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KÄHLER MANIFOLDS WITH ALMOST NON-NEGATIVE BISECTIONAL CURVATURE*

FUQUAN FANG†

Abstract. The main purpose of this paper is to study the topology of Kähler manifolds with *almost* non-negative bisectional curvature. Among others we prove that for simply connected n-dimensional Kähler manifolds M of sectional curvature $K \leq \Lambda$, there exists a universal positive constant $\varepsilon(n, \Lambda)$, depending only on the dimension n and Λ, such that if the bisectional curvature H and the diameter of M satisfy, $H \cdot \mathrm{diam}^2(M) \geq -\varepsilon(n, \Lambda)$, then M is diffeomorphic to the product $M_1 \times \cdots \times M_k$, where each M_i is a simply connected $C^{1,\alpha}$-Kähler manifold with second Betti number $b_2(M_i) = 1$ for any prescribed real number $\alpha \in (0, 1)$. Furthermore, if M is Kähler-Einstein, then M_i are all biholomorphic to irreducible Kähler Hermitian symmetric spaces. In the non-simply connected case, we prove that M is a holomorphic fiber bundle over the Jacobian $\mathcal{J}(M)$.

0. Introduction. Let M be a compact complex manifold. We say M has *almost nonnegative* bisectional curvature, if for any positive constant ε, there is a Hermitian metric g on M whose bisectional curvature H satisfies that $H \cdot diam(M_g)^2 \geq -\varepsilon$. Besides Hermitian manifolds of non-negative bisectional curvature, there are many examples of complex manifolds of almost non-negative bisectional curvature but do not admit any Hermitian metric of non-negative bisectional curvature (c.f. Section 1.)

When the Hermitian manifold is Kählerian, the uniformization theorem of Mok [Mo] (generalized Frankel conjecture, compare Siu-Yau [SY]) asserts that a simply connected compact Kähler manifold M with non-negative bisectional curvature is biholomorphic to the product of

$$P(\mathbb{C})^{m_1} \times \cdots \times P(\mathbb{C})^{m_k} \times N_1 \cdots \times N_l$$

where N_i, $1 \leq i \leq l$, are irreducible Kähler Hermitian symmetric spaces of rank at least 2. The Mok theorem depends on an earlier decomposition theorem of Howard-Smyth-Wu [HSW], Mori's celebrated work [Mo] and Hamilton's heat equation technique.

A natural question is whether one can extend the Mok theorem and the Howard-Smyth-Wu theorem to Kähler manifold of almost non-negative bisectional curvature. In this paper we will prove, among others, for simply connected n-dimensional Kähler manifold M with sectional curvature $K \leq \Lambda$, there exists a universal positive constant $\varepsilon(n, \Lambda)$, depending only on the dimension n and Λ, such that if the bisectional curvature H and the diameter of M satisfy, $H \cdot \mathrm{diam}^2(M) \geq -\varepsilon(n, \Lambda)$, then M is diffeomorphic to the product $M_1 \times \cdots \times M_k$, where each M_i is a simply connected $C^{1,\alpha}$-Kähler manifold with second Betti number $b_2(M_i) = 1$ for any prescribed real number $\alpha \in (0, 1)$. Furthermore, if M is Kähler-Einstein, then M_i are all biholomorphic to irreducible Kähler Hermitian symmetric spaces. In the non-simply connected case, we prove that M is a holomorphic fiber bundle over the Jacobian $\mathcal{J}(M)$.

Now we start to state our main results.

For convenience, let $\mathcal{M}(n, \Lambda)$ denote the set of all n-dimensional Kähler manifolds so that $K \leq \Lambda$.

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† Nankai Institute of Mathematics, Nankai University, Tianjin 300071, P.R.C. (ffang@nankai.edu.cn). Instituto de Matematica, Universidade Federal Fluminense, Niteroi 24005, RJ, Brasil (fuquan@mat.uff.br).
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THEOREM A. *Let $M \in \mathcal{M}(n,\Lambda)$ be a simply connected compact Kähler manifold. Then there exists a constant $\varepsilon(n,\Lambda)$, depending only on n and Λ, such that if $H \cdot diam(M)^2 \geq -\varepsilon(n,\Lambda)$, then M is diffeomorphic to the product $M_1 \times \cdots \times M_k$, where M_i, $1 \leq i \leq k$, are simply connected $C^{1,\alpha}$-Kähler manifolds with second Betti number $b_2(M_i) = 1$ for any prescribed real number $\alpha \in (0,1)$.*

If $M \in \mathcal{M}(n,\Lambda)$ is Kähler-Einstein, we get an improved splitting theorem

THEOREM B. *Let $M \in \mathcal{M}(n,\Lambda)$ be a simply connected Kähler-Einstein manifold. Then there exists a constant $\varepsilon(n,\Lambda)$, depending only on n and Λ, such that if $H \cdot diam^2(M) \geq -\varepsilon(n,\Lambda)$, then M is diffeomorphic to the product*

$$P(\mathbb{C})^{m_1} \times \cdots \times P(\mathbb{C})^{m_k} \times N_1 \cdots \times N_l$$

where N_i, $1 \leq i \leq l$, are irreducible Kähler-Einstein Hermitian symmetric spaces of rank at least 2.

Observe that for M in Theorem A with ε sufficiently small, the Ricci curvature of M is almost non-negative. By [FY] and [CC] we know that the fundamental group $\pi_1(M)$ is virtually nilpotent. In other words, M has a finite cover whose fundamental group is nilpotent. Note that for a finitely generated infinite nilpotent group, its first Betti number is nonzero. The following is a complex analogue of [Ya], which proves M is a bundle over a torus T^b.

THEOREM C. *Let $M \in \mathcal{M}(n,\Lambda)$. Then there is a positive constant $\varepsilon(n,\Lambda)$ such that if $H \cdot diam^2(M) \geq -\varepsilon$, then there is a holomorphic fibration $\pi : M \to \mathcal{J}(M)$, where $\mathcal{J}(M)$ is the Jacobian of M, a complex torus of dimension $\frac{1}{2}b_1$.*

Our next result gives an estimate for the Hodge number $h^{1,1}$, which may be viewed as a complex analogue of the Bochner-Gallot-Gromov first Betti number estimate (cf. [Ga]).

THEOREM D. *Let $M \in \mathcal{M}(n,\Lambda)$. Then there is a positive constant $\varepsilon(n,\Lambda)$ such that if $H \cdot diam^2(M) \geq -\varepsilon(n,\Lambda)$, then $h^{1,1}(M) \leq n$.*

Now let us start to indicate the idea for the proof of the Theorem A.

