AN EXTENSION OF A FUNCTIONAL EQUATION OF POINCARÉ AND MANDELBROT*

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Abstract. We consider the distributional equation $Z \overset{d}{=} A_1 Z_1 + ... + A_N Z_N$, where N, A_i, Z, Z_i are independent real random variables with $N \in \mathbb{N}$ and $A_i \geq 0$. Each A_i is equally distributed as A_1 and each Z_i as Z; the distribution of Z is unknown. After establishing theorems about existence and uniqueness of solution, we show asymptotic properties at 0 of the distribution function $P[Z \leq x]$, and those at ∞ of the characteristic function Ee^{itZ}; such properties are also found for a random difference equation. In particular, we prove the absolute continuity of the distribution of Z under simple moment conditions. As applications, new results are established for multiplicative cascades, for age-dependent branching processes and for branching random walks.

0. Introduction. Let N be a random variable with law $\{p_n\}$ on $\mathbb{N} = \{0, 1, ...\}$, and let A be a random variable with values in $\mathbb{R}_+ = [0, \infty)$. We consider the distributional equation

$$(E) \qquad\qquad Z \overset{d}{=} \sum_{i=1}^{N} A_i Z_i,$$

where $\overset{d}{=}$ means the equality in distribution, the real random variables Z, Z_i, N, A, A_i are all independent to each other, each A_i is equally distributed as A and each Z_i as Z; the distribution of Z is unknown. By convention the empty sum is taken to be zero. In terms of the characteristic function $\phi(t) = E(e^{itZ})$ $(t \in \mathbb{R})$, the equation reads

$$(E') \qquad\qquad \phi(t) = f(E\phi(At)),$$

where $f(t) = \sum_{n=0}^{\infty} p_n t^n$ is the probability generating function of N.

The equation (E) generalizes considerably the notion of stable laws (and semi-stable laws) where both N and A are constants. If A is a constant with $A = 1/EN$ and $1 < EN < \infty$, then the equation (E') reduces to the Poincaré's functional equation for Galton-Watson processes, whose study is a classical subject: see e.g. [20, 44, 2, 14, 9]. If N is a constant ≥ 2, it becomes the Mandelbrot's functional equation about multiplicative cascades [39, 40], and has been studied by many authors: see e.g. [27, 19, 15, 26, 41, 3]; see also [15, 25, 5, 21, 12, 46, 30, 37, 38] for some other (but not all) references. The general form of the equation is the basic equation in typical branching random walks (see Section 8), and can be applied to the study of Bellman-Harris processes (see Section 7). For many other applications of the equation, see e.g. [42] and [31].

Conditions for existence and uniqueness of positive solutions have been established in Liu [32]; the special case where N is constant has been studied by Kahane and Peyrire [27], Durrett and Liggett [15] and Guivarc'h [19]. In Section 1 below we shall extend some of their results to solutions which are not necessarily positive. However, our main object is the study of asymptotic properties of the characteristic function

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and the absolute continuity for any solution, and the asymptotic behavior at 0 of the distribution function for any positive solution. To this end we first establish a result for a random difference equation which is interesting by its own (see Theorem 2.1). The main results are Theorems 3.2, 3.4, 3.5, 4.1 and 4.2 where a polynomial decay rate is shown for the characteristic function Ee^{itZ} ($|t| \to \infty$) and for the distribution function $P(Z \leq x)$ ($x \to 0$). Our results show that the behavior of Z differs according to $P(A = 0) = p_0 = p_1 = 0$ or not. These results extend those of Harris [20] (see also Bingham [8]), Athreya [2] and Dubuc [14] about the Galton-Watson process, and complete those of Bingham and Doney [9] (also for the Galton-Watson process). Exponential rates will be considered in Theorem 5.1. A description of the right tail probability $P(Z > x)$ ($x \to \infty$) will also be given at the end of Sections 4 and 5, in order to compare it with the left tail probability $P(Z \leq x)$ ($x \to 0$).

Specified to limit variables of Mandelbrot's martingales, in addition to new results about the characteristic function and the absolute continuity, our results on the asymptotic behavior near 0 of the distribution function and on the existence of moments of negative orders make precise and complete those of Kahane [26], Molchan [41] and Barral [3, 4]: see Section 6; as has been shown by Molchan [41] and Barral [3, 4], such results are essential in the study of multi-fractal structure of the Mandelbrot's measure. Applied to age-dependent branching processes, our theorems enable us to improve an early result by the author: see Theorem 7.1. In Section 8, we shall see that our theorems can also be applied to obtain new results for limit variables in branching random walks.

In closing this section, let us point out that some results of this paper can be extended to the more general case where $\{N, A_1, A_2, ...\}$ is of arbitrary joint distribution [36], and let us introduce some notations. If E is a set or a statement, we write 1_E for its indicator function; if X is a random variable, we write P_X for its probability law; if $f(t)$ and $g(t)$ are two (real or complex) functions defined for $t > 0$ large (resp. small) enough, we write $f(t) = O(g(t)), t \to \infty$ (resp. $t \to 0$) to mean that for some constant $0 < C < \infty$ and all $t > 0$ large (resp. small) enough, $|f(t)| \leq Cg(t)$, and we write $f(t) = o(g(t))$ if $\lim f(t)/g(t) = 0$, and $f(t) \sim g(t)$ if $\lim f(t)/g(t) = 1$.

1. Existence and Uniqueness of Solutions. Let $\mathbb{M}$ (resp. $\mathbb{M}_+$) be the set of all Borel probability measures on $\mathbb{R}$ (resp. $\mathbb{R}_+$). Put

$$(1.1) \qquad \alpha_0 = P(A = 0) \text{ and } \tilde{N} = \sum_{i=1}^{N} 1_{\{A_i > 0\}}.$$

Then $\tilde{N}$ is the number of non-zero terms of A_i, $1 \leq i \leq N$, whose probability generating function is

$$(1.2) \qquad \tilde{f}(t) = f(\alpha_0 + (1 - \alpha_0)t), \qquad t \geq 0.$$

To avoid unimportant discussion, we suppose throughout the paper that

$$(1.3) \qquad P(\tilde{N} = 0 \text{ or } 1) < 1 \text{ and } P(A = 0 \text{ or } 1) < 1,$$

the contrary case being easy [32, Lemma 1.1]. For simplicity, we also assume

$$(1.4) \qquad f'(1) \equiv EN < \infty \text{ and } EA \log^+ A < \infty,$$

although this is not always necessary. Let $\mathbb{F}$ (resp. $\mathbb{F}_+$) be the set of all non-trivial solutions in $\mathbb{M}$ (resp. $\mathbb{M}_+$). We say that a random variable Z or a characteristic function ϕ is an element of $\mathbb{F}$ or $\mathbb{F}_+$ if so is the corresponding probability measure.

The following result gives necessary and sufficient conditions for (E) to have non-trivial and non-negative solutions, and can be easily obtained from known results. As usual, we put $\log^+ x = \log x$ if $x \geq 1$, and $\log^+ x = 0$ otherwise. Recall that a positive and measurable function $l(t)$ is called slowly varying at 0 (resp. ∞) if it is defined for all $t > 0$ small (resp. large) enough such that as $t \to 0$ (resp. ∞), $l(\lambda t)/l(t) \to 1$ for each given $\lambda > 0$.

THEOREM 1.1.
(i) $\mathbb{F}_+ \neq \emptyset$ if and only if

$$(1.5) \qquad (1 - \alpha_0)EN > 1, \quad ENEA^\alpha = 1 \text{ and } EA^\alpha \log A \leq 0 \text{ for some } \alpha \in (0, 1].$$

(ii) Assume (1.5). There is $\mu \in \mathbb{F}_+$ with finite mean if and only if

$$(1.6) \qquad\qquad ENEA = 1, EN \log^+ N < \infty \text{ and } EA \log A < 0;$$

when (1.6) holds, all $\mu \in \mathbb{F}_+$ have finite mean, and are parametrized by their means; for any given function l slowly varying at 0, there is at most one element $\mu \in \mathbb{F}_+$ whose Laplace transform $\psi(t) = \int e^{-xt} \mu(dx)$ satisfies $\lim_{t \to 0+} \frac{1 - \psi(t)}{t^\alpha l(t)} = 1$.

(iii) If $\mu \in \mathbb{F}_+$, then the extinction probability $q = \mu\{0\}$ is the unique fixed point in $[0, 1)$ of $\tilde{f}$.

Notice that the number α determined by (1.5) is necessarily unique, and is the least solution of the equation $ENEA^x = 1, x > 0$.

Proof of Theorem 1.1. Parts (i) and (iii) follow from Theorem 1.1 and Lemma 3.1 of Liu [32]. In part (ii), the third assertion is just Corollary 1.4(a) of Liu [32], the second assertion is a consequence of the first and the uniqueness theorem of Biggins and Kyprianou [7, Theorem 1.5], while the first comes from Theorem 1.1 of Liu [31] and the fact that

$$(1.7) \qquad E(\sum_{i=1}^N A_i) \log^+ (\sum_{i=1}^N A_i) < \infty \Leftrightarrow EA \log^+ A < \infty \text{ and } EN \log^+ N < \infty,$$

which can be proved as follows. Evidently,

$$E(\sum_{i=1}^N A_i) \log^+ (\sum_{i=1}^N A_i) = p_0 + p_1 EA_1 \log^+ A_1 + \sum_{n \geq 2} p_n nE(A_1 \log^+ \sum_{i=1}^n A_i).$$

Since $\log(x + y) \leq \log x + \log y$ if $x \geq 2$ and $y \geq 2$, we have $E(A_1 \log^+ \sum_{i=1}^n A_i) \leq E(A_1 \log \max(2, A_1)) + EA_1 E \log \max(2, \sum_{i=2}^n A_i)$ for each fixed $n \geq 2$; as $E(\log \max(2, \sum_{i=2}^n A_i)) \leq \log E \max(2, \sum_{i=2}^n A_i)] \leq \log(2 + nEA_1)$, this gives the implication "$\Leftarrow$" in (1.7). To see the opposite implication, it suffices to remark that, by the law of large numbers, for some $n_0 \in \mathbb{N}$ and all $n > n_0$, $P(\sum_{i=2}^n A_i \geq nEA/2) > 1/2$, so that $E[\log(\sum_{i=2}^n A_i)] \geq \frac{1}{2} \log(nEA/2)$. $\square$

We now give a theorem about existence and uniqueness of *any* non-trivial solution.

THEOREM 1.2.
(i) The following condition is sufficient for $\mathbb{F} \neq \emptyset$:

$$(1.8) \qquad (1 - \alpha_0)EN > 1, \quad ENEA^\alpha = 1 \text{ and } EA^\alpha \log A \leq 0 \text{ for some } \alpha \in (0, 2].$$

(ii) If (1.8) holds with $EA^\alpha \log A < 0$, then for any given complex number c, there is at most one element $\phi \in \mathbb{F}$ with $\lim_{t\to0+} \frac{1-\phi(t)}{t^\alpha} = c$;

(iii) Let $\delta > 0$ and let $l : (0, \delta) \to (0, \infty)$ be a function slowly varying at 0 and bounded away from 0 and ∞ on each compact subset of $(0, \delta)$. If there is an element $\phi \in \mathbb{F}$ with $\lim_{t\to0+} \frac{1-\phi(t)}{t^\alpha l(t)} = c$ for some real $\alpha \in (0, 2]$ and some complex $c \neq 0$, and $E[A^{\alpha+\epsilon}] < \infty$ for some $\epsilon > 0$, then $ENEA^\alpha = 1$.

