ASIAN J. MATH.
Vol. 6, No. 1, pp. 101–122, March 2002

006

SINGULARITIES IN CRYSTALLINE CURVATURE FLOWS*

BEN ANDREWS†

Abstract. This paper discusses the behaviour of polygonal convex curves in the plane moving under crystalline curvature flows, in which the speed of motion of each edge is determined by a function of its length. The behaviour depends on the rate of growth of the speed as the length of the edge approaches zero: For slow growth — including the homogeneous case where speed is inversely proportional to a power $\alpha \in (0,1)$ of the length — there are always solutions for which the enclosed area approaches zero while the length remains positive. If $\alpha > 1$, then all solutions are asymptotic to homothetically contracting solutions, and if $\alpha = 1$ then there is a range of different kinds of singularity that occur.

1. Crystalline curvature flows. Several authors have considered crystalline curvature flows of polygonal curves in the plane, since their introduction in [T]. We refer the reader to [TCH] and [AG] for a discussion of the geometric and physical motivation for such flows. For present purposes we consider only convex curves, although the flows can be defined much more generally. In this case the flows can be defined in the following way: Let γ be a closed convex N-sided polygon in the plane, and label the edges $\gamma_0, \ldots, \gamma_{N-1}$ in an anticlockwise order. Let $\theta_i \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ be the angle of the exterior normal of γ_i, and let ℓ_i be the length of γ_i. Moving γ by a crystalline curvature flow consists of finding a continuous family of polygonal curves $\gamma(t)$ starting from γ so that each edge keeps the same direction but moves in the outward normal direction with speed v_i determined by its length:

$$v_i(t) = g_i(\ell_i). \tag{1}$$

Here g_i is a smooth function defined on $(0, \infty)$ which is monotone increasing for each i. This paper mostly concerns contraction flows, for which $g_i < 0$, and the condition $g_i(z) \to -\infty$ as $z \to 0$ will be assumed. The later parts of the paper are concerned particularly with the homogeneous case, defined by

$$g_i(z) = -f_i z^{-\alpha} \tag{2}$$

where $\alpha > 0$ and f_i is a positive real number for each i.

A simple geometric calculation shows that the side lengths $\ell_i(t)$ satisfy an autonomous system of ordinary differential equations:

$$\frac{d}{dt}\ell_i = \frac{g_{i+1}(\ell_{i+1})}{\sin(\theta_{i+1} - \theta_i)} + \frac{g_{i-1}(\ell_{i-1})}{\sin(\theta_i - \theta_{i-1})} - \frac{g_i(\ell_i)\sin(\theta_{i+1} - \theta_{i-1})}{\sin(\theta_{i+1} - \theta_i)\sin(\theta_i - \theta_{i-1})} \tag{3}$$

where the index i is to be read mod N. The original geometric evolution (1) can now be discarded and replaced with the ODE system (3), as long as one bears in mind that each of the side lengths ℓ_i must be non-negative, and that in order to define a closed curve the conditions $\sum_{i=0}^{N-1} \ell_i \sin(\theta_i) = \sum_{i=0}^{N-1} \ell_i \cos(\theta_i) = 0$ must be satisfied. Note that these remain true under (3) if they hold initially. To this end, given a collection of angles $\theta_0 < \theta_1 < \cdots < \theta_{N-1} < \theta_N = \theta_0 + 2\pi$ with $\theta_{i+1} - \theta_i < \pi$, define $\mathcal{L} = \{(\ell_0, \ldots, \ell_{N-1}) : \ \ell_i > 0, \ \sum_{i=0}^{N-1} \ell_i \sin(\theta_i) = \sum_{i=0}^{N-1} \ell_i \cos(\theta_i) = 0\}$.

* Received August 6, 2001; accepted for publication October 25, 2001.
† Centre for Mathematics and its Applications, Australian National University, A.C.T. 0200, Australia (andrews@maths.anu.edu.au).

We review some results that have been obtained for these systems: A simple ODE and comparison argument shows that for any initial data in $\mathcal{L}$ there exists a smooth solution on some finite maximal time interval $[0, T)$, and $\min_i \ell_i(t) \to 0$ as $t \to T$. It is known that there are only two possibilities as $t \to T$ (see [GG]): Either $\max_i \ell_i(t) \to 0$, so the curve shrinks to a point, or there are two parallel edges which have strictly positive length as $t \to T$ while all others shrink to zero (the latter behaviour is called *degenerate pinching*).

A necessary criterion for degenerate pinching was given in [GG] in terms of the growth rate of the speed $g_i(\ell)$ as $\ell \to 0$. It was shown there that for symmetric flows with $N = 4$ this condition is also sufficient, but it was conjectured that for larger N this should not be the case, and in particular in the special case of homogeneous flows (2) with $\theta_i = \frac{\pi i}{k}$, $f_i = 1$, degenerate pinching should occur only when $0 < \alpha < \alpha_k = \frac{1}{1+2\cos(\pi/k)}$. This is suggested by the local stability of the homothetically shrinking regular $2k$-gon solution of the flow. This conjecture is disproved in Section 4 of this paper, where it is shown that every crystalline flow which satisfies the growth condition of [GG] exhibits degenerate pinching, if there is a pair of edges which are parallel (i.e. $\theta_j = \theta_i + \pi$ for some i, j).

