ASIAN J. MATH.
Vol. 6, No. 2, pp. 195–226, June 2002

ON THE REPRESENTATION OF NUMBERS IN THE FORM $a_1X_1^2 + a_2X_2^2 + a_3X_3^2 + a_4W^l$*

C. HOOLEY†

1. Introduction. In a recent paper ([8], to which we refer as I for convenience) we established the previously unproved asymptotic formula for the number of ways a large integer could be expressed as the sum of three squares and a positive (non-linear) ℓth power, deducing from it known criteria for the representability of a number in this form. Having a conformation that a heuristic application of the circle method of Hardy and Littlewood would foretell, the formula was nevertheless beyond the power of the method to deliver and was instead proved by using the theory of Dirichlet's L-functions in conjunction with various formulae for the number $r_3(n_1)$ of representations of an integer n_1 as the sum of three squares.

Since it has been traditional to follow up successful investigations into topics of pure Waring's type by what are usually easy generalizations in which the powers are affected by integral coefficients, it is natural that we here should now widen our sphere of inquiry by contemplating the problem of finding an asymptotic formula for the number $\nu(n)$ of solutions in integers of the equation

$$a_1X_1^2 + a_2X_2^2 + a_3X_3^2 + a_4W^l = n, \tag{1.1}$$

where a_1, a_2, a_3, a_4 are positive integers and W^l is a non-negative power having exponent l exceeding 1. Yet, although at first sight the change to our problem might seem innocuous, a moment's reflection reveals that the widening of the terms of reference may introduce a significant new difficulty. This is because the generalization of the formula for the number $r_3(n_1)$ of representations of an integer n_1 as the sum of three squares has been in the classical theory, not a formula for the number $r_{a_1,a_2,a_3}(n_1)$ of representations of n_1 as $f(X_1, X_2, X_3) = a_1X_1^2 + a_2X_2^2 + a_3X_3^2$, but a formula for a weighted average of representations of n_1 through a set of inequivalent members of the genus to which $f(X_1, X_2, X_3)$ belongs. To establish a corresponding formula for $r_{a_1,a_2,a_3}(n_1)$ itself or, indeed, its parallel for any positive ternary form had long been a goal of the theory and was fraught with difficulty, particularly as it was appreciated that there were exceptions when the expected conclusion was false. Although by now matters have advanced to a point where the lacuna has been substantially filled (see, especially, the paper [2] by Duke and Schulze - Pillot, which employs the theory of modular forms), the newer theory still does not supply an entirely appropriate instrument for our purposes. Consequently we shall handle the problem of the presence of $r_{a_1,a_2,a_3}(n_1)$ by approaching it via a different avenue that we briefly describe.

The genesis of this approach is Heath-Brown's important new version of the circle method ([3]; we refer to this as H in what follows), which affords a particularly convenient initial expression for $r_{a_1,a_2,a_3}(n_1)$ after we incorporate a refinement to suit the present occasion. From the resulting formula we isolate an element that corresponds to the usually expected asymptotic value, while the other element is expressed through exponential sums that are evaluated in such a way that their moduli can be satisfactorily averaged over the values $n - a_4W^l$ of n_1. The influence of the

*Received October 17, 2001; accepted for publication November 16, 2001.
†Cardiff University, School of Mathematics, Senghennydd Road, Cardiff CF24 4YH, U.K. (MillsME@Cardiff.ac.uk).

former term on $\nu(n)$ is treated by a modification of the method of I, it being necessary both here and in the latter part of the analysis to depend on results about L–functions formed from real characters. Thus there emerges a treatment of our problem that is relatively accessible and shuns any reference, implicit or explicit, to modular forms or the difficult theory of the arithmetic of ternary quadratic forms.

In the last part of the paper we use the asymptotic formula to discuss the conditions under which a large number is expressible in the form $a_1X_1^2 + a_2X_2^2 + a_3X_3^2 + a_4W^l$. The discussion is naturally more complicated than it was in I owing to the greater generality of the situation, and we therefore limit our deliberations to the case where the coefficients a_i are odd and relatively prime in pairs.

Some remarks on the circle method should be added. First, Heath-Brown's method, which is not truly a circle method but which has a closely allied structure, casts $r_{a_1,a_2,a_3}(n_1)$ into an especially favourable form, since its formulation does not involve a certain class of exponential sums that normally appear in the circle method with Kloosterman refinement, or, in other words, it acts much as the latter would if all Farey arcs related to a given denominator were of equal length. In fact, the way in which such sums normally appear would compromise our estimations, although this difficulty would be circumvented by means of our smoothed version of the circle method used in [7] at the expense of considerably more calculations. Also, as in I, whether of the conventional type or of the Heath-Brown variety, the circle method cannot apparently act as a complete envelope for the estimation of $\nu(n)$.

2. Notation. Although the meaning of most of the notation is self evident from the context, the following guide may be helpful. The letters n and n_1 are usually positive integers, the former being regarded as tending to infinity in the later part of the work; X_i, X_i' are integers and W is a non-negative integer; the given exponent l exceeds 1 and should normally be thought of as exceeding 2, since the case $l = 2$ is covered by the comprehensive literature on quaternary quadratic forms; ϵ is an arbitrarily small number that is not necessarily the same at each occurrence; B_i is a positive constant depending at most on $a_1, ..., a_4$, and l; $B_i(\epsilon)$ is like B_i save that it may also depend on ϵ; the constants implied by the O-notation are of type B_i save when there is an ϵ occurring in the exponent, in which case they are of type $B_i(\epsilon)$; A is a positive absolute constant, not necessarily the same on each occasion, whose value will be determined so that no account need be taken of it when considering the constants in the O-notation, a similar comment on the arbitrary integer m in §4 being apposite.

Ordered triples are indicated by bold type, their components being denoted by the same letter in italic font with subscripts; if $\mathbf{a} = (a_1, a_2, a_3)$, then $||\mathbf{a}||$ is the usual valuation $(a_1^2 + a_2^2 + a_3^2)^{\frac{1}{2}}$; $\mathbf{ab}$ is the scalar product $a_1b_1 + a_2b_2 + a_3b_3$; the notation $0 < \mathbf{a} \le u$ means $0 < a_1, a_2, a_3 \le u$; in three dimensional integrals $d\mathbf{x}$ is a shorthand for $dx_1dx_2dx_3$.

The highest common factor of integers $U_1, ..., U_r$ is denoted by $(U_1, ..., U_r)$ when it is defined; here $r \neq 3$ so no confusion arises over the previous usage for triples ; $\sigma_{-\alpha}(n) = \sum_{d|n} d^{-\alpha}$ and $d(n) = \sigma_o(n)$; for odd $k, (\frac{m}{k}) = (m|k)$ is the Jacobi symbol of quadratic residuacity.

3. Initial formula for $r_3(n_1)$. We use Heath-Brown's new form of the circle method to formulate an expression for the number $r_3(n_1) = r_{a_1,a_2,a_3}(n_1)$ of representations of a positive number n_1 by the positive definite ternary quadratic form

$$f(\mathbf{X}) = a_1X_1^2 + a_2X_2^2 + a_3X_3^2, \tag{3.1}$$

whose reciprocal or adjoint

$$a_2a_3m_1^2 + a_3a_1m_2^2 + a_1a_2m_3^2 \tag{3.2}$$

we straightway denote by $F(\mathbf{m})$. In setting up this preliminary apparatus, we should note that we depart somewhat from Heath-Brown's notation in order to reserve some appropriate symbolism for the later part of the exposition, an incidental advantage being that some of our language is brought into line with our earlier usages for connected themes (see, for example, [7]).

The source of the formula for $r_3(n_1)$ is the second statement in Theorem 2 of H. Translated into our idiom, this asserts that, if $w(\mathbf{x})$ be an infinitely differentiable function, then the cardinality of the solutions of $f(\mathbf{X}) = n_1$ to which the weight $w(\mathbf{X})$ is attached is equal to

$$\frac{c_M}{M^2}\sum_{\mathbf{m}}\sum_{k=1}^{\infty}\frac{1}{k^3}Q_{n_1}(\mathbf{m},k)I_M(\mathbf{m},k) \tag{3.3}$$

for any[1] $M \geq 1$, in which

$$c_M = 1 + O(M^{-A}) \tag{3.4}$$

depends only on $M, Q_{n_1}(\mathbf{m}, k)$ is the exponential sum

$$\sum_{\substack{0<h\leq k\\(h,k)=1}}\ \sum_{0<\mathbf{b}\leq k} e^{2\pi i\{h(f(\mathbf{b})-n_1)+\mathbf{mb}\}/k}, \tag{3.5}$$

and $I_M(\mathbf{m}, k)$ is the integral

$$\int w(\mathbf{x})h\left(\frac{k}{M}, \frac{f(\mathbf{x})-n_1}{M^2}\right)e^{-2\pi i\mathbf{mx}/k}d\mathbf{x}. \tag{3.6}$$

Then, following the procedures laid down on pp 153 and 154 of H with particular reference to his Corollary 1, we introduce the function

$$w_0(x) = \begin{cases} e^{-1/(1-x^2)} & \text{, if } |x| < 1,\\ 0 & \text{, otherwise,}\end{cases}$$

and use it to define $w(\mathbf{x})$ in (3.6) by

$$ew_0\left(\frac{2(f(\mathbf{x})-n_1)}{n_1}\right)$$

so that (3.3) becomes $r_3(n_1)$ itself, whence, setting

$$M = M_1 = M_1(n_1) = n_1^{\frac{1}{2}}, \quad \mathbf{x} = M_1\mathbf{x}' \tag{3.7}$$

and then removing the prime from the notation after the substitution, we deduce that

$$r_3(n_1) = c_{M_1}M_1\sum_{\mathbf{m}}\sum_{k=1}^{\infty}\frac{1}{k^3}Q_{n_1}(\mathbf{m},k)J_{M_1}\left(\frac{M_1\mathbf{m}}{k}, k\right) \tag{3.8}$$

[1]The stipulation that $M > 1$ given in H is easily weakened to a non-strict inequality.

where

$$J_{M_1}(\mathbf{u},t) = e\int w_0\{2(f(\mathbf{x})-1)\}h\left(\frac{t}{M_1}, f(\mathbf{x})-1\right)e^{-2\pi i\mathbf{ux}}d\mathbf{x}. \tag{3.9}$$

Here, since we may assume that $|f(\mathbf{x})-1| \le \frac{1}{2}$ in the integrand above, the trivial part of Lemma 4 in H implies that

$$J_{M_1}(\mathbf{u},k) = 0 \tag{3.10}$$

for $k > M_1$ so that the summation over k may be limited to the range $k \le M_1$ when desired.

If we anticipate the evaluation of $J_{M_1}(\mathbf{u},k)$ to be shortly undertaken, it can be seen that the component of $r_3(n_1)$ answering to the determination of $\mathbf{m}$ as 0 resembles the principal term in the asymptotic formula for $r_3(n_1)$ we would usually expect, even though the identification is necessarily inexact because of the absence of terms related to values of k exceeding M_1. Foreseeing therefore that the contribution due to other $\mathbf{m}$ should at least be commonly negligible, we write (3.8) as

$$\begin{aligned} &r_3(n_1) \\ &= c_{M_1}M_1\left(\sum_{k=1}^{\infty}\frac{1}{k^3}\ Q_{n_1}(0,k)J_{M_1}(0,k) + \sum_{\mathbf{m}\ne 0}\sum_{k=1}^{\infty}\frac{1}{k^3}\ Q_{n_1}(\mathbf{m},k)J_{M_1}\left(\frac{M_1\mathbf{m}}{k},k\right)\right) \\ &= C_{M_1}M_1\Big(\theta_A(n_1)+\theta_B(n_1)\Big), \end{aligned} \tag{3.11}$$

in which, apart from a multiplicative constant, $\theta_A(n_1)$ may be regarded as an analogue of $\theta(n_1)$ defined in I(2) but in which $\theta_B(n_1)$ introduces an element having no parallel in the previous analysis.

The preliminary study of $r_3(n_1)$ being complete, we treat the integrals $J_{M_1}(\mathbf{u},k)$ in the next section before going on to the sum $Q_{n_1}(0,k)$; however, we reserve the study of $Q_{n_1}(\mathbf{m},k)$ for $\mathbf{m}\ne 0$ till later, since in this case these sums are very different from $Q_{n_1}(0,k)$ and relate to the influence of $\theta_B(n_1)$ on the proceedings.

