THE RESOLVENT TRACE FORMULA FOR RANK ONE LIE GROUPS*

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1. Introduction .

1.1. Introduction. Let $X = G/K$ be a Riemannian symmetric space of non-compact type with G a connected simple Lie group of real rank one and K a maximal compact subgroup of G. In the paper [17], Miatello-Wallach introduced a family of bi-K-invariaint functions Q_s, $s \in \mathbf{C}$ on G, which satisfies the same differential equation as the elementary spherical function ϕ_s of Harish-Chandra on the open set $G^+ = G - K$ but has singularities along K. By making the r-fold convolution of Q_s, they defined a function $Q_{r,s}$ on G, which is less singular than Q_s itself. Then, given a cofinite lattice Γ of G, they introduced the distribution $\mathbf{P}_{r,s}(\dot{x}, \dot{y})$ by forming the Poincaré series

$$\mathbf{P}_{r,s}(\dot{x}, \dot{y}) = c_r(s) \sum_{\gamma \in \Gamma} Q_{r,s}(x^{-1}\gamma y), \quad \dot{x}, \ \dot{y} \in \Gamma \backslash X \qquad (1.1)$$

with a suitable normalizing factor $c_r(s)$ and proved, among other things, that it is smooth on the complement of the diagonal in $(\Gamma \backslash X) \times (\Gamma \backslash X)$ and satisfies the differential equation

$$(\triangle + \rho_0^2 - s^2)^r \mathbf{P}_{r,s}(\dot{x}, -) = \delta(\dot{x}) \qquad (1.2)$$

with $\triangle$ the Laplacian of $\Gamma \backslash X$, $\delta(\dot{x})$ the Dirac delta supported at $\dot{x}$. In the classical situation that X is the upper half plane, the distribution $\mathbf{P}_{1,s}(\dot{x}, \dot{y})$, the resolvent kernel function of Laplacian for the Riemannian surface $\Gamma \backslash X$, was intensively investigated by several German mathematicians from the view point of real analytic automorphic forms ([3], [19]). Based on these works, J. Fischer deduced the resolvent trace formula by computing the integral

$$\int_{\Gamma \backslash X} (\mathbf{P}_{1,s}(\dot{x}, \dot{x}) - \mathbf{P}_{1,s'}(\dot{x}, \dot{x})) \, d\dot{x}, \quad s, \ s' \in \mathbf{C} \qquad (1.3)$$

in two different ways ([5]).

In this paper, we show that the same type of procedure is possible for a higher dimensional X by considering the integral $\int_{\Gamma \backslash X} \mathbf{P}_{r,s}(\dot{x}, \dot{x})d\dot{x}$ with r greater than a half of $\dim X$ instead of (1.3). As a result, following Fischer, we can obtain another proof of the meromorphic continuation of the Selberg zeta function for $\Gamma \backslash X$ and its functional equation, which was originally proved by Selberg, Gangolli and Gangolli-Warner ([7], [8], [20]).

Although a handy formula of $Q_{r,s}$ in the 'polar coordinate '(Cartan decomposition) is desirable for our purpose, it seems rather difficult to have such a formula directly from the definition of $Q_{r,s}$ recalled above. Our strategy is as follows. We first

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have an explicit formula of $Q_s = Q_{1,s}$ in terms of Gaussian hypergeometric series as in the classical case, and then use the system of differential equations among $Q_{r,s}$'s to show that $Q_{r,s}$ is obtained from Q_s by applying the differential operator $(\frac{-1}{2s}\frac{d}{ds})^r$ to it (Proposition 3.1.7, Theorem 3.2.1). Thus a formula of $Q_{r,s}$ in terms of a derivative of the hypergeometric series becomes available, which enables us to compute the integral $\int_{\Gamma \backslash X} \mathbf{P}_{r,s}(\dot{x}, \dot{x}) d\dot{x}$ by dividing it into the local contributions for Γ-conjugacy classes and by using various formulas involving the beta function and the hypergeometric series. Consequently we can evaluate the integral by means of the logarithmic derivative of the Selberg zeta function for $\Gamma \backslash X$. On the other hand, by the spectral expansion of $\mathbf{P}_{r,s}(\dot{x}, -)$ given in [17], we compute the same integral in terms of the eigenvalues of Laplacian on $L^2(\Gamma \backslash X)$. Combining these two expressions of $\int_{\Gamma \backslash X} \mathbf{P}_{r,s}(\dot{x}, \dot{x}) d\dot{x}$, we arrive at the resolvent trace formula, which was studied in [5, Theorem 2.5.2, p.108] for $G = PSL_2(\mathbf{R})$, in [4] for $G = PSL_2(\mathbf{C})$ and in [1] for Jacobi forms.

Finally, we would like to say a few words on the status of our results. The resolvent trace formula (RTF for short) for a general compact locally symmetric space $\Gamma \backslash X$ with rank one X is more or less known, because it is essentially the same as the determinant expression of the Selberg zeta function obtained already in [15] together with its explicit gamma factor. But we believe that our method, that is a slight extension of Fischer's, provides a more direct and elementary way to have the RTF than the traditional method employed in [20], [7] and [8], which necessitates difficult tools such as the Paley-Wiener theorem and the Plancherel formula for $X = G/K$. We also believe that our Theorem 3.2.1, that gives an expression of $Q_{r,s}$ in terms of the derivative of the hypergeometric series, is new and is interesting itself.

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2. Preliminaries.
In this section we introduce basic objects and fix notations.

2.1. Notations.
We denote by $\mathbf{N}$ the set of natural numbers, i.e. $\mathbf{N} = \{1, 2, 3, \ldots\}$. Put $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. The cardinality of a finite set S is denoted by $\#S$.
Let X be a set and f, g two positive real valued functions defined on X. We write

$$f(x) \prec g(x), \quad x \in X$$

if there exists a positive constant c such that $f(x) \leqslant cg(x)$ for all $x \in X$.

2.2. Lie groups and Lie algebras.
Let G be a connected semisimple Lie group of real rank one with finite center. Put $\mathfrak{g} = \mathrm{Lie}(G)$, the real Lie algebra of G. Let K be a maximal compact subgroup of G and θ the Cartan involution of $\mathfrak{g}$ corresponding to K, then we have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ with $\mathfrak{k} = \mathrm{Lie}(K)$. We fix an Iwasawa decomposition $G = NAK$ of G; A is a maximal split torus in G whose Lie algebra $\mathfrak{a}$ is orthogonal to $\mathfrak{k}$ with respect to the Killing form B of G and N a maximal unipotent subgroup of G normalized by A. Since $\dim A = 1$ by assumption, there exists a unique root $\alpha \in \mathfrak{a}^*$ such that $\mathfrak{n}_{j\alpha} = \{X \in \mathfrak{g}| \mathrm{ad}(H)X = j \cdot \alpha(H)X, \ H \in \mathfrak{a}\}$ with $j \in \mathbf{Z}$ is zero if $|j| > 2$, and $\mathrm{Lie}(N) = \mathfrak{n} = \mathfrak{n}_\alpha + \mathfrak{n}_{2\alpha}$.

Let H_0 be the unique element of $\mathfrak{a}$ such that $\alpha(H_0) = 1$. Let $\langle \, , \, \rangle : \mathfrak{a} \times \mathfrak{a} \to \mathbf{R}$ be the inner product induced by B; it gives the identification $\mathfrak{a} \cong \mathfrak{a}^*$. The dual inner product of $\mathfrak{a}^*$ is also denoted by $\langle \, , \, \rangle$. Put $p = \dim_{\mathbf{R}} \mathfrak{n}_\alpha$, $q = \dim_{\mathbf{R}} \mathfrak{n}_{2\alpha}$, $\rho_0 = 2^{-1}(p + 2q)$, $c_0 = (2p + 8q)^{-1}$ and $m = 2^{-1} \dim(G/K)$. Then by the classification, we have the list:

$\mathfrak{g}$	p	q	ρ_0	m
$\mathfrak{su}(l,1)$	$2l-2$	1	l	l
$\mathfrak{so}(l,1)$	$l-1$	0	$2^{-1}(l-1)$	$2^{-1}l$
$\mathfrak{sp}(l,1)$	$4l-4$	1	$2l-1$	$2l-1$
$\mathfrak{f}_{4(-20)}$	8	7	11	8

(l is a natural number greater than one.)

LEMMA 2.2.1. *We have*

$$2m = p + q + 1, \quad \langle H_0, H_0 \rangle = c_0^{-1}, \quad \langle \alpha, \alpha \rangle = c_0.$$

From now on we assume that $m \in \mathbf{N}$, $m \geqslant 2$. In other words, we exclude the case of $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbf{R})$ or $\mathfrak{so}(2l+1,1)$ with $l \geqslant 1$.

2.3. Haar measures. Let dk be the Haar measure of the compact group K with total mass one. Let dt be the standard Lebesgue measure of $\mathbf{R}$; by the identification $\mathbf{R} \cong A = \exp \mathfrak{a}$, $t \mapsto \exp(tH_0)$, it gives the Haar measure of the torus A. Denote by $C_c^0(N)$ the space of compactly supported continuous functions on N. Since $N = \exp(\mathfrak{n}_\alpha + \mathfrak{n}_{2\alpha})$ is a unipotent Lie group we can take its Haar measure dn such that the formula

$$\int_N f(n)dn = \int_{\mathfrak{n}_\alpha} \int_{\mathfrak{n}_{2\alpha}} f(\exp(X+Y)) \, dXdY, \quad f \in C_c^0(N)$$

holds with dX (resp. dY) the Euclidian measure of $\mathfrak{n}_\alpha$ (resp. $\mathfrak{n}_{2\alpha}$). (We regard $\mathfrak{n}_{j\alpha}$ as a Euclidean space by the inner product $-B(Z,\theta Z)$.)

Then we fix the Haar measure dg of $G = ANK$ by $dg = da \cdot dn \cdot dk$. To handle various bi-$K$-invariant functions (distributions) on G, the Cartan decomposition $G = K\exp([0,\infty)H_0)K$ is indispensable. We put

$$G^+ = G - K = KA^+K \text{ with } A^+ = \{\exp(tH_0)|\ t > 0\}.$$

If $g \in G^+$, and $g = k_1(g)a(g)k_2(g)$, with $k_1(g), k_2(g) \in K$ and $a(g) \in A^+$, then $a(g)$ is uniquely determined by g. We choose the Riemannian metric dx on $X = G/K$, induced by the restriction $B|_{\mathfrak{p}}$ of B to $\mathfrak{p}$. We then have that the hyperbolic distance $d(xK, yK) = B(tH_0, tH_0)^{1/2} = t$ if $x, y \in G$ and $a(x^{-1}y) = \exp(tH_0)$, with $t > 0$.

The measure dg on G is decomposed along the Cartan decomposition as follows.

LEMMA 2.3.1. *For any positive measurable function φ on G, the formula*

$$\int_G \varphi(g)dg = c_G \int_K \int_0^\infty \int_K \varphi(k_1 \exp(tH_0)k_2)\mu(t)dk_1 dt dk_2 \qquad (2.1)$$

holds. Here

$$\mu(t) = (\sinh t)^{p+q}(\cosh t)^q,$$

$$c_G = 2\Gamma(m)^{-1}(2^{-1}c_0)^{-m+1/2}\pi^m.$$

Proof. Define the function f on G by

$$f(k_1 \exp(tH_0)k_2) = (\cosh t)^{2s} \qquad (k_1, k_2 \in K, \ t \in \mathbf{R}).$$

Next integrate f in the both hand sides of (2.1). $\qquad \square$

3. Spherical functions. In the first subsection, after recalling the standard properties of zonal spherical functions for G/K, we introduce a bi-K-invariant function $\phi_s^{(2)}$ on G^+ with singularities along K, which is called the secondary spherical function by T. Oda ([18]). We investigate its properties in some detail to show that its r-times derivative with respect to s^2 gives the function $Q_{r,s}$ of Miatello-Wallach ([17]).

3.1. The spherical function with singularities. For $s \in \mathbf{C}$, the zonal spherical function ϕ_s for G/K is defined by the integral

$$\phi_s(g) = \int_K e^{(s+\rho_0)\alpha(H(kg))} dk, \quad g \in G.$$

Here for $g \in G$, $H(g)$ denotes the unique vector in $\mathfrak{a}$ such that $g \in N \exp(H(g))K$. The basic property of ϕ_s is listed below.