The growth condition from [GG] implies that the homogeneous flows (2) with $\alpha \geq 1$ do not admit degenerate pinching. Section 6 of this paper provides a stronger statement about the asymptotic behaviour for flows with $\alpha > 1$: The shrinking curve in fact has a well-defined limiting shape, a curve which evolves by homothetically contracting to some centre. The corresponding result for $\alpha = 1$ is claimed in [S2], but in fact the situation is much more complicated in that case, and singularities of various kinds occur — this is discussed in detail in Section 7, where a fairly complete description of the asymptotic behaviour is given. In particular, for symmetric flows there are two possibilities: Either there exists a symmetric homothetically shrinking solution, in which case the results of [S1] imply that all other convex solutions have this as asymptotic shape as they contract to points (except in the parallelogram case $N = 4$, where every solution evolves homothetically), or there is no such homothetic solution, in which case all solutions contract to points while their isoperimetric ratio approaches infinity. In the latter case the minimum edge length ℓ satisfies either $\ell \sim \sqrt{(T-t)/|\log(T-t)|}$ or $\ell \sim (T-t)^\gamma$ for some $\gamma \in (1/2, 1)$. A simple criterion distinguishes between these cases and determines the asymptotics of the singularity. For non-symmetric flows the situation is more complicated, and some borderline cases are left open. An example shows that these borderline cases include examples where all solutions converge to homothetically shrinking solutions, as well as examples with a variety of other singularities.

2. Preliminary results. Given a collection of angles $\underline{\theta} = (\theta_0, \ldots, \theta_{N-1})$ with $\theta_i < \theta_{i+1} < \theta_i + \pi$ for each i, and any N-tuple $\underline{f} = (f_0, \ldots, f_{N-1})$, denote by $\underline{\ell}(\underline{f}) = (\ell_0(\underline{f}), \ldots, \ell_{N-1}(\underline{f}))$ the N-tuple defined by

$$\ell_i(\underline{f}) = \frac{f_{i+1}}{\sin(\theta_{i+1}-\theta_i)} + \frac{f_{i-1}}{\sin(\theta_i - \theta_{i-1})} - \frac{f_i \sin(\theta_{i+1}-\theta_{i-1})}{\sin(\theta_{i+1}-\theta_i)\sin(\theta_i-\theta_{i-1})}. \tag{4}$$

This is linear in the components of $\underline{f}$, and in particular the variation in $\ell_i(\underline{f})$ induced by a variation in $\underline{f}$ is given by

$$\frac{d}{dt}\ell_i(\underline{f}) = \ell_i\left(\frac{d\underline{f}}{dt}\right). \tag{5}$$

The reason for the notation comes from the following special case: The *support function* of an admissible curve is the N-tuple $\underline{s}$ defined by taking s_i to be the perpendicular distance in the outward normal direction of the ith edge from the origin. Then $\ell_i(\underline{s}) = \ell_i$.

It is immediately clear from the definition of the support function that the enclosed area A of the curve has the expression

$$A = \frac{1}{2}\sum_{i=0}^{N-1} s_i\ell_i. \tag{6}$$

If Ω_1 and Ω_2 are two regions in the plane bounded by admissible convex curves, with support functions $\underline{s}^{(1)}$ and $\underline{s}^{(2)}$ respectively, the Minkowski sum is given by $\Omega_1 + \Omega_2 = \{x + y : x \in \Omega_1, y \in \Omega_2\}$. This is again a convex region with boundary an admissible curve, and the support function is given by $\underline{s}^{(1)} + \underline{s}^{(2)}$.

It follows that the area A behaves as a quadratic function under Minkowski addition of admissible sets, and the *mixed volumes* of Ω_1 and Ω_2 are defined as the coefficients of this quadratic function:

$$V(\Omega_1, \Omega_2) = \frac{1}{2}\frac{d^2}{dt^2}A(\Omega_1 + t\Omega_2)\big|_{t=0}.$$

This can be expressed in terms of the support function as

$$V(\underline{s}^{(1)}, \underline{s}^{(2)}) = \sum_i s_i^{(1)}\ell_i(\underline{s}^{(2)}). \tag{7}$$

In particular $V(\Omega, \Omega) = 2A(\Omega)$, and $V(\Omega, B)$ is the total length of the edges of the boundary curve of Ω if B is the polygonal curve with the same set of edge directions with $s_1 = 1$ for every i (equivalently, B is excribed on the unit circle). It is also convenient to define

$$E(\Omega) = \sum_i \ell_i = V(\Omega, B). \tag{8}$$

$V(\Omega_1, \Omega_2)$ is clearly independent of the order of the arguments. This implies a useful summation formula:

$$\sum_i p_i\ell_i(\underline{q}) = \sum_i q_i\ell_i(\underline{p}). \tag{9}$$

The Brunn-Minkowski theorem states that the square root of the enclosed area A is a concave function under Minkowski addition:

$$A(\Omega_1 + \Omega_2)^{1/2} \geq A(\Omega_1)^{1/2} + A(\Omega_2)^{1/2}, \tag{10}$$

and equality holds for convex Ω_1 and Ω_2 if and only if Ω_2 is a scaled translate of Ω_1. In the special situation of Minkowksi addition of admissible regions, this amounts to the inequality

$$V(\underline{f}, \underline{f}) \leq \frac{V(\underline{f}, \underline{s})^2}{2A} \tag{11}$$

for any function f. Furthermore, if equality holds in (11), then equality holds in (10) with $\underline{s}(\Omega_1) = \underline{s}$ and $\underline{s}(\Omega_2) = \underline{s} + \varepsilon \underline{f}$, for ε sufficiently small. Therefore $f_i = Cs_1 + A\sin\theta_i + B\cos\theta_i$ for some A, B and C, since Ω_1 and Ω_2 are scaled translates. Equivalently, equality holds if and only if $\ell_i(\underline{f}) = C\ell_i$.

The support function is particularly convenient in working with the evolution equation (1), since this can be written as the ODE system

$$\frac{ds_i}{dt} = g_i(\ell_i(\underline{s})). \tag{1'}$$

A comparison principle applies for solutions of Eq. (1'): If $\underline{s}^{(1)}(t)$ and $\underline{s}^{(2)}(t)$ are two solutions with $s_i^{(1)}(0) \geq s_i^{(2)}(0)$ for every i, then $s_i^{(1)}(t) \geq s_i^{(2)}(t)$ for every i and every $t \geq 0$ in the common interval of existence.

It follows from (1') that the rate of change of A under Eq. (1) is given by

$$\frac{dA}{dt} = \sum_i \ell_i g_i(\ell_i). \tag{12}$$

Also, if $\underline{f}$ is any fixed function, then

$$\frac{d}{dt}V(\underline{s}, \underline{f}) = \sum_i g_i(\ell_i)\ell_i(\underline{f}). \tag{13}$$

The later sections of this paper will be concerned with the situation where there are a pair of parallel directions, (labelled 0 and k for convenience), with $\theta_0 = 0$ and $\theta_k = \pi$. In this situation it is convenient to make the following definition:

$$w = V(\Omega, I) = \frac{1}{2}\sum_i \ell_i|\sin\theta_i| = \sum_{i=1}^{k-1} \ell_i \sin\theta_i = -\sum_{i=k+1}^{N-1} \ell_i \sin\theta_i, \tag{14}$$

where I is the degenerate curve with $\ell_0 = \ell_k = 1$ and $\ell_i = 0$ for $i \neq 0, k$. Geometrically, w represents the width of Ω in the direction perpendicular to the edges 0 and k. The variation formula (13) gives a simple evolution equation for w under (1):

$$\frac{dw}{dt} = g_0(\ell_0) + g_k(\ell_k). \tag{15}$$

It is also convenient to define $L = (\ell_0 + \ell_k)/2$. An alternative expression for A is the following, which involves only the lengths $\underline{\ell}$ and not the support function $\underline{s}$:

$$A = Lw + \frac{1}{2}\sum_{0<i<j<k} \ell_i\ell_j \sin(\theta_j - \theta_i) + \frac{1}{2}\sum_{k<i<j<N} \ell_i\ell_j \sin(\theta_j - \theta_i). \tag{16}$$

The following estimates will be useful in the case where L is large compared to w: The expression (14) for w implies that

$$\ell_i \leq Cw \tag{17}$$

for $i \neq 0, k$. Then the expression (16) for A implies that

$$|L - A/w| \leq Cw \tag{18}$$

and the definition (8) of E implies

$$E - Cw \leq L \leq E. \tag{19}$$

The identity

$$0 = \sum_i \ell_i \cos\theta_i = \ell_0 - \ell_k + \sum_{i \neq 0,k} \ell_i \cos\theta_i \tag{20}$$

implies that

$$|\ell_0 - L| \le Cw, \qquad |\ell_k - L| \le Cw. \tag{21}$$

A simple version of the maximum principle applies for systems of the type (1):

PROPOSITION 2.1. *Suppose* $\underline{f} : \{0,\dots,N-1\} \times [0,T) \to \mathbb{R}$ *satisfies an equation of the form*

$$\frac{df_i(t)}{dt} = g_i(\underline{f}(t))$$

where g_i *is locally Lipschitz in each argument, and* $g_i(\underline{\phi}) \le 0$ *whenever* $\phi_i = \max_j \phi_j = 0$. *If* $f_i(0) \le 0$ *for every* i, *then* $f_i(t) \le 0$ *for every* i *and every* $t \in [0,T)$.

3. A gradient estimate. The main result of this section is a gradient estimate for solutions of crystalline curvature flows. This result does not require that the flow be a contraction flow — the estimate applies for any solution of a flow of the form (1) with g_i non-decreasing for each i.