4. The integrals $J_{M_1}(\mathbf{u},k)$. We first consider the case where $\mathbf{u}=0$. Although our main formula could be drawn from the relevant parts of H, it is more illuminating and inherently easier in present circumstances to adopt a different and more direct approach, especially as it serves as a good pathway to the treatment of the other case $\mathbf{u}\ne 0$. We use the substitution

$$x_1 = a_1^{-\frac{1}{2}}\rho\sin\theta\sin\phi, \quad x_2 = a_2^{-\frac{1}{2}}\rho\cos\theta\sin\phi, \quad x_3 = a_3^{-\frac{1}{2}}\rho\cos\phi,$$

that expresses rectangular Cartesian coordinates in terms of modified spherical polar coordinates, the Jacobian being $(a_1a_2a_3)^{-\frac{1}{2}}\rho^2\sin\phi$. Therefore, since $f(\mathbf{x})-1=\rho^2-1$ by (3.1) so that we may assume that $\sqrt{(1/2)}\le\rho\le\sqrt{(3/2)}$, we deduce from (3.9) that

$$\begin{aligned} J_{M_1}(0,k) &= \frac{1}{\sqrt{a_1a_2a_3}}\int_{\sqrt{(1/2)}}^{\sqrt{(3/2)}}\rho^2ew_0(2\rho^2-2)h\left(\frac{k}{M_1},\rho^2-1\right)d\rho\int_0^{2\pi}d\theta\int_0^{\pi}\sin\phi d\phi \\ &= \frac{4\pi}{\sqrt{a_1a_2a_3}}\int_{\sqrt{(1/2)}}^{\sqrt{(3/2)}}\rho^2ew_0(2\rho^2-2)h\left(\frac{k}{M_1},\rho^2-1\right)d\rho \\ &= \frac{2\pi}{\sqrt{a_1a_2a_3}}\int_{-1/2}^{\frac{1}{2}}(\sigma+1)^{\frac{1}{2}}ew_0(2\sigma)h\left(\frac{k}{M_1},\sigma\right)d\sigma. \end{aligned} \tag{4.1}$$

From this, by H, Lemma 9, it follows in the first place that

$$J_{M_1}(0,k) = \frac{2\pi e w_0(0)}{\sqrt{a_1a_2a_3}} + O\left\{\left(\frac{k}{M_1}\right)^A\right\} = \frac{2\pi}{\sqrt{a_1a_2a_3}} + O\left\{\left(\frac{k}{M_1}\right)^A\right\} \tag{4.2}$$

for $k \leq M_1$, which relation remains (trivially) true in the opposite case by (3.10).

For $1 \leq t \leq M_1$, we shall also require an estimate for $J'_{M_1}(0,t)$. Although this could be taken from H, Lemma 16, it is less demanding in principle to begin with the equation

$$\frac{dJ_{M_1}(0,t)}{dt} = \frac{2\pi}{M_1\sqrt{a_1a_2a_3}} \int_{-\frac{1}{2}}^{\frac{1}{2}} (\sigma+1)^{\frac{1}{2}} e w_0(2\sigma) \left(\frac{\partial h(v,\sigma)}{\partial v}\right)_{v=k/M_1} d\sigma,$$

and to use the estimate

$$O\left\{\min^2\left(\frac{M}{t}, \frac{1}{\sigma}\right)\right\}$$

that follows from taking the value 2 for N in H,Lemma 5. The required estimate

$$\frac{dJ_{M_1}(0,t)}{dt} = O\left(\frac{M_1}{t^2}\int_0^{t/M_1} d\sigma\right) + O\left(\frac{1}{M_1}\int_{t/M_1}^{\infty}\frac{d\sigma}{\sigma^2}\right) = O\left(\frac{1}{t}\right) \tag{4.3}$$

then ensues for $1 \leq t \leq M_1$.

So far in this section we have had the option of appealing directly to Heath-Brown's lemmata instead of providing our own treatments. But this luxury is no longer available to us when we consider $J_{M_1}(\mathbf{u},k)$ for $\mathbf{u} \neq 0$ because the results provided in H for this case are not keen enough for our purposes. Therefore, somewhat as before, we take the case $k \leq M_1$ and initially advance by inducing the preliminary substitution

$$x_1 = a_1^{-\frac{1}{2}}x'_1, \quad x_2 = a_2^{-\frac{1}{2}}x'_2, \quad x_3 = a_3^{-\frac{1}{2}}x'_2$$

that transforms $f(\mathbf{x})$ into $x_1'^2 + x_2'^2 + x_3'^2$ and $\mathbf{ux}$ into $\mathbf{u}'\mathbf{x}\,'$ where

$$\mathbf{u}\,' = \left(a_1^{-\frac{1}{2}}u_1, a_2^{-\frac{1}{2}}u_2, a_3^{-\frac{1}{2}}u_3\right). \tag{4.4}$$

Then, interpreting x'_1, x'_2, x'_3 as the rectangular Cartesian coordinates of a point in three dimensions, let us take a new system of coordinates x''_1, x''_2, x''_3 for which the x''_3 plane is the plane $\mathbf{u}\,'\mathbf{x}\,' = 0$ with the consequence that $||\mathbf{u}'||x''_3 = \mathbf{u}\,'\mathbf{x}\,'$ and $f(\mathbf{x}) = x_1''^2 + x_2''^2 + x_3''^2$. Hence, if we express x''_1, x''_2, x''_3 in terms of spherical polar coordinates ρ, θ, ϕ we deduce from (3.9) that

$$\begin{aligned} &J_{M_1}(\mathbf{u},k) \\ &= \frac{1}{\sqrt{a_1a_2a_3}} \int_{\sqrt{(1/2)}}^{\sqrt{3/2}} \rho^2 e w_0(2\rho^2-2) k\left(\frac{k}{M_1}, \rho^2-1\right) d\rho \int_0^{\pi} \sin\phi\, e^{2\pi i \rho ||\mathbf{u}'|| \cos\phi} d\phi \int_0^{2\pi} d\theta \\ &= \frac{2}{||\mathbf{u}\,'||\sqrt{a_1a_2a_3}} \int_{\sqrt{(1/2)}}^{\sqrt{(3/2)}} \rho e w_0(2\rho^2-2) h\left(\frac{k}{M_1}, \rho^2-1\right) \sin 2\pi\rho||\mathbf{u}\,'|| d\rho. \end{aligned} \tag{4.5}$$

To estimate the last integral above on the right, we set $k_1 = k/M_1 \le 1$ for convenience and use successive partial integrations together with the estimate

$$\frac{\partial^m h(r,y)}{\partial y^m} = O\left\{\frac{1}{r^{1+m}} \min\left(1, \frac{r^2}{y^2}\right)\right\}$$

that for $0 \le r < 1$ and $0 < y \le \frac{1}{2}$ is certainly valid by H, Lemma 5, even when $m = 0$. Thence, since all derivatives with respect to ρ of $\rho e w_0(2\rho^2 - 2)h(k_1, \rho^2 - 1)$ vanish at $\rho = \sqrt{1/2}$ and $\rho = \sqrt{3/2}$ and since the mth derivative is

$$O\left\{\frac{1}{k_1^{1+m}} \min\left(1, \frac{k_1^2}{(\rho^2-1)^2}\right)\right\},$$

m partial integrations shew that the integral is

$$O\left\{\frac{1}{||\mathbf{u}\,'||^m} \int_1^{\sqrt{3/2}} \frac{1}{k_1^{1+m}} \min\left(1, \frac{k_1^2}{(\rho^2-1)^2}\right) d\rho\right\}$$
$$= O\left\{\frac{1}{||\mathbf{u}\,'||^m k_1^{m+1}} \left(\int_1^{1+k_1} d\rho + k_1^2 \int_{1+k_1}^{\infty} \frac{d\rho}{(\rho-1)^2}\right)\right\} = O\left(\frac{1}{||\mathbf{u}\,'||^m k_1^m}\right).$$

Combined with (4.5) and (4.4), this yields

$$J_{M_1}(\mathbf{u}, k) = O\left(\frac{1}{||\mathbf{u}||^{m+1} k_1^m}\right),$$

which provides the estimate

$$J_{M_1}\left(\frac{M_1 \mathbf{m}}{k}, k\right) = O\left(\frac{k}{M_1 ||\mathbf{m}||^A}\right) \qquad (\mathbf{m} \ne 0) \tag{4.6}$$

that we shall use (it is of course trivially true when $k > M_1$).

5. The sums $Q_{n_1}(0,k)$ and the singular series for $r_3(n_1)$. If we write

$$\frac{1}{k^3} Q_{n_1}(0,k) = A_3(n_1,k)$$

and define as usual the Gauss sum $S_2(c,k)$ by

$$S_2(c,k) = \sum_{0<b\le k} e^{2\pi i c b^2/k},$$

then (3.5) and (3.1) imply that

$$A_3(n_1,k) = \frac{1}{k^3} \sum_{\substack{0<h\le k \\ (h,k=1)}} S_2(ha_1,k) S_2(ha_2,k) S_2(ha_3,k) e^{-2\pi i h n_1/k} \tag{5.1}$$

is the kth term of the (as yet) formal singular series [2] for $r_3(n_1)$ in accordance with the usual development of the circle method by Hardy and Littlewood. Thus, in particular, $A_3(n_1,k)$ is a multiplicative function of k. Also, as will be essential for

[2]Note the comment about the meaning of the term singular series in footnote 1 in I.

part of our deliberations, $A_3(n_1, k)$ can be evaluated in terms of the number $\rho_3(n_1, d)$ of incongruent solutions, mod d, of the congruence

$$f(X_1, X_2, X_3) \equiv n_1, \quad \text{mod } d,$$

by means of the usual formula

$$A_3(n_1, k) = \sum_{d|k} \mu\left(\frac{k}{d}\right) \frac{\rho_3(n_1, d)}{d^2}. \tag{5.2}$$

This will be applied fairly directly to our problem for relatively small values of k but will need to be superseded by other formulae when k is larger.

In the latter situation it is helpful to have a universal bound for $A_3(n_1, k)$ that will be especially useful when k is a product g, say, of powers of the prime divisors of $2a_1a_2a_3$. This stems from the obvious relation

$$\begin{aligned} S_2(ha_i, k) &= (a_i, k)S_2(ha_i/(a_i, k), k/(a_i, k)). \\ &= O\left\{(a_i, k)^{\frac{1}{2}}k^{\frac{1}{2}}\right\} = O(k^{\frac{1}{2}}) \quad \{(h, k) = 1\} \end{aligned} \tag{5.3}$$

and is the consequential bound

$$A_3(n_1, k) = O\left(\frac{1}{k^{\frac{1}{2}}}\right). \tag{5.4}$$

But, for numbers k prime to $2a_1a_2a_3$ that are denoted by k_1 or k_1', we have another exact formula for $A_3(n_1, k)$ that arises from its comparison with the general term

$$A_1(n_2, k) = \frac{1}{k} \sum_{\substack{0<h\le k \\ (h,k)=1}} S_2(h, k)e^{-2\pi i h n_2/k}$$

of the purely formal singular series associated with the equation $X^2 = n_2$. In fact, since

$$S_2(ha_i, k_1) = \left(\frac{a_i}{k_1}\right) S_2(h, k_1) \qquad (i = 1, 2, 3)$$

when $(h, k_1) = 1$, we have

$$\begin{aligned} A_3(n_1, k_1) &= \frac{1}{k_1^3}\left(\frac{a_1a_2a_3}{k_1}\right) \sum_{\substack{0<h\le k_1, \\ (h,k_1)=1}} S_2^3(h, k_1)e^{-2\pi i h n_1/k_1} \\ &= \frac{1}{k_1^2}\left(\frac{a_1a_2a_3}{k_1}\right)\left(\frac{-1}{k_1}\right) \sum_{\substack{0<h\le k, \\ (h,k_1)=1}} S_2(h, k_1)e^{-2\pi i h n_1/k_1} \\ &= \frac{1}{k_1^2}\left(\frac{a_1a_2a_3}{k_1}\right) \sum_{\substack{0<h\le k_1 \\ (h,k_1)=1}} S_2(-h, k_1)e^{-2\pi i h n_1/k_1} \\ &= \frac{1}{k_1}\left(\frac{a_1a_2a_3}{k_1}\right) A_1(-n_1, k_1) \end{aligned} \tag{5.5}$$

by using further properties of Gauss sums as in the derivation of (9) in I. There is also the formula

$$A_1(n_2,k)=\sum_{d\mid k}\mu\left(\frac{k}{d}\right)\rho_1(n_2,d),$$

in which, being the number of incongruent roots of the congruence

$$X^2\equiv n_2,\quad \bmod d,$$

the entity $\rho_1(n_2,d)$ was systematically evaluated in our paper [4]. Consequently $A_1(-h_1,k_1)/k$ is the coefficient of k_1^{-s} in the formal series

$$\sum_{k_1}\frac{A_1(-n_1,k_1)}{k_1^{1+s}}=\frac{1}{\zeta_{2a_1a_2a_3}(1+s)}\sum_{k_1'}\frac{\rho_1(-n_1,k_1')}{k_1'^{1+s}}$$
$$=\frac{1}{\zeta_{2a_1a_2a_3}(2+2s)}\sum_{\substack{d^2\mid n,\\(d,2a_1a_2a_3)=1}}\frac{d}{d^{2s}}L_{-n_1/d^2}(1+s),$$

where

$$\zeta_{2a_1a_2a_3}(s)=\sum_{\substack{a=1\\(a,2a_1a_2a_3)=1}}^{\infty}\frac{1}{a^s}\text{ and }L_{-n_1/d^2}(s)=\sum_{\substack{b=1\\(b,2a_1a_2a_3)=1}}^{\infty}\left(\frac{-n_1/d^2}{b}\right)\frac{1}{b^s}.$$

Hence, extracting the value of $A_1(-n_1,k_1)$ from this and multiplying by $(a_1a_2a_3|k_1)/k_1$, we find from (5.5) that

$$A_3(n_1,k_1)=\sum_{\substack{a^2d^2b=k_1\\d^2\mid n_1}}\left(\frac{a_1a_2a_3}{a^2d^2b}\right)\frac{\mu^2(a)}{a^2}\cdot\frac{1}{d}\left(\frac{-n_1/d^2}{b}\right)\frac{1}{b}$$
$$=\sum_{\substack{a^2d^2b=k_1\\d^2\mid n_1}}\frac{\mu^2(a)}{a^2}\cdot\frac{1}{d}\left(\frac{-a_1a_2a_3n_1/d^2}{b}\right)\frac{1}{b}\tag{5.6}$$

and then conclude from the multiplicativity of $A(n_1,k)$ that

$$A_3(n_1,k)=\sum_{\substack{ga^2d^2b=k\\(adb,2a_1a_2a_3)=1\\d^2\mid n_1}}A_3(n_1,g)\frac{\mu^2(a)}{a^2}\cdot\frac{1}{d}\left(\frac{-a_1a_2a_3n_1/d^2}{b}\right)\frac{1}{b},\tag{5.7}$$

in which formula there is of course just one value (possibly 1) of g for each value of k.