Suppose not. Then there is a sequence of simply connected Kähler manifolds $M_i \in \mathcal{M}(n,\Lambda)$ such that the bisectional curvature $H_i \geq -(0.1)^i$. By the Gromov precompactness theorem, $\{M_i\}$ has a Gromov-Hausdorff limit X. By [FR] we know that X is a manifold of the same dimension n, and moreover, for every sufficiently large i (say $i \geq N$), there is a diffeomorphism $f_i : M_i \to X$. Moreover, the almost complex structure $\tilde{J}_i = (df_i)J_i(df_i)^{-1}$ converges to an almost complex structure J_∞ on TX which is integrable in $C^{1,\alpha}$-class, for any prescribed real number $\alpha \in (0,1)$. By the Newslander-Nirenberg theorem for $C^{1,\alpha}$-class (cf. Theorem 1.5) (X, J_∞) is a complex manifold. The pullback metrics $((f_i)^{-1})^*g_i$ converge to a $C^{1,\alpha}$ metric g_∞ on X, by the well-known Cheeger-Gromov theorem. This metric is indeed a Kähler metric on X, compatible with the complex structure J_∞.

If $h^{1,1}(M_i) = k$, we will prove there are k linearly independent harmonic $(1,1)$-forms in X of $C^{1,\alpha}$ class which are all parallel. Using these parallel forms we obtain a parallel decomposition of the tangent bundle TX into k distributions. This implies that the holonomy group of X splits into the product of k factors. Now by the de Rham decomposition theorem in $C^{1,\alpha}$-class (cf. Theorem 1.4) we conclude that $X = X_1 \times \cdots \times X_k$, where each factor X_i is a simply connected $C^{1,\alpha}$-Kähler manifold.

On the other hand, we will prove a vanishing theorem for holomorphic p-forms (for any $p > 0$) in simply connected Kähler manifolds with almost non-negative bisectional curvature. Therefore by the Hodge duality theorem $h^{2,0}(M_i) = h^{0,2}(M_i) = 0$ and $b_2(M_i) = h^{2,0}(M_i) + h^{1,1}(M_i) + h^{0,2}(M_i) = k$. This implies that $b_2(X) = k$ and so $b_2(X_1) = \cdots = b_2(X_k) = 1$, since each factor is a compact Kähler manifold. A contradiction. This together implies Theorem A.

The difficulty to prove X_i is a Kähler Hermitian symmetric space occurs since the limit metric on X is not necessarily smooth. However, if M_i is Einstein, by the Einstein equation one can improve the regularity of the convergence. In particular, we get a smooth Kähler metric on X with non-negative bisectional curvature. Therefore the Mok Theorem applies to show that each X_i is an irreducible Kähler Hermitian symmetric space. Therefore Theorem B follows.

Theorem C follows along a similar strategy of [Ya], by using holomorphic forms instead of harmonic forms.

For the proof of Theorem D we show that there are $h^{1,1}$ harmonic $(1,1)$-forms on M which are linearly independent at every point. In particular, this implies a decomposition of the tangent bundle TM into $h^{1,1}$ complex sub-bundles. For the dimension reasoning we know that $h^{1,1} \leq n$.

We should like to remark that our approach in this paper does not imply an estimate for the constant $\varepsilon(n, \Lambda)$. A search for the precise bound will be extremely worthwhile.

In concluding this section we conjecture the following

CONJECTURE E. *There is a positive constant $\varepsilon(n)$ depending only on the dimension, such that if M is a simply connected compact Kähler manifold whose holomorphic bisectional curvature satisfies*

$$H \cdot diam(M)^2 \geq -\varepsilon(n)$$

and the second Betti number $b_2(M) = 1$, then M is either diffeomorphic to a complex projective space or an irreducible Kähler-Hermitian symmetric space of rank ≥ 2.

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1. Preliminaries. In this section we give some necessary preliminary results needed in next sections.

a). Example (manifolds of almost non-negative bisectional curvature)

EXAMPLE 1.1. *Let N be the complex Heisenberg group and Γ the Gaussian integer lattice*

$$N = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} ; x, y, z \in \mathbb{C} \right\}, \quad \Gamma = \left\{ \begin{pmatrix} 1 & c & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} ; a, b, c \in \mathbb{Z} \oplus \mathbb{Z}\sqrt{-1} \right\}$$

The quotient space $M = N/\Gamma$ is a complex analytic variety, and $h^{1,0}(M) = h^{0,1}(M) =$

2. *For each $\varepsilon > 0$, we may define a right invariant Hermitian metric h_ε on M by*

$$\begin{pmatrix} 0 & w & u \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{N}; \quad \| \begin{pmatrix} 0 & w & u \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix} \| = \varepsilon^2 |u|^2 + |v|^2 + |w|^2,$$

where $\mathcal{N}$ is the Lie algebra of N. Note that the sectional curvature and diameter satisfy that $|K(h_\varepsilon)| \leq 24\varepsilon^2$, $diam(M, g_\varepsilon) \leq 2$. In particular, M has almost non-negative bisectional curvature (compare [Ya]).

The above manifold $M = N/\Gamma$ is a complex nil-manifold, whose fundamental group is not virtually abelian. By the Cheeger-Gromoll theorem [CG] we know that M does not admit any metric of non-negative Ricci curvature. On the other hand, by a theorem of Benson-Gordon-Hasegawa [BG][Ha] no nil-manifold other than tori admits a Kähler structure. More generally, it is proved that no fiber bundle over a non-torus nil-manifold with fiber $P(\mathbb{C})^m$ admits a Kähler structure. It seems natural to ask the following problem:

PROBLEM 1.2. *Let M be a Kähler manifold of almost non-negative bisectional curvature. Is its fundamental group virtually abelian?*

b). Gromov-Hausdorff convergence

For subsets A and B of a metric space X, recall that the *Hausdorff distance* $d_H(A, B)$ is the infimum of radii ε so that the ε neighborhood of A (resp. B) include B(resp. A.) For two abstract metric spaces A and B, a metric on the disjoint union $A \coprod B$ is called *admissible* if it restricts to the metrics on A and B respectively. The *Gromov-Hausdorff distance* $d_{GH}(A, B)$ is the infimum of Hausdorff distances of A and B as subsets of $A \coprod B$, with respect to all possible admissible metrics.

The following compactness theorem is important for our applications (cf. [Ch][GLP]).