DEFINITION 3.1. *If* $\varphi_i \in \mathbb{R}$ *for* $i = 0,\dots,N-1$, *we denote by* $\tilde{\varphi}$ *the Lipschitz function on* S^1 *defined by*

$$\tilde{\varphi}(\theta) = \frac{\varphi_i \sin(\theta_{i+1} - \theta) + \varphi_{i+1}\sin(\theta - \theta_i)}{\sin(\theta_{i+1} - \theta_i)}, \qquad \text{for } \theta_i \le \theta \le \theta_{i+1}.$$

The geometric content of this definition is indicated by the following: If Γ is an admissible convex polygon with support function $\underline{s}$, then $\tilde{s}$ satisfies

$$\tilde{s}(\theta) = \sup\{x\cos\theta + y\sin\theta : (x,y) \in \Gamma\}.$$

The main estimate of this section is the following surprising gradient bound for the extension of the speed function:

PROPOSITION 3.2. *Let* $\ell : [0,T) \to \mathcal{L}$ *be a solution of Eq.* (3). *Then for* $t \in [0,T)$

$$\max_{\theta \in S^1}\{\tilde{g}(\theta,t)^2 + \tilde{g}_\theta(\theta,t)^2\} \le \max\left\{\max_{\theta\in S^1, t' \le t}\{\tilde{g}(\theta,t')^2\}, \max_{\theta \in S^1}\{\tilde{g}(\theta,0)^2 + \tilde{g}_\theta(\theta,0)^2\}\right\}.$$

Here the derivative $\tilde{g}_\theta$ is to be interpreted as multi-valued at the points θ_i, taking all values between the left and right-hand derivatives. The result is a direct generalisation of an estimate proved for curvature flows of smooth curves in [A3].

Proof. It suffices to prove $\tilde{g}(\bar\theta,t)^2 + \tilde{g}_\theta(\bar\theta,t)^2$ is non-increasing in t at $t = t_0$ whenever $\tilde{g}(\bar\theta,t_0)^2 + \tilde{g}_\theta(\bar\theta,t_0)^2 = \max_\theta\{\tilde{g}(\theta,t_0)^2 + \tilde{g}_\theta(\theta,t_0)^2\} > \max_\theta\{\tilde{g}(\theta,t_0)^2\}$. First observe the following:

LEMMA 3.3. *For any* $\varphi : \{0,\dots,N-1\} \to \mathbb{R}$,

$$\tilde{\varphi}(\theta)^2 + \tilde{\varphi}_\theta(\theta)^2 = \frac{\varphi_i^2 + \varphi_{i+1}^2 - 2\varphi_i\varphi_{i+1}\cos(\theta_{i+1}-\theta_i)}{\sin^2(\theta_{i+1}-\theta_i)}, \qquad \theta_i < \theta < \theta_{i+1}$$

The Lemma follows by a direct calculation from the definition of $\tilde{\varphi}$. It follows that

$$\max_{\theta\in S^1}\{\tilde{g}(\theta,t)^2+\tilde{g}_\theta(\theta,t)^2\}=\max_i\left\{g_i(t)^2+\left(\frac{g_{i+1}(t)-g_i(t)\cos(\theta_{i+1}-\theta_i)}{\sin(\theta_{i+1}-\theta_i)}\right)^2\right\}.$$

Suppose this maximum is achieved for some value of i, and

$$g_i(t)^2+\left(\frac{g_{i+1}(t)-g_i(t)\cos(\theta_{i+1}-\theta_i)}{\sin(\theta_{i+1}-\theta_i)}\right)^2>\max_{\theta\in S^1}\tilde{g}(\theta,t)^2.$$

In particular this implies that $g_{i+1}(t)-g_i(t)\cos(\theta_{i+1}-\theta_i)\neq 0$, and similarly $g_i(t)-g_{i+1}(t)\cos(\theta_{i+1}-\theta_i)\neq 0$ since

$$g_i(t)^2+\left(\frac{g_{i+1}(t)-g_i(t)\cos(\theta_{i+1}-\theta_i)}{\sin(\theta_{i+1}-\theta_i)}\right)^2=g_{i+1}(t)^2+\left(\frac{g_i(t)-g_{i+1}(t)\cos(\theta_{i+1}-\theta_i)}{\sin(\theta_{i+1}-\theta_i)}\right)^2.$$

LEMMA 3.4. *The quantities $g_{i+1}(t)-g_i(t)\cos(\theta_{i+1}-\theta_i)$ and $\ell_i(g(t))$ do not have opposite signs, and the quantities $g_i(t)-g_{i+1}(t)\cos(\theta_{i+1}-\theta_i)$ and $\ell_{i+1}(g(t))$ do not have opposite signs.*

Proof. By maximality,

$$g_i(t)^2+\left(\frac{g_{i+1}(t)-g_i(t)\cos(\theta_{i+1}-\theta_i)}{\sin(\theta_{i+1}-\theta_i)}\right)^2\geq g_i(t)^2+\left(\frac{g_{i-1}(t)-g_i(t)\cos(\theta_i-\theta_{i-1})}{\sin(\theta_i-\theta_{i-1})}\right)^2$$

and therefore

$$\left|\frac{g_{i+1}(t)-g_i(t)\cos(\theta_{i+1}-\theta_i)}{\sin(\theta_{i+1}-\theta_i)}\right|\geq\left|\frac{g_{i-1}(t)-g_i(t)\cos(\theta_i-\theta_{i-1})}{\sin(\theta_i-\theta_{i-1})}\right|.$$