REMARKS. (1) As a consequence of part (ii) together with Theorem 1.1, if (1.6) holds (so that $\alpha = 1$), then for any given number $c \in \mathbb{R}$, there is only one integrable solution Z satisfying $EZ = c$; moreover almost surely (a.s.) $Z \geq 0$ if $c > 0$, $Z \leq 0$ if $c < 0$, and $Z = 0$ if $c = 0$; so *under (1.6)*, to consider integrable elements of $\mathbb{F}$, it suffices to consider elements of $\mathbb{F}_+$. However, the conclusion does not hold any more if (1.6) fails: for example, when $A = 1/\sqrt{2}$ and $N = 2$ a.s. (so $\alpha = 2$), both the normal law $N(0,1)$ and the Dirac measure δ_0 are solutions of (E) with mean 0.   (2) In the proof, we shall see that in part (iii), when l is constant, the condition $E(A^{\alpha+\epsilon}) < \infty$ can be relaxed to $E(A^\alpha) < \infty$.   (3) If we replace the constraint $\alpha \in (0, 2]$ by $\alpha > 0$, and if instead of $\mathbb{F}$, we consider $\bar{\mathbb{F}} = \{\phi : \phi$ satisfies (E') and $\phi(t) = \phi_+(|t|^\alpha)(\forall t)$, where ϕ_+ is the Laplace transform of some element $\mu \in \mathbb{M}_+ \}$, then all the conclusions (i), (ii) and (iii) still hold. In fact to see that (i) holds it suffices to take ϕ_+ to be the Laplace transform of a non-negative solution of (E) with A_i replaced by A_i^α; the proofs for (ii) and (iii) remain the same as in the proof of Theorem 1.2. Many results in the following sections also hold for elements of $\bar{\mathbb{F}}$, assuming only $\alpha > 0$.

Proof of Theorem 1.2.

(i) We use an argument of Guivarc'h [19]. By Theorem 1.1, there is a non-negative and non-trivial random variable Y satisfying

$$(1.9) \qquad Y \overset{d}{=} A_1^\alpha Y_1 + ... + A_N^\alpha Y_N,$$

$\{Y_k\}$ being independent copies of Y and independent of $\{N, A_1, A_2, ...\}$. Let X, X_k be random variables with stable distribution of index α, all the random variables N, A_k, X, X_k, Y, Y_k being independent each other. We claim that $Z = XY^{1/\alpha}$ is then a solution of (E). In fact, by the preceding equation, we need only to prove that

$$X(A_1^\alpha Y_1 + ... + A_N^\alpha Y_N)^{1/\alpha} \overset{d}{=} A_1 X_1 Y_1^{1/\alpha} + ... + A_N X_N Y_N^{1/\alpha}.$$

Putting $U_i = A_i Y_i^{1/\alpha}$, this just says $X(U_1^\alpha + ... + U_N^\alpha)^{1/\alpha} \overset{d}{=} X_1 U_1 + ... + X_N U_N$, which holds by the property of stable laws with exponent α. $\square$

(ii)   Assume   that ϕ_k are   characteristic   functions   satisfying   (E')   and $\lim_{t\to0+} \frac{1-\phi_k(t)}{t^\alpha} = c$, $k = 1, 2$. Write $\Delta\phi = \phi_1 - \phi_2$. Then by the mean value theorem,

$$|\Delta\phi(t)| \leq f'(1)E|\Delta\phi(At)|, \qquad t > 0.$$

Put $h(t) = \frac{|\Delta\phi(t)|}{t^\alpha}, t > 0$, and let B be a random variable with law $P_B(dx) = (EN)x^\alpha P_A(dx)$. Then $P(B > 0) = 1$ and $h(t) \leq Eh(Bt), t > 0$. By iteration, if $B_1, B_2, ...$ are independent copies of B, then

$$h(t) \leq Eh(B_1...B_n t), \qquad n \geq 1, t > 0.$$

Let $t > 0$ be fixed. As $E\log B = ENEA^\alpha \log A < 0$, we have $B_1...B_n \to 0$ a.s. by the law of large numbers. Since

$$\lim_{x\to0+} h(x) = \lim_{x\to0+} |\frac{1-\phi_1(x)}{x^\alpha} - \frac{1-\phi_2(x)}{x^\alpha}| = 0,$$

using the preceding inequality on $h(t)$ and the dominated convergence theorem, we obtain $h(t) = 0$. This means $\phi_1(t) = \phi_2(t)$ for all $t > 0$, and then for all $t \in \mathbb{R}$ because ϕ_k are hermitian symmetric.

(iii) By considering the distribution of A conditional on $A > 0$, we can assume $A > 0$ a.s. (cf. the proof of Proposition 4.10). Let us extend the definition of l by putting $l(t) = 1$ if $t \geq \delta$. Then for some constant $C_1 > 0$ and all $t > 0$, $1 - \phi(t) \leq Ct^\alpha l(t)$. By the functional equation (E'), we have

$$(1.10) \qquad \frac{1 - \phi(t)}{t^\alpha l(t)} = \frac{E(1 - \phi(At))}{t^\alpha l(t)} \sum_{i=1}^{\infty} p_i \sum_{0 \leq k \leq i-1} (E\phi(At))^k.$$

It is evident that we can assume $\epsilon < \alpha$. Notice that by Potter's theorem [10, p.25, Theorem 1.5.6(ii)], there is a constant $C_2 > 0$ such that for all $x > 0$ and $y > 0$, $l(y)/l(x) \leq C_2 \max\{(y/x)^\epsilon, (x/y)^\epsilon\}$. Therefore

$$\frac{(1 - \phi(At))}{t^\alpha l(t)} = \frac{(1 - \phi(At))}{(At)^\alpha l(At)} \frac{l(At)}{l(t)} A^\alpha \leq C_1 C_2 \max(A^\epsilon, A^{-\epsilon}) A^\alpha,$$

so that by the dominated convergence theorem, $\lim_{t \to 0} \frac{E(1-\phi(At))}{t^\alpha l(t)} = cEA^\alpha$. It follows by (1.10) that $c = cEA^\alpha \sum_{i=1}^{\infty} ip_i$, from which $1 = ENEA^\alpha$. So the proof is finished. $\square$

2. A Theorem on a Random Difference Equation.

In this section, A, B, Z are real random variables which are not necessarily positive. We consider the random difference equation

$$(2.1) \qquad Z \overset{d}{=} AZ + B, \text{ with } A, B, Z \text{ independent of each other,}$$

where the law of Z is unknown. In terms of characteristic functions, the equation reads

$$(2.2) \qquad \phi_Z(t) = \phi_B(t) E[\phi_Z(At)], \qquad t \in R,$$

where $\phi_Y(t) = Ee^{itY}$ if Y is a real random variable. It is well known that if Z is a solution not concentrated at a point, then its distribution is either absolutely continuous or singularly continuous [18], but in general it is not easy to prove the absolute continuity. The following result gives a sufficient condition for the law of Z to have a density function with some regularity properties. For all $k \in \mathbb{N}$, we denote by C^k the class of functions $g : \mathbb{R} \to \mathbb{R}$ with k-fold continuous derivatives, C^0 being the class of continuous functions.

THEOREM 2.1. *Assume* $\zeta := \limsup_{|t| \to \infty} |\phi_B(t)| < 1$ *and let* Z *be a solution of* (2.1). *Let* $a > 0$ *be fixed. If* $\zeta E|A|^{-a} < 1$ *(which reads* $E|A|^{-a} < \infty$ *when* $\zeta = 0$*), then*

$$(2.3) \qquad |\phi_Z(t)| = O(|t|^{-a}) \ (|t| \to \infty) \ and \ \int_{\mathbb{R}} |\phi_Z(t)| \, |t|^{a-1} \, dt < \infty.$$

In particular, if $\zeta E|A|^{-a} < 1$ for some $a > 1/2$, then the law of Z has a square integrable density (with respect to the Lebesgue measure); if the condition holds for some $a \geq 1$, then the law of Z has a density function of class $C^{[a]-1}$, $[a]$ being the integral part of a.

Proof of Theorem 2.1.   Set $g(t) = \sup_{|s| \geq t} |\phi_Z(s)|, t > 0$, and let $\epsilon > 0$ be sufficiently small such that $(\zeta + \epsilon)E|A|^{-a} < 1$. Choose $t_0 > 0$ large enough such that $|\phi_B(t)| \leq \zeta + \epsilon$ for all $|t| > t_0$. Then by the equation (2.2), for all $t > t_0$,

$$g(t) \leq (\zeta + \epsilon)Eg(|A|t).$$

So the conclusions are immediate consequence of the following Lemma 2.2 (with $C = 0$) which extends a result of Barral ([4]) and is also applicable in other contexts.   □

LEMMA 2.2. *(i) Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a bounded and measurable function satisfying*

$$(2.4) \qquad\qquad g(t) \leq pEg(At) + \frac{C}{t^a}, \qquad t > t_0,$$

where $0 < p < 1, a > 0, C \geq 0, t_0 \geq 0$ are constants and A is a positive random variable with $pEA^{-a} < 1$. Then as $t \to \infty$, $g(t) = O(t^{-a})$. (ii) We have moreover $\int_0^\infty t^{a-1}g(t)dt < \infty$ if the inequality (2.4) can be strengthened to

$$(2.5) \qquad\qquad g(t) \leq pEg(At) + \frac{C}{t^a h(t)}, \qquad t > t_0,$$

for some mesurable function $h : (t_0, \infty) \to (0, \infty)$ such that $\int_{t_0}^\infty \frac{1}{h(t)t}dt < \infty$.

Proof. Part (i) has been established in [33] and can be proved as follows. We can assume $t_0 = 0$ by taking C large enough if necessary; then by induction, for all $n \geq 1$ and all $t > 0$,

$$(2.6) \qquad g(t) \leq p^n Eg(A_1...A_n t) + Ct^{-a}[1 + pEA^{-a} + ... + (pEA^{-a})^{n-1}],$$

where A_i are independent copies of A; letting $n \to \infty$ gives $g(t) \leq Ct^{-a}/(1-pEA^{-a})$. To prove part (ii), let $C_1 > 0$ be sufficiently large such that for all $t > 0$,

$$g(t) \leq pEg(At) + \frac{C_1}{(1+t)^a h_1(t)},$$

where $h_1(t) = 1$ if $0 < t \leq t_0$, and $h_1(t) = h(t)$ if $t > t_0$. Put $G(T) = \int_0^T t^{a-1}g(t)dt$ for $T > 0$, and write $C_2 = C_1 \int_0^\infty t^{a-1}(1+t)^{-a}/h_1(t)\, dt$. Then $C_2 < \infty$ and, by the preceding inequality, for all $T > 0$,

$$G(T) \leq pE \int_0^T g(At)t^{a-1}dt + C_2 = pE[A^{-a}]EG(BT)] + C_2,$$

where B is a random variable with distribution $P_B(dx) = x^{-a}P_A(dx)/EA^{-a}$. By iteration, for all $n \geq 1$ and all $T > 0$,

$$(2.7) \qquad G(T) \leq (pEA^{-a})^n EG(B_1...B_n T) + C_2[1 + pEA^{-a} + ... + (pEA^{-a})^{n-1}],$$

where B_i are independent copies of B. Write $\|g\|_\infty = \sup_t |g(t)|$; notice that

$$(pEA^{-a})^n EG(B_1...B_n t) = (pEA^{-a})^n E \int_0^{B_1...B_n} g(t)t^{a-1}dt$$

$$\leq \|g\|_\infty (pEA^{-a})^n (EB^a)^n T^a/a = \|g\|_\infty p^n T^a/a \to 0 \text{ as } n \to \infty,$$

so letting $n \to \infty$ in (2.7) gives $G(T) \leq C_2/(1 - pEA^{-a})$. Since $T > 0$ is arbitrary and the right side is independent of T, we obtain $\int_0^\infty t^{a-1}g(t)\,dt \leq C_2/(1 - pEA^{-a})$, as desired. $\square$