  (a) It is bi-K-invariant C^∞-function on G, i.e., $\phi_s \in C^\infty(K\backslash G/K)$.
  (b) It satisfies the differential equation

$$\Omega\phi_s(g) = (s^2 - \rho_0^2)\phi_s(g), \quad g \in G$$

  with Ω the Casimir element of G corresponding to $c_0 B$.
  (c) If $\mathrm{Re}(s) > 0$, then

$$\lim_{t\to+\infty} e^{t(\rho_0-s)}\phi_s(\exp(tH_0)) = c(s)$$

  with $c(s)$ the c-function for G/K given by

$$c(s) = 2^{\rho_0-s}\Gamma(m)\Gamma(s)\Gamma\left(\frac{s+\rho_0}{2}\right)^{-1}\Gamma\left(\frac{s-\rho_0+2m}{2}\right)^{-1}.$$

Put $u_s^{(1)}(t) = \phi_s(\exp(tH_0))$, $t \in \mathbf{R}$. Then by (a), $u_s^{(1)}(t)$ is a C^∞-function on $\mathbf{R}$ which determines ϕ_s uniquely, and by (b) it satisfies the ordinary second order differential equation

$$(D)_s: \quad \frac{d^2u}{dt^2} + \left(\frac{p}{\tanh t} + \frac{q}{\tanh(2t)}\right)\frac{du}{dt} + (\rho_0^2 - s^2)u = 0$$

which has the regular singularity at $t = 0$ with characteristic exponents $\{0, 2 - 2m\}$. Change the variable by $z = \tanh^2 t$ and consider the function $w(z) = (\cosh t)^{\rho_0-s}u_s^{(1)}(t)$. Then it turns out that w is a solution of the Gaussian hypergeometric differential equation $z(1-z)\frac{d^2w}{dz^2} + \{c - (1+a+b)z\}\frac{dw}{dz} - abw = 0$ with $a = 2^{-1}(-s+\rho_0)$, $b = 2^{-1}(-s+\rho_0-q+1)$ and $c = m$. Thus we have

$$u_s^{(1)}(t) = (\cosh t)^{s-\rho_0} {}_2F_1\left(\frac{-s+\rho_0}{2}, \frac{-s+\rho_0-q+1}{2} \; ; m \; ; \tanh^2 t\right), \quad t \in \mathbf{R}.$$

We are interested in another class of solutions of $(D)_s$ which admit a singularity at $t = 0$. Among them, the one with the fastest decay at infinity, which we now define, is of particular importance: For $s \in \mathbf{C} - \{-1, -2, -3, \ldots\}$, put

$$u_s^{(2)}(t) = \gamma(s)(\cosh t)^{-(s+\rho_0)} {}_2F_1\left(\frac{s+\rho_0}{2}, \frac{s-\rho_0+2m}{2} \; ; s+1 \; ; \frac{1}{\cosh^2 t}\right), t \in \mathbf{R} - \{0\},$$

$$\gamma(s) = \Gamma\left(\frac{s+\rho_0}{2}\right)\Gamma\left(\frac{s-\rho_0+2m}{2}\right)\Gamma(s+1)^{-1}\Gamma(m-1)^{-1} = 2^{-s+\rho_0}m(sc(s))^{-1}.$$

PROPOSITION 3.1.1.

(i) *If $s \in \mathbf{C}$ is not a pole of $\gamma(s)$ and $\gamma(s) \neq 0$, then the family $\{u_s^{(1)}, u_s^{(2)}\}$ gives a system of fundamental solutions of $(D)_s$ around $t = 0$.*

(ii) *There exists a unique family of functions $\{\phi_s^{(2)} | \operatorname{Re}(s) \geqslant 0\}$ in $C^\infty(K \backslash G^+ / K)$ such that*
   (a)

$$\Omega \phi_s^{(2)}(g) = (s^2 - \rho_0^2) \phi_s^{(2)}(g), \quad g \in G^+.$$

   (b)

$$\phi_s^{(2)}(\exp(tH_0)) = O(e^{-t(\operatorname{Re}(s)+\rho_0)}), \quad (t \to +\infty),$$

   (c)

$$\lim_{t \to +0} t^{2m-2} \phi_s^{(2)}(\exp(tH_0)) = 1.$$

For a given $g \in G^+$ the function $s \mapsto \phi_s^{(2)}(g)$ is holomorphic on $\operatorname{Re}(s) \geqslant 0$. We have $\phi_s^{(2)}(\exp(tH_0)) = u_s^{(2)}(t)$ for $t \in \mathbf{R} - \{0\}$.

Proof. (i) Let us consider the Wronskian of the $\{u_s^{(1)}, u_s^{(2)}\}$, put

$$W[u_s^{(1)}, u_s^{(2)}](t) = u_s^{(1)}(t) \cdot \frac{d}{dt} u_s^{(2)}(t) - \frac{d}{dt} u_s^{(1)}(t) \cdot u_s^{(2)}(t). \tag{3.1}$$

It is easy to verify that $\lim_{t \to +0} u_s^{(1)}(t) = 1$, $\lim_{t \to +0} \frac{d}{dt} u_s^{(1)}(t) = 0$, $\lim_{t \to +0} t^{2m-2} u_s^{(2)}(t) = 1$ and $\lim_{t \to +0} t^{2m-1} \frac{d}{dt} u_s^{(2)}(t) = 2 - 2m$. So we have

$$\lim_{t \to +0} t^{2m-1} W[u_s^{(1)}, u_s^{(2)}](t)$$

$$= \lim_{t \to +0} \left\{ u_s^{(1)}(t) \cdot t^{2m-1} \frac{d}{dt} u_s^{(2)}(t) - t \frac{d}{dt} u_s^{(1)}(t) \cdot t^{2m-2} u_s^{(2)}(t) \right\}$$

$$= 2 - 2m \neq 0.$$

Hence there exists a t_0, $0 < t_0 < \epsilon$ with a small $\epsilon > 0$, such that $W[u_s^{(1)}, u_s^{(2)}](t_0) \neq 0$. This means that the Wronskian $W[u_s^{(1)}, u_s^{(2)}](t)$ does not vanish identically and we have the assertion .

(ii) By the condition (a) and the fact (i), $\phi_s^{(2)}$ can be written as a linear combination of $u_s^{(1)}$ and $u_s^{(2)}$ as

$$\phi_s^{(2)}(\exp(tH_0)) = \alpha_1 \cdot u_s^{(1)}(t) + \alpha_2 \cdot u_s^{(2)}(t) \quad (t > 0).$$

Since $\lim_{t \to +0} t^{2m-2} u_s^{(1)}(t) = 0$ and $\lim_{t \to +0} t^{2m-2} u_s^{(2)}(t) = 1$, the condition (c) implies that $\alpha_2 = 1$. Let us check $\alpha_1 = 0$. Firstly we assume that $\operatorname{Re}(s) > 0$. By the explicit formula, we have $\lim_{t \to +\infty} e^{(\rho_0-s)t} u_s^{(1)}(t) = c(s) \neq 0$ and $\lim_{t \to +\infty} e^{(\rho_0-s)t} u_s^{(2)}(t) = 0$. So we obtain $\alpha_1 = 0$ by the condition (b). Secondly we assume that $\operatorname{Re}(s) = 0$. Then we have $u_s^{(1)}(t) = O(\log(\cosh t) e^{-\rho_0 t})$ and $u_s^{(2)}(t) = O(e^{-\rho_0 t})$ when t tends to $+\infty$. Thus we obtain $\alpha_1 = 0$ by (b). $\qquad \square$

PROPOSITION 3.1.2. *Put*

$$c(s)$$

$$= \frac{(-1)^m \pi}{\Gamma(m)\Gamma(m-1)} \cdot \prod_{j=1}^{m-1} \left\{ \left(\frac{s}{2}\right)^2 - \left(\frac{\rho_0}{2} - j\right)^2 \right\} \cdot \left\{ \cot\left(\frac{s-\rho_0}{2}\pi\right) + \cot\left(\frac{s+\rho_0}{2}\pi\right) \right\}.$$

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Then for $s \in \mathbf{C}$ with $|\mathrm{Re}(s)| < 1$, we have

$$\phi_{-s}^{(2)}(g) = \phi_s^{(2)}(g) + \mathsf{c}(s)\phi_s(g), \quad g \in G^+. \tag{3.2}$$

Proof. The function $u_{-s}^{(2)}(t)$ on $t > 0$ is a solution of $(D)_s$. Hence by Proposition 3.1.1, it can be written as a linear combination of $u_s^{(1)}$ and $u_s^{(2)}$ as

$$u_{-s}^{(2)}(t) = c_1 \cdot u_s^{(1)}(t) + c_2 \cdot u_s^{(2)}(t) \quad (t > 0)$$

Since $\lim_{t \to +0} t^{2m-2} u_s^{(1)}(t) = 0$ and $\lim_{t \to +0} t^{2m-2} u_s^{(2)}(t) = \lim_{t \to +0} t^{2m-2} u_{-s}^{(2)}(t) = 1$, we have $c_2 = 1$.

Let us determine c_1. First we assume that $0 < \mathrm{Re}(s) < 1$. By the explicit formula, we have $\lim_{t \to +\infty} e^{(\rho_0-s)t} u_s^{(2)}(t) = 0$ and $\lim_{t \to +\infty} e^{(\rho_0-s)t} u_{-s}^{(2)}(t) = \gamma(-s) \cdot 2^{\rho_0-s}$. So we obtain

$$2^{\rho_0-s}\gamma(-s) = c_1 \cdot 2^{\rho_0-s}\Gamma(s)\Gamma(m)\Gamma\left(\frac{s+\rho_0}{2}\right)^{-1}\Gamma\left(\frac{s-\rho_0+2m}{2}\right)^{-1}.$$

Thus

$$c_1 = \left(\Gamma(m)\Gamma(m-1)\Gamma(s)\Gamma(1-s)\right)^{-1}$$
$$\times\Gamma\left(\frac{-s+\rho_0}{2}\right)\Gamma\left(\frac{s+\rho_0}{2}\right)\Gamma\left(\frac{-s-\rho_0+2m}{2}\right)\Gamma\left(\frac{s-\rho_0+2m}{2}\right). \tag{3.3}$$

Note that

$$\Gamma(s)\Gamma(1-s) = \pi\sin(\pi s)^{-1},$$
$$\Gamma\left(\frac{s-\rho_0+2m}{2}\right) = \Gamma\left(m - \frac{-s+\rho_0}{2}\right)$$
$$= \prod_{j=1}^{m-1}\left(m - j - \frac{-s+\rho_0}{2}\right)\cdot\Gamma\left(1 - \frac{-s+\rho_0}{2}\right), \tag{3.4}$$

$$\Gamma\left(\frac{s+\rho_0}{2}\right) = \Gamma\left(m - \frac{-s-\rho_0+2m}{2}\right)$$
$$= \prod_{j=1}^{m-1}\left(m - j - \frac{-s-\rho_0+2m}{2}\right)\cdot\Gamma\left(1 - \frac{-s-\rho_0+2m}{2}\right). \tag{3.5}$$

Then we have

$$c_1 = \frac{1}{\Gamma(m)\Gamma(m-1)}\frac{\sin(\pi s)}{\pi}$$
$$\times\prod_{j=1}^{m-1}\left\{\left(m - j - \frac{-s+\rho_0}{2}\right)\left(m - j - \frac{-s-\rho_0+2m}{2}\right)\right\}$$
$$\times\Gamma\left(\frac{-s+\rho_0}{2}\right)\Gamma\left(1 - \frac{-s+\rho_0}{2}\right)\Gamma\left(\frac{-s-\rho_0+2m}{2}\right)\Gamma\left(1 - \frac{-s-\rho_0+2m}{2}\right)$$
$$= \frac{1}{\Gamma(m)\Gamma(m-1)}\frac{\sin(\pi s)}{\pi}\prod_{j=1}^{m-1}\left\{\left(\frac{s}{2}\right)^2 - \left(\frac{\rho_0}{2} - j\right)^2\right\}$$
$$\times\frac{\pi}{\sin(\frac{-s+\rho_0}{2}\pi)}\frac{\pi}{\sin(\frac{-s-\rho_0+2m}{2}\pi)}.$$