It follows that

$$\ell_i(g)=\frac{g_{i+1}(t)-g_i(t)\cos(\theta_{i+1}-\theta_i)}{\sin(\theta_{i+1}-\theta_i)}+\frac{g_{i-1}(t)-g_i(t)\cos(\theta_i-\theta_{i-1})}{\sin(\theta_i-\theta_{i-1})}$$

is either zero or has the same sign as $g_{i+1}(t)-g_i(t)\cos(\theta_{i+1}-\theta_i)$. Similarly,

$$g_{i+1}(t)^2+\left(\frac{g_i(t)-g_{i+1}(t)\cos(\theta_{i+1}-\theta_i)}{\sin(\theta_{i+1}-\theta_i)}\right)^2\geq g_{i+1}(t)^2+\left(\frac{g_{i+2}(t)-g_{i+1}(t)\cos(\theta_{i+2}-\theta_{i+1})}{\sin(\theta_{i+2}-\theta_{i+1})}\right)^2,$$

so that

$$\ell_{i+1}(g)=\frac{g_{i+2}(t)-g_{i+1}(t)\cos(\theta_{i+2}-\theta_{i+1})}{\sin(\theta_{i+2}-\theta_{i+1})}+\frac{g_i(t)-g_{i+1}(t)\cos(\theta_{i+1}-\theta_i)}{\sin(\theta_{i+1}-\theta_i)}$$

is either zero or has the same sign as $g_i(t)-g_{i+1}(t)\cos(\theta_{i+1}-\theta_i)$. □

LEMMA 3.5. *The signs of the two quantities $g_{i+1}(t)-g_i(t)\cos(\theta_{i+1}-\theta_i)$ and $g_i(t)-g_{i+1}(t)\cos(\theta_{i+1}-\theta_i)$ are opposite.*

Proof. The function $\tilde{g}(\theta,t)$ on the interval $[\theta_i,\theta_{i+1}]$ satisfies

$$\tilde{g}_\theta(\theta,t)=\frac{-g_i(t)\cos(\theta_{i+1}-\theta)+g_{i+1}(t)\cos(\theta-\theta_i)}{\sin(\theta_{i+1}-\theta_i)}$$

and therefore

$$\tilde{g}_\theta(\theta_i, t) = \frac{g_{i+1}(t) - g_i(t)\cos(\theta_{i+1} - \theta_i)}{\sin(\theta_{i+1} - \theta_i)}$$

and

$$\tilde{g}_\theta(\theta_{i+1}, t) = -\frac{g_i(t) - g_{i+1}(t)\cos(\theta_{i+1} - \theta_i)}{\sin(\theta_{i+1} - \theta_i)}.$$

If the Lemma does not hold, then $\tilde{g}_\theta$ changes sign on the interval $[\theta_i, \theta_{i+1}]$, and therefore there exists $\bar{\theta} \in (\theta_i, \theta_{i+1})$ such that $\tilde{g}_\theta(\bar{\theta}, t) = 0$. Then

$$\tilde{g}(\bar{\theta}, t)^2 = \tilde{g}(\bar{\theta}, t)^2 + \tilde{g}_\theta(\bar{\theta}, t)^2 = \max_{\theta \in S^1}\{\tilde{g}(\theta, t)^2 + \tilde{g}_\theta(\theta, t)^2\},$$

by Lemma 3.3. This contradicts the hypotheses of Proposition 3.2. □

The time derivative of $\tilde{g}^2 + \tilde{g}_\theta^2$ at a maximum point may be computed as follows:

$$\begin{aligned} &\frac{d}{dt}\left(\frac{g_i(t)^2 + g_{i+1}(t)^2 - 2g_i(t)g_{i+1}(t)\cos(\theta_{i+1} - \theta_i)}{\sin^2(\theta_{i+1} - \theta_i)}\right) \\ &\quad = \frac{2g_i'(t)\ell_i(g(t))}{\sin(\theta_{i+1} - \theta_i)}\left(\frac{g_i(t) - g_{i+1}(t)\cos(\theta_{i+1} - \theta_i)}{\sin(\theta_{i+1} - \theta_i)}\right) \\ &\quad + \frac{2g_{i+1}'(t)\ell_{i+1}(g(t))}{\sin(\theta_{i+1} - \theta_i)}\left(\frac{g_{i+1}(t) - g_i(t) cos(\theta_{i+1} - \theta_i)}{\sin(\theta_{i+1} - \theta_i)}\right). \end{aligned}$$

Now observe that $g_i'(t)$ and $g_{i+1}'(t)$ are non-negative since g_i is non-decreasing. Lemmas 3.4 and 3.5 imply that $\ell_i(g(t))$ and $g_i(t) - g_{i+1}(t)\cos(\theta_{i+1} - \theta_i)$ have opposite signs, and that $\ell_{i+1}(g(t))$ and $g_{i+1}(t) - g_i(t)\cos(\theta_{i+1} - \theta_i)$ have opposite signs. Therefore the time derivative is non-positive, and Proposition 3.2 follows by an application of the maximum principle (Proposition 2.1). □