REMARKS. (1) It may be useful to remark that if $EA^{-a} < \infty$ but $pE[A^{-a}] \geq 1$ and A is not a.s. constant, then the assertions (i) and (ii) also hold with a replaced by $b := a\log(1/p)/\log EA^{-a}$. In fact, because $b/a \leq 1$, by Jensen's inequality we have $pEA^{-b} = pE[(A^{-a})^{b/a}] < p[EA^{-a}]^{b/a} = 1$, so applying the proved results for b gives the desired assertions.   If A is a.s. a constant, say, $A = \delta \in (0,1)$, then the number b defined above becomes $b = \log p/\log \delta$; this time we still have $g(t) = O(t^{-b})$ if the condition in (ii) holds with a replaced by b and with h non-decreasing (for example if the condition in (i) holds with $C = 0$). In fact we can assume that g is non-increasing by considering $\bar{g}(t) = \sup_{s \geq t} g(s)$ instead of $g(t)$ if necessary. By iteration, for all $n = 1, 2, ...$, $\phi(t) \leq p^n \phi(\delta^n t) + \frac{C}{t^b}\sum_{k=0}^{n-1}\frac{1}{h(\delta^k t)}$ if $\delta^n t > t_0$. Notice that the sum in the last display is bounded by $(1 - \delta)^{-1}\sum_{k=0}^{n-1}\int_{\delta^{k+1}t}^{\delta^k t}\frac{1}{xh(x)}dx$, which is smaller than $(1 - \delta)^{-1}\int_{t_0}^\infty \frac{1}{xh(x)}dx$ if $\delta^n t > t_0$, so that $g(\delta^{-n}2t_0) \leq C_1[\delta^{-n}2t_0]^{-b}$ for some constant $C_1 > 0$ and all $n \geq 1$; as g is monotone, this gives $\phi(t) = O(t^{-b})$ $(t \to \infty)$.   (2) If a. s. $A > 0$ and $B \geq 0$, then the conclusion of Theorem 2.1 also holds when the Fourier Transform ϕ_Z is replaced by the Laplace transform $\psi_Z(t) = Ee^{-tZ}, t > 0$. This can be easily seen by the proof.   (3) By the remark (1) above, if $|A|$ is not constant and $\zeta^{-1} \leq E(|A|^{-a}) < \infty$, then the conclusions hold with a replaced by $b := a\log(1/\zeta)/\log EA^{-a}$; when $|A|$ is constant, say $|A| = \delta \in (0,1)$, then $|\phi_Z(t)| = O(|t|^{-b})$ $(|t| \to \infty)$, where $b = \log\zeta/\log\delta$. This remark have evident consequences in the following two sections, but for simplicity we shall not mention them.

Let us give some comments for the case where $A = \delta$ is a constant in $(0,1)$. In this case a solution of (2.1) can be expressed as the sum of a random Taylor series:

$$(2.8) \qquad\qquad Z = \sum_{n=0}^\infty B_n \delta^n,$$

where $\{B_n\}$ is a sequence of independent and identically distributed (i.i.d.) real random variables; by the formula of the radius of convergence of a Taylor series, the series converges a.s. for each $\delta \in (0,1)$ if $E\log^+|B_0| < \infty$ (which implies that a.s. $\limsup_{n\to\infty}\frac{\log^+|B_n|}{n} = 0$ by the law of large numbers).

Assume $\delta \in (0,1)$ and $E\log^+|B_0| < \infty$, and let Z be defined by (2.8). An interesting question is to study asymptotic properties of the characteristic function of Z, and to know when its distribution is absolutely continuous. By the second remark after the proof of Theorem 2.1, we know that if

$$\zeta := \limsup_{|t|\to\infty}|Ee^{itB_0}| < 1,$$

then as $|t| \to \infty$, $|Ee^{itZ}| = O(|t|^{-b})$, where $b = \log\zeta/\log\delta$, so that the distribution of Z has a square integrable density if $\zeta^2 < \delta$. Of particular interest is the case where the law of B_0 is a Cantor-like measure, for which K.S.Lau and T.Y.Hu [23] have recently obtained some interesting results about the value of ζ. When $\{B_n\}$ is a Rademacher sequence, the study of the series (2.8) is then a classical subject: see for example [16], [24], [22] and [45]. However in this case our result gives no information (because $\zeta = 1$).

3. Asymptotic Properties of $\phi \in \mathbb{F}$ and Absolute Continuity . In this section, for simplicity we assume $\alpha_0 = p_0 = 0$; this condition may be removed by an argument of reduction (cf. Proposition 4.10). Recall that we always assume (1.3), (1.4), and that $\mathbb{F}$ (resp. $\mathbb{F}_+$) stands for the set of all non-trivial solutions (resp. non-negative solutions) of (E).

For positive solutions with finite mean, the following result is a consequence of a theorem of Biggins and Grey [6]. The basic idea in the following proof is inspired by an argument of Athreya [1]; some similar arguments were used in [6].

LEMMA 3.1. *Assume (1.8) and $\alpha_0 = p_0 = 0$. Let $\mu \in \mathbb{F}$ and let ϕ be its characteristic function. Then $\lim_{|t| \to \infty} \phi(t) = 0$.*

Proof. Set $l = \limsup_{t \to \infty} |\phi(t)|$. Since ϕ is Hermitian symmetric, it suffices to prove that $l = 0$.

(i) We first prove that $l = 0$ or 1. By (E'), $|\phi(t)| \le f(E|\phi(At)|)$, $t \in \mathbb{R}$. Letting $t \to \infty$ and using Fatou's lemma, we obtain $l \le f(l)$. Therefore $l = 0$ or 1, noting that $f(x) < x$ if $0 < x < 1$.

(ii) We next prove that for all $t \ne 0$, $|\phi(t)| < 1$. Otherwise, by Lemma 4 of Chap.IV.1 of Feller [17], there is some $h > 0$ such that $|\phi(h)| = 1$ and $|\phi(t)| < 1$ if $0 < t < h$. So $1 = |\phi(h)| = f(E\phi(Ah)) \le E\phi(Ah)|$. Therefore, a.s. $|\phi(Ah)| = 1$. Since $P(0 < A < 1) > 0$, it follows that for some $0 < a < 1$, $|\phi(ah)| = 1$, which is a contradiction with the definition of h.

(iii) We then show that $l < 1$. Assume $l = 1$. Let $0 < t_0 < \infty$ be arbitrary fixed, and let $0 < \epsilon < 1 - |\phi(t_0)|$. Choose $t_1 = t_1(\epsilon)$ and $t_2 = t_2(\epsilon)$ such that $0 < t_1 < t_0 < t_2 < \infty$, with

$$|\phi(t_1)| = |\phi(t_2)| = 1 - \epsilon \text{ and } |\phi(t)| \le 1 - \epsilon \text{ if } t \in [t_1, t_2].$$

This is possible since ϕ is continuous and $|\phi(0)| = \limsup_{t \to \infty} |\phi(t)| = 1$. By the equation (E'), for all $t \in \mathbb{R}$, $|\phi(t)| \le E|\phi(At)|$. It follows that if A_i $(i \ge 1)$ are independent copies of A, then: $|\phi(t)| \le E|\phi(A_1...A_n t)|$, $E|\phi(At)| \le E|\phi(A_1...A_n t)|$, and

$$|\phi(t)| \le f(E|\phi(At)|) \le f(E|\phi(A_1...A_n t)|) \qquad (n \ge 1, t \in \mathbb{R}).$$

Therefore, using $|\phi(A_1...A_n t_2)| \le 1 - \epsilon$ if $t_1 < A_1...A_n t_2 \le t_2$ and $|\phi(A_1...A_n t_2)| \le 1$ otherwise, we obtain:

$$E|\phi(A_1...A_n t_2)| \le (1 - \epsilon)P[t_1 < A_1...A_n t_2 \le t_2] + 1 - P[t_1 < A_1...A_n t_2 \le t_2]$$
$$= 1 - \epsilon P[t_1 < A_1...A_n t_2 \le t_2],$$

$$1 - \epsilon = |\phi(t_2)| \le f(E|\phi(A_1...A_n t_2)|) \le f(1 - \epsilon P[t_1 < A_1...A_n t_2 \le t_2]),$$

$$\frac{1 - f(1 - \epsilon P[t_1 < A_1...A_n t_2 \le t_2])}{\epsilon P[t_1 < A_1...A_n t_2 \le t_2]} \le \frac{1}{P[t_1 < A_1...A_n t_2 \le t_2]}.$$

Since $\lim_{\epsilon \to 0} t_1(\epsilon) = 0$ (this can be easily verified) and $t_1(\epsilon)/t_2(\epsilon) \le t_1(\epsilon) \to 0$ $(\epsilon \to 0)$, letting $\epsilon \to 0$ in the above inequality gives

$$f'(1) \le 1/P[A_1...A_n \le 1].$$

Let $\alpha \in (0, 2]$ be defined as in (1.8), so that $EA^\alpha = 1/EN < 1$. Then by Markov's inequality,

$$P[A_1...A_n > 1] = P[(A_1...A_n)^\alpha > 1] \le [E(A^\alpha)]^n \to 0 \quad \text{as } n \to \infty.$$

So $\lim_{n\to\infty} P[A_1...A_n \le 1] = 1$. Therefore, letting $n \to \infty$ in the preceding inequality on $f'(1)$ gives $f'(1) \le 1$. This contradicts the hypothesis that $f'(1) = EN > 1$. So $l < 1$. $\square$

THEOREM 3.2. *Assume (1.8) and $\alpha_0 = p_0 = 0$. Let $\mu \in \mathbb{F}$ and let ϕ be its characteristic function. Write $m = \min\{i \ge 1 : p_i > 0\}$ and let $0 < a < \infty$ be a positive number.*
  (i) If $p_1 EA^{-a} < 1$ (which reads $EA^{-a} < \infty$ when $p_1 = 0$), then

$$(3.1) \qquad \phi(t) = O(|t|^{-ma}) \ (|t| \to \infty) \ and \ \int_{-\infty}^{\infty} |\phi(t)| \, |t|^{ma-1} dt < \infty.$$

  (ii) When $p_1 = 0$, the condition $P(A \le x) = O(x^{-a}) \ (x \to 0)$ suffices for $\phi(t) = O(|t|^{-ma}) \ (|t| \to \infty)$.

Proof. (i) We notice that the equation (E) can be regarded as a special case of the random difference equation (2.1) with $B = \sum_{2 \le i \le N} A_i Z_i$ (the empty sum is taken to be 0), whose characteristic function is $\phi_B(t) = f_1(E\phi(At))$, f_1 being the probability generating function of $N - 1$. By Lemma 3.1, $\zeta := \lim_{|t|\to\infty} |\phi_B(t)| = p_1$; so by Theorem 2.1, $p_1 EA^{-a} < 1$ implies

$$(3.2) \qquad \phi(t) = O(|t|^{-a}) \ (|t| \to \infty) \ and \ \int_{-\infty}^{\infty} |\phi(t)| \, |t|^{a-1} dt < \infty.$$

This ends the proof of (3.1) in the case where $p_1 > 0$. Assume $p_1 = 0$. Then $m \ge 2$ and by the equation (E'),

$$(3.3) \qquad |\phi(t)| \le f(E|\phi(At)|) \le (E|\phi(At)|)^m.$$

By (3.2), there is a constant $C > 0$ such that $|\phi(t)| \le C|t|^{-a}$ for all $t \ne 0$, so that $E|\phi(At)| \le E|CAt|^{-a} = CEA^{-a}|t|^{-a}$. Consequently by (3.3), $|\phi(t)| \le C^m (EA^{-a})^m |t|^{-ma}$ and

$$\int_{-\infty}^{\infty} |\phi(t)| \, |t|^{ma-1} \, dt \le \int_{-\infty}^{\infty} (E|\phi(At)| \, |t|^a)^{m-1} E|\phi(At)| \, |t|^{a-1} \, dt$$

$$\le (CEA^{-a})^{m-1} E \int_{-\infty}^{\infty} |\phi(At)| \, |t|^{a-1} \, dt$$

$$(3.4) \qquad = (CEA^{-a})^{m-1} EA^{-a} \int_{-\infty}^{\infty} |\phi(x)| \, |x|^{a-1} \, dx.$$

The last integral is finite by (3.2). So the proof of (i) is finished.