THEOREM 1.3 [CHEEGER-GROMOV]. *Let M_i be a sequence of compact Riemannian manifolds whose sectional curvature, diameter, and injectivity radius satisfy*

$$\lambda \leq K \leq \Lambda, \quad diam \leq d, \quad i_M > i_0,$$

where the constants are independent of i. Then, replacing M_i by a subsequence if necessary, M_i converges to a metric space X, such that
(i) X is a differentiable manifold;
(ii) there is a diffeomorphsim $f_i : X \to M_i$ for all sufficiently large i;
(iii) the pullback metrics $f_i^*(g_i)$ converges in $C^{1,\alpha}$-class to a $C^{1,\alpha}$ (resp. $L^{2,p}$) Riemannian metric g_∞ in X, for any prescribed real number $\alpha \in (0,1)$ (resp. positive integer $p > 1$).

For the sake of simplicity, in the rest of the paper we fix the real number $\alpha \in (0,1)$ (resp. the interger $p > n$).

The classical de Rham decomposition theorem holds for Kähler manifolds in $C^{1,\alpha}$-class with identical proof.

THEOREM 1.4 [LIC]. *If M is a simply connected complete $C^{1,\alpha}$-Kähler manifold whose holonomy representation splits into a direct sum $A_1 \times A_2 \times \cdots \times A_k \subset U(n_1) \times U(n_2) \times \cdots \times U(n_k) \subset U(n_1 + \cdots + n_k)$, then M is isometric to a product $M_1 \times \cdots \times M_k$ of $C^{1,\alpha}$-Kähler manifolds $M_1, \cdots, M_k$ of dimensions $n_1, \cdots, n_k$.*

If every M_i in Theorem 1.3 is a complex Hermitian manifold, then the complex structure $\tilde{J}_i = (df_i)^{-1} J_i (df_i)$ on the tangent bundle TX converges to an almost complex structure J_∞ in X, which is integrable in $C^{1,\alpha}$-class. The following version of the well-known Newlander-Nirenberg theorem implies that J_∞ is a complex structure.

THEOREM 1.5 [NW]. *Let (X, J) be an almost complex manifold with an integrable almost complex structure J in $C^{1,\alpha}$-class. Then (X, J) is a complex manifold.*

Therefore, the limit X is also a complex manifold with a $C^{1,\alpha}$ Hermitian metric g_∞. In general, the metric g_∞ is not C^2. However, if M_i are all Einstein, then from the ellipticity regularity of the Einstein equation one can prove the limit metric g_∞ is smooth, by the work of M. Anderson [An].

THEOREM 1.6 [ANDERSON]. *Let M_i be a sequence of compact Einstein manifolds whose sectional curvature, diameter, and injectivity radius satisfy*

$$\lambda \leq K \leq \Lambda, \quad diam \leq d, \quad i_M > i_0,$$

where the constants are independent of i. Then M_i has a subsequence converging to a C^∞ Riemannian manifold X, such that
(i) there is a diffeomorphsim $f_i : X \to M_i$ for all sufficiently large i;
(ii) the pullback metrics $f_i^*(g_i)$ converges in C^∞-class to a Riemannian metric g_∞.

Let us continue to use the same notations in the Introduction. Observe that for any flat manifold M we have $H \cdot \mathrm{diam}^2(M) = 0 \geq -\varepsilon$. Obviously the diameter can be made arbitrarily large or small by scaling and so is for the injectivity radius. This also happens for simply connected manifold. For example, the Berger spheres provide an example of simply connected positively curved manifold of bounded curvature whose injectivity radius can be arbitrarily small. However, for Kähler manifolds we have a uniform positive lower bound for the injectivity radii by the following theorem in [FR].

THEOREM 1.7. *Let $M \in \mathcal{M}(n, \Lambda)$. If M has finite fundamental group, and $H \cdot diam^2(M) \geq -\varepsilon$, then the injectivity radius i_M has a positive lower bound, depending only on n, Λ and ε.*

2. Convergence of harmonic forms.

a). Harmonic coordinates

A key step in this paper is to establish a convergence theorem for harmonic $(1,1)$-forms in Kähler manifolds. To start with, let us recall some necessary preliminary on harmonic coordinates.

By definition, a local coordinate $(h^1, ..., h^n)$ is *harmonic* if each component is a harmonic function, i.e., $\triangle h^i = 0$ for $i = 1, ..., n$, where $\triangle$ is the Laplacian opertator. In a harmonic coordinate, the Ricci curvature of the metric tensor g satisfies the equation

$$(\mathrm{Ric}_g)_{ij} = -\frac{1}{2} \triangle g_{ij} + Q(g, \partial g). \tag{2.0}$$

here $Q(.,.)$ is a quadratic form of its variables (c.f. [Pe]).

THEOREM 2.1 [ANDERSON, JOST, KARCHER-JOST]. *Given $\epsilon_0 > 0$ and $\alpha \in (0,1)$. Assume the Riemannian manifold (M, g) satisfies the conditions that the injectivity radius $i_g \geq \epsilon_0$ and the sectional curvature $|K_g| \leq 1$. Then there exist $r > 0$*

depending only on ϵ_0, n *and a constant* C *depending only on* ϵ_0, n, α *(resp.* ϵ_0, n, p*) such that there is a harmonic coordinate system* $\{h^i, i = 1, \cdots, n\}$ *on* $B(x, r)$ *satisfying*

$$\|g_{ij}\|_{C^{1,\alpha}} \leq C \text{ (resp. } \|g_{ij}\|_{L^{2,p}} \leq C \text{)}$$

where $g_{ij} = g(\frac{\partial}{\partial h^i}, \frac{\partial}{\partial h^j})$ *and* $L^{2,p}$ *is the Sobolev norm.*

For a Kähler manifold, the holomorphic coordinate is the most natural harmonic coordinate.

b). Convergence of harmonic real $(1,1)$-forms

Let M be a Kähler manifold. Let $\{V_i\}$ be a local frame field of type $(1,0)$ and let $\{\omega^i\}$ be its dual coframe field. Let $\Delta = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ be the complex Laplacian operator on (p,q)-forms. On a Kähler manifold, $\Delta = \partial\partial^* + \partial^*\partial$.

Let $D^2_{XY} = D_X D_Y - D_{D_X Y}$ denote the second order covariant differential, $R_{XY} = -D_X D_Y + D_Y D_X + D_{[X,Y]}$ denote the curvature tensor acting on forms of all degrees as a derivation and $i(X)$ denotes the interior product.