4. Degenerate pinching. This section addresses the phenomenon of degenerate pinching in crystalline curvature flows. The speed functions g_i in Equation (3) are assumed to be negative, locally Lipschitz continuous, and increasing on $(0, \infty)$ for each i, with $\lim_{z \to 0} g_i(z) = -\infty$ for all i. It was shown in [GG] that if

$$\int_0^1 g_i(z)\, dz = -\infty \tag{22}$$

for every i, then degenerate pinching does not occur. The main result of this section is almost converse to that statement:

PROPOSITION 4.1. *Suppose that there exists a pair of parallel edges, so that $\theta_0 = 0$ and $\theta_k = \pi$, and assume the growth restriction*

$$\int_0^1 g_i(z)\, dz > -\infty \tag{23}$$

except possibly for $i = 0, k$. Then for any $L_0 > 0$ there exists a constant $W_1(L_0, \underline{\theta}, \underline{g})$ such that for any $\underline{\ell}_0 \in \mathcal{L}$ with $L = L_0$ and $w \leq W_1$, the solution of (3) with initial data $\underline{\ell}_0$ has a degenerate pinching singularity at the final time T: $\lim_{t \to T} \ell_i(t) = 0$ for $i \neq 0, k$, while $\lim_{t \to T} \ell_0(t) = \lim_{t \to T} \ell_k(t) > 0$.

Proof. A first step is to prove the following more restricted result:

PROPOSITION 4.2. *Under the conditions of Proposition 4.1, there exists for any* $L_0 > 0$ *a constant* $W_0(L_0, \underline{\theta}, \underline{g})$ *such that for any* $\underline{\ell}_0 \in \mathcal{L}$ *with* $L = L_0$ *and* $w \leq W_0$ *and satisfying*

$$\sup_{\theta \in S^1} \left\{ \tilde{g}(\theta)^2 + \tilde{g}_\theta(\theta)^2 \right\} = \sup_{\theta \in S^1} \tilde{g}(\theta)^2,$$

the solution of (3) *with initial data* $\underline{\ell}_0$ *has a degenerate pinching singularity with* $\lim_{t\to T} \ell_0(t) = \lim_{t\to T} \ell_k(t) \geq L_0/2$ *while* $\lim_{t\to T} \ell_i(t) = 0$ *for* $i \neq 0, k$.

Proof. By Proposition 3.2, for any $t \in [0, T)$

$$\sup_{\theta \in S^1} \left\{ \tilde{g}(\theta, t)^2 + \tilde{g}_\theta(\theta, t)^2 \right\} = \sup_{\theta \in S^1, 0 \leq t' \leq t} \tilde{g}(\theta, t')^2.$$

Given any $t > 0$ for which $L \geq L_0/4$, let $\bar{t} \in [0, t]$ such that

$$\sup_{\theta \in S^1} \tilde{g}(\theta, \bar{t})^2 = \sup_{\theta \in S^1, 0 \leq t' \leq t} \tilde{g}(\theta, t')^2.$$

In particular at $t = \bar{t}$,

$$\sup_{\theta \in S^1} \left\{ \tilde{g}(\theta)^2 + \tilde{g}_\theta(\theta)^2 \right\} = \sup_{\theta \in S^1} \tilde{g}(\theta)^2. \tag{24}$$

LEMMA 4.3. *There exists* $W_2 > 0$ *and* C_2, $C_3 > 0$ *depending only on* L_0, $\underline{g}$ *and* $\underline{\theta}$ *such that for any* $\underline{\ell} \in \mathcal{L}$ *satisfying* $L(\underline{\ell}) \geq L_0/3$, $w(\underline{\ell}) \leq W_2$ *and* (24),

$$\max_i |g_i(\ell_i)| \leq C_2 \max_{j \neq 0,k} |g_j(C_3 w(\underline{\ell}))|.$$

Proof. First note that $|g_0(\ell_0)| \leq |g_0(L_0/4)|$ and $|g_k(\ell_k)| \leq |g_k(L_0/4)|$ since $\ell_0 \geq L - Cw \geq L_0/4$ and $\ell_k \geq L_0/4$ for W_2 small enough. Let $\bar{\theta}$ be such that $\sup_\theta |\tilde{g}(\theta)| = |\tilde{g}(\bar{\theta})|$. Then $|\tilde{g}(\bar{\theta})| \geq \max_i |g_i(\ell_i)| \geq \max_{i \neq 0,k} |g_i(Cw)|$, since

$$\ell_i \leq \frac{\sum_{j \neq 0,k} \ell_j |\sin \theta_j|}{|\sin \theta_i|} \leq Cw \leq CW_2$$

for $i \neq 0, k$.

