfrak{p}_m$.

(ii) Using E. Cartan's description of symmetric spaces one can also relate the Lie algebras $\mathfrak{h}_m, \mathfrak{k}_m$ to the homogenous structure of the symmetric space F_m , the fiber of π through $\pi(m)$.

We will now relate the Lie algebra \mathfrak{h}_m to the holonomy group of the base manifold. We define the Lie algebra \mathfrak{hol}_m as the Lie subalgebra of $\mathfrak{so}(H_m)$ generated by skewsymmetric endomorphisms of the form

$$\tau_c^{-1} \circ \mathcal{R}(\tau_c v, \tau_c w) \circ \tau$$

where v, w belong to H_m, c is a horizontal curve in M starting at m and τ_c is parallel transport along c with respect to the connection $\bar{\nabla}$. Using standard considerations in holonomy theory we find that \mathfrak{hol}_m is in fact isomorphic with the Lie algebra of the holonomy group of (N, h) through πm .

LEMMA 6.1. *The Lie algebra \mathfrak{h}_m is an ideal of \mathfrak{hol}_m .*

Proof. Let us first note that (3.3) gives us, after some computations

$$(6.1) \quad [\bar{R}(X, Y), \nabla_V J] = \nabla_{\bar{R}(X, Y)V} J$$

for all X, Y in H and V in \mathcal{V} . It follows that $[\mathcal{R}(v, w), \mathfrak{p}_m] \subseteq \mathfrak{h}_m$ and since $[\mathfrak{p}_m, \mathfrak{p}_m] = \mathfrak{k}_m$ by the Jacobi identity we also get $[\mathcal{R}(v, w), \mathfrak{k}_m] \subseteq \mathfrak{h}_m$ for all v, w in H_m . Let c be a horizontal path in M starting at m . Using (3.3) we obtain easily that $\tau_c \circ \nabla_v J = (\nabla_{\tau_c v} J) \circ \tau_c$ whenever v belongs to H_m . The conclusion now follows. \square

Let us consider now the bundle \mathfrak{h} whose fiber at a point m of M equals \mathfrak{h}_m . Note that \mathfrak{h} can be identified, in a standard way, with a subbundle of $\Lambda^2(H)$. We do this identification tacitly in the rest of this section. In the same way we obtain bundles $\mathfrak{k}, \mathfrak{p}$. We need now the following metric fact.

PROPOSITION 6.2. *We have $\rho^{\mathcal{R}} = \frac{k(n-d)}{d}$ on \mathfrak{h} where we denoted by $\rho^{\mathcal{R}}$ the curvature operator of \mathcal{R} .*

Proof. That the claimed formula holds on \mathfrak{p} was proved in lemma 4.2 (see also table 5.3). Let us show that it holds on \mathfrak{k} . We must compute $\sum_{e_i \in H} \mathcal{R}(e_i, qe_i, X, Y)$ where e_i is a local orthonormal basis in H and $q = [\nabla_V J, \nabla_W J]$ with V, W in \mathcal{V} . The definition of C implies easily that our quantity equals $\sum_{e_i \in H} \bar{R}(e_i, qe_i, X, Y) + \sum_{e_i \in H} \nabla_{(\nabla_{e_i} J)qe_i} J$. Or $\sum_{e_i \in H} (\nabla_{e_i} J)qe_i = 0$ (one uses the base Je_i). Now, using the first Bianchi identity for the Hermitian connection $\bar{\nabla}$ (see [21]) we get :

$$\begin{aligned} & \sum_{e_i \in H} \bar{R}(e_i, qe_i, X, Y) + \sum_{e_i \in H} \bar{R}(qe_i, X, e_i, Y) + \sum_{e_i \in H} \bar{R}(X, e_i, qe_i, Y) = \\ & = Tr_H[\nabla_V J, \nabla_X J]q. \end{aligned}$$

But $\sum_{e_i \in H} \bar{R}(qe_i, X, e_i, Y) = \sum_{e_i \in H} \langle [\bar{R}(X, e_i), q]e_i, Y \rangle + Ric_{\bar{\nabla}}(X, qY)$ where $Ric_{\bar{\nabla}}$ denotes the Ricci tensor of the Hermitian connection $\bar{\nabla}$. The use of (6.1) implies that $\langle [\bar{R}(X, e_i), q]e_i, Y \rangle = \langle [\nabla_V J, \nabla_{\bar{R}(X, e_i)W} J]e_i, Y \rangle - \langle [\nabla_W J, \nabla_{\bar{R}(X, e_i)V} J]e_i, Y \rangle$.