The well-known Weitzenböck formula for Hodge-Laplacian operator on Kähler manifold (cf. [Wu2]) reads

WEITZENBÖCK FORMULA.

$$\Delta = -\sum_i D^2_{V_i\bar V_i} - \sum_{i,j} \bar\omega^i \wedge i(\bar V_j) R_{V_j \bar V_i}$$

LEMMA 2.2. *Let* $M_i \in \mathcal{M}(n, \Lambda)$. *Let* $\xi_i \in H^{1,1}(M_i)$ *be a sequence of harmonic* $(1,1)$*-forms with* L^2*-norm* $\|\xi_i\|_{0,2} = 1$. *If the injectivity radii* $i_{M_i} \geq i_0$, *a uniform constant, then there exists a positive constant* $\varepsilon(n, \Lambda)$ *so that if* $H_i \cdot diam^2(M_i) \geq -\varepsilon(n, \Lambda)$, *then the pullback forms* $f_i^*(\xi_i)$ *contains a converging subsequence in* $C^{1,\alpha}$*-class (for any* $\alpha < 1$*) whose limit is a non-trivial parallel* $(1,1)$*-form* $\hat\xi$ *with respect to the limit metric* g_∞, *where* f_i *are as in Theorem 1.3.*

Proof. Observe that the sectional curvature of M_i are uniformly bounded from below. By Theorem 2.1 we can assume $B_i^1(r), \cdots, B_i^m(r)$ be a cover of harmonic coordinates of M_i with radii r, independent of i. For all $i > N$ large and all q, we get diffeomorphisms $f_i^q : B^q = B_N^q(r) \to B_i^q(r)$. The pullback metric tensors $(f_i^q)^*(g_i)$ converges to a $C^{1,\alpha}$ (resp. $L^{2,p}$) metric g_∞ on B^q. The $(1,1)$-forms $\eta_l = (f_l^q)^*(\xi_l|_{B_l^q(r)})$ on B^q, satisfy the equation

$$\sum_i D^2_{V_i\bar V_i}\eta_l + \sum_{i,j} \bar\omega^i \wedge i(\bar V_j) R_{V_j\bar V_i}(\eta_l) = 0 \tag{2.2.1}$$

since $\Delta(\eta_l) = 0$, by the above Weitzenböck formula.

This is an elliptic system of order 2. It is easy to check that

(i) the second order terms coefficients are uniformly C^0-bounded (independent of the indices l);

(ii) the first order terms coefficients are uniformly $C^{0,\alpha}$-bounded;

(iii) the zero order terms coefficients are uniformly L^p-bounded, by Theorem 2.1.

By the standard elliptic estimate (c.f. [ADN]) we conclude that the Sobolev $L^{2,p}$-norms of η_l are uniformly bounded, for any $p < \infty$. Therefore, η_l has a subsequence converging in $L^{2,p'}$ for any $p' < p$. Note that $L^{2,p} \subset C^{1,1-\frac{n}{p}}$. Therefore there is a $C^{1,\alpha}$-convergence subsequence $\eta_l \to \eta$.

To prove $\hat{\xi}$ is parallel with respect to the metric g_∞, it suffices to show that η is parallel with respect to the $C^{1,\alpha}$-limit metric of $(f_i^q)^* g_i$ on B^q.

Note that for a real $(1,1)$-form, the Weitzenböck formula implies that (c.f. [Wu2])

$$-\Delta|\eta_l|^2 = \sum_i (|D_{\bar{V}_i}\eta_l|^2 + |D_{V_i}\eta_l|^2) - 2\langle \sum_{i,j} \bar{\omega}^i \wedge i(\bar{V}_j) R_{V_i \bar{V}_j}\eta_l, \eta_l\rangle \tag{2.2.2}$$

Let

$$F(\eta_l) = \langle \sum_{i,j} \bar{\omega}^i \wedge i(\bar{V}_j) R_{V_i \bar{V}_j}\eta_l, \eta_l\rangle$$

We may write η_l locally as

$$\eta_l = \sum_i \sqrt{-1}\eta_l^i \omega^i \wedge \bar{\omega}^i$$

By [Wu2] one has the following formula

$$F(\eta_l) = \frac{1}{2}\sum_{j,k} R^l_{jjkk}(\eta_l^j - \eta_l^k)^2 \tag{2.2.3}$$

where $-R^l_{jjkk}$ is the bisectional curvature of M_l at the complex plane spanned by V_j, V_k. Therefore we get

$$-\Delta|\eta_l|^2 = \sum_i |D_{\bar{V}_i}\eta_l|^2 + \sum_i |D_{V_i}\eta_l|^2 - \sum_{j,k} R^l_{jjkk}(\eta_l^j - \eta_l^k)^2 \tag{2.2.4}$$

By integrating both sides and taking limit we get

$$\lim_l (\sum_i |D_{\bar{V}_i}\eta_l|^2 + \sum_i |D_{V_i}\eta_l|^2) = 0$$

since the integration of $-F(\eta_l)$ has limit non-negative, by the assumption of almost non-negativity of the bisectional curvature. Therefore $\lim_l D_{V_i}\eta_l = 0$ and $\lim_l D_{\bar{V}_i}\eta_l = 0$. This implies that $D^\infty_{V_i}\eta = D^\infty_{\bar{V}_i}\eta = 0$, where D^∞ is the covariant derivative of the the limit metric g_∞. This proves that η is parallel in the metric g_∞, so $\hat{\xi}$ is also parallel.

Obviously, $\|\hat{\xi}\|_{0,2} = \lim \|\xi_i\|_{0,2} = 1$. In particular, $\hat{\xi}$ is nontrivial. The desired result follows. □

As a direct corollary we have

PROPOSITION 2.3. *Let $M_i \in \mathcal{M}(n,\Lambda)$. Suppose that $h^{1,1}(M_i) = k$ for all i. Let $\{\xi_i^1, \cdots, \xi_i^k\}$ be a sequence of orthonormal basis for $(H^{1,1}(M_i), \|\cdot\|_{0,2})$. If the injectivity radii $i_{M_i} \geq i_0$, a uniform constant, then there exists a positive constant $\varepsilon(n,\Lambda)$ so that if $H_i \cdot diam^2(M_i) \geq -\varepsilon(n,\Lambda)$, then, passing to a subsequence if necessary, $\{f_i^*(\xi_i^1), \cdots, f_i^*(\xi_i^k)\}$ converges in $C^{1,\alpha}$-class to orthonormal parallel $(1,1)$-forms $\{\hat{\xi}^1, \cdots, \hat{\xi}^k\}$ with respect to the limit metric g_∞.*

Next we consider a sequence of manifolds whose injectivity radii tend to zero.