The identity (24) implies

$$|\tilde{g}(\theta)| \geq |\tilde{g}(\bar{\theta})| \cos(\theta - \bar{\theta}) \tag{25}$$

for all θ. The cases $\theta = 0, \pi$ imply that

$$|\cos \bar{\theta}| \leq \frac{\max\{|g_0(L_0/4)|, |g_k(L_0/4)|\}}{\max_{i \neq 0,k} |g_i(CW_2)|}. \tag{26}$$

Now choose W_2 sufficiently small that

$$\frac{\max\{|g_0(1/4)|, |g_k(1/4)|\}}{\max_{i \neq 0,k} |g_i(CW_2)|} \leq \sin\left(\frac{1}{2} \min\{\theta_1, \pi - \theta_{k-1}, \theta_{k+1} - \pi, 2\pi - \theta_{N-1}\}\right).$$

Suppose that $\bar{\theta} \in (0, \pi)$ (the other possibility can be treated similarly). The estimate (26) yields

$$|\bar{\theta} - \pi/2| \leq \frac{1}{2} \min\{\theta_1, \pi - \theta_{k-1}, \theta_{k+1} - \pi, 2\pi - \theta_{N-1}\}.$$

Then (25) with $\theta = \theta_i$, $1 \le i \le k-1$ yields

$$\begin{aligned} |g_i(\ell_i)| &\ge |\tilde{g}(\bar{\theta})| \cos(\theta_i - \bar{\theta}) \\ &\ge |\tilde{g}(\bar{\theta})| \sin(\theta_i - (\bar{\theta} - \pi/2)) \\ &\ge |\tilde{g}(\bar{\theta})| \sin\left(\frac{1}{2} \min\{\theta_1, \pi - \theta_{k-1}, \theta_{k+1} - \pi, 2\pi - \theta_{N-1}\}\right) \end{aligned}$$

for $1 \le i \le k-1$. But $\max_{i=1,\dots,k-1} \ell_i \ge Cw$, since $\sum_{i=1}^{k-1} \ell_i \sin\theta_i = w$. Therefore $|\tilde{g}(\bar{\theta}| \le C \max_{i \ne 0,k} |g_i(Cw)|$. Since $|g_i(\ell_i)| \le |\tilde{g}(\bar{\theta})|$ for every i, the proof of the Lemma is complete. □

COROLLARY 4.4. *If $t > 0$ is such that $L \ge L_0/3$, then (for W_0 sufficiently small)*

$$\max_i |g_i(\ell_i(t))| \le C_2 \max_{j \ne 0,k} |g_j(C_3 w(t))|.$$

Proof. By the choice of $\bar{t}$,

$$\max_i |g_i(\ell_i(t))| \le \max_i |g_i(\ell_i(\bar{t}))| \le C_2 \max_{j \ne 0,k} |g_j(C_3 w(\bar{t}))| \le C_2 \max_{j \ne 0,k} |g_j(C_3 w(t))|$$

where the fact that $w(t)$ is decreasing in t was used to obtain the last inequality. □

The proof of Proposition 4.2 can now be completed: Equation (15) gives

$$\frac{dw}{dt} = g_0(\ell_0) + g_k(\ell_k) \le -|g_0(2L_0) + g_k(2L_0)|$$

since $\ell_0 \le L + Cw \le 2L_0$ for W_0 sufficiently small (by (21)), and

$$\frac{dE}{dt} = \sum_i c_i g_i(\ell_i) \tag{27}$$

by (13), where

$$c_i = \ell_i(\underline{1}) = \frac{\sin(\theta_i - \theta_{i-1}) + \sin(\theta_{i+1} - \theta_i) - \sin(\theta_{i+1} - \theta_{i-1})}{\sin(\theta_i - \theta_{i-1}) \sin(\theta_{i+1} - \theta_i)} > 0.$$

For any t such that $L > L_0/4$, Corollary 4.4 implies

$$\frac{dE}{dt} \ge -c_0|g_0(L_0/4)| - c_k|g_k(L_0/4)| - C \max_{j \ne 0,k} |g_j(C_3 w)| \ge -C \max_{j \ne 0,k} |g_j(C_3 w)|$$

if W_0 is small enough. w is decreasing in time, so a new time variable may be defined by $\tau = -w(t)$. Then

$$\frac{dE}{d\tau} \ge -C \max_{j \ne 0,k} |g_j(-C_3\tau)|,$$

and

$$E(t) \ge E(0) - C \max_{j \ne 0,k} \int_{-W_0}^{\tau} |g_j(-C_3 s)| ds \ge L_0 - CW_0 - C \max_{j \ne 0,k} \int_0^{W_0/C_3} |g_j(s)| ds,$$

as long as $L \ge L_0/3$ (by (19)). In particular, for W_0 sufficiently small

$$L(t) \ge E(t) - Cw(t) \ge L_0/2$$

for all t such that $\tau < 0$. But this implies that $L(t) \ge L_0/2$ while $w(t) \to 0$, so degenerate pinching occurs. Note also that E is nonincreasing, and $|\ell_0 - E| \le Cw$

and $|\ell_k - E| \le Cw$, so the limits of $\ell_0(t)$ and $\ell_k(t)$ exist as $t \to T$ and are at least $L_0/2$. $\square$

It has not yet been demonstrated that the hypotheses of Proposition 4.2 can be achieved. However explicit examples of initial data satisfying these conditions will be provided in the course of the proof of Proposition 4.1, which follows.