A particular corollary of (5.6) will be needed during the preliminary study of the singular series for $\nu(n)$. This is that, if $p\nmid 2a_1a_2a_3$, then

$$\rho_3(n_1,p)=p^2\{1+A_3(n_1,p)\}=p^2\left(1+\frac{1}{p}\left(\frac{-a_1a_2a_3n_1}{p}\right)\right),\tag{5.8}$$

for which formula a companion will be produced for the case $p|a_1, p\nmid 2a_2a_3$ when the investigation of the singular series for $\nu(n)$ is resumed at the end.

In using the above work for larger values of k we shall need as in I to study the properties of characters defined by the Jacobi symbol. However, since these are needed for the disparate entities $A_3(n_1,k)$ and $\theta_B(n_1)$, it is appropriate to hold back the relevant analysis involving L-functions until a later section.

6. The singular series for $\nu(n)$. We have reached the stage corresponding to §4 in I where some results on the singular series $\mathfrak{S}(n)$ for $\nu(n)$ are needed in anticipation of the estimation of the first piece to be cut off from $\nu(n)$. Being a simple generalization of what was previously obtained in the special case $a_1 = a_2 = a_3 = a_4 = 1$, our requirements are easily met and therefore need not detain us for long.

The kth term in the singular series $\mathfrak{S}(n)$ being now

$$A(n,k) = \frac{1}{k^4} \sum_{\substack{0<h\leq k \\ (h,k)=1}} S_2(a_1h,k)S_2(a_2h,k)S_2(a_3h,k)S_l(a_4h,k)e^{-2\pi ihn/k} \tag{6.1}$$

where

$$S_l(c,k) = \sum_{0<b\leq k} e^{2\pi icb^l/k},$$

we let $\tau(n,d)$ denote the number of incongruent solutions of the congruence

$$a_1X_1^2 + a_2X_2^2 + a_3X_3^2 + a_4W^l \equiv n, \quad \bmod d,$$

and deduce in the customary way that $A(n,k)$ is a multiplicative function of k and that

$$A(n,k) = \sum_{d|k} \mu\left(\frac{k}{d}\right)\frac{\tau(n,d)}{d^3}. \tag{6.2}$$

From this and (5.8), it follows for $p \nmid 2a_1a_2a_3$ that

$$\begin{aligned} p^3A(n,p) = \tau(n,p) - p^3 &= \sum_{0<W\leq p} \{\rho_3(n - a_4W^l, p) - p^2\} \\ &= p\left(\frac{a_1a_2a_3}{p}\right) \sum_{0<W\leq p} \left(\frac{a_4W^l - n}{p}\right), \end{aligned}$$

in which the sum is (i) $O(p^{\frac{1}{2}})$ by a theorem due to Weil when $p \nmid a_4n$, (ii) is never more than p in absolute value. Hence we always have

$$A(n,p) = O\left(\frac{1}{p^{\frac{3}{2}}}\right) \quad (p \nmid n), \quad A(n,p) = O\left(\frac{1}{p}\right) \quad (p|n), \tag{6.3}$$

since the estimates are trivial when $p|2a_1a_2a_3a_4$.

When $\alpha > 1$ sharp bounds for $A(n,p^\alpha)$ are not required and we therefore act as in the second part of §4, I, using the special case

$$S_2(a_ih,p^\alpha) = O\left(p^{\frac{1}{2}\alpha}\right) \qquad ((h,p) = 1, i = 1,2,3)$$

of (5.3) and its analogue

$$S_l(a_4h,p^\alpha) = O(p^{\alpha-1}) \qquad (\alpha > 1, (h,p) = 1)$$

that proceeds from a well-known estimate for generalized Gauss sums. Absorbed in (6.1), these yield the adequate bound

$$A(n,p^\alpha) = O\left(p^{-\frac{1}{2}\alpha-1}\right), \tag{6.4}$$

which with (6.3),(6.2), and the Möbius inversion formula implies that

$$\begin{aligned}\frac{\tau(n,p^\alpha)}{p^{3\alpha}} &\leq 1+\sum_{1\leq\beta\leq\alpha}|A(n,p^\beta)| \\ &\leq 1+\frac{B_1(p,n)^{\frac{1}{2}}}{p^{\frac{3}{2}}}+B_1\sum_{\beta\geq 2}\frac{1}{p^{\frac{1}{2}\beta+1}} \\ &< 1+\frac{B_2(p,n)^{\frac{1}{2}}}{p^{\frac{3}{2}}}.\end{aligned}$$

Thus we deduce that

$$\tau(n,l)=O\left\{l^3\prod_{p|l}\left(1+\frac{B_2}{p}\right)\right\}=O\left\{l^3\sigma_{-\frac{1}{2}}(l)\right\} \tag{6.5}$$

and confirm through Euler's multiplicative principle that the singular series $\mathfrak{S}$ is absolutely convergent.

7. Decomposition of $\nu(n)$ and estimation of $\nu_1(n)$. We are ready to estimate the first tranche $\nu_1(n)$ of the sum $\nu(n)$, the definition of which follows from setting

$$N=n^{\delta_1} \tag{7.1}$$

for a suitably small positive number δ_1 and writing $\theta_A(n_1)$ in (3.11) for $n_1\leq n$ as [3]

$$\sum_{k\leq N}A_3(n_1,k)J_{M_1}(0,k)+\sum_{k>N}A_3(n_1,k)J_{M_1}(0,k)=\theta_1(n_1)+\theta_2(n_1) \tag{7.2}$$

by analogy with I(14). Then $\nu_1(n)$ emerges as one of the constituents in the equation

$$\begin{aligned}\nu(n) &= \sum_{a_1X_1^2+a_2X_2^2+a_3X_3^2+a_4W^l=n} 1 \\ &= \sum_{W\leq(n/a_4)^{\frac{1}{l}}} r_3(n-a_4W^l) \\ &= \sum_{W<(n/a_4)^{\frac{1}{l}}} r_3(n-a_4W^l)+O(1) \\ &= \sum_{W<(n/a_4)^{\frac{1}{l}}} c_{M_1}(n-a_4W^l)^{\frac{1}{2}}\theta_1(n-a_4W^l) \\ &\quad+\sum_{W<(n/a_4)^{\frac{1}{l}}} c_{M_1}(n-a_4W^l)^{\frac{1}{2}}\theta_2(n-a_4W^l) \\ &\quad+\sum_{W<(n/a_4)^{\frac{1}{l}}} c_{M_1}(n-a_4W^l)^{\frac{1}{2}}\theta_B(n-a_4W^l)+O(1) \\ &= \nu_1(n)+\nu_2(n)+\nu_B(n)+O(1),\text{ say,}\end{aligned} \tag{7.3}$$

that is stated on the understanding that $n_1=n-a_4W^l$ and that the first part of (3.7) shall hold. Already clearly analogous to its namesake in I, this entity must now

[3]Note that $\theta_1(n_1)$ and $\theta_2(n_1)$ depend on both n_1 and n.

be thrown into a slightly different form to enhance the resemblance and thus promote its estimation by previous methods.

First, by (4.2) and the succeeding comment, we may replace $J_{M_1}(0,k)$ in the formula for $r_3(n - a_4W^l)$ by

$$\frac{2\pi}{\sqrt{a_1a_2a_3}}\left\{1 + O\left(\frac{k}{(n-a_4W^l)^{\frac{1}{2}}}\right)\right\}$$

even when $k > n - a_4W^l \geq 1$, while also using the special case

$$c_{M_1} = 1 + O\left(\frac{1}{(n-a_4W^l)^{\frac{1}{2}}}\right)$$

of (3.4). Consequently, by (7.3) and (7.2),

$$\begin{aligned}
\nu_1(n) &= \frac{2\pi}{\sqrt{a_1a_2a_3}} \sum_{W<(n/a_4)^{\frac{1}{l}}} (n-a_4W^l)^{\frac{1}{2}} \sum_{k\leq N} A_3(n-a_4W^l, k) \\
&\quad + O\left(\sum_{W<(n/a_4)^{\frac{1}{l}}} \sum_{k\leq N} k|A_3(n-a_4W^l,k)|\right) \\
&= \frac{2\pi}{\sqrt{a_1a_2a_3}} \sum_{k\leq N} \sum_{W<(n/a_4)^{\frac{1}{l}}} (n-a_4W^l)^{\frac{1}{2}} A_3(n-a_4W^l,k) \\
&\quad + O\left(\sum_{W<(n/a_4)^{\frac{1}{l}}} \sum_{k\leq N} k^{\frac{1}{2}}\right) \\
&= \frac{2\pi}{\sqrt{a_1a_2a_3}} \sum\nolimits_1 + O\left(n^{\frac{1}{l}} N^{\frac{3}{2}}\right), \text{say,}
\end{aligned} \tag{7.4}$$

because of the universal bound (5.4).

The main part of $\nu_1(n)$ having been identified, its estimation follows that of the parallel item in I almost verbatim if one substitute the new meanings of $\rho_3(n,d), A_3(n,k), \tau(n,d)$, and $A(n,k)$ given by (5.2) and (6.2) for those previously assigned. Indeed, by the latter cited equation we now have, as the analogue of I(26),

$$\sum\nolimits_1 = \sum_{k\leq N} \sum_{d|k} \mu\left(\frac{k}{d}\right) \frac{1}{d^2} \sum\nolimits_{d,1}, \tag{7.5}$$

where the inner sum in the last term of the equation

$$\begin{aligned}
\sum\nolimits_{d,1} &= \sum_{W<(n/a_4)^{\frac{1}{l}}} \rho_3(n-a_4W^l,d)(n-a_4W^l)^{\frac{1}{2}} \\
&= \sum_{0<c\leq d} \rho_3(n-a_4c^l,d) \sum_{\substack{W<(n/a_4)^{\frac{1}{l}} \\ W\equiv c, \mod d}} (n-a_4W^l)^{\frac{1}{2}}
\end{aligned}$$

is estimated as

$$\frac{1}{d}\int_0^{(n/a_4)^{\frac{1}{l}}}(n-a_4u^l)^{\frac{1}{2}}du+O\left(n^{\frac{1}{2}}\right)=\frac{1}{da_4^{\frac{1}{l}}}\int_0^{n^{1/l}}(n-u'^l)^{\frac{1}{2}}du'+O(n^{\frac{1}{2}})$$
$$=\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{l}+1\right)}{da_4^{\frac{1}{l}}\Gamma\left(\frac{3}{2}+\frac{1}{l}\right)}n^{\frac{1}{2}+\frac{1}{l}}+O\left(n^{\frac{1}{2}}\right).$$

Having gained the counterpart of (27) in I, we continue as in I by deducing via the definition of $\tau(n,d)$ that

$$\sum\nolimits_{d,1}=\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{l}+1\right)n^{\frac{1}{2}+\frac{1}{l}}}{a_4^{\frac{1}{l}}\left(\frac{3}{2}+\frac{1}{l}\right)}\cdot\frac{\tau(n,d)}{d}+O\left\{n^{\frac{1}{2}}\tau(n,d)\right\}$$

and then infer from (7.5) and (6.2) that

$$\sum\nolimits_1=\frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{l}+1)n^{\frac{1}{2}+\frac{1}{l}}}{a_4^{\frac{1}{l}}\Gamma\left(\frac{3}{2}+\frac{1}{l}\right)}\sum_{k\le N}\sum_{d\mid k}\mu\left(\frac{k}{d}\right)\frac{\tau(n,d)}{d^3}+O\left(n^{\frac{1}{2}}\sum_{k\le N}\sum_{d\mid k}\frac{\tau(n,d)}{d^2}\right)$$
$$=\frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{l}+1)n^{\frac{1}{2}+\frac{1}{l}}}{a_4^{\frac{1}{l}}\Gamma\left(\frac{3}{2}+\frac{1}{l}\right)}\sum_{k\le N}A(n,k)+O\left(n^{\frac{1}{2}}N\sum_{d\le N}\frac{\tau(n,d)}{d^3}\right) \tag{7.6}$$

just as in the derivation of I(28).