  (ii) Under the given conditions, we have $m \ge 2$ and $EA^{-a_1} < \infty$ for all $0 < a_1 < a$, so the conclusion in part (i) implies $\phi(t) = O(|t|^{-b})$ for for all $0 < b < ma$. Now choose $b \in (a, ma)$ and let C_1 be such that $|\phi(t)| \le C_1(1 + |t|)^{-b}$ for all $t \ne 0$. Then a similar argument as above gives

$$\phi(t) \le [E\phi(At)]^m \le C_1^m (E[(1 + A|t|)^{-b}])^m = O(|t|^{-ma}),$$

where the last step holds by the following Lemma.   □

LEMMA 3.3. *[33, Lemma 4.3]* If $P(A \leq x) = O(x^a)$ $(x \to 0)$ *for some* $a > 0$, *then for all* $b > a$,

$$E[(1 + At)^{-b}] = O(t^{-a}) \text{ as } t \to \infty.$$

When $\mu \in \mathbb{F}_+$ has finite mean, a similar argument as in the proof of Theorem 3.2 gives the decay rate of the derivative $\phi'(t)$, ϕ being the characteristic function of μ:

THEOREM 3.4. *Assume (1.6) and* $\alpha_0 = p_0 = 0$. *Let* $\mu \in \mathbb{F}_+$ *and let* ϕ *be its characteristic function. Write* $m = \inf\{i \geq 1 : p_i > 0\}$ *and let* $0 \leq a < \infty$ *be a non-negative number.*

 (i) If $p_1 E A^{-a} < 1$ *(which reads* $E A^{-a} < \infty$ *when* $p_1 = 0$*), then*

$$(3.5) \qquad \phi'(t) = O(|t|^{-(1+ma)}) \ (|t| \to \infty) \ and \ \int_{-\infty}^{\infty} |\phi'(t)| \, |t|^{ma} dt < \infty.$$

 (ii) When $p_1 = 0$, *the condition* $P(A \leq x) = O(x^{-a})$ *(* $x \to 0$*) suffices for*
 $\phi'(t) = O(|t|^{-(1+ma)}) \ (|t| \to \infty)$.

Proof. Without loss of generality, we assume $\int x\mu(dx) = 1$. Still by the equation (E'), we have

$$(3.6) \qquad \phi'(t) = f'(E\phi(At))E\phi'(At)A, \qquad t \in \mathbb{R}.$$

Let $\tilde{Z}$ and $\tilde{A}$ be random variables with distributions $x\mu(dx)$ and $f'(1)xP_A(dx)$ respectively, and let $\tilde{B}$ be a random variable with characteristic function $\phi_{\tilde{B}}(t) = f'(E\phi(At))/f'(1)$; $\tilde{Z}, \tilde{A}$ and $\tilde{B}$ being independent of each other. Then $\tilde{Z}$ has characteristic function $\phi_{\tilde{Z}}(t) = \phi'(t)/i$, and the preceding identity reads

$$(3.7) \qquad \phi_{\tilde{Z}}(t) = \phi_{\tilde{B}}(t)E\phi_{\tilde{Z}}(\tilde{A}t), \qquad t \in \mathbb{R}.$$

 (i) Notice that $\tilde{\zeta} := \lim_{|t| \to \infty} |\phi_{\tilde{B}}(t)| = f'(0)/f'(1) = p_1/f'(1)$ and $\tilde{\zeta} E|\tilde{A}|^{-(a+1)} = p_1 E A^{-a} < 1$, so by Theorem 2.1,

$$(3.8) \qquad \phi'(t) = O(|t|^{-(1+a)}) \ (|t| \to \infty) \ and \ \int_{-\infty}^{\infty} |\phi'(t)| \, |t|^a dt < \infty.$$

This completes the proof of (3.5) in the case where $p_1 > 0$. Assume $p_1 = 0$. Then $m \geq 2$. Because $|\phi_{\tilde{B}}(t)| \leq (E|\phi(At)|)^{m-1}$, it follows that for all $t \in \mathbb{R}$,

$$(3.9) \qquad |\phi_{\tilde{Z}}(t)| \leq (E|\phi(At)|)^{m-1} E[|\phi_{\tilde{Z}}(\tilde{A}t)|].$$

Recall that $|\phi(t)| = O(|t|^{-ma})$ by Theorem 3.2 and $\phi'(t) = O(|t|^{-(1+a)})$ by (3.8); so for some constant $C > 0$ and all $t \in \mathbb{R}$, $|\phi(t)| \leq C|t|^{-a}$ and $|\phi_{\tilde{Z}}| = |\phi'(t)| \leq C|t|^{-(1+a)}$. Therefore by (3.9),

$$(3.10) \qquad |\phi_{\tilde{Z}}(t)| \leq (EC|At|^{-a})^{m-1} E[C|\tilde{A}t|^{-(1+a)}] = C_1|t|^{-(ma+1)},$$

where $C_1 = C^m (E A^{-a})^{m-1} E[\tilde{A}^{-(1+a)}] = f'(1)C^m (E A^{-a})^m < \infty$. This gives the first assertion in (3.5). For the second assertion, we remark that for all t, $(E|\phi(At)|t^a)^{m-1} \leq (CEA^{-a})^{m-1} =: C_2$, so that by (3.9),

$$(3.11) \qquad \int_{-\infty}^{\infty} |\phi_{\tilde{Z}}(t)| \, |t|^{ma} dt \leq C_2 E \int_{-\infty}^{\infty} |\phi_{\tilde{Z}}(\tilde{A}t)| \, |t|^a dt$$

$$= C_2 E\tilde{A}^{-(a+1)} \int_{-\infty}^{\infty} |\phi_{\tilde{Z}}(x)| \, |x|^a dx.$$

Because $E\tilde{A}^{-(a+1)} = f'(1)EA^{-a} < \infty$, the second assertion in (3.5) follows from (3.11) and (3.8).

(ii) Under the given conditions, we have $m \geq 2$ and $EA^{-a_1} < \infty$ for all $a_1 \in (0, a)$, so by the conclusion in part (i), we know that $\phi_{\tilde{Z}}(t) = O(|t|^{-(1+mb)})$ $(|t| \to \infty)$ for all $0 < b < a$. Choose $b \in (a/m, a)$ and let $C_3 > 0$ be a constant such that for all $t \in \mathbb{R}$, $|\phi(t)| \leq C_3|t|^{-a}$ and $|\phi'(t)| \leq C_3(1 + |t|)^{-(1+mb)}$. Then for all $t \in \mathbb{R}$,

$$|\phi_{\tilde{Z}}(t)| \leq (E|\phi(At)|)^{m-1}E[|\phi_{\tilde{Z}}(\tilde{A}t)|] \leq (EC_3|At|^{-a})^{m-1}E[C_3(1 + |\tilde{A}t|)^{-(1+mb)}]$$

Since $P(\tilde{A} \leq x) = O(x^{1+a})$ $(x \to 0)$ and $1 + mb > 1 + a$, we have $E[(1 + |\tilde{A}t|)^{-(1+mb)}] = O(|t|^{-(1+a)})$ by Lemma 3.3; it follows by the preceding inequality on $\phi_{\tilde{Z}}(t)$ that $|\phi_{\tilde{Z}}(t)| = O(|t|^{-(ma+1)})$. $\square$

Notice that the condition $p_1EA^{-a} < 1$ always holds for $a = 0$, so that by Theorem 3.4(i), under (1.6), all $\phi \in \mathbb{F}_+$ satisfies

$$(3.12) \qquad \phi'(t) = O(|t|^{-1}) \; (|t| \to \infty) \text{ and } \int_{-\infty}^{\infty} |\phi'(t)|dt < \infty.$$

As an application of Theorems 3.2 and 3.4, we obtain the absolute continuity of $\mu \in \mathbb{F}$ under simple moment conditions:

THEOREM 3.5. *Assume (1.8) and $\alpha_0 = p_0 = 0$. Let $\mu \in \mathbb{F}$ and let ϕ be its characteristic function. Write $m = \min\{i \geq 1 : p_i > 0\}$. Then μ has a density function with respect to the Lebesgue measure if one of the following conditions holds* [1]: *(a) $p_1 = 0$ and $EA^{-a} < \infty$ for some $a > 1/(2m)$; (b) $p_1 > 0$ and $p_1EA^{-a} < 1$ for some $a > 1/2$; (c) (1.6) and $\mu \in \mathbb{F}_+$; (d) $\mu \in \mathbb{F}_+$, $\alpha < 1$, A is not a.s. concentrated on a geometric series, and $EN^{1+\epsilon} + EA^{1+\epsilon} < \infty$ for some $\epsilon > 0$. Moreover, if (a) or (b) holds with $a \geq 1/m$, then μ has a density function of class $C^{[ma]-1}$, given by*

$$(3.13) \qquad x \mapsto \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(t)e^{-itx}dt;$$

when (c) holds, μ has always a continuous density function on $(0, \infty)$, given by

$$(3.14) \qquad x \mapsto \frac{1}{2\pi ix}\int_{-\infty}^{\infty} \phi'(t)e^{-itx}dt,$$

which is of class $C^{[ma]}$ if additionally (a) or (b) holds with $a \geq 1/m$.

Proof. If (a) or (b) holds, then ϕ is square integrable by the first assertion in part (i) of Theorem 3.2, so that μ has a square integrable density function; if the condition holds with $a \geq 1/m$, then by the second assertion in part (i) of the same theorem, ϕ is integrable and μ has a density function of class $C^{[ma]-1}$ given by (3.13). If (c) holds, then the measure $x\mu(dx)$ has Fourier transform $t \mapsto \phi'(t)/i$, which is always integrable by Theorem 3.4 (i) (cf. (3.12)), so that the measure $x\mu(dx)$ has a density function given by $x \mapsto \frac{1}{2\pi i}\int_{-\infty}^{\infty} \phi'(t)e^{-itx}dt$; this density function is of class $C^{[ma]}$ again by Theorem 3.4 (i). If (d) holds, then μ is absolutely continuous because, by [32, Theorem 7.3], μ is the distribution of $XY^{1/\alpha}$, where Y is a non-negative and non-trivial solution of (1.9), X is of stable law with index α and independent of Y. $\square$

We remark that in Case (c), Theorem 3.4 gives more information about the existence of a density and its regularity than what we can obtain from Theorem 3.2.

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[1] Since the completion of this paper, the author has recently proved in [36] (and in a more general setting) that μ has a density function on $(0, \infty)$ whenever $EA^{-\epsilon} 1\{A > 0\} < \infty$ for some $\epsilon > 0$.

4. Polynomial Tail Behavior of $\mu \in \mathbb{F}_+$. In this section, we consider the case where $P(0 < Z \le x)$ decays at a polynomial rate as $x \to 0+$. The main results of this section are the following two theorems. We distinguish two cases according as $\alpha_0 = p_0 = p_1 = 0$ or not. Recall again that we always assume (1.3) and (1.4).