Let $M_i \in \mathcal{M}(n,\Lambda)$. Let $\xi_i \in \Lambda^{1,1}(M_i)$ be real harmonic $(1,1)$-forms with normalized L^2-norm $\|\xi_i\|_{0,2}/\sqrt{\mathrm{vol}(M_i)} = 1$, where $\mathrm{vol}(M_i)$ is the volume of M_i. If $H_i \cdot \mathrm{diam}^2(M_i) \geq -1$, then the sectional curvature of M_i are uniformly bounded from both sides. Therefore the conjugate radii have a uniform lower bound, say $2r$. Let $B_i^1(2r), \cdots, B_i^m(2r)$ be the radii $2r$ balls in the tangent spaces $T_{p_k}M_i$, $1 \leq k \leq m$, so that the exponential maps $\exp_{p_k} : B_i^k(2r) \subset T_{p_k}M_i \to M_i$ are immersions and $\exp_{p_1}(B_i^1(r)), \cdots, \exp_{p_m}(B_i^m(r))$ is a cover of M_i. For every fixed k, consider the pullback metrics $\exp_{p_k}^*(g_i)$ in $B_i^k(2r)$. Note that the injectivity radius at every point of $B_i^k(r)$ with the metric $\exp_{p_k}^*(g_i)$ have a uniform lower bound, independent of the index i. By Theorem 1.3 there is an integer N such that for all $i > N$, there are diffeomorphisms $f_i^k : B^k := B_N^k(r) \to B_i^k(r)$ so that the pullback metric tensors $(f_i^k)^*(\exp_{p_k}^*(g_i))$ converge in $C^{1,\alpha}$ (resp. $L^{2,p}$) class. Let $\eta_i = (f_i^k)^*(\exp_{p_k}^*(\xi_i))$ be the pulled back $(1,1)$-forms in B^k.

LEMMA 2.4. *Let $M_i \in \mathcal{M}(n,\Lambda)$ and let η_i be as above. There exists a positive constant $\varepsilon(n,\Lambda)$ so that if $H_i \cdot diam^2(M_i) \geq -\varepsilon(n,\Lambda)$, then η_i contains a $C^{1,\alpha}$ converging subsequence whose limit is a parallel $(1,1)$-form with respect to the limit metric in B^k, $1 \leq k \leq m$.*

Proof. Let $\xi_i \in \Lambda^{1,1}(M_i)$ be the real harmonic $(1,1)$-forms as above. By Peter Li [Li] Lemma 8 the pointwise C^0-norms $|\xi_i|$ satisfy the inequalities

$$\Delta|\xi_i| \leq C|\xi_i|$$

where C is a constant depending only on the bound $\Lambda, \varepsilon(n,\Lambda)$ and n. This together with Lemma 20 in [Ga] implies that

$$|\xi_i| \leq C\|\xi_i\|_{0,2}/\sqrt{\mathrm{vol}(M_i)} = C$$

Therefore the C^0-norms $|\eta_i| \leq C$ for all i.

By (2.2.1) and the standard elliptic estimate [ADN] it follows that

$$\|\eta_i\|_{2,p} \leq C_{2,p}$$

for any $p < \infty$. As in the proof of Lemma 2.2, this implies that η_i contains a convergence subsequence in $C^{1,\alpha}$ class.

By the technique of Yamaguchi (cf. [Ya] the proof of Prop. 2.2) and the formula (2.2.2) it follows that η_i converges to a parallel $(1,1)$-form in $C^{1,\alpha}$-class. The desired result follows. □

c). A vanishing theorem for holomorphic p-forms

It is well-known that a compact Kähler m-manifold with positive Ricci form has no non-trivial holomorphic p-forms, $p = 1, \cdots, m$ (cf. [Be] page 323). The same result holds for simply connected compact Kähler manifold with non-negative bisectional curvature. The following vanishing result is a generalization of this fact to simply connected compact Kähler manifold with almost non-negative bisectional curvature.

THEOREM 2.5. *Let $M \in \mathcal{M}(n,\Lambda)$ be a Kähler manifold. Suppose the fundamental group of M is finite. Then there is a positive constant $\varepsilon(n,\Lambda)$ such that if $H \cdot diam^2(M) \geq -\varepsilon(n,\Lambda)$, then M does not have nonzero holomorphic p-forms for $p > 0$, i.e. $h^{p,0}(M) = 0$.*

In the above theorem the finiteness of the fundamental group of M is necessary, since flat Kähler manifold fits in the class of the manifolds but with $h^{p,0}(M)$ nonzero. The idea in proving Theorem 2.5 is by argument by contradiction, to show otherwise the fundamental group can never be finite.

For a compact Kähler manifold $M \in \mathcal{M}(n,\Lambda)$ with metric g, a real harmonic $(1,1)$-form $\xi \in H^{1,1}(M)$ induces a linear self-adjoint transformation

$$S : TM \to TM \tag{2.6}$$

by setting $\xi(v,w) = g(S(Jv),w)$ for all $v,w \in TM$, where J is the complex structure. Let $\alpha_1(x),\cdots,\alpha_n(x)$ denote the eigenvalues of $S|_x$, the restriction of S on the tangent space T_xM. Obviously, if the $(1,1)$-form ξ is parallel, then $\alpha_1(x),\cdots,\alpha_n(x)$ are constants. Moreover, if ξ is not a multiple of ω, these constants can not be all the same.

LEMMA 2.7. *Let $M_i \in \mathcal{M}(n,\Lambda)$ and let $\{\xi_i^1,\cdots,\xi_i^k\}$ be as in Proposition 2.3. Let (X,J_∞,g_∞) be the Gromov-Hausdorff limit of the sequence (M_i,J_i,g_i). Then there exist J_i-invariant distributions $E_i^1,\cdots,E_i^k$ on M_i such that*
(2.7.1) the tangent bundle $TM_i = E_i^1 \oplus \cdots \oplus E_i^k$ for sufficiently large i.
(2.7.2) the limits of E_i^j, $1 \le j \le k$, are parallel J_∞-invariant distributions of the $C^{1,\alpha}$-Kähler manifold (X,J_∞,g_∞).