Since

$$\frac{\sin(\pi s)}{\sin(\frac{-s+\rho_0}{2}\pi)\sin(\frac{-s-\rho_0+2m}{2}\pi)} = (-1)^{m-1}\left\{\cot\left(\frac{-s+\rho_0}{2}\pi\right) + \cot\left(\frac{-s-\rho_0+2m}{2}\pi\right)\right\},$$

the formula (3.2) is now established for $0 < \mathrm{Re}(s) < 1$. By the analytic continuation, the formula remains valid on $|\mathrm{Re}(s)| < 1$. $\square$

The 'bad 'behavior of the function $\phi_s^{(2)}(\exp(tH_0))$ near $t = 0$ is controlled by a simple function. Indeed, we have

PROPOSITION 3.1.3.   *There exists a function $(s,t) \mapsto \mathsf{Y}_s(t)$ on $\mathbf{C} \times (\mathbf{R} - \{0\})$ with the following properties*

(a) We can write

$$\mathsf{Y}_s(t) = \sum_{j=0}^{m-1} \frac{\mathsf{a}_j(s)}{(\sinh t)^{2j}} + \mathsf{b}(s)\log(\sinh^2 t)$$

$$+ \frac{\mathsf{b}(s)}{\Gamma(m)\Gamma(m-1)}\left\{\psi\left(\frac{s+\rho_0}{2}\right) + \psi\left(\frac{s-\rho_0}{2}+1\right)\right\}$$

with polynomial functions $\mathsf{a}_j(s)$ and $\mathsf{b}(s)$ such that

$$\mathsf{a}_j(-s) = \mathsf{a}_j(s), \quad \deg(\mathsf{a}_j(s)) \leqslant 2(m-j-1), \quad j = 0, \dots, m-1,$$
$$\mathsf{a}_{m-1}(s) = 1,$$
$$\mathsf{b}(s) = (-1)^m \prod_{j=1}^{m-1}\left\{\left(\frac{s}{2}\right)^2 - \left(\frac{\rho_0}{2}-j\right)^2\right\}.$$

Here $\psi(s)$ is the digamma function, i.e., the logarithmic derivative of the Gamma function.

(b) There exists a family of polynomial functions $\{\mathsf{c}_n(s)\}_{n\geqslant 1}$ and $\{\mathsf{d}_n(s)\}_{n\geqslant 1}$ such that $\sum_{n=1}^{\infty}\mathsf{c}_n(s)t^n$ and $\sum_{n=1}^{\infty}\mathsf{d}_n(s)t^n$ have positive radius of convergence and such that

$$\phi_s^{(2)}(\exp(tH_0)) = \mathsf{Y}_s(t) + \sum_{n=1}^{\infty}\mathsf{c}_n(s)t^n + \log(t)\sum_{n=1}^{\infty}\mathsf{d}_n(s)t^n$$

on $0 < t < \epsilon$ with a small $\epsilon > 0$.

Proof. Put $u = \sinh^2 t$, $a = 2^{-1}(s-\rho_0)+1$ and $b = 2^{-1}(s+\rho_0-2m)+1$. Then

$$u_s^{(2)}(t) = \gamma(s)(1+u)^{(2m-\rho_0-s)/2-1}\,{}_2F_1\left(a, b\,;a+b+m-1\,;\frac{1}{1+u}\right).$$

By the formula [16, (9.7.5), p.257], we have

$$
\begin{aligned}
u_s^{(2)}(t) = \gamma(s) \cdot (1+u)^{\frac{2m-\rho_0-s}{2}-1} u^{1-m} \\
\times \Bigg\{ \frac{\Gamma(a+b+m-1)}{\Gamma(a+m-1)\Gamma(b+m-1)} \sum_{k=0}^{m-2} \frac{(-1)^k(m-k-2)!}{k!}(a)_k(b)_k \Big(\frac{u}{1+u}\Big)^k \\
+ \frac{(-1)^{m-1}\Gamma(a+b+m-1)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a+m-1)_k(b+m-1)_k}{(m+k-1)!k!} \\
\times \Big(\psi(k+1) + \psi(m+k) - \psi(a+m+k-1) - \psi(b+m+k-1) \\
- \log\Big(\frac{u}{1+u}\Big) \Big) \cdot \Big(\frac{u}{1+u}\Big)^{k+m-1} \Bigg\}
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
= \sum_{k=0}^{m-2} \frac{(-1)^k(m-k-2)!}{(m-2)!k!}(a)_k(b)_k u^{k-m+1}(1+u)^{\frac{2m-\rho_0-s}{2}-1-k} \\
+ (-1)^{m-1} \sum_{k=0}^{\infty} \frac{(a+m-1)_k(b+m-1)_k(a)_{m-1}(b)_{m-1}}{k!(m-2)!(m+k-1)!} \\
\times \Big\{ \psi(k+1) + \psi(m+k) - \psi(a+m+k-1) - \psi(b+m+k-1) \\
- \log\Big(\frac{u}{1+u}\Big) \Big\} u^k (1+u)^{-\frac{\rho_0+s}{2}-k} \qquad (0 < |u| < \epsilon)
\end{aligned}
\tag{3.7}
$$

with a small ϵ.

Note that $\gamma(s) = \Gamma(a+m-1)\Gamma(b+m-1)\Gamma(a+b+m-1)^{-1}\Gamma(m-1)^{-1}$. Now put

$$
A_k(s) = \frac{(-1)^k(m-k-2)!}{k!(m-2)!}(a)_k(b)_k \qquad (k = 0, \ldots, m-2),
\tag{3.8}
$$

$$
B_k(s) = \frac{(a+m-1)_k(b+m-1)_k(a)_{m-1}(b)_{m-1}}{k!(m-2)!(m+k-1)!} \qquad (k \in \mathbf{N}_0),
\tag{3.9}
$$

$$
\begin{aligned}
\gamma_k = \psi(k+1) + \psi(m+k) \\
- \psi(a+m+k-1) - \psi(b+m+k-1) \qquad (k \in \mathbf{N}_0)
\end{aligned}
\tag{3.10}
$$

and

$$
(1+u)^{-b-k} = \sum_{n=0}^{\infty} c_{n,k}^+(s)\, u^n \quad (|u| < 1),
$$

$$
(1+u)^{-\frac{s+\rho_0}{2}-k} = \sum_{n=0}^{\infty} c_{n,k}^-(s)\, u^n \quad (|u| < 1),
$$

with $\quad c_{n,k}^+(s) = \dfrac{(-1)^n}{n!}\Big(\dfrac{s+\rho_0}{2} - m + k + 1\Big)_n, \qquad c_{n,k}^-(s) = \dfrac{(-1)^n}{n!}\Big(\dfrac{s+\rho_0}{2} + k\Big)_n.$

Then

$$u_s^{(2)}(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{m-2} A_k(s)\, c_{n,k}^+(s)\, u^{k+n-m+1}$$

$$+ (-1)^{m-1} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} B_k(s)\, c_{n,k}^-(s) \left\{ \gamma_k - \log\left(\frac{u}{1+u}\right) \right\} u^{k+n} \qquad (3.11)$$

$$= \sum_{k=1}^{m-1} \frac{\mathsf{a}_k(s)}{u^k} + \mathsf{b}(s)\log(u) + \tilde{\mathsf{a}}_0(s)$$

$$+ \left\{ \sum_{k=1}^{\infty} p_k(s)\, u^k + \log(u) \sum_{k=1}^{\infty} q_k(s) u^k \right\}. \qquad (3.12)$$

with $\mathsf{b}(s) = (-1)^m B_0(s)$,

$$\mathsf{a}_j(s) = \sum_{\substack{n \geqslant 0,\, 0 \leqslant k < m-1 \\ k+n-m+1=-j}} A_k(s)\, c_{n,k}^+(s) \qquad (j = 1,\dots,m-1), \qquad (3.13)$$

$$\tilde{\mathsf{a}}_0(s) = \sum_{k=0}^{m-2} A_k(s)\, c_{m-1-k,k}^+(s) + (-1)^{m-1} B_0(s)\, \gamma_0, \qquad (3.14)$$

and $p_k(s)$, $q_k(s)$ are some polynomials. By the foumulae of A_k and $c_{n,k}^+$, we see that $A_k(s)\, c_{n,k}^+(s)$ with $n \geqslant 0$, $0 \leqslant k < m-1$, $k+n-m+1 = -j$ is a polynomial of s such that $\deg(A_k(s)\, c_{n,k}^+(s)) = 2k+n = k-j+m-1$. Hence $\mathsf{a}_j(s)$ is also a polynomial with

$$\deg(\mathsf{a}_j(s)) \leqslant \max_{\substack{n \geqslant 0,\, 0 \leqslant k < m-1 \\ k+n-m+1=-j}} \left\{ k-j+m-1 \right\} = 2m-2j-2 = 2(m-j-1).$$

We have

$$B_0(s) = \frac{(a)_{m-1}(b)_{m-1}}{(m-2)!\,(m-1)!}$$

$$= \frac{1}{\Gamma(m-1)\Gamma(m)} \left(\frac{s-\rho_0}{2} + 1 \right)_{m-1} \left(\frac{s+\rho_0-2m}{2} + 1 \right)_{m-1}$$

$$= \frac{1}{\Gamma(m-1)\Gamma(m)} \prod_{j=0}^{m-2} \left(\frac{s-\rho_0}{2} + j + 1 \right) \left(\frac{s+\rho_0-2m}{2} + j + 1 \right)$$

$$= \frac{1}{\Gamma(m-1)\Gamma(m)} \prod_{j=1}^{m-1} \left(\frac{s-\rho_0}{2} + j \right) \left(\frac{s+\rho_0}{2} - m + j \right)$$

$$= \frac{1}{\Gamma(m-1)\Gamma(m)} \prod_{j=1}^{m-1} \left\{ \left(\frac{s}{2} \right)^2 - \left(\frac{\rho_0}{2} - j \right)^2 \right\}, \qquad (3.15)$$

$$\tilde{\mathsf{a}_0}(s) = \frac{(-1)^m}{\Gamma(m-1)\Gamma(m)} \prod_{j=1}^{m-1} \left\{ \left(\frac{s}{2}\right)^2 - \left(\frac{\rho_0}{2} - j\right)^2 \right\}$$

$$\times \left\{ \psi\left(\frac{s-\rho_0}{2} + m\right) + \psi\left(\frac{s+\rho_0}{2}\right) - \psi(1) - \psi(m) \right\}$$

$$+ \sum_{k=0}^{m-2} A_k(s)\, c_{m-1-k,k}^+(s). \tag{3.16}$$

Using the formula

$$\psi(y+m) = \psi(y+1) + \sum_{j=1}^{m-1} \frac{1}{y+m-j} \qquad (m \in \mathbf{N}_0),$$

we have

$$\tilde{\mathsf{a}_0}(s) = \frac{(-1)^m}{\Gamma(m-1)\Gamma(m)} \prod_{j=1}^{m-1} \left\{ \left(\frac{s}{2}\right)^2 - \left(\frac{\rho_0}{2} - j\right)^2 \right\} \cdot \left\{ \psi\left(\frac{s-\rho_0}{2} + 1\right) + \psi\left(\frac{s+\rho_0}{2}\right) \right\}$$

$$+ \mathsf{a}_0(s) \tag{3.17}$$

with $\mathsf{a}_0(s) = \displaystyle\prod_{j=1}^{m-1} \left\{ \left(\frac{s}{2}\right)^2 - \left(\frac{\rho_0}{2} - j\right)^2 \right\} \cdot \left(\sum_{j=1}^{m-1} \frac{2}{s - \rho_0 + 2m - 2j} - \psi(1) - \psi(m) \right)$

$$+ \sum_{k=0}^{m-2} A_k(s)\, c_{m-1-k,k}^+(s). \tag{3.18}$$

By the formulae $A_k(s)$ and $c_{n,k}^+(s)$, we have that $\mathsf{a}_0(s)$ is a polynomial function with degree no more than $2(m-1)$. Put

$$\mathsf{Y}_s(t) := \sum_{j=0}^{m-1} \frac{\mathsf{a}_j(s)}{(\sinh t)^{2j}} + \mathsf{b}(s) \log(\sinh^2 t)$$

$$+ \frac{\mathsf{b}(s)}{\Gamma(m)\Gamma(m-1)} \left\{ \psi\left(\frac{s+\rho_0}{2}\right) + \psi\left(\frac{s-\rho_0}{2} + 1\right) \right\}. \tag{3.19}$$