To round off this formula we still adhere to previous practice. By (6.5) the remainder term is

$$O\left(n^{\frac{1}{2}}N\sum_{d\le N}\sigma_{-\frac{1}{2}}(d)\right)=O\left(n^{\frac{1}{2}}N^2\right), \tag{7.7}$$

while the tail

$$\sum_{k>N}A(n,k)$$

of the singular series $\mathfrak{S}(n)$ is majorized by

$$\frac{1}{N^{\frac{1}{2}-\epsilon}}\sum_{k=1}^{\infty}k^{\frac{1}{2}-\epsilon}|A(n,k)|<\frac{1}{N^{\frac{1}{2}-\epsilon}}\prod_p\left(1+\frac{B_1(p,n)^{\frac{1}{2}}}{p^{1+\epsilon}}+\frac{B_1}{p}\sum_{\alpha=2}^{\infty}\frac{1}{p^{\alpha\epsilon}}\right)$$
$$<\frac{1}{N^{\frac{1}{2}-\epsilon}}\prod_p\left(1+\frac{B_2(\epsilon)}{p^{1+\epsilon}}\right)\prod_{p\mid n}B_3(\epsilon)$$
$$<\frac{B_4(\epsilon)\{B_3(\epsilon)\}^{\omega(n)}}{N^{\frac{1}{2}-\epsilon}}<\frac{B_5(\epsilon)}{N^{\frac{1}{2}-\epsilon}}$$

because of (6.3), (6.4), and the argument near the end of §5, I. Therefore, summing up the influence of this, (7.7), and (7.6) on (7.4), we complete the estimation of $\nu_1(n)$

by concluding that

$$\begin{aligned}\nu_1(n) &= \frac{2\pi\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{l}+1\right)}{\sqrt{a_1a_2a_3}\sqrt[l]{a_4}\,\Gamma\left(\frac{3}{2}+\frac{1}{l}\right)}\mathfrak{S}(n)n^{\frac{1}{2}+\frac{1}{l}} + O\left(n^{\frac{1}{l}}N^{\frac{3}{2}}\right) + O\left(n^{\frac{1}{2}+\frac{1}{l}}N^{-\frac{1}{2}+\epsilon}\right) \\ &\quad + O\left(n^{\frac{1}{2}}N^2\right) \\ &= \frac{2\pi\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{l}+1\right)}{\sqrt{a_1a_2a_3}\sqrt[l]{a_4}\,\Gamma\left(\frac{3}{2}+\frac{1}{l}\right)}\mathfrak{S}(n)n^{\frac{1}{2}+\frac{1}{l}} + O\left(n^{\frac{1}{2}+\frac{4}{5l}+\epsilon}\right) \qquad (7.8)\end{aligned}$$

after putting

$$\delta_1 = \frac{2}{5l} \qquad (7.9)$$

in (7.1).

8. Real characters and the Dirichlet's L-functions formed therefrom. The treatments of both $\nu_2(n)$ and $\nu_B(n)$ involve, in rather different ways, the real Dirichlet characters defined by the Jacobi symbol and the properties of the L-functions associated with them. Therefore, slightly widening the previous context of §3, I, we express a given integer n_2, positive or negative, as $D\Omega^2$ where D is square-free and, for any odd positive divisor d of Ω, enumerate some attributes of the function [4]

$$\chi(b, n_2/d^2) = \begin{cases} ((-n_2/d^2)|b), & \text{if } b \text{ odd,} \\ 0, & \text{if } b \text{ even.} \end{cases} \qquad (8.1)$$

These are:

(i) $\chi(b, n_2/d^2)$ is a non-principal character to a modulus not exceeding $4|n_2|$ unless $D = -1$, in which case it is a principal character;

(ii) if $D \neq -1$, then $\chi(b, n_2/d^2)$ is associated with a primitive character, the modulus of which is $2|D|, 4|D|,$ or $4|D|$ according as $D \equiv 3, \mod 4, D \equiv 1, \mod 4,$ or D is even; consequently, any such primitive character cannot correspond to different values of D of the same sign (nor, indeed, of the opposite sign, since a change in the character accompanies a change of D into $-D$);

(iii) a unique primitive character is associated with all the characters $\chi(b, n_2/d^2)$ for any given value of n_2 for which $D \neq -1$.

The properties required of the Dirichlet's series formed with these characters are covered by the following two results, the first of which is a slight restatement of Lemma 2 in I.

LEMMA 1. *Let η_1, η_2 be any positive constants (less than 1) and suppose that $\eta_3 = \eta_3(\eta_1, \eta_2)$ is a sufficiently small positive constant. Then, save when the non-principal character χ_k, mod k, is associated with at most $O(Y^{\eta_1})$ exceptional primitive characters χ_q^*, mod q, we have*

$$\sum_{y_1 < m \leq y_2} \frac{\chi_k(m)}{m} = O\left(\frac{1}{Y^{\eta_3}}\right)$$

[4] Note the change from the notation in I to avoid any confusion in the interpretation of Lemmata 1 and 2 below.

for $k \leq 4Y$ *and* $AY^{\eta_2} \leq y_1 < y_2 \leq Y$.

Our second lemma will be needed during the estimation of $\nu_B(n)$ and depends on a similar order of ideas, which, being familiar to practicians in the subject, need only be treated briefly.

LEMMA 2. *Let* η_1 *be any positive constant (less than* 1*). Then, save when the non-principal character* χ_k *to a modulus* k *not exceeding* $4Y$ *is associated with at most* $O(Y^{\eta_1})$ *exceptional primitive characters, we have*

$$\sum_{p\leq u}\frac{\chi_k(p)}{p} = O(\log\log\log Y) \qquad (Y > 1000)$$

for $u \leq Y$.

We stay in the zero-free region found in the proof of Lemma 2 in I for functions $L(s,\chi_k)$ not associated with functions $L(s,\chi_q^*)$ appertaining to an exceptional set of moduli q of cardinality $O(Y^{\eta_1})$. Thus, confining attention throughout to the non-excluded L-functions and taking $T = 2Y^3$ as before, we may assume that $L(s,\chi_k)$ is regular and subject to the inequality [5]

$$|\log L(s,\chi_k)| < A\log Y$$

in the region

$$\sigma \geq 1 - \frac{1}{14}\eta_1, \quad |t| \leq \frac{1}{2}T.$$

In this environment let $\Lambda(n)$ be the von Mangoldt function and use the formulae

$$\sum_{p\leq u}\frac{\chi_k(p)}{p} = \sum_{n\leq u}\frac{\chi_k(n)\Lambda(n)}{n\log n} + O\left(\sum_{p\leq u}\frac{1}{p^2}\right) = \sum_{n\leq u}\frac{\chi_k(n)\Lambda(n)}{n} + O(1)$$

and

$$\sum_{n\leq u}\frac{\chi_k(n)\Lambda(n)}{n\log n} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\log L(s+1,\chi_k)\frac{u^s}{s}ds \qquad (c > 1).$$

Under the condition

$$\log^{30/\eta_1} Y \leq u \leq Y \tag{8.2}$$

and accompanied by a deformation of the contour of integration, these imply that

$$\begin{aligned}\sum_{p\leq u}\frac{\chi_k(p)}{p} &= \log L(1,\chi_k) + \frac{1}{2\pi i}\int_{-\frac{1}{14}\eta_1-\frac{1}{2}iT}^{-\frac{1}{14}\eta_1+\frac{1}{2}iT}\log L(s+1,\chi_k)\frac{u^s}{s}ds\\ &+O\left(\frac{u^2}{T}\max_{-\frac{1}{14}\eta_1\leq\sigma\leq 2}|\log L(1+\sigma+\frac{1}{2}iT,\chi_k)|\right)\\ &+O\left(\frac{u^2}{T}\sum_{m=1}^{\infty}\frac{|\chi_k(m)|}{m^3|\log(u/m)|}\right) + O(1),\end{aligned}$$

[5]We only need what was extracted from the Borel-Carathéodory theorem in I; the improvement rendered by Hadamard's three circles theorem is superfluous here.

whence, by (8.2), we infer that

$$\sum_{p\le u} \frac{\chi_k(p)}{p} = \log L(1,\chi_k) + O\left(u^{-\frac{1}{14}\eta_1} \log Y \log T\right)$$
$$+O\left(\frac{Y^2 \log Y}{T}\right) + O\left(\frac{Y^2}{T} \sum_{m=1}^{\infty} \frac{1}{m^2}\right) + O(1)$$
$$= \log L(1,\chi_k) + O\left(\log^{-\frac{1}{7}\eta_1} Y\right) + O\left(\frac{\log Y}{Y}\right) + O(1)$$
$$= \log L(1,\chi_k) + O(1)$$

subject to the initial restriction, then seen to be unnecessary, that $u - \frac{1}{2}$ be an integer. Therefore, by first selecting the lower limit for u, we conclude that

$$\log L(1,\chi_k) = O(\log\log\log Y)$$

and then that

$$\sum_{p\le u} \frac{\chi_k(p)}{p} = O(\log\log\log Y) + O(1) = O(\log\log\log Y)$$

when (8.2) is in place, the result being trivial for smaller values of u.

9. Estimation of $\nu_2(n)$. The estimation of $\nu_2(n)$ is sufficiently close to its counterpart in I that it is enough to portray it with a broad brush. Apart from the additional factor $a_1a_2a_3$ in the Jacobi symbol, the formula for $A_3(n - W^l, k)$ in I is mainly changed into our formula for $A_3(n - a_4W^l, k)$ in (5.7) above by letting the special number g take over the rôle previously played by powers of 2. As before, to dissect $A_3(n - a_4W^l, k)$ we set $n_1 = n - a_4W^l$, writing the right-side of (5.7) as

$$\sum_{a^2gd^2 > N^{\frac{1}{2}}} + \sum_{a^2gd^2 \le N^{\frac{1}{2}}}$$

and letting the respective contributions of these portions to $\theta_2(n_1)$ in (7.2) be $\theta_2'(n_1)$ and $\theta_2''(n_1)$. The effect of the former on $\nu_2(n)$ by way of (7.3) is then easily dismissed because the above cited equations, (5.4), (4.2), and (3.10) imply in succession that [6]

$$\theta_2'(n_1) = O\Bigg(\sum_{\substack{a^2gd^2 > N^{\frac{1}{2}} \\ d^2 | n_1}} \frac{|A_3(n_1, g)|}{a^2db} |J_{M_1}(0, a^2gd^2b)| \Bigg) = O\Bigg(\sum_{\substack{a^2gd^2b \le n_1^{\frac{1}{2}} \\ a^2gd^2 > N^{\frac{1}{2}} \\ d | n_1}} \frac{1}{a^2g^{\frac{1}{2}}db} \Bigg)$$
$$= O\Bigg(\log n \sum_{\substack{a^2gd^2 > N^{\frac{1}{2}} \\ d|n_1}} \frac{1}{a^2g^{\frac{1}{2}}d} \Bigg) = O\Bigg(\frac{\log\, n}{N^{\frac{1}{8}}} \sum_{\substack{a,g \\ d|n_1}} \frac{1}{a^{\frac{3}{2}}g^{\frac{1}{4}}d^{\frac{1}{2}}} \Bigg)$$
$$= O\left\{ \frac{\log nd(n_1)\sigma_{-\frac{1}{8}}(2a_1a_2a_3)}{N^{\frac{1}{8}}} \right\} = O\left(\frac{1}{N^{\frac{1}{9}}}\right), \tag{9.1}$$

[6]We note that Lemma 1 of I is not needed here and in similar places because the presence of $J_M(0, k)$ in the workings restricts the size of b.

the contribution of which to $\nu_2(n)$ is

$$O\left(\frac{1}{N^{\frac{1}{9}}}\sum_{W<(n/a_4)^{\frac{1}{l}}}(n-a_4W^l)^{\frac{1}{2}}\right)=O\left(n^{\frac{1}{2}+\frac{1}{l}-\frac{2}{45l}}\right)=O\left(n^{\frac{1}{2}+\frac{43}{45l}}\right). \tag{9.2}$$

by (7.1) and (7.9).