Now using lemma 3.1 we obtain after computing at some length that

$$\sum_{e_i \in H} \langle [\nabla_V J, \nabla_{\overline{R}(X, e_i)W} J] e_i, Y \rangle = - \langle (\nabla_W J)X, G(\nabla_V J)Y \rangle - Tr_H(\nabla_X J)(\nabla_Y J)(\nabla_V J)(\nabla_W J).$$

where $G : H \rightarrow H$ is defined by $G = - \sum_{e_i \in H} (\nabla_{e_i} J)^2$. It follows that

$$\sum_{e_i \in H} \langle [\overline{R}(X, e_i), q] e_i, Y \rangle = -Tr_H(\nabla_X J)(\nabla_Y J)q + \langle (\nabla_V J)X, G(\nabla_W J)Y \rangle - \langle (\nabla_W J)X, G(\nabla_V J)Y \rangle.$$

Since $G = k1_H$ we finally obtain that

$$\sum_{e_i \in H} \overline{R}(e_i, qe_i, X, Y) + Ric_{\nabla}(X, qY) - Ric_{\nabla}(Y, qX) + 2k \langle qX, Y \rangle = 0.$$

We conclude by the fact (see [14]) that $Ric_{\nabla} = \frac{1}{4}(3Ric + Ric^*)$ on M and by using the proposition 5.2, (i). \square

We are now ready to prove :

THEOREM 6.1. *Let (M, g, J) be a nearly Kähler manifold with special algebraic torsion and let us suppose that the fiber of the canonical fibration is irreducible. If the tensor F has a single eigenvalue and the base space of the canonical fibration is not symmetric then M is the twistor space over a quaternionic-Kähler manifold of positive scalar curvature.*

Proof. The base space (N^{2m}, h) of the canonical fibration is an irreducible, Einstein manifold of strictly positive scalar curvature. Then using the Berger holonomy theorem [31] and the well known fact that metrics with holonomy $SU(m), Sp(q), G_2$ or $Spin(7)$ are Ricci flat we find that that they are only three possibilities for the holonomy group of (N, h) , namely $SO(2m), U(m)$ and $Sp(q) \cdot Sp(1)$ when $m = 2q$. We will now treat each case separately.

If $Hol(N, h) = SO(2m)$, then if $m \neq 3$ the Lie algebra $\mathfrak{so}(2m)$ is simple and by lemma 6.1 we get easily that $\mathfrak{h} = \Lambda^2(H)$ so (N, h) has constant sectional curvature by proposition 6.2. Hence (N, h) is a symmetric space (in fact a round sphere), an absurdity. If $m = 2$ then $\mathfrak{so}(4)$ has two simple factors, each of dimension 3. Thus if $\mathfrak{h} \neq \Lambda^2(H)$ it has dimension 3 and it follows that \mathcal{V} has dimension 2. We found that (M, g, J) is 6-dimensional, Hermitian reducible nearly Kähler manifold, hence (N, h) is symmetric by [1].

We consider now the case when (N, h) has holonomy equal to $U(m)$. Then (N, h) is a Kähler manifold with associated complex structure I . The holonomy bundle of (N, h) is then $\Lambda^{1,1}(N)$. Denote by $\Lambda_0^{1,1}(N)$ the space of traceless two forms of type $(1, 1)$. Using lemma 4.1 and the fact that \mathfrak{h} has rank at least 2 we have either that $\mathfrak{h} = \pi^* \Lambda^{1,1}(N)$ or $\mathfrak{h} = \pi^* \Lambda_0^{1,1}(N)$. As the curvature tensor of a Kähler Einstein manifold is determined by its restriction to $\Lambda_0^{1,1}(N)$ it is easy to conclude using proposition 6.2 that (N, h) is symmetric, a contradiction.