THEOREM 4.1. *Assume (1.5) and $\alpha_0 = p_0 = p_1 = 0$, and let Z be a non-trivial and non-negative solution of (E). Write $m = \min\{i \ge 2 : p_i > 0\}$. Then for each fixed $0 < a < \infty$, the following assertions hold:*

(i) *As $x \to 0, P(A \le x) = O(x^a)$ if and only if $P(Z \le x) = O(x^{ma})$.*

(ii) *If for some constants $C_1 > 0$, $C_2 > 0$ and all $x > 0$ small enough, $C_1 x^a \le P(A \le x) \le C_2 x^a$, then there are some constants $C_3 > 0$, $C_4 > 0$ such that, for all $x > 0$ small enough,*

$$C_3 x^{ma} \le P(Z \le x) \le C_4 x^{ma}.$$

(iii) *If $E(A^{-a}) < \infty$, then $E[Z^{-ma}] < \infty$; conversely, if $E[Z^{-ma}] < \infty$, then $E(A^{-b}) < \infty$ for all $0 < b < a$.*

(iv) *Let $\delta > 0$ and let $l : (0,\delta] \to (0,\infty)$ be a function slowly varying at 0 and bounded away from 0 and ∞ on each compact subset of $(0,\delta]$. If $P(A \le x) \sim x^a l(x) \, (x \to 0)$, then*

$$P(Z \le x) \sim p_m x^{ma}[E(Z^{-a})]^m[l(x)]^m \, (x \to 0).$$

REMARK. *The converse in part (iii) cannot be improved to "$E[Z^{-ma}] < \infty \Rightarrow E(A^{-a}) < \infty$."* To see this, let A be such that for some $a > 0, b \in \mathbb{R}$ and $C > 0$, $P(A \le x) \sim Cx^a |\log x|^{-b} \, (x \to 0)$. Then by part (iv) of Theorem 4.1, $P(Z \le x) \sim C^m x^{ma} E^m(Z^{-a})|\log x|^{-mb} \, (x \to 0)$. So for some constants $C_1 > 0$ and $C_2 > 0$, $E(Z^{-ma}) = ma \int_0^\infty P(Z \le 1/y) y^{ma-1} dy \le C_1 + C_2 \int_2^\infty y^{-ma}|\log y|^{-mb} y^{ma-1} dy = C_1 + C_2 \int_2^\infty \frac{dy}{y(\log y)^{mb}}$, which is finite if $mb > 1$. Similarly, for some constant $C_3 > 0$, $E(A^{-a}) = a \int_0^\infty P(A \le 1/y) y^{a-1} dy \ge C_3 \int_2^\infty \frac{dy}{y(\log y)^b} = \infty$ if $b \le 1$. So for $1/m < b \le 1$, $E(Z^{-ma}) < \infty$ but $E(A^{-a}) = \infty$.

THEOREM 4.2. *Assume (1.5) and $\alpha_0 + p_0 + p_1 > 0$, and let Z be a non-trivial and non-negative solution of (E). Write $\tilde{p}_1 = (1-\alpha_0)f'(\alpha_0 + (1-\alpha_0)q)$. Then $0 < \tilde{p}_1 < 1$, and for each fixed $0 < a < \infty$, the following assertions hold:*

(i) *If $\tilde{p}_1 E[A^{-a} | A > 0] < 1$, then $E[Z^{-b} | Z > 0] < \infty$ for all $0 < b < a$; conversely, if $E[Z^{-a} | Z > 0] < \infty$, then $\tilde{p}_1 E[A^{-a} | A > 0] < 1$;*

(ii) *In the case where $A \le 1$ almost surely, $\tilde{p}_1 E[A^{-a} | A > 0] < 1$ if and only if $E[Z^{-a} | Z > 0] < \infty$.*

Our result gives explicitly the critical value for existence of moments of negative orders:

COROLLARY. Under the assumptions of Theorem 4.2, If $\tilde{p}_1 E[A^{-\lambda} | A > 0] = 1$ for some $\lambda > 0$, then

$$E[Z^{-b} | Z > 0] \begin{cases} < \infty & \text{when } b \in (0,\lambda), \text{ and} \\ = \infty & \text{when } b \in [\lambda,\infty). \end{cases}$$

Theorems 4.1 and 4.2 show that there are two essentially different cases, according to $\alpha_0 = p_0 = p_1 = 0$ or not. For the Mandelbrot's equation, the fact that the situation differs according to $P(A = 0) = 0$ or not has already been remarked by Barral [3, 4],

although his description was not so precise as is given by our results above. For the Poincaré's equation, the results show that Z has moments of all negative orders if $p_0 = p_1 = 0$, and that there is an critical value for existence of negative orders if $p_0 + p_1 > 0$; in fact it is known [8] that as $x \to 0$, $P(0 < Z \le x)$ decays at an exponential rate if $p_0 = p_1 = 0$ (see also Theorem 5.1 in Section 5), and at a polynomial rate if $p_0 + p_1 > 0$.

We proceed towards the proof of Theorems 4.1 and 4.2 by a series of lemmas and propositions. In the following, we always assume (1.5) and let Z be a non-trivial and non-negative solution of (E).

LEMMA 4.3. *[33] Let $X > 0$ be a positive random variable. For each fixed $0 < a < \infty$, consider the following statements: (i) $E[X^{-a}] < \infty$; (ii) $E[e^{-tX}] = O(t^{-a})$ $(t \to \infty)$; (iii) $P[X \le x] = O(x^a)$ $(x \to 0)$; (iv) $\forall b \in (0, a), E[X^{-b}] < \infty$. Then the following implications hold: (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv).*

PROPOSITION 4.4. *Assume $\alpha_0 = p_0 = 0$ and write $m = \min\{i \ge 1 : p_i > 0\}$. Let $0 < a < \infty$ be a positive number. (i) If $Ee^{-Zt} = O(t^{-a})$ $(t \to \infty)$, then $Ee^{-At} = O(t^{-a/m})$ $(t \to \infty)$. (ii) If $EZ^{-a} < \infty$, then $EA^{-a} < \infty$ when $m = 1$, and $EA^{-b/m} < \infty$ when $b \in (0, a)$ and $m > 1$.*

Proof. Write $\phi_A(t) = E[e^{-At}]$. Let $0 < c < \infty$ be such that $P(Z \le c) > 0$. Then by (E), for all $t > 0$,

$$E[e^{-tZ}] = E[e^{-t \sum_{i=1}^{N} A_i Z_i}]$$
$$= \sum_{i=m}^{\infty} p_i [E\phi_A(Zt)]^i \ge p_m [E\phi_A(Zt)]^m \ge p_m [P(Z \le c)\phi_A(ct)]^m.$$

Therefore $E[e^{-tZ}] = O(t^{-a})$ implies $\phi_A(t) = O(t^{-a/m})$ $(t \to \infty)$, which ends the proof of (i). To prove (ii), we assume $EZ^{-a} < \infty$ and write $Z' = A_1 Z_1 + ... + A_N Z_N$. Then we have consecutively,

$$Z' \le \max\{A_1, ..., A_N\}(Z_1 + ... + Z_N),$$
$$(Z')^{-a} \ge (\max\{A_1, ..., A_N\})^{-a}(Z_1 + ... + Z_N)^{-a},$$
$$EZ^{-a} = E(Z')^{-a} \ge E[(\max\{A_1, ..., A_N\})^{-a}(Z_1 + ... + Z_N)^{-a} 1\{N = m\}]$$
$$= E[\max\{A_1, ..., A_m\}]^{-a} E(Z_1 + ... + Z_m)^{-a} p_m.$$

Therefore $E(\max\{A_1, ..., A_m\})^{-a} < \infty$. If $m = 1$, this just says $EA^{-a} < \infty$; if $m > 1$, then as $x \to 0$, $P[\max\{A_1, ..., A_m\} \le x] = O(x^a)$, so that $P[A \le x] = O(x^{a/m})$ $(x \to 0)$ and $EA^{-b/m} < \infty$ for all $0 < b < a$ by Lemma 4.3. $\square$

PROPOSITION 4.5. *Assume $\alpha_0 = p_0 = p_1 = 0$ and write $m = \min\{i \ge 1 : p_i > 0\}$. Let $0 < a < \infty$ be a positive number. If either (i) $p_1 = 0$ and $P(A \le x) = O(x^a)$ as $x \to 0$, or (ii) $p_1 > 0$ and $p_1 EA^{-a} < 1$, then $Ee^{-Zt} = O(t^{-ma})$ as $t \to \infty$.*

Proof. The same as in the proof of Theorem 4.2, using the Laplace transform instead of the characteristic function, together with the remark (2) following the proof of Theorem 2.1. $\square$

LEMMA 4.6. *Assume $p_0 = 0$ and put $m = \min\{k \ge 1 : p_k > 0\}$. Then for all fixed $K > 0$ and all $x > 0$, $P(Z \le x) \ge p_m [P(Z \le K)]^m [P(A \le \frac{x}{Km})]^m$.*

Proof. It suffices to use (E) and the fact that

$$P(A_1 Z_1 + ... + A_N Z_N \le x) \ge P(N = m, Z_i \le K \text{ and } A_i K \le x/m \text{ for all } i = 1, ..., m).$$

□

LEMMA 4.7. *Let $n \geq 2$ be an integer and let $X_1, ..., X_n$ be a sequence of i.i.d. positive random variables. Then for all $a > 0$, $E[(X_1 + ... + X_n)^{-na}] \leq n[E(X_1^{-a})]^n$.*

Proof. If $E(X_1^{-a}) = \infty$, there is nothing to prove. Assume that $E(X_1^{-a}) < \infty$. Then

$$E(X_1^{-a}) = \int_0^\infty P(X_1^{-a} > x) dx = a \int_0^\infty P(X_1 < 1/y) y^{a-1} dy.$$

Similarly, $E[(X_1 + ... + X_n)^{-na}] = na \int_0^\infty P(X_1 + ... + X_n < 1/y) y^{na-1} dy \leq na \int_0^\infty [P(X_1 < 1/y)]^n y^{na-1} dy$. Since $P(X_1 < 1/y) = P(X_1^{-a} > y^a) \leq y^{-a} E(X_1^{-a})$, it follows that

$$E[(X_1 + ... + X_n)^{-na}] \leq na \int_0^\infty [y^{-a} E(X_1^{-a})]^{n-1} [P(X_1 < 1/y)] y^{na-1} dy$$

$$= n[E(X_1^{-a})]^{n-1} a \int_0^\infty P(X_1 < 1/y) y^{a-1} dy = n[E(X_1^{-a})]^n.$$

□

PROPOSITION 4.8. *Assume $\alpha_0 = p_0 = p_1 = 0$ and write $m = \min\{i \geq 2 : p_i > 0\}$. Let $0 < a < \infty$ be a positive number. Then: (i) If $E(A^{-a}) < \infty$, then $E(Z^{-ma}) < \infty$; (ii) Let $t_0 > 0$ and let $l : [t_0, \infty) \to (0, \infty)$ be a function slowly varying at ∞ and bounded away from 0 and ∞ on every compact subset of $[t_0, \infty)$. If $Ee^{-At} \sim t^{-a} l(t)$ $(t \to \infty)$, then*

$$Ee^{-Zt} \sim p_m t^{-ma} [E(Z^{-a})]^m [l(t)]^m, \qquad t \to \infty.$$

Proof. (i) Assume $E(A^{-a}) < \infty$. Then $P(A \leq x) = O(x^a)$ $(x \to 0)$. So by Lemma 4.3 and Proposition 4.5, $P(Z \leq x) = O(x^{ma})$ $(x \to 0)$ and $E(Z^{-mb}) < \infty$ for all $0 < b < a$. In particular, $E[(AZ)^{-a}] = E[A^{-a}]E[Z^{-a}] < \infty$. So by the equation (E) and Lemma 4.7,

$$E[Z^{-ma}] = E[(A_1 Z_1 + ... + A_N Z_N)^{-ma}]$$

$$\leq E[(A_1 Z_1 + ... + A_m Z_m)^{-ma}] \leq m(E[(AZ)^{-a}])^m < \infty.$$

(ii) Write $\phi(t) = Ee^{-tZ}$ and $\phi_A(t) = Ee^{-tA}$. For simplicity we can assume $t_0 = 0$; otherwise we consider $\tilde{l}$ instead of l, where $\tilde{l}(t) = 1$ if $0 \leq t < t_0$ and $\tilde{l}(t) = l(t)$ if $t \geq t_0$.