Alongside the earlier part of (9.1), there is the equation

$$\begin{aligned}&\theta_2''(n_1)\\ &=\sum_{\substack{a^2gd^2\leq N^{\frac{1}{2}}\\ (ad,2a_1a_2a_3)=1\\ d^2|n_1}}\frac{\mu^2(a)A_3(n_1,g)}{a^2d}\sum_{\substack{b>N/a^2gd^2\\ (b,2)=1}}\frac{1}{b}\left(\frac{-a_1a_2a_3n_1/d^2}{b}\right)J_{M_1}(0,a^2gd^2b)\\ &=\sum_{\substack{a^2gd\leq N^{\frac{1}{2}}\\ (ad,2a_1a_2a_3)=1\\ d^2|n_1}}\frac{\mu^2(a)A_3(n_1,g)}{a^2d}\sum_{\substack{N/a^2gd^2<b\leq n_1^{\frac{1}{2}}/a^2gd^2\\ (b,2)=1}}\frac{1}{b}\left(\frac{-a_1a_2a_3n_1/d^2}{b}\right)J_{M_1}(0,a^2gd^2b),\end{aligned} \tag{9.3}$$

for whose application we need only consider the inner sum for $n_1>N^2$ after reminding ourselves that $N=n^{\frac{2}{5l}}$. In this case let us first use Lemma 1 with the values $Y=a_1a_2a_3n$, $\eta_1=\frac{1}{3l}$, $\eta_2=\frac{1}{5l}$, and $\eta=\eta_3\left(\frac{1}{3l},\frac{1}{5l}\right)$. Then, since the lower bound y_1 for b lies between $N^{\frac{1}{2}}=n^{\eta_2}=(a_1a_2a_3)^{-\eta_2}Y^{\eta_2}$ and $n_1^{\frac{1}{2}}/a^2gd^2$, the related sum

$$s(y_1,y_2)=\sum_{\substack{y_1<b\leq y_2\\ (b,2)=1}}\left(\frac{-a_1a_2a_3n_1/d^2}{b}\right)\frac{1}{b}$$

is $O\left(n^{-\eta}\right)$ for $y_1<y_2\leq y_3=n_1^{\frac{1}{2}}/a^2gd^2$ provided that the (unique) primitive character associated with all the characters $\chi(b,a_1a_2a_3n_1/d^2)$ for given n_1 in (8.1) do not belong to an exceptional set with $O(n^{\frac{1}{3l}})$ members. In this situation the inner sum in (9.3) equals

$$\begin{aligned}&\int_{y_1}^{y_3}J_{M_1}(0,a^2gd^2t)ds(y_1,t)\\ &=\Big[J_{M_1}(0,a^2gd^2t)s(y_1,t)\Big]_{y_1}^{y_3}-\int_{y_1}^{y_3}\frac{dJ_{M_1}(0,a^2gd^2t)}{dt}s(y_1,t)dt\\ &=O\left(\frac{1}{n^\eta}\right)+O\left(\frac{1}{n^\eta}\int_{y_1}^{y_3}\frac{a^2gd^2dt}{a^2gd^2t}\right)\\ &=O\left(\frac{\log n}{n^\eta}\right)=O\left(\frac{1}{n^{\frac{2}{3}\eta}}\right)\end{aligned}$$

by (4.2) and (4.3). Hence, much as before,

$$\theta_2''(n_1) = O\left(\frac{1}{n^{\frac{2}{3}\eta}} \sum_{\substack{a^2gd^2 \le N^{\frac{1}{2}} \\ d^2|n_1}} \frac{\mu^2(a)}{a^2g^{\frac{1}{2}}d}\right) = O\left(\frac{1}{n^{\frac{2}{3}\eta}} \sum_{\substack{a,g \\ d^2|n_1}} \frac{1}{a^2g^{\frac{1}{2}}d}\right)$$
$$= O\left(\frac{d(n_1)}{n^{\frac{2}{3}\eta}}\right) = O\left(\frac{1}{n^{\frac{1}{2}\eta}}\right),$$

and the effect of this on $\nu_2(n)$ due to the relevant values of W is

$$O\left(\frac{1}{n^{\frac{1}{2}\eta}} \sum_{W<(n/a_4)^{\frac{1}{l}}} (n - a_4W^l)^{\frac{1}{2}}\right) = O\left(n^{\frac{1}{2}+\frac{1}{l}-\frac{1}{2}\eta}\right). \tag{9.4}$$

On the other hand, (9.3) implies that we always have the assessment

$$\theta_2''(n_1) = O\left(\log n \sum_{\substack{a^2gd^2 \le N^{\frac{1}{2}} \\ d^2|n_1}} \frac{1}{a^2g^{\frac{1}{2}}d}\right) = O\left(\log n d(n_1)\right) = O(n^{\epsilon}), \tag{9.5}$$

of which we avail ourselves when $\chi(b, a_1a_2a_3n_1)$ is associated with an exceptional primitive character of the type previously described. In this case, if

$$a_1a_2a_3n_1 = a_1a_2a_3(n - a_4W^l) = D\Omega^2, \tag{9.6}$$

then by (ii) in §8 there are at most $O(n^{\frac{1}{3l}})$ possible values of D, to each of which there will answer those solutions in W of the absolutely irreducible equation (9.6) for which $0 \le W < (n/a_4)^{\frac{1}{l}}$ and $0 < \Omega \le (a_1a_2a_3n)^{\frac{1}{2}}$. Since the number of these solutions is $O(n^{\frac{1}{2l}+\epsilon})$ by a theorem due to Bombieri and Pila [1], the set of W for which we must use (9.5) has cardinality $O(n^{\frac{5}{6l}+\epsilon})$ and therefore makes a donation of

$$O\left(n^{\frac{1}{2}+\frac{6}{7l}}\right) \tag{9.7}$$

to $\nu_2(n)$.

In summation, we deduce from (9.2), (9.4), and (9.7) that

$$\nu_2(n) = O\left(n^{\frac{1}{2}+\frac{1}{l}-\delta_2}\right) \tag{9.8}$$

for a suitably small positive constant δ_2.

10. The sum $Q_{n_1}(\mathbf{m}, k)$ for $\mathbf{m} \ne \mathbf{0}$. The sum $Q_{n_1}(\mathbf{m}, k)$ occurring in (3.5) belongs to the class of sums described in Lemma 3 of [6] that have a multiplicative property. From this, or from Lemma 23 of H, we have

LEMMA 3. *For coprime moduli k_1, k_2 let $\bar{k}_1, \bar{k}_2$ be defined,* modulis k_2 and k_1, *respectively,* by $k_1\bar{k}_1 \equiv 1, \mod k_2$ and $k_2\bar{k}_2 \equiv 1, \mod k_1$. *Then*

$$Q_{n_1}(\mathbf{m}, k_1k_2) = Q_{n_1}(\bar{k}_2\mathbf{m}, k_1)Q_{n_1}(\bar{k}_1\mathbf{m}, k_2).$$

Our examination of $Q_{n_1}(\mathbf{m}, k)$ for $\mathbf{m} \neq 0$ depends first on our recollecting the definition of $F(\mathbf{m})$ in (3.2) and using the lemma in respect of the representation $k = k'g$, where, in a modification to the notation in §5, $k' = k'_{\mathbf{m}}$ (with or without extra subscripts) denotes a number prime to $2a_1a_2a_3F(\mathbf{m})$ and $g = g_{\mathbf{m}}$ is a number whose prime factors all divide $2a_1a_2a_3F(\mathbf{m})$. Of the two sums that occur, the first of type $Q_{n_1}(\mathbf{m}, k')$ can be handled explicitly very accurately, whereas the second of type $Q_{n_1}(\mathbf{m}, g)$ is not amenable (at least fully) to the same method and is treated by our finding an all embracing bound for the sums $Q_{n_1}(\mathbf{m}, k)$ that serves our needs for the special moduli g. Both methods, however, rest on the properties of the generalized Gaussian sum

$$S(u, v; k) = \sum_{0<l\leq k} e^{2\pi i(ul^2+vl)/k} \tag{10.1}$$

in terms of which the formula (5.1) is extended to

$$\frac{1}{k^3}Q_{n_1}(\mathbf{m}, k) = \frac{1}{k^3}\sum_{\substack{0<h\leq k\\(h,k)=1}}\prod_{1\leq j\leq 3} S(a_jh, m_j; k)e^{-2\pi ihn_1/k} \tag{10.2}$$

in virtue of (3.5).

The universal bound for $Q_{n_1}(\mathbf{m}, k)$ depends on the assessment [7]

$$S(u, v; k) = O\{(u, k)^{\frac{1}{2}}k^{\frac{1}{2}}\};$$

this is not easily traceable in the literature but is easily verified by Weyl's method. Accordingly we at once gain the estimate

$$\frac{1}{k^3}Q_{n_1}(\mathbf{m}, k) = O\left(\frac{(a_1, k)^{\frac{1}{2}}(a_2, k)^{\frac{1}{2}}(a_3, k)^{\frac{1}{2}}}{k^{\frac{1}{2}}}\right) = O\left(\frac{1}{k^{\frac{1}{2}}}\right) \tag{10.3}$$

that actually includes (5.4).

But in the case where k is of type k' the formula (10.2) can be exploited more effectively because then the sum within it can be calculated by means of the Gauss sum through an obvious transformation, to facilitate which we let $\bar{b}$ denote a solution of $b\bar{b} \equiv 1, \mod k$, when $(b, k) = 1$. Indeed, only initially assuming that $(k, 2a_1a_2a_3) = 1$ and following a not unfamiliar line of attack, we express the argument in the summand of $S(ah, m; k)$ when $(2a, k) = 1$ as

$$ahl^2 + ml = ah(l^2 + 2.\bar{2}\bar{a}\bar{h}ml) = ah(l + \bar{2}\bar{a}\bar{h}m)^2 - \bar{4}\bar{a}\bar{h}m^2$$

and deduce that

$$S(ah, m; k) = S_2(ah, k)e^{-2\pi i\bar{4}\bar{a}\bar{h}m^2/k}.$$

Hence, since

$$S_2(ah, k) = \left(\frac{ah}{k}\right)S_2(1, k) = \left(\frac{ah}{k}\right)i^{\frac{1}{4}(k-1)^2}k^{\frac{1}{2}},$$

[7]The better bound $O\{(u, v, k)^{\frac{1}{2}}k^{\frac{1}{2}}\}$ is easily confirmed but does not confer any extra benefit here.

we find from (10.2) that

$$\frac{1}{k^3}Q_{n_1}(\mathbf{m},k) = \frac{1}{k^{\frac{3}{2}}}\left(\frac{a_1a_2a_3}{k}\right) i^{\frac{3}{4}(k-1)^2} \sum_{\substack{0<h\le k\\(h,k)=1}} \left(\frac{h}{k}\right) e^{2\pi i(-hn_1-\bar{4}(\bar{a}_1m_1^2+\bar{a}_2m_2^2+\bar{a}_3m_3^2)\bar{h})/k}$$

$$= \frac{1}{k^{\frac{3}{2}}}\left(\frac{a_1a_2a_3}{k}\right) i^{\frac{3}{4}(k-1)^2} T\left(-n_1, -\bar{4}\bar{a}_1\bar{a}_2\bar{a}_3F(\mathbf{m})\right), \qquad (10.4)$$

where

$$T(u,v;k) = \sum_{\substack{0<h\le k\\(h,k)=1}} \left(\frac{h}{k}\right) e^{2\pi i(uh+v\bar{h})/k}$$

is a generalized Kloosterman sum of Salié type. This has been known since the time of Salié to be amenable under various conditions to an explicit evaluation, of which we find it convenient to give our own quick account.

Here it is enough to suppose merely that $(2v,k)=1$ although what we find would be equally valid when $(2u,2v,k)=1$. First, having changed h into hv, mod k, to shew that

$$T(u,v;k) = \left(\frac{v}{k}\right) T(uv,1;k) = \pm T(uv,1;k), \qquad (10.5)$$

we arrive at $T(w,1;k)$ by treating the sum

$$Z(w,k) = \sum_{\substack{\nu^2\equiv w, \mod k\\0<\nu\le k}} e^{4\pi i\nu/k},$$

which equals

$$\frac{1}{k}\sum_{0<r\le k}\sum_{0<\nu\le k} e^{4\pi i\nu/k}e^{2\pi ir(\nu^2-w)/k} = \frac{1}{k}\sum_{0<r\le k} e^{-2\pi irw/k}\sum_{0<\nu\le k} e^{2\pi i(r\nu^2+2\nu)/k}.$$

If $(r,k)=k^*>1$, the inner sum equals

$$\sum_{0<\nu^*\le k/k^*} e^{2\pi ir\nu^{*2}/k} \sum_{\substack{\nu\equiv\nu^*, \mod k/k^*\\0<\nu\le k}} e^{4\pi i\nu/k} = 0$$

so that

$$Z(w,k) = \frac{1}{k}\sum_{\substack{0<r\le k\\(r,k)=1}} e^{-2\pi irw/k}\sum_{0<\nu\le k} e^{2\pi i\{r(\nu+\bar{r})^2-\bar{r}\}/k}$$

$$= \frac{1}{k}\sum_{\substack{0<r\le k\\(r,k)=1}} e^{-2\pi i(rw+\bar{r})/k}S_2(r,k)$$

$$= \frac{i^{\frac{1}{4}(k-1)^2}}{k^{\frac{1}{2}}}\sum_{\substack{0<r\le k\\(r,k)=1}} \left(\frac{r}{k}\right) e^{-2\pi i(rw+\bar{r})/k}$$

$$= \pm\frac{i^{\frac{1}{4}(k-1)^2}}{k^{\frac{1}{2}}}T(w,1;k)$$

by the usual determination of the Gauss sum. Thus

$$|T(w,1;k)| = k^{\frac{1}{2}}|Z(w,k)|. \tag{10.6}$$

On assuming that k is not only prime to $2a_1a_2a_3$ but also to $F(\mathbf{m})$, we infer from (10.4), (10.5), and (10.6) that

$$\frac{1}{k^3}|Q_{n_1}(\mathbf{m},k)| = \frac{1}{k}\Upsilon_{n_1}(1,k), \quad (\mathbf{m}\neq 0)$$

where

$$\Upsilon_{n_1}(h,k) = \Upsilon_{n_1,\mathbf{m}}(h,k) = |\sum_{G(\nu)\equiv 0, \mod k} e^{4\pi i h\nu/k}| \tag{10.7}$$

and

$$G(\nu) = G_{n_1,\mathbf{m}}(\nu) = 4a_1a_2a_3\nu^2 - n_1F(\mathbf{m}). \tag{10.8}$$

This is valid for numbers k of type k' but must be supplanted by (10.3) for those of type g. Hence, if $k = k'g$ and $g\bar{g}\equiv 1, \mod k'$, Lemma 3 gives

$$\frac{1}{k^2}Q_{n_1}(\mathbf{m},k) = O\left(\frac{g^{\frac{1}{2}}}{k'^2}|Q_{n_1}(\bar{g}\mathbf{m},k')|\right) = O\left(g^{\frac{1}{2}}\Upsilon_{n_1}(\bar{g},k')\right). \tag{10.9}$$

Moreover, it also shews that $\Upsilon_{n_1}(h,k')$ has the property that, if $(k_1',k_2') = (h,k_1'k_2') = 1$, then

$$\Upsilon_{n_1}(h,k_1'k_2') = \Upsilon_{n_1}(h\bar{k}_2',k_1')\Upsilon_{n_1}(h\bar{k}_1',k_2'),$$

as may be otherwise seen without restriction on h from Lemma 3 in our paper [5] whose method we shall shortly follow.