Lastly, we suppose that $m = 2q$ and $Hol(N, h) = Sp(q) \cdot Sp(1)$. Thus, (N, h) is a quaternionic Kähler manifold and let us denote by Q the 3-dimensional subbundle of $\Lambda^2(N)$ giving the quaternionic structure. Then we have a decomposition $\Omega^2(N) = E_1 \oplus E_2 \oplus Q$ where fibers of E_1 and Q are isomorphic in each point with $\mathfrak{sp}(q)$ and

$\mathfrak{sp}(1)$ respectively. Furthermore, the holonomy bundle of the metric h equals $E_1 \oplus Q$. Now the curvature transformation of the metric h is determined by its restriction to E_1 as shown in [29]. Hence if it is constant on E_1 , then (N, h) is symmetric, in fact a quaternionic projective space. It follows that the only possibility is that $\mathfrak{h} = \pi^*Q$ and this implies that \mathcal{V} is of rank two. Thus, the canonical Hermitian connexion of (M, g, J) has holonomy contained in $U(1) \times U(2q)$ and we conclude by using a result from [24]. \square

It remains to investigate the case when the manifold (N, h) is a symmetric space. We set the following :

DEFINITION 6.1. *A simply connected, nearly Kähler manifold with special algebraic torsion and such that the canonical fibration has irreducible fiber is called homogenous of type IV if the tensor F has exactly two eigenvalues and the base space of the canonical fibration is a symmetric space.*

Let us remark that the terminology "homogenous" is not yet justified. However, using the relation between the tensors \overline{R} and R^h given by O'Neill's relations, one easily gets that when N is symmetric, $\overline{\nabla}$ is an Ambrose-Singer connection, hence M is homogenous.

Now the proof of the theorem 1.1 is a consequence of the material previously presented.

7. Nearly Kähler metrics from Riemannian foliations. The main purpose of this section is to give a proof of the theorem 1.2, stated in the introduction. In order to do this let us consider a Kähler manifold (M, g, J) together with a Riemannian foliation \mathcal{F} . Let \mathcal{V} be the integrable distribution induced by \mathcal{F} . We recall that the metric g is bundle-like with respect to our foliation, that is $\mathcal{L}_V g$ vanishes on H , the orthogonal complement of \mathcal{V} , for every V in \mathcal{V} . As in the statement of the theorem 1.3 we make the assumption that the leaves of \mathcal{F} are complex submanifolds of M , which gives that $J\mathcal{V} \subseteq \mathcal{V}$.

We consider now the Riemannian metric on M defined by

$$\hat{g}(X, Y) = \frac{1}{2}g(X, Y) \text{ if } X, Y \in \mathcal{V}, \hat{g}(X, Y) = g(X, Y) \text{ for } X, Y \text{ in } H.$$

The metric \hat{g} admits a compatible almost complex structure \hat{J} given by $\hat{J}|_{\mathcal{V}} = -J$ and $\hat{J}|_H = J$. This almost complex structure was introduced in [6] for the case of twistor spaces over 4-manifolds.

PROPOSITION 7.1. *The manifold (M, \hat{g}, \hat{J}) is nearly Kähler. The distributions \mathcal{V} and H are parallel with respect to the canonical Hermitian connection of (M, \hat{g}, \hat{J}) which thus has reduced holonomy.*

Proof. Let $A : TM \times TM \rightarrow TM$ be the O'Neill tensor of the Riemannian foliation induced by \mathcal{V} . As g is Kähler we must have $A_X J = J A_X$ for all X in TM . Using the relations between the Levi-Civita connections of \hat{g} and g given in [27] we obtain after a standard computation :

$$\begin{aligned} (\hat{\nabla}_X \hat{J})V &= -(\hat{\nabla}_V \hat{J})X = -A_X(JV) \\ (\hat{\nabla}_V \hat{J})W &= 0, (\hat{\nabla}_X \hat{J})Y = 2A_X(JY) \end{aligned}$$

for every X, Y in \mathcal{V} and V, W in H . It is now straightforward to conclude \square

COROLLARY 7.1. *The twistor space of a positive quaternionic-Kähler manifold of dimension $4k$ admits a canonical NK structure with reducible holonomy, contained in $U(1) \times U(2k)$.*