Proof. Following Proposition 2.3 let $\hat\xi^1,\cdots,\hat\xi^k \in H^{1,1}(X)$ denote the limits of $\hat\xi_i^1,\cdots,\hat\xi_i^k \in H^{1,1}(M_i)$. For the sake of simplicity we assume $\xi_i^1 = \omega_i$ be the Kähler form of M_i, and $\hat\xi^1 = \hat\omega$ the Kähler form of X.

Let $S_{i,2},\cdots,S_{i,k}$ be the endomorphisms in (2.6) associated to $\xi_i^2,\cdots,\xi_i^k$ with respect to the Kähler form ω_i. Since $\hat\omega,\hat\xi^2,\cdots,\hat\xi^k$ are all parallel, all the eigenvalue functions of $S_{i,j}$, $j=2,\cdots,k$, converge to constants. Indeed, they converge to the eigenvalue functions of $\hat S_j$, $j=2,\cdots,k$, the endomorphisms associated to $\hat\xi^2,\cdots,\hat\xi^k$ with respect to $\hat\omega$.

Note that every $\hat S_j$, $j=2,\cdots,k$, must have at least two different eigenvalues. Otherwise, $\hat\xi^j$ must be a multiple of $\hat\omega$. This is impossible by Proposition 2.3. Therefore $\hat S_j$ gives a decomposition of TX into at least two parallel eigen distributions. Therefore, if $k \ge 2$ the eigen distributions of $\hat S_2$ gives a parallel J_∞-invariant decomposition $TX = \tilde E^1 \oplus \cdots \oplus \tilde E^r$ with $r \ge 2$. If $r < k$, the restrictions of $\hat\xi^2,\hat\xi^3,\cdots,\hat\xi^k$ on some factor, $\hat E^s$, must have rank > 1. By the above $\hat E^s$ can be further splitted into parallel eigen distributions. This implies $r \ge k$. Correspondingly, $S_{i,j}$ gives an eigen decomposition $TM_i = \tilde E_i^1 \oplus \cdots \oplus \tilde E_i^r$ for i sufficiently large so that $\tilde E_i^j$ converges to $\tilde E^j$. Note that all the distributions are J_i-invariant.

It suffices to prove $r = k$.

Note that the parallel distributions $\tilde E^1,\cdots,\tilde E^r$ are integrable. By Theorem 1.4 $X = X_1 \times \cdots \times X_r$ accordingly so that $\tilde E^j$, $j = 1\cdots,r$, are the distributions given by the product foliations. Observe that $h^{1,1}(X_j) \ge 1$ since X_j is a compact $C^{1,\alpha}$-Kähler manifold.

Let $\omega_{i,j}$ denote the Kähler forms of the distributions $\tilde E_i^j \subset TM_i$, $j = 1\cdots,r$. Note that $\omega_{i,j}$ is not necessarily a closed form. Clearly, $\omega_{i,j}$ converges to $\hat\omega_j$, the closed Kähler form of X_j, for each $j = 1,\cdots,k$. Let $\bar\omega_{i,j}$ denote the harmonic component of $\omega_{i,j}$ in $H^{1,1}(M_i)$. Obviously, $\bar\omega_{i,j}$ also converges to $\hat\omega_j$. If $r > k$, then $h^{1,1}(X) > k$ and so the vectors $\bar\omega_{i,j}$, $j = 1,\cdots,r$, has rank greater than k for i large by Proposition 2.3. This is impossible since $h^{1,1}(M_i) = k$ for all i. □

Let $M \in \mathcal{M}(n,\Lambda)$ be a Kähler manifold. Assume $h^{1,1}(M) = k$. By the above there is a constant $\varepsilon(n,\Lambda)$ such that, if $H\cdot\mathrm{diam}^2(M) \ge \varepsilon(n,\Lambda)$, then Lemma 2.7 implies

k distributions $E^1, \cdots, E^k$. Let $\omega_1, \cdots, \omega_k$ be the Kähler forms of the distributions and let $\bar{\omega}_1, \cdots, \bar{\omega}_k \in H^{1,1}(M)$ be their $(1,1)$-harmonic components. From the proof of Lemma 2.7 we know that $\bar{\omega}_1, \cdots, \bar{\omega}_k$ are linear independent. Therefore $\bar{\omega}_1, \cdots, \bar{\omega}_k \in H^{1,1}(M)$ is a basis.

Let ϕ be the Ricci form of M. We may write $\phi = a_1\bar{\omega}_1 + \cdots + a_k\bar{\omega}_k + d\theta$, where $d\theta$ is an exact form.

LEMMA 2.8. *Let $M \in \mathcal{M}(n, \Lambda)$ be a Kähler manifold with finite fundamental group. Suppose $h^{1,1}(M) = k$. Then there is a positive constant $\varepsilon(n, \Lambda)$ such that if $H \cdot diam^2(M) \geq -\varepsilon(n, \Lambda)$, then $a_i > c(n, \Lambda)$ for $i = 1, \cdots, k$, where $c(n, \Lambda)$ is a positive constant depending only on n, Λ.*

Proof. We first prove that

$$\int_M \phi \wedge \omega_1^{n_1} \wedge \cdots \wedge \omega_i^{n_i - 1} \wedge \cdots \wedge \omega_k^{n_k} > c(n, \Lambda) \tag{2.8.0}$$

for some positive constant as above.

Suppose not. For simplicity we assume a sequence of Kähler manifolds M_i with Ricci forms ϕ_i converging to a $C^{1,\alpha}$-Kähler manifold X such that:
(2.8.1) $H_i \geq -1/i$, the sectional curvature $K_{M_i} \leq \Lambda$, and diameter $\operatorname{diam}(M_i) \leq D$;
(2.8.2) $h^{1,1}(M_i) = k$ and $\pi_1(M_i)$ are all finite;
(2.8.3) A sequence of J_i-invaraint distributions $E_i^1, \cdots, E_i^k$ in TM_i with limit $E^1, \cdots, E^k$ parallel J_∞-distributions of dimensions $n_1, \cdots, n_k$ respectively.
(2.8.4) The lower limit $\underline{\lim}_i \int_{M_i} \phi_i \wedge \omega_{i,1}^{n_1 - 1} \wedge \cdots \wedge \omega_{i,k}^{n_k} \leq 0$, where $\omega_{i,1}, \cdots, \omega_{i,k}$ are the Kähler forms of the distributions $E_i^1, \cdots, E_i^k$.