11. The sum $\nu_B(n)$ - the initial treatment. It is opportune to decompose the sum $\nu_B(n)$ in (7.3) into constituents upon which the theory of the previous section can be brought to bear. Consequently, returning to (3.11) and using (3.10) and (4.6), we have

$$\theta_B(n_1) = O\left(\sum_{\mathbf{m}\neq 0}\frac{1}{M_1||\mathbf{m}||^A}\sum_{k\leq M_1}\frac{1}{k^2}|Q_{n_1}(\mathbf{m},k)|\right)$$

and then, [8] by (10.9),

$$\theta_B(n_1) = O\left(\sum_{\mathbf{m}\neq 0}\frac{1}{M_1||\mathbf{m}||^A}\sum_{gk'\leq M_1} g^{\frac{1}{2}}\Upsilon_{n_1}(\bar{g},k')\right) = O\left(\frac{1}{M_1}\sum_{\mathbf{m}\neq 0}\frac{\theta_{\mathbf{m}}(n_1)}{||\mathbf{m}||^A}\right), \text{say}, \tag{11.1}$$

[8]Whether or not the series occurring below were finite, the majoration would be valid under usual conventions regarding series with positive terms even though we would obviously only use it in the foreknowledge that the former case was in place.

where it is still to be understood that $M_1 = n_1^{\frac{1}{2}}$ and that n_1 will become $n - a_4W^l$. Hence, by (7.3) and (3.4),

$$\nu_B(n) = O\left(\sum_{\mathbf{m}\neq 0}\frac{1}{||\mathbf{m}||^A}\sum_{W<(n/a_4)^{\frac{1}{l}}}\theta_{\mathbf{m}}(n-a_4W^l)\right) = O\left(\sum_{\mathbf{m}\neq 0}\frac{\nu_{\mathbf{m}}(n)}{||\mathbf{m}||^A}\right), \text{say.} \tag{11.2}$$

Having completed the dissection of the sum, we go on to the sum $\theta_{\mathbf{m}}(n-a_4W^l)$ whose treatment depends on the size of $||\mathbf{m}||$ and whether or not W belong to an exceptional set.

12. Estimations of $\theta_{\mathbf{m}}(n_1)$ and $\nu_B(n)$. In anticipation of these estimations, we let $\rho(k) = \rho_{n_1,\mathbf{m}}(k)$ (not to be confused with $\rho_3(n_1, d)$ and $\rho_1(n_2, d)$) be the number of incongruent roots, mod k, of the congruence

$$G(\nu) \equiv 0, \quad \text{mod } k,$$

defined by (10.8), still assuming as in §11 that $\mathbf{m} \neq 0$. Then, by the theory of quadratic congruences as described for example in [4], we have

LEMMA 4. *If $p \nmid 2a_1a_2a_3F(\mathbf{m})$, then*

$$\rho(p^\alpha) \leq \begin{cases} 2, & \text{if } \alpha = 1, \\ 2p^{\frac{1}{2}\alpha}, & \text{if } \alpha > 1, \end{cases}$$

always, while

$$\rho(p) = 1 + \left(\frac{a_1a_2a_3n_1F(\mathbf{m})}{p}\right)$$

and, under the additional condition $p \nmid n_1, \rho(p^\alpha) = \rho(p)$ for $\alpha \geq 1$. Also $\rho(k)$ is a multiplicative function.

We first find an always valid bound for the sums

$$R(g, u) = \sum_{k'\leq u}\Upsilon(\bar{g}, k')$$

that will be both an auxiliary tool and also a surrogate for more accurate bounds which may fail when either $||\mathbf{m}|| > n^{\frac{1}{8}}$ or W is exceptional. This follows from the inequality $\Upsilon(g, k') \leq \rho(k')$ and unfolds as

$$\begin{aligned} R(g,u) &\leq \sum_{k'\leq u}\rho(k') \leq u\sum_{k'\leq u}\frac{\rho(k')}{k'} \leq u\prod_{p\leq u}\left(1+\frac{\rho(p)}{p}+\frac{\rho(p^2)}{p^2}+.....\right) \\ &\leq u\prod_{p\leq u}\left\{1+\frac{4}{p}+O\left(\frac{1}{p^{\frac{3}{2}}}\right)\right\} \\ &= O\left\{u\prod_{p\leq u}\left(1+\frac{1}{p}\right)^4\right\} = O(u\log^4 2u) \end{aligned} \tag{12.1}$$

in virtue of Lemma 4. Incorporated in the equation

$$\theta_{\mathbf{m}}(n_1) = \sum_{g\leq M_1} g^{\frac{1}{2}}R(g, M_1/g) \tag{12.2}$$

implicit in (11.1), this yields

$$\begin{aligned}\theta_{\mathbf{m}}(n_1) &= O\left(M_1\log^4 2n_1 \sum_{g\le M}\frac{1}{g^{\frac12}}\right) = O\left\{M_1\log^4 2n_1 \prod_{p|2a_1a_2a_3F(\mathbf{m})}\left(1-\frac{1}{p^{\frac12}}\right)^{-1}\right\}\\ &= O\left\{M_1\log^4 2n_1\ \sigma_{-\frac14}(2a_1a_2a_3F(\mathbf{m}))\right\} = O\left(M_1||\mathbf{m}||^{\epsilon}\log^4 2n_1\right) \qquad (12.3)\end{aligned}$$

as a bound without restricting conditions. Also, in preparation for a more accurate assessment of $\theta_{\mathbf{m}}(n_1)$, we rewrite (12.2) as

$$\theta_{\mathbf{m}}(n_1) = \sum_{g\le M_1^{\frac12}} g^{\frac12}R(g,M_1/g) + \sum_{M_1^{\frac12}<g\le M_1} g^{\frac12}R(g,M_1/g) = \sum\nolimits_2 + \sum\nolimits_3, \text{say}, \quad (12.4)$$

wherein

$$\begin{aligned}\sum\nolimits_3 &= O\left(M_1\log^4 2n_1 \sum_{g>M_1^{\frac12}}\frac{1}{g^{\frac12}}\right) = O\left(\frac{M_1\log^4 2n_1}{M_1^{\frac18}}\sum_g \frac{1}{g^{\frac14}}\right)\\ &= O\left(M_1^{\frac78}\log^4 2n_1||\mathbf{m}||^{\epsilon}\right) = O\left(M_1^{\frac{15}{16}}||\mathbf{m}||^{\epsilon}\right) \qquad (12.5)\end{aligned}$$

by a virtual repetition of previous arguments.

The crux in the method occurs when we size up $R(g,x)$ for values of x between $n_1^{\frac14}$ and $n_1^{\frac12}$ that correspond to the range of M_1/g in $\sum_2$. Save for a preliminary transformation, this is performed much as in our proof of our Theorem 1 in our paper [5] on the distribution of the roots of polynomial congruences, wherein the polynomial $f(u)$ is replaced here by the quadratic $G(u)$ for given values of n_1 and $\mathbf{m}$ and wherein $|S(h,k)|$ becomes $\Upsilon(\bar g,k')$. The degree n in [5] is now 2 and the summation over k is initially only limited to one over k'. But, since the possible presence in k' of prime divsors of n_1 means that part (iv) of Lemma 4 in [5] is no longer available, the treatment requires some remodelling, which for brevity of description is best instituted here by letting the symbols k'' and g_1 denote numbers of type k' that are, respectively, those that are prime to n_1 and those that are composed entirely of prime factors of n_1. Then

$$\begin{aligned}R(g,x) &= \sum_{g_1k''\le x}\Upsilon(\bar g\bar k'',g_1)\Upsilon(\bar g\bar g_1,k'') \le \sum_{g_1k''\le x}\rho(g_1)\Upsilon(\bar g\bar g_1,k'')\\ &= \sum_{g_1\le x^{\frac18}} + \sum_{g_1>x^{\frac18}},\end{aligned}$$

where we have utilized the fact that $\bar g$, mod k', is also a multiplicative inverse of g, mod k''. Here, by (12.1) and Lemma 4 again, the second sum does not exceed

$$\sum_{g_1 > x^{\frac{1}{8}}} \rho(g_1) \sum_{k'' \le x/g_1} \rho(k'') = O\left(x \log^4 x \sum_{g_1 > x^{\frac{1}{8}}} \frac{\rho(g_1)}{g_1} \right)$$
$$= O\left(x^{\frac{31}{32}} \log^4 x \sum_{g_1} \frac{\rho(g_1)}{g_1^{\frac{3}{4}}} \right)$$
$$= O\left\{ x^{\frac{31}{32}} \log^4 x \prod_{p|n_1} \left(1 + \frac{A}{p^{\frac{1}{2}}}\right) \right\}$$
$$= O\left(x^{\frac{31}{32}} n_1^{\epsilon} \log^4 x \right) = O\left(x^{\frac{32}{33}} \right),$$

whence, on setting

$$R_1(gg_1, y) = \sum_{k'' \le y} \Upsilon(\bar{g}\bar{g}_1, k'')$$

for values of y between $x^{\frac{7}{8}}$ and x and hence between $n_1^{\frac{7}{32}}$ and $n_1^{\frac{1}{2}}$, we have

$$R(g, x) = \sum_{g_1 \le x^{\frac{1}{8}}} \rho(g_1) R_1(gg_1, x/g) + O(x^{\frac{32}{33}}). \tag{12.6}$$

The method of [5] can now be applied because $R_1(gg_1, y)$ is a sum over numbers that are prime to n_1. Starting with numbers k_1'', k_2'' that stand in the same relation to k'' as k_1, k_2 did to k in [5], we follow the procedure in that paper from its Lemma 7 until we reach (11) therein, whereupon we are confronted by the sum

$$\sum_{k_1'' \le y} \frac{\rho^{\frac{1}{2}}(k_1'')}{k_1''^{\frac{1}{2}} \phi^{\frac{1}{2}}(k_1'')}, \tag{12.7}$$

whose analysis begins with its majorization as

$$O\left((\log\log n)^{\frac{1}{2}} \sum_{k_1'' \le n_1} \frac{\rho^{\frac{1}{2}}(k_1'')}{k_1''} \right)$$
$$= O\left\{ (\log\log n)^{\frac{1}{2}} \prod_{\substack{p \le n_1; p \ne 2 \\ (a_1a_2a_3n_1F(\mathbf{m})|p)=1}} \left(1 + \frac{\sqrt{2}}{p}\left(1 - \frac{1}{p}\right)^{-1}\right) \right\}$$
$$= O\left\{ (\log\log n)^{\frac{1}{2}} \prod_{\substack{p \le n_1; p \ne 2 \\ (a_1a_2a_3n_1F(\mathbf{m})|p)=1}} \left(1 + \frac{1}{p}\right)^{\sqrt{2}} \right\} \tag{12.8}$$

by Lemma 4. We then go on to scrutinize the last product through the principles of §8, assuming that $0 < ||\mathbf{m}|| \le n^{\frac{1}{8}}$.