Proof. We have only to recall [29] that such a twistor space is the total space of a Riemannian submersion with totally geodesic fibers of dimension 2 and that it admits a compatible Kähler structure \square

REMARK 7.1. *If one drops the condition that the distribution \mathcal{V} be totally geodesic, the construction above produces only a quasi-Kähler manifold.*

We are now going to prove theorem 1.2. We remark that the nearly Kähler manifold (M, \hat{g}, \hat{J}) has the property that $\mathcal{V} \bullet \mathcal{V} = 0$ and $H \bullet H \subseteq \mathcal{V}$. Using the decomposition result of the proposition 2.1 we obtain that (under the simple connectivity assumption) M is the Riemannian product of a Kähler manifold and a nearly Kähler manifold with special algebraic torsion. Hence by the structure results of sections 4,5 and 6 the de deRham decomposition of the second factor consists only of spaces of type III and IV and twistor spaces over positive quaternionic Kähler manifolds equipped with their canonical nearly Kähler metric. The proof of the theorem comes now by reversing the construction done in the beginning of the section, and by using proposition 4.3. However, we postpone until the next section the proof of the fact that homogeneous space (M, g, J) of type III or IV is a homogeneous Kähler manifold when endowed with the metric \bar{g} and the complex structure \bar{J} .

8. Homogeneous nearly Kähler manifolds. In this section we will investigate some properties of homogeneous nearly Kähler manifolds that are introduced by the following definition.

DEFINITION 8.1. *Let (M^{2n}, g, J) a strict and complete nearly Kähler manifold. It is called homogeneous if it admits a transitive action of a connected, closed subgroup of the group of holomorphic isometries of (g, J) .*

Let us now prove that what we called homogeneous spaces of type I, II, III, IV are in fact homogeneous nearly Kähler manifolds and that for the spaces of type III and IV the natural Kähler metric is homogeneous.

PROPOSITION 8.1. *Let (M^{2n}, g, J) be a strict and complete nearly Kähler manifold.*

(i) *If $\bar{\nabla} \bar{R} = 0$ then (M, g, J) is a homogeneous nearly Kähler space. In particular M is naturally reductive and if moreover, M is simply connected, then (M, g, J) is a Riemannian 3-symmetric space.*

(ii) *If (M, g, J) is a space of type III or IV then (M, \bar{g}, \bar{J}) is a homogeneous Kähler manifold.*

Proof. (i) In the terminology of [28], the $\bar{\nabla}$ -parallel tensor $T_X Y = -\frac{1}{2}(\nabla_X J)JY$ is a homogeneous structure. The theorem of Ambrose and Singer (see [28]) gives now an explicit construction of M as a homogeneous space and from this it is easy to conclude using a result of Gray (see [13], proposition 5.5, page 358).

(ii) Using the relation between the Levi-Civita of \bar{g} and $\bar{\nabla}$ given in the proof of proposition 4.3 one can prove directly that the tensor $T_X Y$ below is a homogeneous structure for (M, \bar{g}, \bar{J}) . We omit the details. \square

Let us give now a corollary of theorem 1.1 related to homogeneous nearly Kähler spaces.

PROPOSITION 8.2. *Let (M^{2n}, g, J) be a homogeneous nearly Kähler manifold which is also strict and simply connected. Then (M, g, J) is isometric and biholomor-*

phic to a Riemannian product whose factors belongs to the following list :

-homogeneous 6-dimensional

-spaces of type I,II,III or IV

-twistor spaces over positive quaternionic Kähler manifolds which are homogeneous as nearly Kähler manifolds.

Proof. It is not hard to see that each of the manifolds occurring in the decomposition of theorem 1.1 is irreducible in the usual Riemannian sense. Hence, that decomposition is the deRham decomposition of the nearly Kähler manifold (M, g, J) and by a result of Hano [21] it follows that the connected isometry group of (M, g) is the direct product of the isometry groups of the factors. But this implies that each factor is a homogeneous space and using that J preserves our decomposition it is straightforward to conclude. \square

The rest of this section will be devoted to the proof of the following classification result.