Let $0 < \epsilon < a$. So $EZ^{-(a+\epsilon)} < \infty$ by part (i). By Potter's theorem [10, p.25, Theorem 1.5.6(ii)], there exists $C_1 = C_1(\epsilon) > 0$ such that for all $x > 0$ and $y > 0$,

$$l(y)/l(x) \leq C_1 \max\{(y/x)^\epsilon, (x/y)^\epsilon\}.$$

Since $\phi_A(t) \sim t^{-a} l(t)$, we can suppose that there is some constant $C_2 > 0$ such that, for all $t > 0$, $\phi_A(t) t^a / l(t) \leq C_2$. As $\phi_A(Zt) t^a / l(t) \to Z^{-a}$ $(t \to \infty)$ and

$$\frac{\phi_A(Zt) t^a}{l(t)} = \frac{\phi_A(Zt)(Zt)^a}{l(Zt)} \frac{l(Zt)}{l(t)} Z^{-a}$$

$$\leq C_2 C_1 \max(Z^\epsilon, Z^{-\epsilon}) Z^{-a} \leq C_2 C_1 (Z^{-(a-\epsilon)} + Z^{-(a+\epsilon)}),$$

by the dominated convergence theorem we obtain $\lim_{t\to\infty}\frac{E\phi_A(Zt)t^a}{l(t)} = EZ^{-a}$. So $E\phi(At) = E\phi_A(Zt) \sim t^{-a}E(Z^{-a})l(t)$. Therefore, by the equation (E'), as $t \to \infty$,

$$\phi(t) \sim p_m[E\phi(At)]^m \sim p_m t^{-am}[E(Z^{-a})]^m[l(t)]^m.$$

□

PROPOSITION 4.9. *Assume $\alpha_0 = p_0 = 0$ and $p_1 > 0$. Let $0 < a < \infty$ be a positive number. Then: (i) If $p_1 E(A^{-a}) < 1$, then for all $0 < b < a$, $EZ^{-b} < \infty$; if additionally $A \le 1$ a.s., then $EZ^{-a} < \infty$. (ii) Conversely, if $E(Z^{-a}) < \infty$, then $p_1 E(A^{-a}) < 1$.*

Proof. (i) If $p_1 E(A^{-a}) < 1$, then $Ee^{-tZ} = O(t^{-a})\,(t \to \infty)$ by Lemma 3.5. So for all $0 < b < a$, $EZ^{-b} < \infty$. This gives the first conclusion. Now assume $P(A \le 1) = 1$. Then for all $0 < b < a$, $E(A_i Z_i)^{-b} < \infty$, $Ee^{-t(A_1 Z_1 + A_2 Z_2)} = O(t^{-2b})$, so that $E[(A_1 Z_1 + A_2 Z_2)^{-a}] < \infty$ by Lemma 4.3. Now for all $T > 0$,

$$EZ^{-a}1\{Z > T\} = E(\sum_{i=1}^{N} A_i Z_i)1\{\sum_{i=1}^{N} A_i Z_i > T\}$$

$$\le E[(A_1 Z_1)^{-a}1\{A_1 Z_1 > T)\}1\{N = 1\}] + E[(A_1 Z_1 + A_2 Z_2)^{-a}]P(N > 1).$$

Because a. s. $A_1 \le 1$, $1\{A_1 Z_1 > T)\} \le 1\{Z_1 > T)\}$, it follows that

$$(1 - p_1 EA^{-a})E[Z^{-a}1\{Z > T\}] \le (1 - p_1)E[(A_1 Z_1 + A_2 Z_2)^{-a}].$$

Since $T > 0$ is arbitrary, this gives $EZ^{-a} \le (1-p_1)E[(A_1 Z_1 + A_2 Z_2)^{-a}]/(1 - p_1 EA^{-a})$.

(ii) Conversely, if $E(Z^{-a}) < \infty$, then by the equation (E),

$$E[Z^{-a}] = E[(A_1 Z_1 + ... + A_N Z_N)^{-a}] > E[(A_1 Z_1)^{-a}1\{N = 1\}] = p_1 E[A^{-a}]E[Z^{-a}].$$

Therefore $p_1 E[A^{-a}] < 1$.   □

We now give a *principle of reduction*, which says that the case where $\alpha_0 > 0$ or $p_0 > 0$ can be reduced to the case where $\alpha_0 = p_0 = 0$.

PROPOSITION 4.10. *Let $\bar{A}$ (resp. $\bar{Z}$) be a random variable whose distribution is that of A (resp. Z) conditional on $A > 0$ (resp. $Z > 0$). Let $\bar\phi$ be the characteristic function of $\bar{Z}$ and put*

$$\bar{f}(t) = \frac{\tilde{f}(q + (1 - q)t) - q}{1 - q},$$

where $q = P(Z = 0)$ is the unique fixed point in $[0, 1)$ of $\tilde{f}(t) = f(\alpha_0 + (1 - \alpha_0)t)$. Then the functional equation (E') reduces to

(4.1) $$\bar{\phi}(t) = \bar{f}(E\bar{\phi}(\bar{A}t)),   t \in \mathbb{R},$$

with $P(\bar{A} = 0) = \bar{f}(0) = 0$. Moreover, $\bar{f}'(0) = (1 - \alpha_0)f'(\alpha_0 + (1 - \alpha_0)q)$, so that $\bar{f}'(0) = 0$ if and only if $\alpha_0 = p_0 = p_1 = 0$.

Proof. It suffices to remark that $E\phi(At) = E\phi(At)1\{A > 0\} + E\phi(At)1\{A = 0\} = \alpha_0 + (1 - \alpha_0)E\phi(\bar{A}t)$, that $\phi(t) = q + (1 - q)\bar\phi(t)$, and that $f'(\alpha_0 + (1 - \alpha_0)q) = 0 \Leftrightarrow \bar{f}'(0) = 0$ and $\alpha_0 + (1 - \alpha_0)q = 0$.   □

For the Mandelbrot's equation (N is constant), the idea to consider the distribution of A conditional on $A > 0$ has been used by Barral [3]; for the Poincaré's equation (A is constant), the principle reduces to the famous transformation of Harris [20].

Proof of Theorem 4.1. Part (i) is a combination of Lemma 4.3, Proposition 4.5(i) and Lemma 4.6; part (ii) is a sequence of part (i) and Lemma 4.6; part (iii) comes from Propositions 4.5 and 4.8(i), together with Lemma 4.3; part (iv) follows from Proposition 4.8(ii) and a classical Tauberian theorem [10, Theorem 1.7.1]. □

Proof of Theorem 4.2. By the principle of reduction (Proposition 4.10), we can assume $p_0 = \alpha_0 = 0$; the theorem is then Proposition 4.9. □

It will be interesting to compare the left tail behavior with the right tail behavior; to this end let us state a theorem about the right tail behavior. The problem is called *lattice* if for some $h > 0$, $\log A$ is almost surely an integer multiple of h whenever $A > 0$; otherwise, it is called *non-lattice*. The following result is a direct consequence of Theorems 1 and 2 of Liu [35]:

PROPOSITION 4.11. *Assume (1.6) and let Z be a non-trivial and non-negative solution of (E).*
(i) *For each fixed $p > 1$, $E[Z^p] < \infty$ if and only if $E[N^p] < \infty$ and $ENE[A^p] < 1$.*
(ii) *Suppose that for some $\chi > 1$, $ENE[A^\chi] = 1$, $E[A^\chi \log^+ A] < \infty$ and $E[N^\chi] < \infty$. If the problem is non-lattice, then $\lim_{x\to\infty} x^\chi P(Z > x)$ exists and is strictly positive and finite; if the problem is lattice, then $0 < \liminf_{x\to+\infty} x^\chi P(Z > x) \leq \limsup_{x\to+\infty} x^\chi P(Z > x) < \infty$.*

The result shows that if $P(A \leq 1) = 1$, then the right tail of Z behaves like that of N: for all $p > 1$, $E[Z^p] < \infty$ if and only if $E[N^p] < \infty$; if $P(A \leq 1) < 1$, then there is a critical value $\chi > 1$ determined by the equation $ENE[A^\chi] = 1$ such that the right tail of Z behaves like $x^{-\chi}$ as $x \to \infty$. Theorems 4.1 and 4.2 tell us that there is a similar phenomenon for the left tail of Z: if $\alpha_0 + p_0 + p_1 = 0$, then the left tail of Z behaves like that of A; if $\alpha_0 + p_0 + p_1 > 0$ then there is a critical value $\lambda > 0$ determined by $\tilde{p}_1 E[A^{-\lambda}|A > 0] = 1$, such that the left tail of Z behaves like x^λ as $x \to 0$.

5. Exponential Tail Behavior of $\mu \in \mathbb{F}_+$.

This section deals with the case where $P(Z \leq x)$ decays exponentially when $x \to 0+$. As usual, we always assume (1.3) and (1.4).

THEOREM 5.1. *Suppose that*

$$(5.1) \qquad m := \operatorname{ess\,inf} N > 1 \ and \ a := \operatorname{ess\,inf} A > 0,$$

and put $\gamma = -\log m / \log a$. Then the following assertions hold:
(i) *Assume (1.5) and that either N or A is not a. s. constant, and let Z be a non-trivial and non-negative solution of (E). Then $0 < \gamma < 1$ and, for all $\epsilon > 0$, there are some constants $C_1 > 0$ and $C_2 > 0$ such that for all $x > 0$ small enough,*

$$(5.2) \qquad \exp\{-C_1 x^{-(\frac{\gamma}{1-\gamma}+\epsilon)}\} \leq P(Z \leq x) \leq \exp\{-C_2 x^{-(\frac{\gamma}{1-\gamma})}\};$$

moreover, the conclusion is also valid for $\epsilon = 0$ if additionally $P(A = \|A\|_{-\infty}) > 0$.
(ii) *Assume (1.8) and let Z be any non-trivial solution of (E). Then for some constant $K > 0$ and all $t \in \mathbb{R}$ with $|t| \geq 1$,*

$$|Ee^{itZ}| \leq e^{-K|t|^\gamma}.$$

The theorem can be applied to the study of exact packing measures of some random Cantor- like sets [34]. In the context of Galton-Watson process (thus A is constant), part (ii) is due to Harris (1948), and part (i) was deduced from a result of Harris (1948) by Bingham (1988), using Tauberian theorems. For the proof, we need the following proposition.