Then all the assertions in Proposition 3.1.3, except that $\mathsf{a}_j(s)$'s are even functions are established. To have $\mathsf{a}_j(s) = \mathsf{a}_j(-s)$, we use the functional equation in Proposition 3.1.2. From that equation

$$\mathsf{a}_j(-s) = \mathsf{a}_j(s) \qquad (j = 1, \ldots, m-1), \tag{3.20}$$

$$\tilde{\mathsf{a}_0}(-s) = \tilde{\mathsf{a}_0}(s) + \mathsf{c}(s). \tag{3.21}$$

From (3.17) and (3.21), we have

$$\mathsf{a}_0(-s) - \mathsf{a}_0(s)$$

$$= \mathsf{c}(s) + \frac{(-1)^m}{\Gamma(m-1)\Gamma(m)} \prod_{j=1}^{m-1} \left\{ \left(\frac{s}{2}\right)^2 - \left(\frac{\rho_0}{2} - j\right)^2 \right\}$$

$$\times \left\{ \psi\left(\frac{s-\rho_0}{2} + 1\right) + \psi\left(\frac{s+\rho_0}{2}\right) - \psi\left(\frac{-s-\rho_0}{2} + 1\right) - \psi\left(\frac{-s+\rho_0}{2}\right) \right\}$$

$$= \frac{(-1)^m}{\Gamma(m-1)\Gamma(m)} \prod_{j=1}^{m-1} \left\{ \left(\frac{s}{2}\right)^2 - \left(\frac{\rho_0}{2} - j\right)^2 \right\} \cdot \Delta(s) \tag{3.22}$$

with

$$\triangle(s) = \pi \cot\left(\frac{s - \rho_0}{2}\pi\right) + \pi \cot\left(\frac{s + \rho_0 - 2m}{2}\pi\right) + \psi\left(\frac{s - \rho_0}{2} + 1\right)$$
$$+ \psi\left(\frac{s + \rho_0}{2}\right) - \psi\left(\frac{-s - \rho_0}{2} + 1\right) - \psi\left(\frac{-s + \rho_0}{2}\right).$$
$$(3.23)$$

We claim $\triangle(s) \equiv 0$. Indeed, the formula $\psi(y) - \psi(1 - y) = -\pi \cot(\pi y)$ gives

$$-\psi\left(\frac{-s + \rho_0}{2}\right) + \psi\left(1 - \frac{-s + \rho_0}{2}\right) + \pi \cot\left(\frac{s - \rho_0}{2}\pi\right) = 0,$$
$$\psi\left(\frac{s + \rho_0}{2}\right) - \psi\left(1 - \frac{s + \rho_0}{2}\right) + \pi \cot\left(\frac{s + \rho_0}{2}\pi\right) = 0.$$

Since $\pi \cot(\pi y + m\pi) = \pi \cot(\pi y)$, we have $\triangle(s) = 0$. □

We introduce a family of functions $\phi_s^{[r]}$ ($\text{Re}(s) > -1$, $r \in \mathbf{N}_0$) as

DEFINITION 3.1.4. *For* $r \in \mathbf{N}_0$, *we put*

$$\phi_s^{[r]}(g) = \frac{c_G^{-1}}{(2 - 2m)r!}\left(-\frac{1}{2s}\frac{d}{ds}\right)^r \phi_s^{(2)}(g), \quad g \in G^+, \quad \text{Re}(s) > -1.$$

The basic property of $\phi_s^{[r]}$ we need is as follows.

PROPOSITION 3.1.5. *Let* $r \in \mathbf{N}_0$ *and* $s \in \mathbf{C}$ *with* $\text{Re}(s) > 0$.
(i) The function $\phi_s^{[r]}$ *belongs to* $C^\infty(K \backslash G^+ / K)$.
(ii) We have

$$\phi_s^{[r]}(\exp(tH_0)) = O(e^{-t(\text{Re}(s) + \rho_0)}),$$

on $t > R$ *with a large* $R > 0$.
(iii) If $r \geqslant m$, *then the function* $\phi_s^{[r]}$ *has a continuous extension to all of* G. *We have*

$$\lim_{\substack{g \to e, g \in G^+}} \phi_s^{[r]}(g)$$
$$= \frac{-c_G^{-1}}{2r!}\left(-\frac{1}{2s}\frac{d}{ds}\right)^r\left[\frac{\mathbf{b}(s)}{\Gamma(m)^2}\left\{\psi\left(\frac{s + \rho_0}{2}\right) + \psi\left(\frac{s - \rho_0}{2} + 1\right)\right\}\right].$$

(iv) Let $\text{Re}(s) > \rho_0$. *Then we have*

$$(\Omega - s^2 + \rho_0^2)\phi_s^{[r+1]} = -\phi_s^{[r]}, \quad r \in \mathbf{N}_0,$$
$$(\Omega - s^2 + \rho_0)\phi_s^{[0]} = \delta$$

in the sense of distributions on G/K *with* δ *the Dirac delta supported at the origin of* G/K.

Proof.
(i)   By the explicit formula of $\phi_s^{(2)}(\exp(tH_0)) = u_s^{(2)}(t)$, it follows that $\phi_s^{[r]}(\exp(tH_0)) = (-\frac{1}{2s}\frac{d}{ds})^r u_s^{(2)}(t)$ is C^∞ on $t > 0$ and $\phi_s^{[r]}(\exp(tH_0)) = \phi_s^{[r]}(\exp(-tH_0))$. From this, the assertion follows.
(ii)   Since $_2F_1(\frac{s+\rho_0}{2}, \frac{s-\rho_0}{2} + m\,; s + 1\,; z)$ is holomorphic with respect to (s, z) on the domain $\{(s, z) \,|\, \text{Re}(s) > 0, |z| < 1\}$, the function $z \mapsto (\frac{d}{dz})^j {}_2F_1(\frac{s+\rho_0}{2}, \frac{s-\rho_0}{2} +$

with $F(g) = \int_K f(xkg)\,dk \quad (g \in G)$. For $t > 0$, put $h_r(t) = \phi_s^{[r]}(\exp(tH_0))$ and $F_0(t) = F(\exp(tH_0))$. Then by integration by parts, the last formula of (3.26) becomes

$$\int_0^{+\infty} h_r(t) \cdot D_t F_0(t) \cdot \mu(t)\,dt$$

$$= \lim_{\zeta \to \infty} \lim_{\epsilon \to +0} \left\{ \mu(\zeta)h_r(\zeta)F_0'(\zeta) - \mu(\zeta)h_r'(\zeta)F_0(\zeta) - \mu(\epsilon)h_r(\epsilon)F_0'(\epsilon) + \mu(\epsilon)h_r'(\epsilon)F_0(\epsilon) \right\}$$

$$+ \int_0^{+\infty} D_t h_r(t) \cdot F_0(t) \cdot \mu(t)\,dt \qquad (3.27)$$

with $D_t = \mu(t)^{-1}\frac{d}{dt}(\mu(t)\frac{d}{dt}) + \rho_0^2 - s^2$. By Proposition 3.1.5 (i), we have $h_r(t) \prec e^{-(\mathrm{Re}(s)+\rho_0)\zeta}$ as $\zeta \to +\infty$ and $h_r'(t) \prec e^{-(\mathrm{Re}(s)+\rho_0)\zeta}$ as $\zeta \to +\infty$. By the formula of $\mu(t)$, we have, $\mu(\zeta) \prec e^{2\rho_0 s}$ as $\zeta \to +\infty$. Using the property of the norm $\|\cdot\|$, we have some constants T and $M_0 > 0$ such that

$$\|xk\exp(tH_0)\| \leqslant \|x\| \cdot \|\exp(tH_0)\| \leqslant M_0\|x\|e^t \quad (\forall t \geqslant T, \forall k \in K). \qquad (3.28)$$

Since $f \in C_\mu^\infty(G/K)$, we thus have

$$|F_0(\zeta)| \leqslant p_\mu(f) \cdot M_0\|x\|e^{\zeta\mu} \qquad (\forall \zeta \geqslant T),$$
$$|F_0'(\zeta)| \leqslant p_{H_0,\mu}(f) \cdot M_0\|x\|e^{\zeta\mu} \qquad (\forall \zeta \geqslant T). \qquad (3.29)$$

Therefore the first and the second terms in the bracket of the right-hand side of (3.27) is estimated as

$$|\mu(\zeta)h_r(\zeta)F_0'(\zeta)| \prec e^{(-\mathrm{Re}(s)+\rho_0+\mu)\zeta},$$
$$|\mu(\zeta)h_r'(\zeta)F_0(\zeta)| \prec e^{(-\mathrm{Re}(s)+\rho_0+\mu)\zeta}, \qquad (3.30)$$

so they tend to zero as $\zeta \to +\infty$. We evaluate the third and the fourth term in the right-hand side of (3.27). By the expression of $\phi_s^{[r]}(\exp(tH_0))$ around $t = 0$ given in Proposition 3.1.3, we have

$$|h_r(\epsilon)| \prec \epsilon^{-2(m-1)} \quad (\epsilon \to +0),$$
$$h_r'(\epsilon) = \frac{c_G^{-1}}{\epsilon^{2m-1}}(1 + O(\epsilon\log\epsilon)) \quad (\text{if } r = 0),$$
$$|h_r'(\epsilon)| \prec \epsilon^{-(2m-2r-1)} \quad (\text{if } r > 0).$$

Since $\mu(\epsilon) \prec \epsilon^{p+q} = \epsilon^{2m-1}$ and

$$F_0(\epsilon) = F_0(0) + O(\epsilon), \quad F_0'(\epsilon) \prec 1,$$

We have

$$|\mu(\epsilon)h_r(\epsilon)F_0'(\epsilon)| \prec \epsilon \quad (\epsilon \to +0),$$
$$|\mu(\epsilon)h_r'(\epsilon)F_0(\epsilon)| \prec \epsilon^{2r} \quad (\epsilon \to +0) \text{ if } r > 0,$$
$$\mu(\epsilon)h_r'(\epsilon)F_0(\epsilon) = c_G^{-1}(1 + O(\epsilon\log\epsilon))F_0(0) \quad (r = 0).$$

Therefore, the third and the fourth term of (3.27) are equal to zero if $r = 1$ and are equal to $c_G^{-1}F_0(0)$ if $r = 0$. Suppose $r = 0$. Since the last integral in the right-hand

side of (3.27) vanishes due to the differential equation $D_t h(t) = 0$ $(t > 0)$, we have

$$c_G^{-1} \int_G \phi_s^{[0]}(x^{-1}y)\big((\Omega - s^2 + \rho_0^2)f\big)(y)\, dy$$
$$= F_0(0)c_G^{-1} = f(x)c_G^{-1} \tag{3.31}$$

When $r > 1$, all the terms in the right-hand side of (3.27) except the last one vanish. Consequently we have

$$c_G^{-1} \int_G \phi_s^{[r]}(x^{-1}y)\big((\Omega - s^2 + \rho_0^2)f\big)(y)\, dy$$
$$= \int_0^\infty D_t h_r(t) \cdot F_0(t) \cdot \mu(t)\, dt. \tag{3.32}$$

To compute $D_t h_r(t)$, we need a formula:

CLAIM 3.1.6. *For $r \geqslant 1$, we have*

$$\left[D_t, \left(-\frac{1}{2s}\frac{d}{ds}\right)^r\right] = -r\left(-\frac{1}{2s}\frac{d}{ds}\right)^{r-1}.$$

Proof of the claim. Obviously it is sufficient to prove

$$\left[s^2, \left(-\frac{1}{2s}\frac{d}{ds}\right)^r\right] = r\left(-\frac{1}{2s}\frac{d}{ds}\right)^{r-1}.$$