THEOREM 8.1. *Let (M^{2n}, g, J) be a homogeneous nearly Kähler manifold which is also strict and simply connected. Then (M, g, J) is isometric and bihomorphic to a Riemannian product whose factors belong to the following list :*

-homogeneous 6-dimensional nearly Kähler manifolds

-spaces of type I,II,III or IV

Before getting into the proof let us make some comments on theorem 8.1. As any Riemannian 3-symmetric space is nearly Kähler as soon as it is naturally reductive the previous result yields a geometric version of the classification of naturally reductive 3-symmetric spaces. Note that the general classification theory of 3-symmetric spaces developed in [13] was obtained using quite involved Lie algebra arguments. It also follows from theorem 8.1 that Wolf&Gray's conjecture (see [39], page 158) asserting that any strict homogenous nearly Kähler manifold is 3-symmetric holds true if and only if it holds in 6-dimensions. Concerning the types of homogenous spaces appearing in the theorem let us recall that those of type I are compact, simply connected isotropy irreducible homogeneous spaces, of naturally reductive type, carrying a nearly Kähler structure. Here our geometric results do not allow classification. For spaces of type II, further work is required to localize completely this class in Gray's list (see section 3, remark 3.1). Finally, spaces of types III and IV can be classified using on the one hand the classification of compact and simply connected, irreducible Hermitian symmetric spaces and other the other hand the description of the fiber of the canonical fibration in the terms of the Lie algebra \mathfrak{h} together with lemma 6.1 (see [25] for details). Note that the spaces of type III and IV were already studied from various points of view in [30] and [4].

Let us now prove the theorem 8.1. It suffices to show that if a twistor space with its canonical nearly Kähler metric is homogeneous then the base quaternionic manifold is symmetric. For, let us consider such a twistor space; it is a strict nearly Kähler manifold with special algebraic torsion (M, g, J) with decomposition $TM = \mathcal{V} \oplus H$ where \mathcal{V} is of rank two and such the tensor F has a single eigenvalue of the form $\frac{k}{4}, k > 0$.

Let us set now the following notation. If (Z, h) is a Riemannian manifold we denote by $\iota(Z, h)$ the Lie algebra of Killing vector fields. Furthermore, if I is an almost complex structure giving (Z, h) the structure of a almost complex manifold we define $\iota(Z, h, J)$ to be the Lie algebra of holomorphic Killing vector vector fields.

Our aim is to obtain a comparison result between $\iota(M, \bar{g}, \bar{J})$ and $\iota(M, g, J)$. Let $X = V + K$ be an element of $\iota(M, g, J)$ split into its horizontal and vertical components. We use the parallelism of the decomposition $TM = \mathcal{V} \oplus H$ with respect to the connection $\bar{\nabla}$ in order to see that the Killing equation for X (the skew-symmetry of ∇X) is equivalent with the following system of equations for its components :

$$(8.1) \quad \begin{aligned} & \langle \bar{\nabla}_Y K, Z \rangle + \langle \bar{\nabla}_Z K, Y \rangle = 0 \\ & \langle \bar{\nabla}_W K, Y \rangle + \langle \bar{\nabla}_Y V, W \rangle = 0 \\ & \langle \bar{\nabla}_{W_1} V, W_2 \rangle + \langle \bar{\nabla}_{W_2} W_1 \rangle = 0 \end{aligned}$$

where Y, Z are in H and W, W_1, W_2 are in \mathcal{V} . Using the same arguments the condition that X be a J -holomorphic vector field projects onto the following two systems :

$$(8.2) \quad \begin{aligned} \bar{\nabla}_{JY} K - J(\bar{\nabla}_Y K) &= -2(\nabla_Y J)V \\ \bar{\nabla}_{JY} V - J(\bar{\nabla}_Y V) &= -2(\nabla_Y J)K \end{aligned}$$

and

$$(8.3) \quad \begin{aligned} \bar{\nabla}_{JW} K - J(\bar{\nabla}_W K) &= -2(\nabla_W J)K \\ \bar{\nabla}_{JW} V - J(\bar{\nabla}_W V) &= 0 \end{aligned}$$

whenever Y and W are in H and \mathcal{V} respectively.

REMARK 8.1. *Neither all the equations of the systems (8.1), (8.2), (8.3) are independent, nor all will be used in what follows. We gave all of them for completeness reasons.*

We will need now the following auxiliary result.