Let $V(M_i) = \int_{M_i} \omega_{i,1}^{n_1} \wedge \cdots \wedge \omega_{i,k}^{n_k}$. By Theorem 1.7 the injectivity radius of M has a uniform positive lower bound. Clearly, $V(M_i)$ converges to $V(X)$, the volume of X, since $\omega_{i,1}^{n_1} \wedge \cdots \wedge \omega_{i,k}^{n_k}$ converges in $C^{1,\alpha}$ (resp. $L^{2,p}$) class to the volume form of X. Moreover, $\int_{M_i} \phi_i \wedge \omega_{i,1}^{n_1 - 1} \wedge \omega_{i,2}^{n_2} \wedge \cdots \wedge \omega_{i,k}^{n_k}$ converges to $\lim_i a_{i,1} V(X)$. Therefore the desired result follows from (2.8.0).

Consider the restriction of the Ricci transformation $\operatorname{Ric} : TM_i \to TM_i$ on the distribution E_i^1. Let λ_i^j, $1 \leq j \leq n_1$, denote the eigenvalues of this restriction. Let $\lambda_i^{j,-} = \chi \lambda_i^j$ where χ is the character function of the set $\{x \in M_i : \lambda_i^j(x) < 0\}$ (i.e. χ has value 1 for x in the set and zero otherwise). Since E_i^j converges in $L^{2,p}$-class to a totally geodesic foliation in X for any integer $p > 0$, by (2.8.1) it follows that

$$\int_{M_i} |\lambda_i^{j,-}|^p \leq b_i(p, n, \Lambda) \tag{2.8.5}$$

where $\{b_i(p, n, \Lambda)\}$ is a sequence of positive constants depending only on p, n and Λ which converges to zero when i goes to infinity.

Let $A_i = \int_{M_i} \phi_i \wedge \omega_{i,1}^{n_1 - 1} \wedge \cdots \wedge \omega_{i,k}^{n_k}$. Note that

$$A_i = \frac{1}{(n_1 - 1)!} \int_{M_i} (\lambda_i^1 + \cdots + \lambda_i^{n_1}) \omega_{i,1}^{n_1} \wedge \cdots \wedge \omega_{i,k}^{n_k}, \tag{2.8.6}$$

By (2.8.5) it is clear that $\underline{\lim}_i A_i \geq 0$. If $\underline{\lim}_i A_i = 0$, passing to a subsequence if necessary we may assume that $\lim_i A_i = 0$.

Let $\lambda_i(x) = \max\{\lambda_i^1(x), \cdots, \lambda_i^{n_1}(x)\}$ for $x \in M_i$. Using Hölder inequality, (2.8.5) and (2.8.6) imply that

$$\int_{M_i} |\lambda_i|^p \le c_1(p,n,\Lambda)A_i + c_2(p,n,\Lambda)b_i(p,n,\Lambda) \tag{2.8.7}$$

for all large i, where $c_1(p,n,\Lambda)$ (resp. $c_2(p,n,\Lambda)$) is a positive constant depending only on p, n and Λ. Therefore λ_i converges in L^p-class to zero for any integer $p > 0$. This implies that the limit metric $g = g_\infty$ on X_1 (the limit of the distribution E_i^1) gives an $L^{2,p}$-weak solution to the Ricci flat equation (2.0)

$$-\frac{1}{2}g^{ij}\frac{\partial^2 g_{rs}}{\partial x_i \partial x_j} + Q(\frac{\partial g_{kl}}{\partial x_m}) = 0$$

By the elliptic regularity we know that the restriction of g_{ij} on X_1 is a smooth metric with flat Ricci curvature (cf. [An] and Theorem 1.6). Note that the bisectional curvature of (X_1, g) is non-negative. Therefore (X_1, g) has zero bisectional curvature and so (X_1, g) is flat. By the well-known Bieberbach theorem a finite covering space of X_1 is a torus. In particular, $\pi_1(X_1)$ is infinite. Thus $\pi_1(X) = \pi_1(X_1)\times\cdots\times\pi_1(X_k)$ is also infinite. By Theorems 1.3 and 1.7 it follows that $\pi_1(M_i) \cong \pi_1(X)$ is infinite for i large. A contradiction. The desired result follows. □

LEMMA 2.9. *Let $M \in \mathcal{M}(n,\Lambda)$ be as in Lemma 2.8. Let $\mu(x) = min\{\lambda_1(x),\cdots,\lambda_n(x)\}$ for $x \in M$, where $\lambda_1,\cdots,\lambda_n$ are the eigenvalue functions of the Ricci transformation $Ric : TM \to TM$. Then*

$$\int_M \mu > c_0(n,\Lambda)$$

a positive constant depending only on n and Λ.

Proof. Suppose not, there is a sequence of manifolds $M_i \in \mathcal{M}(n,\Lambda)$ satisfying (2.8.1) and (2.8.2). Write the Ricci form $\phi_i = a_1\bar\omega_{i,1} + \cdots + a_k\bar\omega_{i,k} + d\theta_i$, where $d\theta_i$ is an exact form. By Lemma 2.8 $a_j > c > 0$, $j = 1,\cdots,k$. Therefore

$$\lim_i \int_{M_i} \phi_i^n \ge c^n\alpha(n)\mathrm{V}(X) > 0$$

where $\alpha(n)$ is a positive function depending only on n.

Let $\lambda_{i,1},\cdots,\lambda_{i,n}$ be the eigenvalue functions of Ric : $TM_i \to TM_i$. Note that

$$\lim_i \int_{M_i} \lambda_{i,1}\cdots\lambda_{i,n} \ge c^n\beta(n)\mathrm{V}(X) > 0 \tag{2.9.1}$$

where $\beta(n)$ is a positive function depending only on n.

Let $M_i^{\ge 0} = \{x \in M_i : \mu_i(x) \ge 0\}$ and $M_i^{<0}$ be the complement of $M_i^{\ge 0}$ in M_i. Since the sectional curvature $K \le \Lambda$, it holds that

$$|\int_{M_i^{<0}} \lambda_{i,1}\cdots\lambda_{i,n}| \le \frac{(n-1)}{i}(n\Lambda)^{n-1}V(M_i) \tag{2.9.2}$$

$$\int_{M_i^{\ge 0}} \lambda_{i,1}\cdots\lambda_{i,n} \le (n\Lambda)^{n-1}\int_{M_i^{\ge 0}} \mu_i \tag{2.9.3}$$

Together with (2.9.1) we get that

$$\lim_i \int_{M_i^{\geq 0}} \lambda_{i,1}\cdots\lambda_{i,n} = \lim_i \int_{M_i} \lambda_{i,1}\cdots\lambda_{i,n} - \lim_i \int_{M_i^{<0}} \lambda_{i,1}\cdots\lambda_{i,n} \tag{2.9.4}$$

$$\geq c^n\beta(n)\mathrm{V}(X) - \lim_i \frac{(n-1)}{i}(n\Lambda)^{n-1}\mathrm{V}(M_i)$$

Now (2.9.3) and (2.9.4) together implies that

$$\int_{M_i} \mu_i = \int_{M_i^{\geq 0}} \mu_i + \int_{M_i^{<0}} \mu_i \geq \int_{M_i^{\geq 0}} \mu_i - \frac{n-1}{i}V(M_i) > c_0(n,\Lambda) > 0$$

for i sufficiently large. A contradiction. This proves the desired result. □

Now we are ready to prove the Theorem 2.5.