PROPOSITION 5.2. *Assume (1.5) and (5.1), and let Z be a non-trivial and non-negative solution of (E). Then the following assertions hold:*
(i) For some constant $K_1 > 0$ and all $t \geq 1$, $Ee^{-tZ} \leq e^{-K_1 t^\gamma}$.
(ii) For all $\epsilon > 0$, there exists some constant $K_2 > 0$ such that for all $t \geq 1$,
$Ee^{-tZ} \geq e^{-K_2 t^{\gamma+\epsilon}}$; *the conclusion also holds for $\epsilon = 0$ if $P(A = a) > 0$.*

Proof. We first remark that $a < 1$ by (1.5). Write $\phi(t) = Ee^{-tZ}$. By (E'), we have

$$\phi(t) \leq [\phi(at)]^m \quad \text{for all } t > 0.$$

So for $b = 1/a$ and $t > 0$, $\phi(bt) \leq [\phi(t)]^m$ and, by iteration, for all $k = 0, 1, ...$, $\phi(b^k) \leq [\phi(1)]^{m^k}$. Since $m = a^{-\gamma} = b^\gamma$, it shows that, for all $k = 0, 1, ...$,

$$-\log \phi(b^k) \geq K(b^k)^\gamma, \quad \text{where } K = -\log \phi(1) > 0.$$

For all $t \geq 1$, let $k \in \{0, 1, ...\}$ be such that $b^k \leq x < b^{k+1}$. Then by the monotonicity of ϕ,

$$-\log \phi(t) \geq -\log \phi(b^k) \geq K(b^k)^\gamma \geq Ka^\gamma(b^{k+1})^\gamma \geq Ka^\gamma t^\gamma.$$

This ends the proof of part (i). We now come to the proof of part (ii). Let $\epsilon > 0$ be such that $a + \epsilon < 1$. Again by the equation (E'), we have

$$\phi(t) \geq p_m (E[\phi(At)])^m \geq p_m (E[\phi(At)1\{A \leq a + \epsilon\}])^m$$
$$\geq p_m (P\{A \leq a + \epsilon\})^m [\phi((a+\epsilon)t)]^m.$$

Therefore, for $b = b_\epsilon := 1/(a+\epsilon) > 1$, $p = p_\epsilon := p_m(P\{A \leq a+\epsilon\})^m \in (0,1)$ and all $t > 0$, $\phi(bt) \geq p[\phi(t)]^m$. Iterating, we obtain

$$\phi(b^k) \geq [\phi(1)]^{m^k} p^{\sum_{1 \leq i \leq k} m^i}, \quad k = 0, 1, ...$$

(where the empty sum is taken to be 0). It follows that for all $k = 0, 1, ...$,

$$-\log \phi(b^k) \leq m^k[-\log \phi(1) + (-\log p)m^{-k} \sum_{1 \leq i \leq k} m^i]$$
$$= m^k[-\log \phi(1) + (-\log p)m^{-k}m(m^k - 1)/(m-1)] \leq (b^k)^{\gamma_\epsilon} K_\epsilon,$$

where $\gamma_\epsilon := \log m / \log \frac{1}{a+\epsilon}$ and $K_\epsilon := -\log \phi(1) + (-\log p)m/(m-1)$. As in the proof of part (i), together with the monotonicity of ϕ, this implies that, for all $t \geq 1$,

$$-\log \phi(t) \leq K_\epsilon a^{-\gamma_\epsilon} t^{\gamma_\epsilon}.$$

As $\epsilon > 0$ is arbitrary, this gives the first assertion in part (ii) of the theorem. If additionally $P(A = a) > 0$, it is easily seen that the preceding argument also holds for $\epsilon = 0$, giving the second assertion in part (ii). So the proof is finished. $\quad \square$

In the context of Mandelbrot's cascades (thus N is constant), part (i) was established by Holley and Waymire [21]; in the context of Galton-Watson process (thus A is constant), both parts are due to Harris [20].

Proof of Theorem 5.1. By (1.5) and the condition that either N or A is not a. s. constant, we know that $\gamma < 1$. So part (i) follows from Tauberian theorems of exponential type [28, Theorem 3 (ii) & (iii)]. Part (ii) can be shown in a similar way as in the proof of part (i) of Proposition 5.2 by considering $\bar{\phi}(t) = \sup_{|s| \geq t} |Ee^{isZ}|$ instead of $\phi(t)$.

The following results are consequences of Theorem 5.1, again by Tauberian theorems [30].

COROLLARY. Assume (1.5), (5.1) and that either N or A is not a. s. constant, and let Z be a non-trivial and non-negative solution of (E). Then:
   (i) For all $\epsilon > 0$, there exist some constants $C_i > 0$ $(i = 3, 4)$ such that for all $k \geq 1$, $C_3 k^{\frac{1-\gamma}{\gamma}-\epsilon} \leq \|Z^{-1}\|_k \leq C_4 k^{\frac{1-\gamma}{\gamma}}$, where $\|Z^{-1}\|_k = (E[Z^{-k}])^{1/k}$. Consequently, $\lim_{k\to\infty} \log \|Z^{-1}\|_k / \log k = (1-\gamma)/\gamma$.
   (ii) If $\gamma > 1/2$, then for all $\epsilon > 0$, there are some constants $C_5 > 0$ and $C_6 > 0$ such that, for all $t > 0$ large enough, $C_5 t^{\frac{\gamma}{2\gamma-1}-\epsilon} \leq \log Ee^{tZ^{-1}} \leq C_6 t^{\frac{\gamma}{2\gamma-1}}$ (so that Z^{-1} has an entire characteristic function of order $\frac{\gamma}{2\gamma-1} > 1$).
   (iii) $E \exp(Z^{-q}) < \infty$ if $q < \frac{\gamma}{1-\gamma}$, and $E \exp(Z^{-q}) = \infty$ if $q > \frac{\gamma}{1-\gamma}$.
   (iv) If $P(A = a) > 0$, then the conclusions in (i) and (ii) also hold for $\epsilon = 0$, and the assertion (iii) can be improved to the following: for some but not all $r > 0$, $E \exp(rZ^{-\gamma/(1-\gamma)}) < \infty$.

Again it is interesting to compare exponential left tail with exponential right tail. The corresponding results for the right tail are the following:

PROPOSITION 5.3. *Assume (1.6) and let Z be a non-trivial and non-negative solution of (E). Suppose that either N or A is not almost surely constant, and that $\bar{m} := \operatorname{ess\,sup} N < \infty$ and $\bar{a} := \operatorname{ess\,sup} A < 1$. Write $\bar{\gamma} = -\log \bar{m}/\log \bar{a}$. Then $\bar{\gamma} > 1$ and for any $\epsilon > 0$, there exist some constants $C_i > 0 \, (1 \leq i \leq 6)$ such that the following assertions hold:*
   (i) *For all $t \geq 1$, $C_1 t^{\bar{\gamma}+\epsilon} \leq \log Ee^{tZ} \leq C_2 t^{\bar{\gamma}}$ (therefore Z has an entire characteristic function of regular growth with order $\bar{\gamma}$).*
   (ii) *For all $x > 0$ large enough, $\exp\{-C_3 x^{(\frac{\bar{\gamma}}{\bar{\gamma}-1}+\epsilon)}\} \leq P(Z > x) \leq \exp\{-C_4 x^{(\frac{\bar{\gamma}}{\bar{\gamma}-1})}\}$.*
   (iii) *For all $k \geq 1$, $C_5 k^{\frac{\bar{\gamma}-1}{\bar{\gamma}}-\epsilon} \leq \|Z\|_k \leq C_6 k^{\frac{\bar{\gamma}-1}{\bar{\gamma}}}$, where $\|Z\|_k = (E[Z^k])^{1/k}$. Consequently, $\lim_{k\to\infty} \log \|Z\|_k / \log k = (\bar{\gamma}-1)/\bar{\gamma}$.*
   (iv) *$E \exp(Z^q) < \infty$ if $q < \frac{\bar{\gamma}}{\bar{\gamma}-1}$, and $E \exp(Z^q) = \infty$ if $q > \frac{\bar{\gamma}}{\bar{\gamma}-1}$.*
   (v) *If additionally $P(Z = \bar{a}) > 0$, then the conclusions in (i), (ii) and (iii) also hold for $\epsilon = 0$, and the conclusions in (iv) can be improved to the following: for some but not all $r > 0$, $E \exp(rZ^{\bar{\gamma}/(\bar{\gamma}-1)}) < \infty$.*

Part (i) can be obtained in a similar way as in the proof of Proposition 5.2 by considering Ee^{tZ} (which can be shown finite) instead of Ee^{-tZ}, $t > 0$; other parts are consequences of Part (i) by Tauberian Theorems [30]. In fact, by [30, Corollary 1.2], the condition $P(Z = \bar{a}) > 0$ in (v) can be relaxed to the following: for some constants $c > 0$, $a \geq 0$ and all $x > 0$ small enough,

$$(5.3) \qquad\qquad P(A/\bar{a} > 1 - x) \geq cx^a.$$

Because the proof in [30] is not easy, the new approach seems to be of interest.

6. Martingales in Self-similar Cascades. Fix an integer $r \geq 2$, and write $T = \cup_{n=0}^{\infty} \{1, ..., r\}^n$ for the set of finite sequences of integers between 1 and r. Let $\{A_u\}_{u \in T}$ be a family of independent random variables, each distributed as $A \geq 0$ with $EA = 1/r$. Put

$$(6.1) \qquad Y_n = \sum_{u \in T, |u|=n} X_u, \text{ where } X_u = A_{u_1}...A_{u_1...u_n} \text{ if } u = u_1...u_n, n \geq 1$$

($|u|$ being the length of the sequence u). Then $\{Y_n\}$ is a non-negative martingale (associated with the natural σ-algebra), which converges (when $n \to \infty$) almost surely to a random variable $Y_\infty \geq 0$ satisfying the equation (E) with $N = r$ (we shall always assume this in this section) and $EY_\infty \leq 1$. The study of the limit variable Y_∞ and the equation (E) plays an essential role in the investigation of the Mandelbrot's cascade [39, 40, 27, 19]. It has been known [27] that $EY_\infty = 1$ if

$$(6.2) \qquad EA \log A < 0,$$

and $Y_\infty = 0$ a.s. otherwise. It has also been known that [35] if

$$(6.3) \qquad EA \log A = 0 \text{ and } EA^{1+\epsilon} < \infty \text{ for some } \epsilon > 0,$$

then

$$(6.4) \qquad Y_n^* = \sum_{u \in T, |u|=n} X_u \log \frac{1}{X_u}$$

is a martingale, and converges to a a solution $Y_\infty^* \geq 0$ of (E) with $\lim_{t \to 0+} \frac{1 - Ee^{-tY_\infty^*}}{t|\log t|} = 1$, so that $EY_\infty^* = \infty$. Since both Y_∞ and Y_∞^* are non-negative solutions of (E), our theorems can be applied to get new results on them; as examples, let us deduce some of these results in the following. Remark that by our notations, we have: (i) $\tilde{p}_1 = (1 - \alpha_0)rq/[\alpha_0 + (1 - \alpha_0)q]$ where $\alpha_0 = P(A = 0)$, $q = P(Z = 0)$ is the unique fixed point in $[0, 1)$ of $\tilde{f}(t) = [\alpha_0 + (1 - \alpha_0)t]^r$; (ii) $m = r$ if $\alpha_0 = 0$ and $m = 1$ if $\alpha_0 > 0$.

THEOREM 6.1.
   We assume (6.2) if Y_∞ is concerned and (6.3) if Y_∞^* is concerned.
 (i) *Let ϕ_∞ be the characteristic function of Y_∞, and let ϕ_∞' be its derivative. Then $\phi_\infty'(t) = O(|t|^{-1})$ ($|t| \to \infty$) and $\int_{\mathbb{R}} |\phi_\infty'(t)|dt < \infty$, so that the distribution of Y_∞ has a continuous density function on $(0, \infty)$, given by $x \mapsto \frac{1}{2\pi ix} \int_{\mathbb{R}} \phi_\infty'(t)e^{-ixt}dt$. If for some $a \geq 0$, either (a) $\alpha_0 = 0$ and $EA^{-a} < \infty$ or (b) $\alpha_0 > 0$ and $\tilde{p}_1 E[A^{-a}|A > 0] < 1$, then $\phi'(t) = O(|t|^{-(ma+1)})$ ($|t| \to \infty$) and $\int_{\mathbb{R}} |\phi_\infty'(t)| |t|^{ma}dt < \infty$, so that the density function given above is of class $C^{[ma]}$.*
 (ii) *If for some $a > 0$ either (a) or (b) of (i) holds, then $E[e^{itY_\infty^*}|Y_\infty^* > 0] = O(|t|^{-ma})$ ($|t| \to \infty$) and $\int_{\mathbb{R}} |E[e^{itY_\infty^*}|Y_\infty^* > 0]| |t|^{ma-1}dt < \infty$, so that the distribution of Y_∞^* has a square integrable density function on $(0, \infty)$ if $a > 1/(2m)$, and has a density function of class $C^{[ma]-1}$ if $a \geq 1/m$.*
 Notice that the assertion (ii) also holds for Y_∞ instead of Y_∞^*. The results follow from Theorems 3.2, 3.4, 3.5 and the principle of reduction (Proposition 4.10). The

fact that Y_∞ has always a continuous density function on $(0, \infty)$ is also a consequence of a result of Biggins and Grey [6].