Let us now suppose, for such a given $\mathbf{m}$, that

$$a_1a_2a_3(n - a_4W^l)F(\mathbf{m}) = -D\Omega^2 \quad (0 \le W < (n/a_4)^{\frac{1}{l}}) \tag{12.9}$$

so that, under the discussion of §8, $\chi(b, -a_1a_2a_3n_1F(\mathbf{m})$ is a non-principal character, mod $2|D|$ or $4|D|$, when $D < -1$. Then, setting $Y = Aa_1a_2a_3n^{\frac{5}{4}}$ in Lemma 2, we know there are at most $O(Y^{\frac{4}{45l}}) = O(n^{\frac{1}{9l}})$ exceptional values of such D for which the inequality

$$\sum_{2<p\le n_1} \left(\frac{a_1a_2a_3n_1F(\mathbf{m})}{p}\right)\frac{1}{p} = O\left(\log\log\log n\right)$$

and hence the inequality

$$\sum_{\substack{2<p\le n_1\\(a_1a_2a_3n_1F(\mathbf{m})|p)=1}} \frac{1}{p} \le \frac{1}{2}\sum_{2<p\le n_1}\left\{1+\left(\frac{a_1a_2a_3n_1F(\mathbf{m})}{p}\right)\right\}\frac{1}{p}$$
$$\le \frac{1}{2}\log\log 2n_1 + A\log\log\log n$$

are false. Thus, save in the case of failure, the product in (12.8) and thence the sum (12.7) are of the form

$$O\left\{\log^{\frac{1}{2}\sqrt{2}}2n_1(\log\log n)^A\right\},$$

wherefore, completing the argument on p.47 of [5], we find that

$$R_1(gg_1,y) = O\left(\frac{y(\log\log n)^A}{\log^{1-\frac{1}{2}\sqrt{2}}2n_1}\right)$$

and deduce from (12.6) and Lemma 4 that we can replace (12.1) by

$$\begin{aligned} R(g,x) &= O\left(\frac{x(\log\log n)^A}{\log^{1-\frac{1}{2}\sqrt{2}}2n_1}\sum_{g_1}\frac{\rho(g_1)}{g_1}\right) + O\left(x^{\frac{32}{33}}\right)\\ &= O\left\{\frac{x(\log\log n)^A}{\log^{1-\frac{1}{2}\sqrt{2}}2n_1}\prod_{p|n_1}\left(1+\frac{3}{p}+\frac{5}{p^{\frac{3}{2}}}\right)\right\}\\ &= O\left(\frac{x(\log\log n)^A}{\log^{1-\frac{1}{2}\sqrt{2}}2n_1}\right)\end{aligned}$$

when the circumstances are favourable. This we put in $\sum_2$ to yield

$$\sum\nolimits_2 = O\left(\frac{M_1(\log\log n)^A}{\log^{1-\frac{1}{2}\sqrt{2}}2n_1}\sum_g\frac{1}{g^{\frac{1}{2}}}\right) = O\left(\frac{M_1||\mathbf{m}||^{\epsilon}(\log\log n)^A}{\log^{1-\frac{1}{2}\sqrt{2}}2n_1}\right)$$

by the reasoning associated with (12.5), and we deduce that

$$\theta_{\mathbf{m}}(n_1) = O\left(\frac{M_1||\mathbf{m}||^{\epsilon}(\log\log n)^A}{\log^{1-\frac{1}{2}\sqrt{2}}2n_1}\right) \tag{12.10}$$

in the light of (12.4) and (12.5). Finally, the contribution to $\nu_{\mathbf{m}}(n)$ in (11.2) for those values of W for which (12.10) is successful is

$$O\left(n^{\frac{1}{2}}||\mathbf{m}||^{\epsilon}(\log\log n)^A \sum_{W<(n/a_4)^{\frac{1}{l}}} \frac{1}{\log^{1-\frac{1}{2}\sqrt{2}} 2(n-a_4W^l)}\right) = O\left(\frac{n^{\frac{1}{2}+\frac{1}{l}}||\mathbf{m}||^{\epsilon}}{\log^{\delta_3} n}\right) \tag{12.11}$$

for any exponent δ_3 less than $1 - \frac{1}{2}\sqrt{2}$.

In the situations not yet covered for a given value of $\mathbf{m}$, there are $O(n^{\frac{1}{9l}})$ possible values of D even when we add the previously excluded value -1, to each of which there will correspond those W in the absolutely irreducible equation (12.9) for which $0 \le W < (n/a_4)^{\frac{1}{l}}$ and $0 < \Omega \le (a_1a_2a_3F(\mathbf{m})n)^{\frac{1}{2}} < An^{\frac{5}{8}}$. Since the number of such solutions is $O(n^{\frac{5}{8l}+\epsilon})$ by the Bombieri-Pila theorem already used after (9.6), the number of W for which we are obliged to use the estimate (12.3) is $O(n^{\frac{3}{4l}})$, whose contribution to $\nu_{\mathbf{m}}(n)$ is therefore

$$O\left(n^{\frac{1}{2}+\frac{3}{4l}}||\mathbf{m}||^{\epsilon}\log^4 n\right)$$

by (11.2). With (12.11), this then implies that

$$\nu_{\mathbf{m}}(n) = O\left(\frac{n^{\frac{1}{2}+\frac{1}{l}}||\mathbf{m}||^{\epsilon}}{\log^{\delta_3} n}\right) \tag{12.12}$$

for $0 < ||\mathbf{m}|| \le n^{\frac{1}{8}}$.

On the other hand, for $||\mathbf{m}|| > n^{\frac{1}{8}}$, we have the trivial estimate

$$\nu_{\mathbf{m}}(n) = O\left(n^{\frac{1}{2}}||\mathbf{m}||^{\epsilon}\log^4 n \sum_{W<(n/a_4)^{\frac{1}{l}}} 1\right) = O\left(n^{\frac{1}{2}+\frac{1}{l}}||\mathbf{m}||^{\epsilon}\log^4 n\right)$$

that flows from (12.3). Hence, choosing [9] A to exceed 7, we conclude that

$$\begin{aligned}\nu_B(n) &= O\left(\frac{n^{\frac{1}{2}+\frac{1}{l}}}{\log^{\delta_3} n}\sum_{0<||\mathbf{m}||\le n^{\frac{1}{8}}}\frac{1}{||\mathbf{m}||^6}\right) + O\left(n^{\frac{1}{2}+\frac{1}{l}}\log^4 n\sum_{||\mathbf{m}||>n^{\frac{1}{8}}}\frac{1}{||\mathbf{m}||^7}\right)\\ &= O\left(\frac{n^{\frac{1}{2}+\frac{1}{l}}}{\log^{\delta_3} n}\sum_{||\mathbf{m}||>0}\frac{1}{||\mathbf{m}||^6}\right)\\ &= O\left\{\frac{n^{\frac{1}{2}+\frac{1}{l}}}{\log^{\delta_3} n}\left(\sum_{m>0}\frac{1}{m^2}\right)^2\right\} = O\left(\frac{n^{\frac{1}{2}+\frac{1}{l}}}{\log^{\delta_3} n}\right).\end{aligned} \tag{12.13}$$

13. The primary asymptotic formula. If we combine (7.8),(9.8), and (12.13) in (7.3), we obtain at once the required asymptotic formula in its primary form. This we state in

[9] A less generous value of A would obviously suffice.

THEOREM 1. *Let a_1, a_2, a_3, a_4 be positive integers and $\nu(n)$ the number of representations of a (large) integer n in the form $a_1X_1^2 + a_2X_2^2 + a_3X_3^2 + a_4W^l$. Then*

$$\nu(n) = \frac{8\Gamma^3(\frac{3}{2})\Gamma(\frac{1}{l}+1)}{\sqrt{a_1a_2a_3}\sqrt[l]{a_4}\,\Gamma(\frac{3}{2}+\frac{1}{l})}\, n^{\frac{1}{2}+\frac{1}{l}}\mathfrak{S}(n) + O\left(\frac{n^{\frac{1}{2}+\frac{1}{l}}}{\log^{\delta_3} n}\right), \tag{13.1}$$

for any positive exponent δ_3 less than $1 - \frac{1}{2}\sqrt{2}$.

However, just as at the corresponding place in I, it would be premature to attempt to draw conclusions from this formula until we have studied the singular series in appropriate circumstances, since we do not yet know when the explicit term dominates the remainder. Indeed, as shewn in I, there are cases where the formula would not be efficacious unless the remainder could be reduced in size in the appropriate context.

14. Return to the singular series. In the examination of $\nu(n)$ and $\mathfrak{S}(n)$, as is usual in this sort of subject, the necessary divisibility of n by (a_1, a_2, a_3, a_4) for the existence of the proposed representations means that we can always assume that this highest common factor is 1. Under this simplifying supposition, any individual case one cares to choose can in theory be so analyzed that an appropriate conclusion can be reached about $\nu(n)$ and the representation of large numbers. Yet so many different situations arise that it would be both impracticable and dreary to attempt to cover them all here exhaustively, a difficulty, moreover, that could not be instructively removed by our assaying conditions in terms of solubility over p-adic rings. We therefore confine our attention to the case where

$$a_1, a_2, a_3, a_4 \text{ are odd and relatively prime in pairs;} \tag{14.1}$$

this affords the most natural widening of the scope of I and is a good indication of how one might proceed within a wider frame of reference.

We shall need the analogue

$$\mathfrak{S}(n) = \prod_p \lim_{\alpha\to\infty} \frac{\tau(n, p^\alpha)}{p^{3\alpha}} = \prod_p \Theta(n,p), \text{say}, \tag{14.2}$$

of (45) in I that stems here from §6, applying it by means of the following two principles.

PRINCIPLE A. *A contribution of $p^{3(\alpha-1)}$ to $\tau(n,p^\alpha)$ is due from each solution,* mod p, *of*

$$a_1X_1'^2 + a_2X_2'^2 + a_3X_3'^2 + a_4W'^l \equiv n, \quad \text{mod}\, p, \tag{14.3}$$

for which either $a_iX_i' \not\equiv 0$, mod p, *for some i when $p \neq 2$* or *$a_4W' \not\equiv 0$,* mod p, *when $p \nmid l$, all such solutions being primitive in the sense that $p \nmid (a_1X_1', a_2X_2', a_3'X_3', a_4'W')$.*

PRINCIPLE B. *If the congruence $Y_1^2 \equiv H_1$,* mod 8, *with odd H_1 have a solution U_1,* mod 8, *then the congruence $Y \equiv H$,* mod 2^α, *with $H \equiv H_1$,* mod 8, *has a solution Y,* mod 8, *that is congruent to U_1,* mod 8, *when $\alpha > 3$.*

We also now require the promised analogue of (5.8) in the case where p divides just one of a_1, a_2, a_3 and where, therefore, we may assume for illustration that $p|a_1$

and $p \nmid 2a_2a_3$. Here $\rho_3(n_1, p)$ is now p times the number of solutions, $\bmod p$, of $a_2X_2^2 + a_3X_3^2 \equiv n_1, \bmod p$, which equals

$$\sum_{0<X_3\leq p} \left\{1 + \left(\frac{a_2}{p}\right)\left(\frac{-a_3X_3^2 + n_1}{p}\right)\right\}.$$

Hence, by a formula due to Jacobsthal,

$$\rho_3(n_1, p) = \begin{cases} p^2 - (-\frac{a_2a_3}{p})p, & \text{, if } p \nmid n_1, \\ p^2 + (p-1)p(-\frac{a_2a_3}{p}) & \text{, if } p|n_1, \end{cases} \tag{14.4}$$

a result that could otherwise be derived less simply by forming an analogue of (5.5). From this, under the same condition where therefore $a_4 \not\equiv 0, \bmod p$ by (14.1), we then deduce the equation

$$\begin{aligned}\tau(n,p) &= \sum_{0<W\leq p} \rho_3(n - a_4W^l, p) \\ &= p^3 - \left(-\frac{a_2a_3}{p}\right)p \sum_{0<W\leq p} 1 + \left(-\frac{a_2a_3}{p}\right)p^2 \sum_{\substack{0<W\leq p \\ a_4W^l\equiv n, \bmod p}} 1 \\ &= p^3 - \left(-\frac{a_2a_3}{p}\right)p^2 + \left(-\frac{a_2a_3}{p}\right)p^2H,\end{aligned}$$

where H, the number of incongruent solutions of $a_4W^l \equiv n, \bmod p$, does not exceed $p-1$. Thus

$$\begin{aligned}\tau(n,p) - pH &= p^3 - p^2\left(-\frac{a_2a_3}{p}\right) + \left\{p^2\left(-\frac{a_2a_3}{p}\right) - p\right\}H \\ &= p\left\{p - \left(-\frac{a_2a_3}{p}\right)\right\}\left\{p + \left(-\frac{a_2a_3}{p}\right)H\right\} \geq p^2 + p > p^2,\end{aligned} \tag{14.5}$$

which inequality immediately enables us to dismiss the case where $p|a_1a_2a_3$ in the product in (14.2). If for example $p|a_1$, the number of incongruent solutions of (14.3) for which $X_2' \equiv X_3' \equiv 0, \bmod p$, is pH with the consequence that the number of solutions satisfying the data in Principle A is not less than $\tau(n,p) - pH > p^2$, whence $\Theta(n,p) > 1/p$ and

$$\prod_{p|a_1a_2a_3} \Theta(n,p) > \frac{1}{a_1a_2a_3}. \tag{14.6}$$

In the opposite situation for odd primes p where $p \nmid a_1a_2a_3$, we merely use a simplified version of the procedure in the corresponding part of I. Since here the number of incongruent solutions of (14.3) not satisfying the criteria in Principle A is not greater than the number H of those for which $X_1' \equiv X_2' \equiv X_3' \equiv 0, \bmod p$, the remaining solutions have cardinality not less than $\tau(n,p) - H \geq \tau(n,p) - p$, where by the statements (i) and (ii) in §6

$$\tau(n,p) = p^3 + O(p^{\frac{3}{2}}) \;\; (p \nmid n) \text{ and } \tau(n,p) \geq p^3 - p^2 \text{ always.}$$