If $r = 1$, then this is checked by a direct computation. Let $r > 1$, and assume that the formula for $r-1$ is valid. Then,

$$s^2\left(-\frac{1}{2s}\frac{d}{ds}\right)^r$$
$$= s^2\left(-\frac{1}{2s}\frac{d}{ds}\right)^{r-1} \cdot \left(-\frac{1}{2s}\frac{d}{ds}\right)$$
$$= \left\{\left(-\frac{1}{2s}\frac{d}{ds}\right)^{r-1} \cdot s^2 + (r-1)\left(-\frac{1}{2s}\frac{d}{ds}\right)^{r-2}\right\}\left(-\frac{1}{2s}\frac{d}{ds}\right)$$
$$= \left(-\frac{1}{2s}\frac{d}{ds}\right)^{r-1} \cdot \frac{-s}{2}\frac{d}{ds} + (r-1)\left(-\frac{1}{2s}\frac{d}{ds}\right)^{r-1}.$$

To have the second equality, we used the induction-assumption. Since $-\frac{s}{2}\frac{d}{ds} = 1 + (-\frac{1}{2s}\frac{d}{ds})s^2$, we obtain

$$s^2\left(-\frac{1}{2s}\frac{d}{ds}\right)^r$$
$$= \left(-\frac{1}{2s}\frac{d}{ds}\right)^{r-1}\left(1 - \frac{1}{2s}\frac{d}{ds} \cdot s^2\right) + (r-1)\left(-\frac{1}{2s}\frac{d}{ds}\right)^{r-1}$$
$$= r\left(-\frac{1}{2s}\frac{d}{ds}\right)^{r-1} + \left(-\frac{1}{2s}\frac{d}{ds}\right)^r \cdot s^2.$$

This completes the proof of the formula for r.   □

Let us finish the proof of the proposition. We apply the claim to the function $\phi_s^{(2)}(\exp(tH_0))$ in s. Then

$$
\begin{aligned}
& D_t h_r(t) \\
& = D_t \left(-\frac{1}{2s}\frac{d}{ds} \right)^r \frac{c_G^{-1}}{(2-2m)r!} \phi_s^{(2)}(\exp(tH_0)) \\
& = \left\{ -r\left(-\frac{1}{2s}\frac{d}{ds} \right)^{r-1} + \left(-\frac{1}{2s}\frac{d}{ds} \right)^r D_t \right\} \frac{c_G^{-1}}{(2-2m)r!} \phi_s^{(2)}(\exp(tH_0)) \\
& = -h_{r-1}(t) + \frac{c_G^{-1}}{(2-2m)r!} \left(-\frac{1}{2s}\frac{d}{ds} \right)^r D_t \phi_s^{(2)}(\exp(tH_0)). \qquad (3.33)
\end{aligned}
$$

Since $D_t \phi_s^{(2)}(\exp(tH_0)) = 0$ for $t > 0$, we have

$$
D_t h_r(t) = -h_{r-1}(t) \quad \text{for } t > 0. \qquad (3.34)
$$

By (3.32), (3.34) and (3.26), we consequently have

$$
\begin{aligned}
& c_G^{-1} \int_G \phi_s^{[r]}(x^{-1}y)\Big((\Omega - s^2 + \rho_0^2)f \Big)(y)\,dy \\
& = -c_G^{-1} \int_G \phi_s^{[r-1]}(x^{-1}y) \cdot f(y)\,dy \qquad (3.35)
\end{aligned}
$$

for $r \geqslant 1$. By (3.31) and (3.35), we have the desired formula. $\qquad \square$

We have a characterization of the family $\{\phi_s^{[r]}\}$.

PROPOSITION 3.1.7. *Let $\{\varphi_{r,s}|\ r \in \mathbf{N}_0,\ \mathrm{Re}(s) > \rho_0\}$ be a family of bi-K-invariant distributions on G with the following properties.*

  (i) *For $r \in \mathbf{N}_0$, $\mathrm{Re}(s) > \rho_0$ the distribution $\varphi_{r,s}$ is represented by a C^∞-function on G^+.*

  (ii) *For $\mathrm{Re}(s) > \rho_0$,*

$$
\lim_{t\to +0} t^{2m-2}\varphi_{0,s}(\exp(tH_0)) = 1.
$$

  (iii) *For $r \in \mathbf{N}_0$, $\mathrm{Re}(s) > \rho_0$,*

$$
\varphi_{r,s}(\exp(tH_0)) = O(e^{-t(\mathrm{Re}(s)+\rho_0)}), \quad t \to +\infty.
$$

  (iv) *Let $\mathrm{Re}(s) > \rho_0$. If we regard $\varphi_{r,s}$'s as distributions on G/K, they satisfy the differential equations*

$$
\begin{aligned}
(\Omega - s^2 + \rho_0^2)\varphi_{r+1,s} &= -\varphi_{r,s}, \quad r \in \mathbf{N}_0, \\
(\Omega - s^2 + \rho_0^2)\varphi_{0,s} &= \delta.
\end{aligned}
$$

Then for $r \in \mathbf{N}_0$ and $s \in \mathbf{C}$, $\mathrm{Re}(s) > \rho_0$ we have $\varphi_{r,s}(g) = \phi_s^{[r]}(g)$ on G/K in the sense of distributions.

Proof. The difference $\varphi_{0,s} - \phi_s^{[0]}$, considered to be a distribution on G/K, is annihilated by the elliptic differential operator $\Omega - s^2 + \rho_0^2$ on G/K (see Proposition 3.1.5 (iv)). Hence by the elliptic regularity theorem $\varphi_{0,s} - \phi_s^{[0]}$ is smooth on all of G. By Proposition 3.1.1 (ii) the two distributions $\varphi_{0,s}$ and $\phi_s^{[0]}$ coincide with each other on

the open dense set G^+. Therefore $\varphi_{0,s} = \phi_s^{[0]}$ on all of G. Let $r > 0$. We use induction on r to show $\varphi_{r,s} = \phi_s^{[r]}$. Assume $\varphi_{r-1,s} = \phi_s^{[r-1]}$. Then from the property (iv) of $\varphi_{r,s}$ and Proposition 3.1.4 (iv), the distribution $\varphi_{r,s} - \phi_s^{[r]}$ (on G/K) is annihilated by the elliptic differential operator $\Omega - s^2 + \rho_0^2$; hence it becomes smooth on all of G and is actually a constant (say C_0) multiple of the zonal spherical function ϕ_s. But the property (iii) and Proposition 3.1.5 (ii) give that $(\varphi_{r,s} - \phi_s^{[r]})(\exp(tH_0))$ is of exponential decay as $t \to +\infty$. Since $\phi_s(\exp(tH_0))$ with $\text{Re}(s) > \rho_0$ grows exponentially as $t \to +\infty$, the proportionality constant C_0 should be zero. This completes the proof. $\square$

3.2. Miatello-Wallach's spherical functions. We recall some basic properties of the functions $Q_{r,s}$, $r \in \mathbf{N}$ which Miatello-Wallach introduced and studied in detail ([17]).

   (i) For $s \in \mathbf{C}$, $\text{Re}(s) > 0$, $Q_{1,s} \in C^\infty(K \backslash G^+ / K)$ ([17, Theorem 1.1 (a)]) .

   (ii) For a fixed $g \in G^+$, the function $s \mapsto Q_{1,s}(g)$ is holomorphic on $\text{Re}(s) > 0$ and has a meromorphic continuation to $\mathbf{C}$ ([17, Theorem 1.1,(b)]).

   (iii)
   $$Q_{1,s}(\exp(tH_0)) \sim \frac{c_G^{-1} sc(s)}{m-1} \cdot t^{2-2m}, \quad t \to +0$$

   ([17, Theorem 1.1, (d)]).

   (iv) Let $\text{Re}(s) > \rho_0$ and $r \in \mathbf{N}$. Then $Q_{r,s}$ is bi-K-invariant and integrable function on G satisfying the formula
   $$Q_{r+1,s} = Q_{1,s} \star Q_{r,s}.$$

   Here $\star$ means the convolution on G with respect to the measure dg. (see [17, page 678]).

   (v) Let $\text{Re}(s) > \rho_0$ and $r \in \mathbf{N}$. Then
   $$Q_{r,s}(\exp(tH_0)) = O(e^{-t(\text{Re}(s)+\rho_0)}), \quad t \to +\infty$$

   ([17, Lemma 2.4]).

   (vi) Let $\text{Re}(s) > \rho_0$. Then the distributions $Q_{r,s}$ on G/K satisfy the differential equations
   $$(\Omega - s^2 + \rho_0^2)Q_{r+1,s} = -2sc(s)Q_{r,s}$$

   for $r \in \mathbf{N}_0$ with the convention that $Q_{0,s} = \delta$, the Dirac delta supported at the origin of G/K ([17, Lemma 2.2, Lemma 2.6]).

Thus the family $\{-(2sc(s))^{-(r+1)}Q_{r+1,s} | \ r \in \mathbf{N}_0, \ \text{Re}(s) > \rho_0\}$ possesses all the properties (i) to (iv) in Proposition 3.1.7. Hence applying that proposition, we have the following theorem, which is one of the main results of this article.

THEOREM 3.2.1. *Let $\text{Re}(s) > \rho_0$ and $r \in \mathbf{N}_0$. Then as distributions on G/K the equality*
$$\phi_s^{[r]}(g) = -\left(\frac{1}{2sc(s)}\right)^{r+1} Q_{r+1,s}(g)$$

holds.

4. Miatello-Wallach's function $\mathbf{P}_{r,s}$ and its spectral expansion.

4.1. The function $\mathbf{P}_{r,s}$. Let $X = G/K$. Let Γ be a neat co-finite lattice of G, that is a discrete torsion-free subgroup of G such that $\Gamma\backslash G$ has finite volume. We assume that if Γ is not cocompact then it satisfies the Langlands' axiom. Here is a notational convention: A point of the double coset space $\Gamma\backslash X$ is denoted by a letter with a dot and any one of the lifts of that point to G is by the same letter without a dot. For example if $x \in G$ then the corresponding coset $\Gamma x K \in \Gamma\backslash X$ is $\dot{x}$.

Let $\triangle$ be the Laplacian of $\Gamma\backslash X$ corresponding to $-\Omega$.

In [17], Miatello-Wallach introduced the functions $\mathbf{P}_{r,s}$ ($r \in \mathbf{N}_0$, $\mathrm{Re}(s) > \rho_0$) by

$$\mathbf{P}_{r,s}(\dot{x}, \dot{y}) = -\left(\frac{1}{2sc(s)}\right)^r \sum_{\gamma \in \Gamma} Q_{r,s}(x^{-1}\gamma y), \quad \dot{x}, \ \dot{y} \in \Gamma\backslash X$$

with $Q_{r,s}$ the spherical function which we recalled in 3.2. Among other things, they proved that

(a) the series $\mathbf{P}_{r,s}(\dot{x}, \dot{y})$ converges absolutely and defines $\mathbf{P}_{r,s}(\dot{x}, \dot{y})$ holomorphic in s on $\mathrm{Re}(s) > \rho_0$ and smooth in $\dot{x}, \dot{y}$ in the complement of the diagonal of $(\Gamma\backslash X) \times (\Gamma\backslash X)$;

(b) for each $\dot{x} \in \Gamma\backslash X$, as a distribution on $\Gamma\backslash X$ $\mathbf{P}_{r,s}(\dot{x}, -)$ satisfies

$$(\triangle + s^2 - \rho_0^2)^r \mathbf{P}_{r,s}(\dot{x}, -) = -\delta(\dot{x})$$

with $\delta(\dot{x})$ the Dirac delta on $\Gamma\backslash X$ supported at $\dot{x}$

([17, page 685, Theorem 3.4], [2, page 621, Theorem 3.2]).