THEOREM 6.2. *Assume (6.2) if Y_∞ is concerned and (6.3) if Y_∞^* is concerned, and let Z be either Y_∞ or Y_∞^*. If $\alpha_0 = 0$, then for each fixed $0 < a < \infty$, the following assertions hold:*

(i) *As $x \to 0$, $P(A \le x) = O(x^a)$ if and only if $P(Z \le x) = O(x^{ra})$.*

(ii) *If for some constants $C_1 > 0$, $C_2 > 0$ and all $x > 0$ small enough, $C_1 x^a \le P(A \le x) \le C_2 x^a$, then there are some constants $C_3 > 0$, $C_4 > 0$ such that, for all $x > 0$ small enough,*

$$C_3 x^{ra} \le P(Z \le x) \le C_4 x^{ra}.$$

(iii) *If $E(A^{-a}) < \infty$, then $E[Z^{-ra}] < \infty$; conversely, if $E[Z^{-ra}] < \infty$, then $E(A^{-b}) < \infty$ for all $0 < b < a$.*

(iv) *Let $\delta > 0$ and let $l : (0, \delta] \to (0, \infty)$ be a function slowly varying at 0 and bounded away from 0 and ∞ on each compact subset of $(0, \delta]$. If $P(A \le x) \sim x^a l(x)$ $(x \to 0)$, then*

$$P(Z \le x) \sim x^{ra} [E(Z^{-a})]^r [l(x)]^r \ (x \to 0).$$

The results follow from Theorem 4.1. By the remark following Theorem 4.1, the converse in (iii) cannot be improved to "$EZ^{-ra} < \infty \Rightarrow EA^{-a} < \infty$". In (i), the "only if" part was shown independently by Kahane [26], Molchan [41] and Barral [3, 4], and the "if" part can be deduced from the proof of Theorem II.A' of Barral [4]; in (iii), the first conclusion was due to Molchan [41, Theorem 4] and the second due to Barral [3, 4]; other results are new.

THEOREM 6.3. *Assume (6.2) if Y_∞ is concerned, and (6.3) if Y_∞^* is concerned. Let Z be either Y_∞ or Y_∞^*. If $\alpha_0 > 0$, then $0 < \tilde{p}_1 < 1$, and for each fixed $0 < a < \infty$, the following assertions hold:*

(i) *If $\tilde{p}_1 E[A^{-a} \, |A > 0] < 1$, then $E[Z^{-b} \, |Z > 0] < \infty$ for all $0 < b < a$; conversely, if $E[Z^{-a} \, |Z > 0] < \infty$, then $\tilde{p}_1 E[A^{-a} \, |A > 0] < 1$;*

(ii) *In the case where $A \le 1$ almost surely, $\tilde{p}_1 E[A^{-a} \, |A > 0] < 1$ if and only if $E[Z^{-a} \, |Z > 0] < \infty$*

The results follow from Theorem 4.2. Barral [3, 4] showed that when $\alpha_0 > 0$, $E[Y_\infty^{-b} | Y_\infty > 0] = \infty$ if b is sufficiently large. Our result gives explicitly the critical value for existence of moments of negative orders:

COROLLARY. *Under the hypotheses of Theorem 6.3, If $\tilde{p}_1 E[A^{-\lambda} \, | \, A > 0] = 1$ for some $\lambda > 0$, then*

$$E[Z^{-b} \, |Z > 0] \begin{cases} < \infty & \text{when } b \in (0, \lambda), \text{ and} \\ = \infty & \text{when } b \in [\lambda, \infty). \end{cases}$$

Notice that this result is useful in the study of multi-fractal structure of Mandelbrot's measures [3, 4, 41].

7. Applications to age-dependent processes . An *age-dependent branching process* - the Bellman-Harris process - can be described as follows. A particle existing at time 0 is assumed to have a life-length, L, with values in $[0, \infty)$ and probability distribution $G(x) = P(L \le x)$. At the end of its life, it is transformed into N particles according as a probability law $\{p_n : n \ge 0\}$ on $\mathbb{N} = \{0, 1, ...\}$: $P(N = $

$n) = p_n, \sum_{n \in \mathbb{N}} p_n = 1$. These new particles are taken to have the same life-length distribution and transformation probabilities as the original one. We assume that the life-length distribution and the transformation probabilities for each particle is independent of its time of birth and the number of other particles existing at the time. Let $Z(t)$ be the number of particles existing at time $t \geq 0$. As usual, we assume

$$(7.1) \qquad p_0 + p_1 < 1, 1 < EN < \infty \text{ and } P(L = 0) < 1.$$

It is well-known that the limit

$$(7.2) \qquad W = \lim_{t \to \infty} Z(t)/EZ(t)$$

exists a.s., and its Laplace transform $\phi(s) = Ee^{-sW}$ satisfies the functional equation

$$(7.3) \qquad \phi(s) = \int_0^\infty f(\phi(se^{-\beta x}))dG(x), \qquad s > 0,$$

where $f(x) = \sum_{n=0}^\infty p_n x^n$ is the probability generating function of N, and β is the Malthus parameter defined by $(EN)\int_0^\infty e^{-\beta x}dG(x) = 1$. Notice that this equation also reads

$$(7.4) \qquad \phi(t) = Ef(\phi(At)),$$

with $A = e^{-\beta L}$ (so that $ENEA = 1$); this corresponds to the equation (E) with all the coefficients A_i replaced by A. It is known that $EW = 1$ if $EN \log^+ N < \infty$, and $W = 0$ almost surely otherwise; when $EN \log^+ N = \infty$, there are positive constants $\{C(t)\}$ such that

$$(7.5) \qquad \lim_{t \to \infty} Z(t)/C(t) = W' \quad \text{almost surely}$$

for some non-degenerate random variable W' whose Laplace transform ϕ satisfies (7.4)(Cf. [13] or [43]). The equations (E') and (7.4) are closely related. In fact, if ϕ is a solution of (7.4), then $\phi_1 = f(\phi(t))$ is a solution of (E'); conversely, if ϕ_1 is a solution of (7.4), then $\phi(t) = E\phi_1(At)$ is a solution of (E). So properties of solutions of (E) can be transfered to solutions of (7.4). Let us give an example to explain this.

THEOREM 7.1. *Assume $p_0 = p_1 = 0$ and let $Z = W$ if $EN \log N < \infty$ and $Z = W'$ if $EN \log N = \infty$. Then for each fixed $0 < a < \infty$, the following assertions hold:*

(i) *As $x \to 0, P(A \leq x) = O(x^a)$ if and only if $P(Z \leq x) = O(x^a)$.*

(ii) *Let $\delta > 0$ and let $l : (0, \delta] \to (0, \infty)$ be a function slowly varying at 0 and bounded away from 0 and ∞ on each compact subset of $(0, \delta]$. If $P(A \leq x) \sim x^a l(x) (x \to 0)$, then*

$$P(Z \leq x) \sim cx^a l(x) (x \to 0),$$

where $0 < c = E(Z_1 + ... + Z_N)^{-a} < \infty$, Z_k being independent copies of Z and independent of N.

Proof. Write $m = \min\{i \geq 2 : p_i > 0\}$ and let ϕ be the Laplace transform of Z. Since $Z' = Z_1 + ... + Z_N$ is a solution of (E) with Laplace transform $f(\phi(t)) \sim p_m(\phi(t))^m$ as $t \to \infty$, the results follow from Theorem 4.1 and Tauberian theorems. $\square$

Part (i) was established in [33]; part (ii) is new. Some other known results on W and W' can also be obtained from our results on the equation (E). The equation (7.4) can also be considered for any given random variable $A \geq 0$ [33].

8. Applications to branching random walks. *An i.i.d. branching random walk* can be described as follows. Let $\mathbb{N}^* = \{1, 2, \cdots\}$ be the set of positive integers and let $\mathbf{U} = \{\emptyset\} \cup \cup_{k=1}^{\infty}(\mathbb{N}^*)^k$ be the set of all finite sequences including the null sequence $\emptyset$. Let $(\Omega, \mathbb{F}, P)$ be a probability space and let $\{N_u : u \in \mathbf{U}\}$ and $\{L_u : u \in \mathbf{U}\}$ be two independent families of independent random variables defined on $(\Omega, \mathbb{F}, P)$, the $N_u's$ are distributed as $N = N_\emptyset$ and with values on $\{0, 1, ...\}$, and the $L_u's$ are distributed as $L = L_\emptyset$) and with values on $\mathbb{R} = (-\infty, \infty)$. ($L$ is now not necessarily non-negative.) Let $\mathbb{T} = \mathbb{T}(\omega)$ be the Galton -Watson tree with defining elements $\{N_u\}$ - we have $\emptyset \in \mathbb{T}$ and, if $u \in \mathbb{T}$ and $i \in \mathbb{N}$, then $ui \in \mathbb{T}$ if and only if $1 \le i \le N_u$ -. The initial particle $\emptyset \in \mathbb{T}$ is placed at $S_\emptyset = 0$ of the real line $\mathbb{R} = (-\infty, \infty)$. It gives birth to $N_\emptyset$ new particles $i\,(1 \le i \le N_\emptyset)$ with displacements L_i, $1 \le i \le N$. In general, if $u = u_1...u_n \in \mathbb{T}$ is a particle in n-th generation ($u = \emptyset$ if $n = 0$), then its position is given by

$$S_u = L_{u_1} + ... + L_{u_1...u_n}.$$

Assume (7.1) and let $\beta \in \mathbb{R}$ be such that $m(\beta) := ENEe^{-\beta L} < \infty$. Then the sequence

$$Y_n := m(\beta)^{-n} \sum_{u=u_1...u_n \in T} e^{-\beta S_u}, \qquad n \ge 1,$$

is a martingale, so that the limit

$$(8.1) \qquad\qquad Z := \lim_{n\to\infty} Y_n$$

exists almost surely. By considering the sub-trees beginning at $i \in \{1, ..., N\}$, we see easily that Z satisfies the distributional equation (E) with $A_i = e^{-\alpha L_i}/m(\alpha)$. In the following, we always write $A = e^{-\beta L}/m(\beta)$, so $ENEA = 1$.

Let Z be the random variable defined by (8.1). It is known (see for example [31]) that the distribution of Z is the unique non-negative solution of (E) with mean 1 if

$$(8.2) \qquad EN\log^+ N < \infty \text{ and } EA\log^+ A < \infty \text{ with } EA\log A < 0,$$

and $Z = 0$ a. s. otherwise. In the case where (8.2) fails, (E) has no non-negative and non-trivial solution with finite mean, but it does have non-negative and non-trivial solutions with infinite mean by Theorem 1.1. In fact, if

$$(8.3) \qquad EN\log^+ N = \infty \text{ and } EA\log^+ A < \infty \text{ with } EA\log A < 0,$$

then there is a sequence of constants $\{c_n\}$ such that Y_n/c_n converges in probability to a solution $Z' \ge 0$ of (E) with $EZ' = \infty$ [7]; if

$$(8.4) \qquad E(N^{1+\delta}) + E(A^{1+\delta}) < \infty \text{ for some } \delta > 0 \text{ and } EA\log A = 0,$$

then

$$(8.5) \qquad Y_n^* := \sum_{u=u_1...u_n \in T(\omega)} X_u \log \frac{1}{X_u}, \text{ where } X_u = m(\alpha)^{-n} e^{-\alpha S_u},$$

is a martingale, and converges to a solution $Z^* \ge 0$ of (E) with $\lim_{t\to 0}(1 - Ee^{-tZ^*})/|t\log t| = 1$, so that $EZ^* = \infty$ [35].

Since Z, Z' and Z^* are all solutions of (E) with $\alpha_0 = P(A = 0) = 0$, our Theorems 3.2, 3.4, 3.5, 4.1, 4.2, 5.1 show asymptotic properties of their distributional functions or characteristic functions, and prove the existence of a density function and its regularity under simple moment conditions. To my knowledge all these results are new, except for the absolute continuity of Z (defined by (8.1)) which was shown by Biggins and Grey [6].

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