LEMMA 8.1. *If Y and W are in H and \mathcal{V} respectively, then :*

- (i) $\bar{\nabla}_W V = \frac{f}{2} JW$ where $f = d^*(JV)$,
- (ii) $\bar{\nabla}_W K = (\nabla_W J)JK$,
- (iii) $\bar{\nabla}_Y V = (\nabla_Y J)JK$.

Proof. The proof of (i), an easy exercise, will be left to the reader. In order to prove (ii) let $\{e_i\}, i = 1, 2$ be a local orthonormal basis in \mathcal{V} . Using the second equation of (8.1) with $W = e_i$ and next deriving it in the direction of e_i we obtain :

$$\langle \bar{\nabla}_{e_i} \bar{\nabla}_{e_i} K, Y \rangle + \langle \bar{\nabla}_{e_i} K, \bar{\nabla}_{e_i} Y \rangle + \langle \bar{\nabla}_{e_i} \bar{\nabla}_Y V, e_i \rangle + \langle \bar{\nabla}_Y V, \bar{\nabla}_{e_i} e_i \rangle = 0.$$

Using again the second equation of (8.1) for the second and third terms of the previous sum we get that $\langle \bar{\nabla}_{e_i, e_i}^2 K, Y \rangle + \langle \bar{\nabla}_{e_i, Y}^2 V, e_i \rangle = 0$. But by the Ricci identity $\bar{\nabla}_{e_i, Y}^2 V = \bar{\nabla}_{Y, e_i}^2 V - \bar{R}(e_i, Y)V + \bar{\nabla}_{(\nabla_Y J)J e_i} V$ and moreover $\bar{R}(e_i, Y)V = 0$ as e_i is in \mathcal{V} and Y belongs to H . Now ,

$$\langle \bar{\nabla}_{Y, e_i}^2 V, e_i \rangle = \langle \bar{\nabla}_Y \bar{\nabla}_{e_i} V - \bar{\nabla}_{\bar{\nabla}_Y e_i} V, e_i \rangle = \frac{1}{2} Y \cdot f \langle J e_i, e_i \rangle = 0$$

by (i). Hence $\langle \bar{\nabla}_{e_i, e_i}^2 K, Y \rangle + \langle \bar{\nabla}_{(\nabla_Y J)J e_i} V, e_i \rangle = 0$ and using the second equation of (8.1) (together with the fact that $\bar{\nabla}_{e_i} J$ is skew-symmetric and J -anticommuting) we arrive at $\langle \bar{\nabla}_{e_i, e_i}^2 K, Y \rangle = \langle (\nabla_{e_i} J)J \bar{\nabla}_{e_i} K, Y \rangle$. We define now

$$S = \sum_{i=1}^2 (\nabla_{e_i} J)J \bar{\nabla}_{e_i} K.$$

Then $S = \sum_{i=1}^2 (\nabla_{J e_i} J) J \bar{\nabla}_{J e_i} K$ and we use that $(\nabla_{JU} J)JT = -(\nabla_U J)T$ for all U, T in TM and also the first equation of (8.3) to show that $S = \sum_{i=1}^2 (\nabla_{e_i} J)^2 K = -\frac{k}{2} K$.

Resuming, we have that $\sum_{i=1}^2 \langle \bar{\nabla}_{e_i, e_i}^2 K, Y \rangle = -\frac{k}{2} \langle K, Y \rangle$. On the other hand, as $\bar{\nabla}_{e_i} e_i = \nabla_{e_i} e_i$ and by the second equation of the system (8.1) we obtain that

$$\begin{aligned} \|\bar{\nabla}_{e_i} K\|^2 &= e_i \cdot \langle \bar{\nabla}_{e_i} K, K \rangle - \langle \bar{\nabla}_{\nabla_{e_i} e_i} K, K \rangle - \langle \bar{\nabla}_{e_i, e_i}^2 K, K \rangle = \\ &= - \left[e_i \cdot \langle \bar{\nabla}_K V, e_i \rangle - \langle \bar{\nabla}_K V, \nabla_{e_i} e_i \rangle \right] - \langle \bar{\nabla}_{e_i, e_i}^2 K, K \rangle. \end{aligned}$$