Proof of Theorem 2.5. Let $\varepsilon(n,\Lambda)$ be the constant in Lemma 2.9. Let $M \in \mathcal{M}(n,\Lambda)$ be a Kähler manifold such that $H{\cdot}\mathrm{diam}^2(M) \geq -\varepsilon(n,\Lambda)$. Let $\{e_1, Je_1, \cdots, e_n, Je_n\}$ be an orthonormal basis of the tangent space M_x with the following property: if $V_i = \frac{1}{\sqrt{2}}(e_i - \sqrt{-1}Je_i)$, then $(\sum_k R_{V_k\bar{V}_k})V_i = \lambda_i V_i$ for all i with $\lambda_i \in \mathbb{R}$. Let $\{\theta^i\}$ be dual of $\{V_i\}$ and of type $(1,0)$. Let ξ be a harmonic form of type $(p,0)$. We may write $\xi = \sum_I \xi_I\theta^I$, where I runs through all strictly increasing multi-indices $(i_1, \cdots, i_p)$ and $\theta^I = \theta^{i_1}\wedge\cdots\wedge\theta^{i_p}$. By the Weitzenböck formula we get (c.f. [Wu2] page 313)

$$-\Delta|\xi|^2 = \sum_i |D_{\bar{V}_i}\xi|^2 + \sum_i |D_{V_i}\xi|^2 + \sum_I (\sum_{i\in I} Ric(e_i,e_i))|\xi_I|^2 \tag{2.10}$$

Normalize ξ so that its L^2-norm is 1. Integrating (2.10) on M we get that

$$\int_M \mu = 0$$

where μ is as in Lemma 2.9. A contradiction to Lemma 2.9. The desired result follows. □

3. Proofs of Theorems A, B, C, D.

Proof of Theorem A. Let $k = h^{1,1}(M) > 1$. Assume not, then we get a sequence of simply connected Kähler manifolds M_i, such that

(3.1) $H_i \cdot \mathrm{diam}^2(M_i) \geq -(0.1)^i$ and $K_i \leq \Lambda$;

(3.2) $h^{1,1}(M_i) = k$;

(3.3) M_i does not diffeomorphic to the product of k simply connected $C^{1,\alpha}$-Kähler manifolds with second Betti number 1 for any i.

By Theorems 1.3, 1.5 and 1.7 the limit X of the sequence M_i is a $C^{1,\alpha}$ Kähler manifold of dimension n. It suffices to prove that X is diffeomorphic to a product $X_1 \times \cdots \times X_k$, where each factor X_i is a $C^{1,\alpha}$-Kähler manifold with $b_2(X_i) = 1$.

By Lemma 2.7 we get k parallel distributions of X, which implies that the $C^{1,\alpha}$ Kähler manifold X is isometric to the product $X_1 \times \cdots \times X_k$ by Theorem 1.4. We now verify that $b_2(X_1) = \cdots = b_2(X_k) = 1$, which clearly implies the desired result.

By Theorem 2.5 we know that $b_2(M_i) = h^{2,0}(M_i) + h^{1,1}(M_i) + h^{0,2}(M_i) = h^{1,1}(M_i) = k$ using Hodge duality. Clearly $b_2(X_j) \geq h^{1,1}(X_j) \geq 1$ since X_j is a $C^{1,\alpha}$-Kähler manifold. Therefore $b_2(X) = b_2(X_1) + \cdots + b_2(X_k) \geq k$ and the equality holds only if $b_2(X_j) = 1$ for $j = 1, \cdots, k$. Since $b_2(M_i) = b_2(X)$ for i sufficiently large, the desired result follows. □

Proof of Theorem B. By the Anderson's regularity theorem we know that g_∞ is a smooth Kähler metric on X. Applying Mok [Mo] the desired result follows. □

Proof of Theorem C. We claim that there is a constant $\varepsilon(n, \Lambda)$ such that if $M \in \mathcal{M}(n, \Lambda)$ with $H \cdot \mathrm{diam}^2(M) \geq -\varepsilon(n, \Lambda)$, then there are $k = h^{1,0}$ pointwisely linearly independent holomorphic 1-forms $\theta_1, \cdots, \theta_k$. The proof of this fact is exactly the same as [Ya] for harmonic 1-forms. For such holomorphic 1-forms $\theta_1, \cdots, \theta_k$, the Albanese map $\pi : M \to \mathcal{J}(M)$ is a holomorphic submersion. Therefore it is a holomorphic bundle since M is compact. This completes the proof. □

Proof of Theorem D. Suppose not. For simplicity we may assume a sequence of compact Kähler manifolds M_i such that $H_i \geq -\frac{1}{i}$ and $\mathrm{diam}(M_i) \leq 1$, $K_i \leq \Lambda$ but $h^{1,1}(M_i) > n$. Let $\xi_{i,1} \cdots, \xi_{i,l}$ be l (where $l > n$) linearly independent real harmonic $(1,1)$-forms such that the normalized L^2-norms, $\|\xi_{i,1}\|_{0,2}/\sqrt{vol(M_i)}$, are all equal to 1.

Consider harmonic coordinate covers of M_i with radii r and uniform number k, $B_i^1(r), \cdots, B_i^k(r)$. The injectivity radii of points in the radii r balls with respect to the lifted metrics on $B_i^1(2r), \cdots, B_i^k(2r)$ have a uniform positive lower bound. Therefore the lifting of the harmonic $(1,1)$-forms $\xi_{i,j}$ on $B_i^1(r), \cdots, B_i^k(r)$ converge to l linearly independent parallel $(1,1)$-forms by Lemma 2.4. By Lemma 2.7 if i is sufficiently large TM_i may be decomposed into the direct sum of l J_i-invariant distributions. For dimension reasoning this clearly implies $l \leq n$. A contradiction. □

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