PROPOSITION 4.1.1. *Let $s \in \mathbf{C}$ with $\mathrm{Re}(s) > \rho_0$. If $r > m$, then $\mathbf{P}_{r,s}(\dot{x}, \dot{y})$ has a unique continuous extension to all of $(\Gamma\backslash X) \times (\Gamma\backslash X)$.*

Proof. This is a consequence of [17, Corollary 2.5]. Indeed, by that corollary, $Q_{r,s} \in L^2(G)$ if $r > m/2$ and $\mathrm{Re}(s) > \rho_0$. Hence $Q_{r,s}$ is continuous on all of G if $r > m$ and $\mathrm{Re}(s) > \rho_0$, since in general the convolution of two L^2-functions on G gives a continuous function. By the discussion in [17, pp.683–687], the continuity of $\mathbf{P}_{r,s}$ follows. $\square$

REMARK. Proposition 4.1.1 is also a corollary of our earlier results. Indeed, from Proposition 3.1.5 (iii) and Theorem 3.2.1, the function $Q_{r,s}$ ($r > m$, $\mathrm{Re}(s) > \rho_0$) has a continuous extension on G.

By this proposition, we can consider the restriction of $\mathbf{P}_{r,s}(\dot{x}, \dot{y})$ to the diagonal $\dot{x} = \dot{y}$ of $(\Gamma\backslash X) \times (\Gamma\backslash X)$. *From now on we assume that Γ is cocompact.* Then $\mathbf{P}_{r,s}(\dot{x}, \dot{y})$ becomes bounded on $(\Gamma\backslash X) \times (\Gamma\backslash X)$ if $r > m$; in particular the function $\mathbf{P}_{r,s}(\dot{x}, \dot{x})$ is integrable on $\Gamma\backslash X$. We want to evaluate the integral

$$\int_{\Gamma\backslash X} \mathbf{P}_{r+1,s}(\dot{x}, \dot{x}) \, d\dot{x} \tag{4.1}$$

with $r \geqslant m$ explicitly.

4.2. Spectral expansion of $\mathbf{P}_{r,s}(\dot{x}, \dot{y})$. In this subsection we compute the integral (4.1) by using the spectral expansion of $\mathbf{P}_{r+1,s}(\dot{x}, \dot{y})$. Since we assume that Γ is cocompact the Laplacian $\triangle$ has no continuous spectrum on $L^2(\Gamma\backslash X)$. The eigenvalues of $\triangle$ forms a countable subset of non-negative reals enumerated as

$$0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n \leqslant \ldots$$

so that each eigenvalue occurs in this sequence with its multiplicity. Let $\{\varphi_n\}_{n \geqslant 0}$ be the orthonormal basis of $L^2(\Gamma \backslash X)$ such that $\varphi_n \in C^\infty(\Gamma \backslash X)$ and $\triangle \varphi_n = \lambda_n \varphi_n$. For each n we fix a complex number s_n such that $\lambda_n = \rho_0^2 - s_n^2$.

PROPOSITION 4.2.1. *Let* $r \in \mathbf{N}_0$ *and* $s \in \mathbf{C}$ *be such that* $r \geqslant m$ *and* $\mathrm{Re}(s) > \rho_0$. *Then*

$$\mathbf{P}_{r+1,s}(\dot{x}, \dot{y}) = -\sum_{n=0}^{\infty} \frac{\overline{\varphi_n(\dot{x})}\varphi_n(\dot{y})}{(s^2 - s_n^2)^{r+1}}, \quad \dot{x}, \ \dot{y} \in \Gamma \backslash X. \qquad (4.2)$$

Here the infinite series in the right-hand side of this identity converges uniformly in $(\dot{x}, \dot{y}) \in (\Gamma \backslash X) \times (\Gamma \backslash X)$.

Proof. Since $r + 1 > m$ and $\Gamma \backslash X$ is compact, [17, Theorem 4.2, page 689] implies that the series

$$\sum_{n=0}^{\infty} \frac{|\varphi_n(\dot{x})|^2}{|s^2 - s_n^2|^{r+1}}$$

converges uniformly in $\dot{x} \in \Gamma \backslash X$. Hence by Cauchy-Schwarz inequality, we have that the series

$$\Phi(\dot{x}, \dot{y}) = -\sum_{n=0}^{\infty} \frac{\overline{\varphi_n(\dot{x})}\varphi_n(\dot{y})}{(s^2 - s_n^2)^{r+1}}$$

converges absolutely and uniformly in $(\dot{x}, \dot{y}) \in (\Gamma \backslash X)^2$, which in turn means the continuity of the function $\Phi(\dot{x}, \dot{y})$ on $(\Gamma \backslash X)^2$. Since $\Gamma \backslash X$ is compact, for a fixed $\dot{x}$, the function $\Phi(\dot{x}, -)$ is bounded on $\Gamma \backslash X$; in particular it is in $L^2(\Gamma \backslash X)$. Moreover

$$\langle \Phi(\dot{x}, -), \varphi_n \rangle = -\frac{\overline{\varphi_n(\dot{x})}}{(s^2 - s_n^2)^{r+1}}$$

for $n \in \mathbf{N}_0$. Since the last formula equals $\langle \mathbf{P}_{r,s}(\dot{x}, -), \varphi_n \rangle$ ([17, page 688]), the difference $\Phi(\dot{x}, -) - \mathbf{P}_{r,s}(\dot{x}, -)$ is orthogonal to all φ_n. Noting that the difference is continuous on $\Gamma \backslash X$ (Proposition 4.1.1) we indeed have $\Phi(\dot{x}, \dot{y}) = \mathbf{P}_{r,s}(\dot{x}, \dot{y})$ for all $\dot{y}$. $\square$

By this proposition we can compute the integral (4.1) in terms of the eigenvalues of $\triangle$.

PROPOSITION 4.2.2. *If* $r \geqslant m$ *and* $\mathrm{Re}(s) > \rho_0$, *then*

$$\int_{\Gamma \backslash X} \mathbf{P}_{r+1,s}(\dot{x}, \dot{x}) \, d\dot{x} = -\sum_{n=0}^{\infty} \frac{1}{(s^2 - s_n^2)^{r+1}}. \qquad (4.3)$$

Proof. This is a direct consequence of Proposition 4.2.1 $\square$

5. Computation of the integral $\int_{\Gamma \backslash X} \mathbf{P}_{r,s}(\dot{x}, \dot{x}) d\dot{x}$ and the resolvent trace formula.

5.1. Computation of hyperbolic term. Let Γ be as in the previous section. Then an element $\gamma \in \Gamma - \{e\}$ is G-conjugate to an element h_γ of $A^+ M$ with $A^+ = \exp((0, +\infty)H_0)$ and M the centralizer of A in K; h_γ is not uniquely determined by γ, but its ambiguity is unimportant for our purpose. We can write

$$h_\gamma = \exp(t_\gamma H_0)m_\gamma, \quad t_\gamma > 0, \ m_\gamma \in M.$$

Let G_γ be the centralizer of γ in G and put $\Gamma_\gamma = \Gamma \cap G_\gamma$. Then G_γ is reductive and $\Gamma_\gamma \backslash G_\gamma$ is compact. We fix a Haar measure dg_γ on G_γ in a manner analogous to the manner in which the Haar measure on G was fixed, following the Iwasawa decomposition of G_γ, and put $d\dot{g}_\gamma$ for the invariant measure on $\Gamma_\gamma \backslash G_\gamma$. The group Γ_γ is known to be isomorphic to $\mathbf{Z}$. Hence there exists a unique generator γ_0 of Γ_γ and a positive integer $j(\gamma)$ (the multiplicity of γ) such that $\gamma = \gamma_0^{j(\gamma)}$. Let $\mathcal{H}(\Gamma)$ be the set of Γ-conjugacy classes in $\Gamma - \{e\}$. We first calculate the orbital integral of $\phi_s^{[r]}$ associated with a hyperbolic conjugacy class.

PROPOSITION 5.1.1. *Let $r \in \mathbf{N}_0$ and $\mathrm{Re}(s) > \rho_0$. For $[\gamma] \in \mathcal{H}(\Gamma)$, put*

$$J^{[r]}([\gamma] \ ; s) = \mathrm{vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} \phi_s^{[r]}(g^{-1}\gamma g) \, dg_\gamma^*,$$

where dg_γ^* is the G-invariant measure on $G_\gamma \backslash G$ normalized so that $dg = dg_\gamma dg_\gamma^*$. Then the integral $J^{[r]}([\gamma] \ ; s)$ converges absolutely and uniformly on $\mathrm{Re}(s) \geqslant \rho_0 + \epsilon$ for any $\epsilon > 0$ and is evaluated as

$$J^{[r]}([\gamma] \ ; s) = \frac{1}{r!}\left(-\frac{1}{2s}\frac{d}{ds}\right)^r \left\{ -j(\gamma)^{-1}|\det(1 - \mathrm{Ad}(h_\gamma)^{-1}|_\mathfrak{n})|^{-1} \frac{t_\gamma e^{-(s+\rho_0)t_\gamma}}{2s} \right\}.$$

Proof. We shall compute the orbital integral

$$I^{[r]}([\gamma] \ ; s) = \int_{G_\gamma \backslash G} \phi_s^{[r]}(g^{-1}\gamma g) \, dg_\gamma^*. \tag{5.1}$$

Let $h = h_\gamma = \exp(t_\gamma H_0)m_\gamma = \exp(tH_0)m \in A^+ M$, to which γ is conjugate. Then $I^{[r]}([\gamma] \ ; s) = I^{[r]}([h] \ ; s)$ and

$$I^{[r]}([h] \ ; s) = \int_{G_h \backslash G} \phi_s^{[r]}(g^{-1}hg) \, dg_h^* = \mathrm{vol}(A \backslash G_h)^{-1} \int_{A \backslash G} \phi_s^{[r]}(g^{-1}hg) \frac{dg}{da}, \tag{5.2}$$

where $\mathrm{vol}(A \backslash G_h) = \int_{A \backslash G_h} \frac{dg_h}{da}$. Moreover

$$\int_{A \backslash G} \phi_s^{[r]}(g^{-1}hg) \frac{dg}{da} = \int_N \int_K \phi_s^{[r]}(k^{-1}n^{-1}hnk) \, dk \, dn$$

$$= \int_N \phi_s^{[r]}(n^{-1}hn) \, dn = |\det(1 - \mathrm{Ad}(h)|_\mathfrak{n})|^{-1} \int_N \phi_s^{[r]}(nh) \, dn. \tag{5.3}$$

We recall a lemma (cf. [12, p.487]) to calculate the integral (5.3). Put $|Z|^2 = -B(X, \theta X)$ for $Z \in \mathfrak{g}$. If $n = \exp(X + Y)$ with $X \in \mathfrak{n}_\beta, Y \in \mathfrak{n}_{2\beta}$ and we write

$$nh = n \cdot \exp(tH_0)m = k_1 \exp(uH_0)k_2 \quad (k_1, k_2 \in K, \ u > 0), \tag{5.4}$$

then we have

$$\cosh^2(u) = \left[\cosh(t) + \frac{c}{2}e^{-t}|X|^2\right]^2 + ce^{-2t}|Y|^2, \tag{5.5}$$

where $c = 4^{-1}(p + 4q)^{-1} = 2^{-1}c_0$. We first treat the case of $r = 0$. We have

$$\int_N \phi_s^{[0]}(nh)\,dn$$

$$= \int_N \phi_s^{[0]}(k_1 \exp(uH_0)k_2)\,dn$$

$$= \frac{c_G^{-1}}{(2-2m)} \int_{\mathfrak{n}_\beta \times \mathfrak{n}_{2\beta}} \phi_s^{(2)}(\exp uH_0)\,dX dY$$

$$= \frac{c_G^{-1}}{(2-2m)}\gamma(s) \int_{\mathfrak{n}_\beta \times \mathfrak{n}_{2\beta}} (\cosh u)^{-(s+\rho_0)}{}_2F_1\left(\alpha, \beta\ ; s+1\ ; \frac{1}{\cosh^2 u}\right) dX dY$$

$$= \frac{c_G^{-1}}{(2-2m)}\gamma(s) \int_{\mathfrak{n}_\beta \times \mathfrak{n}_{2\beta}} \left\{\left(\cosh(t) + \frac{c}{2}e^{-t}|X|^2\right)^2 + ce^{-2t}|Y|^2\right\}^{-\frac{s+\rho_0}{2}}$$

$$\times {}_2F_1\left(\alpha, \beta\ ; s+1\ ; \frac{1}{(\cosh(t) + \frac{c}{2}e^{-t}|X|^2)^2 + ce^{-2t}|Y|^2}\right) dX dY, \qquad (5.6)$$