As the vector field $\bar{\nabla}_K V$ is vertical, we obtain easily that $\sum_{i=1}^2 \|\bar{\nabla}_{e_i} K\|^2 = \frac{k}{2} \|K\|^2 + d^* \alpha$ where α is the 1-form of M dual to $\bar{\nabla}_K V$. By the previous considerations, this implies that

$$\sum_{i=1}^2 \|\bar{\nabla}_{e_i} K - (\nabla_{e_i} J)JK\|^2 = d^* \alpha$$

and the conclusion follows by integration over M . The identity in (iii) is a consequence of (i) and of the second equation of (8.1). \square

We can now prove the following.

PROPOSITION 8.3. *We have $\iota(M, g, J) = \iota(M, \bar{g}, \bar{J})$.*

Proof. Let us prove that $\iota(M, g, J) \subseteq \iota(M, \bar{g}, \bar{J})$. If $\tilde{\nabla}$ -denotes the Levi-Civita connection of (M, \bar{g}) then we recall that $\tilde{\nabla}_X Y = \nabla_X Y, \tilde{\nabla}_X V = \bar{\nabla}_X V - (\nabla_X J)JV, \tilde{\nabla}_V X = \bar{\nabla}_V X$ and $\tilde{\nabla}_V W = \nabla_V W$ for all X, Y in H and V, W in \mathcal{V} respectively. For, if X in $\iota(M, g, J)$ it is easy to see, using the previous lemma that the equation $\tilde{\nabla} X = 0$ follows from the system 8.1 (of course we have to take into account that on a compact Kähler manifold every Killing vector field is holomorphic). The other inclusion can be proven, for example, using the same technique, and will be left to the reader. \square

It follows that $Isom^0(M, g, J) = Isom^0(M, \bar{g}, \bar{J})$, hence if (M, g, J) is a homogeneous nearly Kähler manifold then (M, \bar{g}, \bar{J}) is a homogeneous Kähler manifold. Recall now that being a twistorial space (M, \bar{g}, \bar{J}) has a canonical complex structure (see [29]) and this is not hard to see that it is in fact a homogeneous complex contact manifold. It follows then by the results in [38] that the base quaternionic-Kähler manifold is in fact symmetric and this ends the proof of theorem 8.1.

REFERENCES

[1] F. BELGUN , A. MOROIANU, *Nearly Kähler 6-manifolds with reduced holonomy*, Ann. Glob. An. Geom., 19 (2001), pp. 307-319.
 [2] A. BESSE, *Einstein manifolds*, Springer-Verlag, New York, 1987.
 [3] C.P. BOYER, K. GALICKI, B.M. MANN, E.G. REES, *Compact 3-sasakian manifolds 7-manifolds with arbitrary second Betti number*, Inv. Math., 131 (1998), pp. 321-344.
 [4] F. E. BURSTALL, J. H. RAWNSLEY, *Twistor theory for Riemannian symmetric spaces*, Lect. Notes In Math. 1424, 1990.
 [5] R. CLEYTON, *G-structures and Einstein metrics*, Ph.D. thesis, Odense, 2001.