Hence, by Principle A,

$$\Theta(n,p) > 1 - \frac{B}{p^{\frac{3}{2}}} > 0 (p \nmid n, p > B') \text{ and } \Theta(n) \geq 1 - \frac{1}{p} - \frac{1}{p^2} \quad (p|n \text{ or } p < B')$$

with the result that

$$\prod_{p \nmid 2a_1a_2a_3} \Theta(n,p) > B_3 \prod_{p|n} \left(1 - \frac{1}{p}\right) > \frac{B_4}{\log\log n},$$

which combines with (14.6) to yield

$$\prod_{p>2} \Theta(n,p) > \frac{B_5}{\log\log n}. \tag{14.7}$$

Even if $p = 2$, there is still no difficulty when l is odd because then it is easily verified that there is always a solution of (14.3) for which W' is odd and for which therefore Principle A is applicable. Thus

$$\Theta(n,2) > B_6 \qquad (l \text{ odd}). \tag{14.8}$$

Yet, much more so than in I, the remaining case with $p = 2$ and even l presents us with a greater problem than those that preceded it when we confront the congruence

$$\Psi(X_1, ..., X_4) = a_1X_1^2 + ... + a_4X_4^2 \equiv 2^\beta n_1, \quad \text{mod } 2^\alpha,$$

and its associate

$$\Psi(X_1, ..., X_3, W^{\frac{1}{2}l}) = a_1X_1^2 + ... + a_3X_3^2 + a_4W^l \equiv 2^\beta n_1, \quad \text{mod } 2^\alpha, \tag{14.9}$$

where here $2^\beta \| n$ and $n = 2^\beta n_1$ with n_1 odd. This is due in part to there being a multitude of categories requiring individual analysis and also to the fact that primitive solutions do not always answer to the congruences

$$\Psi(X_1', ..., X_4') \equiv 0, \quad \text{mod } 8,$$

that contain those of the type

$$\Psi(X_1', ..., W'^{\frac{1}{2}l}) \equiv 0, \quad \text{mod } 8. \tag{14.10}$$

It is with these congruences we begin and easily find that the condition that either one be primitively soluble is that either

$$a_1 + a_2 + a_3 + a_4 \equiv 0, \quad \text{mod } 8, \tag{14.11}$$

or

$$a_i + a_j \equiv 0, \quad \text{mod } 4, \text{ for some pair of (necessarily) unequal subscripts} \tag{14.12}$$

because it is required that exactly two or four of the unknowns be odd (one of which is necessarily an X_i' in (14.10) in this instance). Two cases, which are to be separately analyzed, then emerge, namely,

$$\begin{aligned} & A \; — \; \text{(14.12) does not hold but (14.11) does } (l \text{ even}) \\ \text{and} \quad & B \; — \; \text{(14.12) holds } (l \text{ even}). \end{aligned} \tag{14.13}$$

The residual case in which (14.10) is not primitively soluble is then described by

$$C \quad — \quad \text{neither (14.11) nor (14.12) holds } (l \text{ even}). \tag{14.14}$$

As a prelude to the study of cases A and C, we must examine some modular properties of the ternary form $f(X_1', X_2', X_3')$ when property (14.12) is denied. First a_1, a_2, a_3 are not all incongruent, mod 8, because otherwise the even numbers $a_1+a_2, a_2+a_3, a_3+a_1$ would assume three incongruent values, mod 8, one of which would perforce be congruent to 0, mod 4. Next, excluding the case $a_1 \equiv a_2 \equiv a_3$, mod 8 where it is known that $f(X_1', X_2', X_3')$ does not represent all odd residues, mod 8, let us isolate all other eligible forms with a like property. In the typical case where $a_2 \equiv a_1$, mod 8, and $a_3 \not\equiv a_1$, mod 8, the obviously incongruent representable residues $a_1 + 4a_3, a_3 + 4a_1$ cannot both assume values, mod 8, that are neither a_1 nor a_3 so that, for example, $a_3 + 4a_1 \equiv a_1$, mod 8, and $a_3 \equiv 5a_1$, mod 8, and thus, equally well, $a_1 + 4a_3 \equiv a_3$, mod 8. Characteristically $f(X_1', X_2', X_3')$ must be congruent, mod 8, to an odd multiple of $X_1'^2 + X_2'^2 + 5X'^2$, which qualifies because the only odd residual values, mod 8, of the latter form are 1, 5, and 7. Moreover, it is easily verified that $f(X_1', X_2', X_3')$ primitively admits all oddly even values, mod 8, but not those that are evenly even, as is also true for the previously excluded forms that are an odd multiple of $X_1'^2 + X_2'^2 + X_3'^2$, mod 8.

Let us first describe the scene under the first heading A. Since each of $a_1+a_2, a_1+a_3, a_1 + a_4$ is congruent to either 2 or -2, mod 8, two of them are congruent to each other and thus typically $a_1 + a_2 \equiv a_1 + a_3 \equiv \pm 2$, mod 8, with the implication that $a_2 \equiv a_3$, mod 8, and $a_3 + a_4 \equiv \pm 6$, mod 8, by (14.11), the same sign being used in both instances. Therefore there is a class of forms $\Psi(X_1', X_2', X_3', X_4')$ in Category A consisting of those congruent, mod 8, to

$$aX_1'^2 + (\pm 2 - a)X_2'^2 + (\pm 2 - a)X_3'^2 + (4 + a)X_4'^2, \qquad (a \text{ odd})$$

which, on specializing a and reducing, mod 8, we see are all odd multiples of

$$X_1'^2 + X_2'^2 + X_3'^2 + 5X_4'^2,$$

all classes being exhausted by permuting the coefficients. Here, whatever be the position of the coefficient 5, we see that $\Psi(X_1', X_2', X_3', W'^{\frac{1}{2}l})$ can primitively represent all residues, mod 8, save 4 (in the case of 0 we already knew this from the preamble), whence, advancing via Principle B to (14.9) for $\alpha > 3$, we deduce that $\tau(n, 2^\alpha) \geq 2^{3\alpha-3}$ for all $\beta \neq 2$ and thus that

$$\Theta(n, 2) > B_7 \quad (\text{case } A : \beta \neq 2). \tag{14.15}$$

Yet, although (14.9) is not primitively soluble when $\beta = 2$ and $\alpha \geq 5$, there remains the possibility that there may be imprimitive solutions necessarily satisfying $2||(X_1, X_2, X_3, W)$, which, on setting $X_1 = 2X_1', X_2 = 2X_2', X_3 = 2X_3', W = 2W'$, we observe must conform to the congruence $\Psi(X_1', X_2', X_3', 2^l W'^l) \equiv n$, mod $2^{\alpha-2}$, that for $\alpha = 5$ is tantamount to one or other of conditions of the type

$$a(X_1'^2 + X_2'^2 + X_3'^2 + 5.2^{l-2}W'^l) \equiv n_1, \quad \text{mod } 8,$$
$$a(X_1'^2 + X_2'^2 + 5X_3'^2 + 2^{l-2}W'^l) \equiv n_1, \quad \text{mod } 8,$$

containing an odd value of a. Taking these congruences, we first suppose that $l = 4$ and choose W' so that $n_1 - a5.4W'^l$ or $n_1 - a4W'^l$ is not an odd number that escapes primitive representation, mod 8, by the relevant ternary form $a(X_1'^2+X_2'^2+X_3'^2)$ or $a(X_1'^2+X_2'^2+5X_3'^2)$. Thus in this case $\tau(n,p^\alpha)\ge B_8p^{3\alpha}$ for $\alpha\ge 5$ by Principle B, and we deduce that

$$\Theta(n,2)>B_8 \quad (\text{case } A;\beta=2;l=4). \tag{14.16}$$

However, if $l>4$, then our needs are met if and only if $2\nmid(X_1',X_2',X_3')$, in which event by starting with the exponent $\alpha=5$ we find that only three odd residual values of n_1 are admissible. Therefore in this case we deduce that $\tau(n,p^\alpha)\ge B_9p^{3\alpha}$ or $\tau(n,p^\alpha)=0$ according as n_1 does not or does belong to one specific residue class, mod 8, whence

$$\Theta(n_2)>B_9 \quad (\text{case } A;\beta=2;l>4;$$
$$n_1 \text{ belongs to one of three odd residue classes, mod } 8) \tag{14.17}$$

and

$$\Theta(n_2)=0(\text{case } A;\beta=2;l>4;n_1 \text{ belongs to the remaining residue class, mod } 8); \tag{14.18}$$

of course, in the latter case, $\nu(n)$ itself is necessarily zero.

On arrival at Case B, we cease to justify our assertions because the methods of demonstration have been amply rehearsed above and in I. We therefore first merely state the inequality

$$\mathfrak{S}(n,2)>B_{10} \qquad (\text{case } B). \tag{14.19}$$

Case C comprehends the whole situation in I, the only significant difference being that we must discriminate between the cases where the ternary form $f(X_1',X_2',X_3')$ represents all four odd residue classes, mod 8, and where, as in I, it only represents three such classes. We have

$$\Theta(n,2)>\frac{B_{11}n_1}{n}=\frac{B_{11}}{2^\beta}$$

in the following instances:

(i) case C; β odd; (14.20)

(ii) case C; β even; $f(X_1,X_2,X_3)$ neither an odd multiple, mod 8, of $X_1^2+X_2^2+X_3^2$ nor of a form with coefficients 1, 1, 5 (in some order); (14.21)

(iii) case C; β even; $f(X_1,X_2,X_3)$ either an odd multiple, mod 8, of $X_1^2+X_2^2+X_3^2$ or of a form with coefficients 1, 1, 5; either one of $\beta-2$, $\beta,\beta+2$ congruent to 0, mod l, or n_1 is congruent to one of three odd residues, mod 8. (14.22)

In the remaining situation,

(iv) case C; none of (i), (ii), (iii) apply,

both $\Theta(n,p)$ and $\nu(n)$ are zero because $\tau(n,2^\alpha)$ vanishes for large enough α.

Our study of the singular series ends by the insertion in (14.2) of (14.7) and our results for $\Theta(n,2)$. We first find that

$$\mathfrak{S}(n)>\frac{B_{12}}{\log\log n}$$

when

$$\text{one of the conditions in (14.8), (14.15), (14.16), (14.17) or (14.19) holds.} \quad (14.23)$$

Also

$$\mathfrak{S}(n) > \frac{B_{13}n_1}{n \log\log n} = \frac{B_{13}}{2^\beta \log\log n}$$

when

$$\text{one of the conditions (14.20), (14.21), or (14.22) holds.} \quad (14.24)$$

In all other circumstances both $\nu(n)$ and $\mathfrak{S}(n)$ are zero.

15. The final theorems. On applying our results on $\mathfrak{S}(n)$ to Theorem 1, we immediately deduce our

THEOREM 2. *To the data in Theorem 1 let us add the requirement that a_1, a_2, a_3, a_4 be relatively prime in pairs. Then we have*

(i) *if stipulation* (14.23) *hold,*

$$\nu(n) \sim \frac{8\Gamma^3(\frac{3}{2})\Gamma(\frac{1}{l}+1)}{\sqrt{a_1a_2a_3}\sqrt[l]{a_4}\,\Gamma(\frac{3}{2}+\frac{1}{l})} n^{\frac{1}{2}+\frac{1}{l}}\mathfrak{S}(n) \quad (15.1)$$

as $n \to \infty$:

(ii) let $n = 2^\beta n_1$ *where n_1 is odd and $\beta < \delta \log\log n$ for some small positive number δ; then the asymptotic formula* (15.1) *is still valid when* (14.24) *is in place:*

(iii) *in the situations outlined above a large number n is representable as $a_1X_1^2 + a_2X_2^2 + a_3X_3^2 + a_4W^l$; but, if both* (14.23) *and* (14.24) *fail, there are no representations and the asymptotic formula is trivial.*

As it stands, this theorem has the blemish that a limit has been placed on the size of the power of 2 in n in certain circumstances. However, just as in I, this defect can be removed by an initial transformation of (1.1) because only certain cases included in C require attention. After all relevant details have been attended to, we can reach our final inference in the form of

THEOREM 3. *The conclusion in part* (ii) *of Theorem* 2 *is still valid when it is only assumed that $n_1 \to \infty$; part* (iii) *is then to be interpreted in the light of the revised part* (ii).

Thus we have achieved our goal of obtaining a meaningful asymptotic formula for $\nu(n)$ in all cases under (14.1) where the odd constituent n_1 of n tends to infinity. We are therefore also provided in these circumstances with a criterion for deciding when n is representable in the proposed form.

In other situations we still have the asymptotic formula of Theorem 1 but, as previously stated, further work is needed to elucidate it in any individual instance.

REFERENCES

[1] E. BOMBIERI AND J. PILA, *The number of integral points on arcs and ovals,* Duke Math. J., 59 (1989), pp. 337–357.

[2] W. DUKE AND R. SCHULZE-PILLOT, *Representation of integers by positive ternary quadratic forms and equidistribution of lattice points on ellipsoids,* Invent. math., 99 (1990), pp. 49–57.

[3] D.R. HEATH-BROWN, *A new form of the circle method, and its application to quadratic forms,* J. reine angew. Math., 481 (1996), pp. 149–206.

[4] C. HOOLEY, *On the representation of a number as the sum of a square and a product*, Math. Zeitschr., 69 (1958), pp. 211–227.

[5] C. HOOLEY, *On the distribution of the roots of polynomial congruences,* Mathematika, 11 (1964), pp. 39–49.

[6] C. HOOLEY, *On power-free numbers and polynomials: I*, J. reine angew. Math., 293/294 (1977), pp. 67–83.

[7] C. HOOLEY, *On nonary cubic forms,* J. reine angew. Math., 386 (1988), pp. 32–98.

[8] C. HOOLEY, *On Waring's problem for three squares and an lth power*, Asian J. Math., 4 (2000), pp. 885-904.