where $\alpha = (s+\rho_0)/2$ and $\beta = (s-\rho_0+2m)/2$. Substituting $X = \cosh(t)^{\frac{1}{2}}(2^{-1}c)^{-\frac{1}{2}}e^{\frac{t}{2}}\xi$ and $Y = \cosh(t)c^{-\frac{1}{2}}e^t\eta$, we have $dX = (\cosh(t)^{\frac{1}{2}}(2^{-1}c)^{-\frac{1}{2}}e^{\frac{t}{2}})^p d\xi$, $dY = (\cosh(t)c^{-\frac{1}{2}}e^t)^q d\eta$ and $dX dY = (\cosh t)^{\rho_0}e^{\rho_0 t}2^{-\frac{p}{2}}c^{-m+\frac{1}{2}}d\xi d\eta$. So the integral (5.6) equals

$$\frac{c_G^{-1}}{(2-2m)}\gamma(s) \int_{\mathbf{R}^p \times \mathbf{R}^q} \left\{\left(1 + \|\xi\|^2\right)^2 + \|\eta\|^2\right\}^{-\frac{s+\rho_0}{2}} \cdot (\cosh t)^{-(s+\rho_0)+\rho_0}$$

$$\times {}_2F_1\left(\alpha, \beta\ ; s+1\ ; \frac{(\cosh t)^{-2}}{(1 + \|\xi\|^2)^2 + \|\eta\|^2}\right) e^{\rho_0 t}2^{\frac{p}{2}}c^{-m+\frac{1}{2}}\,d\xi d\eta$$

$$= \frac{c_G^{-1}}{(2-2m)}\gamma(s)(\cosh t)^{-s}e^{\rho_0 t}2^{\frac{p}{2}}c^{-m+\frac{1}{2}}I(s). \qquad (5.7)$$

Here, $I(s)$ is given by

$$\int_{\mathbf{R}^p \times \mathbf{R}^q} \left\{\left(1 + \|\xi\|^2\right)^2 + \|\eta\|^2\right\}^{-\frac{s+\rho_0}{2}}{}_2F_1\left(\alpha, \beta\ ; s+1\ ; \frac{(\cosh t)^{-2}}{(1 + \|\xi\|^2)^2 + \|\eta\|^2}\right) d\xi d\eta. \quad (5.8)$$

Using polar coordinates for $\mathbf{R}^p, \mathbf{R}^q$ and the power series expansion of the hypergeometric function, we carry out the integral term by term; this is permissible by dominated convergence theorem. Hence,

$$I(s) = \frac{2\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})}\frac{2\pi^{\frac{q}{2}}}{\Gamma(\frac{q}{2})} \int_0^\infty \int_0^\infty \rho_1^{p-1}\rho_2^{q-1}\left\{\left(1+\rho_1^2\right)^2 + \rho_2^2\right\}^{-\frac{s+\rho_0}{2}}$$

$$\times {}_2F_1\left(\alpha, \beta\ ; s+1\ ; \frac{(\cosh t)^{-2}}{\left(1+\rho_1^2\right)^2 + \rho_2^2}\right) d\rho_1 d\rho_2$$

$$= \frac{2\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})}\frac{2\pi^{\frac{q}{2}}}{\Gamma(\frac{q}{2})} \int_0^\infty \int_0^\infty \rho_1^{p-1}\rho_2^{q-1}\left\{\left(1+\rho_1^2\right)^2 + \rho_2^2\right\}^{-\frac{s+\rho_0}{2}}$$

$$\times \frac{\Gamma(s+1)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{k=0}^\infty \frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(s+1+k)k!}\left(\frac{(\cosh t)^{-2}}{\left(1+\rho_1^2\right)^2 + \rho_2^2}\right)^k d\rho_1 d\rho_2$$

$$= \frac{2\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})}\frac{2\pi^{\frac{q}{2}}}{\Gamma(\frac{q}{2})}\frac{\Gamma(s+1)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{k=0}^\infty I_k(s)\frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(s+1+k)k!}(\cosh t)^{-2k} \qquad (5.9)$$

with

$$I_k(s) = \int_0^\infty \int_0^\infty \rho_1^{p-1} \rho_2^{q-1} \left\{ \left(1 + \rho_1^2\right)^2 + \rho_2^2 \right\}^{-\frac{s+\rho_0}{2}-k} d\rho_1 d\rho_2 \quad (k \in \mathbf{N}_0). \tag{5.10}$$

By making the substitution $\rho_2 \to (1 + \rho_1^2)\rho_2$, we have the expression for $I_k(s)$ as

$$\int_0^\infty \rho_1^{p-1}\left(1 + \rho_1^2\right)^{-(s+\rho_0)-2k+q} d\rho_1 \cdot \int_0^\infty \rho_2^{q-1}\left(1 + \rho_2^2\right)^{-\frac{s+\rho_0}{2}-k} d\rho_2. \tag{5.11}$$

The relation

$$\int_0^\infty \rho^{2x-1}\left(1 + \rho^2\right)^{-x-y} d\rho = \frac{\Gamma(x)\Gamma(y)}{2\Gamma(x+y)}, \qquad \mathrm{Re}(x), \mathrm{Re}(y) > 0$$

gives

$$\begin{aligned}
I_k(s) &= \frac{1}{2} \frac{\Gamma(\frac{p}{2})\Gamma(s + \rho_0 + 2k - q - \frac{p}{2})}{\Gamma(s + \rho_0 + 2k - q)} \cdot \frac{1}{2} \frac{\Gamma(\frac{q}{2})\Gamma(\frac{s+\rho_0}{2} + k - \frac{q}{2})}{\Gamma(\frac{s+\rho_0}{2} + k)} \\
&= \frac{1}{4} \Gamma\left(\frac{p}{2}\right) \Gamma\left(\frac{q}{2}\right) \frac{\Gamma(s + 2k)\Gamma(\frac{s+\rho_0-q}{2} + k)}{\Gamma(s + \rho_0 - q + 2k)\Gamma(\frac{s+\rho_0}{2} + k)} \\
&= \frac{1}{4} \Gamma\left(\frac{p}{2}\right) \Gamma\left(\frac{q}{2}\right) \frac{\Gamma(s + 2k) \cdot \pi^{\frac{1}{2}} 2^{-(s+\rho_0-q+2k-1)}}{\Gamma(\frac{s+\rho_0}{2} + k)\Gamma(\frac{s+\rho_0-q}{2} + k + \frac{1}{2})} \\
&= \frac{1}{4} \Gamma\left(\frac{p}{2}\right) \Gamma\left(\frac{q}{2}\right) \frac{\Gamma(s + 2k) \cdot \pi^{\frac{1}{2}} 2^{-s-\rho_0+q+1} \cdot 2^{-2k}}{\Gamma(\alpha + k)\Gamma(\beta + k)}. \tag{5.12}
\end{aligned}$$

The third equality is obtained by the duplication formula:

$$\Gamma(2z) = \pi^{-\frac{1}{2}} 2^{2z-1} \Gamma(z)\Gamma(z + \tfrac{1}{2}), \tag{5.13}$$

for $2z = s + \rho_0 - q + 2k$ and the last one is by $s + \rho_0 - q + 1 = s + \frac{p}{2} + 1 = s + 2m - \rho_0 = 2\beta$. Substituting (5.12) into (5.9) and using the duplication formula (5.13) for $2z = s + 2k$, we have

$$\begin{aligned}
I(s) &= \pi^{\frac{1}{2}(p+q+1)} 2^{-s-\rho_0+q+1} \frac{\Gamma(s+1)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{k=0}^\infty \frac{\Gamma(s+2k)}{\Gamma(s+1+k)k!} 2^{-2k} (\cosh t)^{-2k} \\
&= \pi^{\frac{1}{2}(p+q)} 2^{-\rho_0+q} \frac{\Gamma(s+1)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{k=0}^\infty \frac{\Gamma(\frac{s}{2}+k)\Gamma(\frac{s+1}{2}+k)}{\Gamma(s+1+k)k!} (\cosh t)^{-2k} \\
&= 2^{-\rho_0+q} \pi^{\frac{1}{2}(p+q)} \frac{\Gamma(s+1)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})}{\Gamma(s+1)} {}_2F_1\left(\frac{s}{2}, \frac{s+1}{2}; s+1; \frac{1}{\cosh^2 t}\right). \tag{5.14}
\end{aligned}$$

By [16, (9.8.3) in p.259]:

$$_2F_1(\alpha, \alpha + \tfrac{1}{2}; 2\alpha + 1; z) = \left(\frac{1 + \sqrt{1-z}}{2}\right)^{-2\alpha}, \tag{5.15}$$

we get

$$I(s) = 2^{-\rho_0+q} \pi^{\frac{1}{2}(p+q)} \frac{\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})}{\Gamma(\alpha)\Gamma(\beta)} e^{-st} (2\cosh t)^s. \tag{5.16}$$

By (5.6), (5.7) and (5.16), we have

$$\int_N \phi_s^{[0]}(nh)\,dn$$

$$= \frac{c_G^{-1}}{(2-2m)} \int_{\mathfrak{n}_\beta \times \mathfrak{n}_{2\beta}} \phi_s^{(2)}(\exp uH_0)\,dX dY$$

$$= \frac{c_G^{-1}}{(2-2m)} \gamma(s)(\cosh t)^{-s} e^{\rho_0 t} 2^{\frac{p}{2}} c^{-m+\frac{1}{2}} \cdot I(s)$$

$$= \frac{c_G^{-1}}{(2-2m)} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(s+1)\Gamma(m-1)}(\cosh t)^{-s} \cdot e^{\rho_0 t} 2^{\frac{p}{2}} c^{-m+\frac{1}{2}}$$

$$\times 2^{-\rho_0+q} \pi^{\frac{1}{2}(p+q)} \frac{\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})}{\Gamma(\alpha)\Gamma(\beta)} e^{-st}(2\cosh t)^s$$

$$= \frac{c_G^{-1}}{(2-2m)} \frac{c^{-m+\frac{1}{2}}\pi^{m-\frac{1}{2}}}{\Gamma(m-1)} \frac{2^s \Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})}{\Gamma(s+1)} e^{-(s-\rho_0)t}$$

$$= \frac{c_G^{-1}}{(2-2m)} \frac{2\pi^m c^{-m+\frac{1}{2}}}{\Gamma(m-1)} \frac{e^{-(s-\rho_0)t}}{s}$$

$$= \frac{c_G^{-1}}{(2-2m)}(2m-2)\frac{\pi^m c^{-m+\frac{1}{2}}}{\Gamma(m)} \frac{e^{-(s-\rho_0)t}}{s} = -\frac{e^{-(s-\rho_0)t}}{2s}. \qquad (5.17)$$

The proof for $r = 0$ is finished by

$$J^{[0]}([\gamma]\,;s) = \mathrm{vol}(\Gamma_\gamma \backslash G_\gamma) I^{[0]}([\gamma]\,;s)$$

$$= \mathrm{vol}(\Gamma_\gamma \backslash G_\gamma)\mathrm{vol}(A\backslash G_h)^{-1}|\det(1 - \mathrm{Ad}(h_\gamma)|_{\mathfrak{n}})|^{-1} \int_N \phi_s^{[0]}(nh)\,dn$$

$$= \mathrm{vol}(\Gamma_\gamma \backslash G_\gamma)\mathrm{vol}(A\backslash G_h)^{-1} e^{-2\rho_0 t}|\det(1 - \mathrm{Ad}(h_\gamma^{-1})|_{\mathfrak{n}})|^{-1} \int_N \phi_s^{[0]}(nh)\,dn \quad (5.18)$$

and [23, Theorem 11.30, p.189] (or [6]),

$$\mathrm{vol}(\Gamma_\gamma \backslash G_\gamma)\mathrm{vol}(A\backslash G_h)^{-1} = t_\gamma j(\gamma)^{-1}. \qquad (5.19)$$

Finally we have the formula for $J^{[r]}([\gamma]\,;s)$ $(r \geqslant 1)$, by interchanging integration and differentiation. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Recall the integral (4.1), which is expressed by eigenvalues of Laplacian in Proposition 4.2.2. Now we obtain another expression of that integral.