- [6] J. EELS, S. SALAMON, *Constructions twistorielles des applications harmoniques*, C. R. Acad. Sc. Paris, 296 (1983), pp. 685–687.
- [7] M. FALCITELLI, A.M PASTORE, *A note on almost Kähler and nearly Kähler submersions*, J. Geom., 69 (2000), pp. 79–87.
- [8] TH. FRIEDRICH, I. KATH, A. MOROIANU, U. SEMMELMANN, *On nearly parallel G_2 -structures*, J. Geom. Phys., 23 (1997), pp. 259–286.
- [9] TH. FRIEDRICH, *Weak Spin(9)-structures on 16-dimensional manifolds*, Asian J. Math., 5 (2001), pp. 129–160.
- [10] TH. FRIEDRICH, S. IVANOV, *Parallel spinors and connections with skew-symmetric torsion in string theory*, preprint 2001, eprint [arXiv:math. DG/0102142](https://arxiv.org/abs/math/0102142).
- [11] A. GRAY, *Nearly Kähler manifolds*, J. Diff. Geom., 4 (1970), pp. 283–309.
- [12] A. GRAY, *Weak holonomy groups*, Math. Z, 125 (1971), pp. 290–300.
- [13] A. GRAY, *Riemannian manifolds with geodesic symmetries of order 3*, J. Diff. Geometry, 7 (1972), pp. 343–369.
- [14] A. GRAY, *The structure of nearly Kähler manifolds*, Math. Ann., 223 (1976), pp. 233–248.
- [15] A. GRAY, L. M. HERVELLA, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl., 123 (1980), pp. 35–58.
- [16] R. GRUNEWALD, *Six-Dimensional Riemannian manifolds with real Killing spinors*, Ann. Global Anal. Geom., 8 (1990), pp. 43–59.
- [17] S. HELGASON, *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, 1978.
- [18] D.L. JOHNSON, *Kähler submersions and holomorphic connections*, J. Diff. Geom., 15 (1980), pp. 71–79.
- [19] V.F. KIRICHENKO, *K-spaces of maximal rank*, Mat. Zametki, 22, no.4 (1977), pp. 465–476.
- [20] S. KOBAYASHI, *On compact Kähler manifolds with positive definite Ricci tensor*, Ann. of Math., 74 (1961), pp. 570–574.
- [21] S. KOBAYASHI, K. NOMIZU, *Foundations of differential geometry*, vol.1, Wiley, 1963.
- [22] C. LEBRUN, S. SALAMON, *Strong rigidity of positive quaternion-Kähler manifolds*, Invent. Math., 118 (1994), pp. 109–132.
- [23] C. LEBRUN, *Fano manifolds, contact structures and quaternionic geometry*, International J. Math., 6 (1995), pp. 419–437.
- [24] P.A. NAGY, *On nearly Kähler geometry*, Ann. Glob. An. Geom., 22 (2002), pp. 167–178.
- [25] P.A. NAGY, *Homogeneous nearly Kähler manifolds*, in preparation, 2002.
- [26] Y. S. POON, S. SALAMON, *Eight-dimensional quaternion Kähler manifolds with positive scalar curvature*, J. Diff. Geom., 33 (1990), pp. 363–378.
- [27] PH. TONDEUR, *Geometry of Foliations*, Birkhäuser Verlag, 1997.
- [28] F. TRICERRI AND L. VANHECKE, *Homogeneous structures on Riemannian manifolds*, London Math. Soc. Lecture Notes, 83 (1982).
- [29] S. SALAMON, *Quaternionic Kähler manifolds*, Invent. Math., 67 (1982), pp. 143–171.
- [30] S. SALAMON, *Harmonic and holomorphic maps*, In : Seminar L. Bianchi, Lect. Notes In Math. 1164, 1986.
- [31] S. SALAMON, *Riemannian geometry and holonomy groups*, Longman Scientific & Technical, 1989.
- [32] S. SALAMON, *Almost parallel structures*, to appear in Contemp. Math., eprint [arXiv:math.DG/0107146](https://arxiv.org/abs/math/0107146)
- [33] K. SEKIGAWA, L. VANHECKE, *Almost hermitian manifolds with vanishing first Chern classes or Chern numbers*, Rend. Sem. Mat. Univ. Pol. Torino, 50 (1992), pp. 195–208.
- [34] R. W. SHARPE, *Differential geometry*, Springer Verlag, 1997.
- [35] A. SWANN *Weakening holonomy*, Proceedings of the second meeting on quaternionic structures, World Scientific, Singapore (2001), pp. 405–415.
- [36] Y. WATANABE, K. TAKAMATSU, *On a K-space of constant holomorphic sectional curvature*, Kodai Math. Sem. Rep., 25 (1973), pp. 297–306.
- [37] B. WATSON, *Almost hermitian submersions*, J. Diff. Geom., 11 (1976), pp. 147–165.
- [38] J.A. WOLF, *Complex homogeneous contact manifolds and quaternionic symmetric spaces*, J.Math.Mech., 14, No.6 (1965), pp. 1033–1047.
- [39] J.A. WOLF, A. GRAY, *Homogeneous spaces defined by Lie group automorphisms I, II*, J. Diff. Geom., 2 (1968), pp. 77–114, pp. 115–59.