PROPOSITION 5.1.2.
(a) *The infinite series*

$$J_{\mathrm{hyp}}(s) = -\sum_{\gamma \in \mathcal{H}(\Gamma)} j(\gamma)^{-1}|\det(1 - \mathrm{Ad}(h_\gamma^{-1})|_{\mathfrak{n}})|^{-1} \frac{t_\gamma e^{-(s+\rho_0)t_\gamma}}{2s}$$

converges absolutely and uniformly on $\mathrm{Re}(s) \geqslant \rho_0 + \epsilon$ *for any* $\epsilon > 0$.
(b) *If* $r \geqslant m$ *and* $\mathrm{Re}(s) > \rho_0$, *then we have*

$$\int_{\Gamma \backslash X} \mathbf{P}_{r+1,s}(\dot{x},\dot{x})d\dot{x} = \mathrm{vol}(\Gamma\backslash G)\left(\lim_{g \to e,\ g \in G^+} \phi_s^{[r]}(g)\right) + \sum_{\gamma \in \mathcal{H}(\Gamma)} J^{[r]}([\gamma]\,;s),$$

$$(5.20)$$

where the series in the right-hand side of (5.20) converges absolutely and uniformly on $\mathrm{Re}(s) \geqslant \rho_0 + \epsilon$ *for any* $\epsilon > 0$.

Proof. The assertion is ensured by the next lemma.   □

LEMMA 5.1.3. *Suppose that Γ is a discrete subgroup of G such that $\Gamma\backslash G$ is compact. Then the counting function*

$$\pi_0(T) := \#\{\{\gamma\}_\Gamma \in \mathcal{H}(\Gamma) \,|\, N(\gamma) = e^{t_\gamma} \leqslant T\}, \quad T > 0$$

satisfies the growth condition

$$\pi_0(T) = O(T^{2\rho_0}) \quad as \quad T \to \infty.$$

Proof. (Similar to [4, Lemma 2.6.3, p.70]) It is well known that $\log N(\gamma) = t_\gamma = \inf\{d(x,\gamma x) \,|\, x \in X\}$. Set $\mathrm{ax}(\gamma) = \{x \in X \,|\, d(x,\gamma x) = t_\gamma\}$, the axis of γ. Since $\Gamma\backslash G$ is compact, Γ has a compact fundamental domain $X_\Gamma \subset X$ such that $\Gamma X_\Gamma = X$. Let $\gamma \in \Gamma$, hyperbolic and suppose that $z \in \mathrm{ax}(\gamma)$. Then there is an element $h_0 \in \Gamma$ such that $h_0 z \in X_\Gamma$ and hence $t_\gamma = d(z,\gamma z) = d(h_0 z, h_0 \gamma h_0^{-1} \cdot h_0 z)$ with $h = h_0 \gamma h_0^{-1} \in \{\gamma\}_\Gamma$. Let $d(A,B) := \inf\{d(x,y) \,|\, x \in A,\, y \in B\}$ denote the hyperbolic distance between the non-empty sets $A, B \subset X$, and $B(x,r)$ the ball with radius $r > 0$ centered at $x \in X$. Put $d_0 := \sup\{d(x,y) \,|\, x, y \in X_\Gamma\}$ and choose a point $x_0 \in X_\Gamma$.

$$\begin{aligned}
\pi_0(T) = &\#\{\{\gamma\}_\Gamma \in \mathcal{H}(\Gamma) \,|\, N(\gamma) = e^{t_\gamma} \leqslant T\} \\
\leqslant &\#\{h \in \Gamma \,|\, d(X_\Gamma, hX_\Gamma) \leqslant \log T\} \\
\leqslant &\#\{h \in \Gamma \,|\, d(x_0, hx_0) \leqslant \log T + 2d_0\} \\
\leqslant &\#\{h \in \Gamma \,|\, hX_\Gamma \subset B(x_0, \log T + 3d_0)\} \\
\leqslant &\frac{\mathrm{vol}(B(x_0, \log T + 3d_0))}{\mathrm{vol}(X_\Gamma)} \\
\leqslant &\frac{c_G}{\mathrm{vol}(X_\Gamma)} \int_0^{\log T + 3d_0} \mu(t)\, dt \\
\leqslant &\frac{2^{-(p+2q)} c_G}{\mathrm{vol}(X_\Gamma)} \int_0^{\log T + 3d_0} (e^t - e^{-t})^{p+q}(e^t + e^{-t})^q \, dt \\
\leqslant &\frac{2^{-(p+2q)} c_G e^{(p+2q)3d_0}}{(p+2q)\mathrm{vol}(X_\Gamma)} T^{(p+2q)}.
\end{aligned}$$

So we have the desired formula $\pi_0(T) = O(T^{2\rho_0})$ as $T \to \infty$.   □

5.2. The resolvent trace formula. From Proposotion 4.2.2, Proposition 3.1.5 (iii) and Proposition 5.1.2, we arrive at the formula.

THEOREM 5.2.1. *If $r \geqslant m$ and $\mathrm{Re}(s) > \rho_0$, then we have*

$$\sum_{n=0}^\infty \frac{1}{(s^2 - s_n^2)^{r+1}} = -\frac{1}{r!}\left(-\frac{1}{2s}\frac{d}{ds}\right)^r \left(J_{\mathrm{id}}(s) + J_{\mathrm{hyp}}(s)\right)$$

with

$$J_{\mathrm{id}}(s) = \mathrm{vol}(\Gamma\backslash G)\frac{2^{-m-3/2}\pi^{-m}c_0^{m-1/2}}{\Gamma(m)}(-1)^{m+1}$$

$$\times \prod_{j=1}^{m-1}\left\{\left(\frac{s}{2}\right)^2 - \left(\frac{\rho_0}{2} - j\right)^2\right\} \cdot \left\{\psi\left(\frac{s+\rho_0}{2}\right) + \psi\left(\frac{s-\rho_0}{2} + 1\right)\right\},$$

$$J_{\mathrm{hyp}}(s) = -\sum_{\gamma \in \mathcal{H}(\Gamma)} j(\gamma)^{-1} |\det(1 - \mathrm{Ad}(h_\gamma^{-1})|_{\mathfrak{n}})|^{-1}\frac{t_\gamma e^{-(s+\rho_0)t_\gamma}}{2s}.$$

6. Selberg zeta function.

6.1. Analytic continuation of the Selberg zeta function. We recall the definition of the Selberg zeta function for $\Gamma\backslash X$ with Γ as in the previous section. Let H be a θ-stable Cartan subgroup of G containing A. Then $H = AH^-$ with $H^- = H \cap K$. Let P be the set of those root β for $(\mathfrak{h}_\mathbf{C}, \mathfrak{g}_\mathbf{C})$ with $\beta(H_0) > 0$, and Λ the set of linear forms on $\mathfrak{h}_\mathbf{C}$ of the form

$$\lambda = \sum_{\beta \in P} n_\beta \beta, \qquad n_\beta \in \mathbf{N}_0. \tag{6.1}$$

For $\lambda \in \Lambda$ let m_λ denote the number of the ways to express it in the form (6.1).

Let $\mathrm{Prim}(\Gamma)$ be the set of primitive conjugacy classes in $\mathcal{H}(\Gamma)$, i.e., the set of non-trivial Γ-conjugacy class which is not a power of any other Γ-conjugacy class. Then for $[\gamma] \in \mathcal{H}(\Gamma)$ there exists a unique $[\gamma_0] \in \mathrm{Prim}(\Gamma)$ such that $[\gamma] = [\gamma_0^{j(\gamma)}]$ with $j(\gamma)$ the multiplicity of γ.

Since H^- is a Cartan subgroup of the compact group M, any element of M is M-conjugate to an element of H^-. Hence the G-conjugacy class of a $[\gamma] \in \mathcal{H}(\Gamma)$ contains an element of H expressed as

$$h_\gamma = \exp(t_\gamma H_0) h_\gamma^-, \qquad t_\gamma > 0, \ h_\gamma^- \in H^-.$$

For $\lambda \in \Lambda$ the associated character of H is denoted by $\xi_\lambda : H \to \mathbf{C}^*$. With these notations, the Selberg zeta function for $\Gamma\backslash X$ is defined as the Euler product

$$Z_\Gamma(s) = \prod_{[\gamma] \in \mathrm{Prim}(\Gamma)} \prod_{\lambda \in \Lambda} (1 - \xi_{-\lambda}(h_\gamma) e^{-st_\gamma})^{m_\lambda}. \tag{6.2}$$

It is easy to see that the logarithmic derivative of $Z_\Gamma(s)$ is related to the function $J_{\mathrm{hyp}}(s)$ by the formula

$$-\frac{1}{2s}\frac{d}{ds}\log Z_\Gamma(s + \rho_0) = J_{\mathrm{hyp}}(s) \tag{6.3}$$

Hence by Proposition 5.1.2 (a), the infinite product (6.2) converges absolutely and locally uniformly on $\mathrm{Re}(s) > 2\rho_0$ defining $Z_\Gamma(s)$ holomorphic in s on that half-plane.

COROLLARY 6.1.1. *The Selberg zeta function $Z_\Gamma(s)$, defined for $\mathrm{Re}(s) > 2\rho_0$, has the analytic continuation as a meromorphic function on the whole complex plane. $Z_\Gamma(s)$ has zeros located at $s = \rho_0 \pm s_n, n \geqslant 0$. If $\lambda_n \neq \rho_0^2$, the order of the zeros at $s = \rho_0 \pm s_n$ equals the multiplicity of the eigenvalue λ_n. If ρ_0^2 is an eigenvalue of the Laplacian $\triangle$, then the order of the zero at $s = \rho_0$ equals twice the multiplicity of the eigenvalue $\lambda_k = \rho_0^2$.*

Proof. By (6.3) and Theorem 5.2.1, we have

$$\sum_{n=0}^{\infty} \frac{1}{(s^2 - s_n^2)^{r+1}} = -\frac{1}{r!}\left(-\frac{1}{2s}\frac{d}{ds}\right)^r \left(J_{\mathrm{id}}(s) - \frac{1}{2s}\frac{d}{ds}\log Z_\Gamma(s + \rho_0)\right). \tag{6.4}$$

The left-hand side of (6.4) is a meromorphic function of $s \in \mathbf{C}$ and its poles are located at the points $s = \pm s_n$ with order $r + 1$ (or $2r + 2$ if $s_n = 0$). It is simple to see that $J_{\mathrm{id}}(s)$ is meromorphic on $\mathbf{C}$ and its all the residues are integers. Hence $\frac{d}{ds}\log Z_\Gamma(s)$ is a meromorphic function with simple poles only at $s = \rho_0 \pm s_n$ and at

points coming from $J_{\mathrm{id}}(s)$. For $\lambda_n \neq \rho_0^2$, the residue of $\frac{d}{ds} \log Z_\Gamma(s)$ at $s = \rho_0 \pm s_n$ is equal to the multiplicity of λ_n, and if ρ_0^2 is an eigenvalue, the residue of the pole at $s = \rho_0$ equals twice the multiplicity of the eigenvalue ρ_0^2. So all the residues at $s = \rho_0 \pm s_n$ are non-negative integers. This completes the proof. $\square$

REMARK 1. For almost all $n \geqslant 0$, s_n is purely imaginary.

REMARK 2. We can also show that there exists a meromorphic function $Z_{\mathrm{id}}(s)$ such that

$$-\frac{1}{2s}\frac{d}{ds} \log Z_{\mathrm{id}}(s + \rho_0) = J_{\mathrm{id}}(s).$$

Since the left-hand side of (6.4) is invariant under $s \to -s$, the completed Selberg zeta function $\widehat{Z}_\Gamma(s) := Z_\Gamma(s) Z_{\mathrm{id}}(s)$ satisfy the symmetric functional equation

$$\widehat{Z}_\Gamma(2\rho_0 - s) = \widehat{Z}_\Gamma(s).$$

The function $Z_{\mathrm{id}}(s)$ is called *gamma factors* (or identity factor) of $Z_\Gamma(s)$. It is known that $Z_{\mathrm{id}}(s)$ is described by the multiple gamma functions. We refer [15], [21] and [9] for this topic.

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