If Proposition 4.1 does not hold, then for any $W > 0$ there exists some initial condition $\underline{\ell} \in \mathcal{L}$ with $w \le W$ and $L = L_0$, for which degenerate pinching does not occur.

It will be shown that there is some time $t_* \in [0, T)$ such that the conditions of Proposition 4.2 are satisfied (with a smaller L_0). First, a bound on the speed may be obtained at some positive time by constructing a barrier $(\ell'_0, \dots, \ell'_{N-1})$: Choose

$$\begin{aligned} g_i(\ell'_i) &= -\alpha \sin\theta_i, \quad i = 1, \dots, k-1; \\ g_i(\ell'_i) &= \beta \sin\theta_i, \quad i = k+1, \dots, N-1; \\ \sum_{i \ne 0,k} \ell'_i |\sin\theta_i| &= \sum_{i\ne 0,k} \ell_i |\sin\theta_i| = w. \end{aligned}$$

Here α and β are determined by w, since $\sum_{i=1}^{k-1} g_i^{-1}(-\alpha\sin\theta_i)\sin\theta_i = w$ and $-\sum_{i=k+1}^{N-1} g_i^{-1}(\beta\sin\theta_i)\sin\theta_i = w$. In each of these equations the left-hand side is monotone decreasing, defined and positive for α or β sufficiently large respectively, and approaches zero as α or β approaches infinity. Thus α and β are uniquely determined for w sufficiently small, and so ℓ'_i is determined for each $i \ne 0, k$. Also, ℓ'_0 and ℓ'_k are determined by the identity (20). In particular, ℓ'_0 and ℓ'_k are positive provided λ is sufficiently small. One can then choose $\frac{\ell'_0 + \ell'_k}{2} \ge L_0 - Cw$ and obtain a curve which can be placed inside our original curve. Denote by $\ell'_i(t)$ the solution of Eq. (3) with this initial data.

LEMMA 4.5. *If w is sufficiently small, then*

$$\sup_{\theta\in S^1} \{\tilde{g}'(\theta)^2 + \tilde{g}'_\theta(\theta)^2\} = \sup_{\theta\in S^1} \tilde{g}'(\theta)^2,$$

where $\tilde{g}'$ is defined by Definition 3.1 from $g_(\ell'_i)$.

Proof. A direct computation gives

$$\tilde{g}'(\theta) = \begin{cases} -\alpha\sin\theta + \dfrac{g_0(\ell'_0)\sin(\theta_1 - \theta)}{\sin\theta_1}, & 0 \le \theta \le \theta_1; \\ -\alpha\sin\theta, & \theta_1 \le \theta \le \theta_{k-1}; \\ -\alpha\sin\theta + \dfrac{g_k(\ell'_k)\sin(\theta - \theta_{k-1})}{\sin\theta_{k-1}}, & \theta_{k-1} \le \theta \le \pi; \\ \beta\sin\theta - \dfrac{g_k(\ell'_k)\sin(\theta_{k+1} - \theta)}{\sin\theta_{k+1}}, & \pi \le \theta \le \theta_{k+1}; \\ \beta\sin\theta, & \theta_{k+1} \le \theta \le \theta_{N-1}; \\ \beta\sin\theta - \dfrac{g_0(\ell'_0)\sin(\theta - \theta_{N-1})}{\sin\theta_{N-1}}, & \theta_{N-1} \le \theta \le 2\pi. \end{cases}$$

This is minus the support function of a convex figure (with at most six sides), provided

$$\alpha \geq \max\left\{\frac{|g_0(\ell_0')|\cos\theta_1}{\sin\theta_1}, -\frac{|g_k(\ell_k')|\cos\theta_{k-1}}{\sin\theta_{k-1}}\right\}$$

and

$$\beta \geq \max\left\{-\frac{|g_0(\ell_0')|\cos\theta_{N-1}}{\sin\theta_{N-1}}, \frac{|g_k(\ell_k')|\cos\theta_{k+1}}{\sin\theta_{k+1}}\right\}.$$

If w approches zero, then α and β approach infinity, while ℓ_0' and ℓ_k' approach 1, and so these conditions can be realized. Thus for w sufficiently small (compare the comments after Definition 3.1),

$$\sup_{\theta\in S^1}\{\tilde{g}'(\theta)^2 + \tilde{g}_\theta'(\theta)^2\} = \sup_{\theta\in S^1}\{\tilde{g}'(\theta)^2\}.$$

□

Proposition 4.2 applies to show that $\ell'(t)$ evolves to a degenerate pinching with $\ell_k'(t)$ and $\ell_0'(t)$ approaching a limit no less than $L_0/2$ as w' approaches zero. Consider the time t_0 at which $w'(t) = w/2$.

From the proof of Proposition 4.2 we have $t_0 \geq Cw$ (from Equation (15)) and

$$E(\underline{\ell}'(t_0)) \geq E(\underline{\ell}'(0)) - C\max_{i\neq 0,k}\int_{w/2}^{w}|g_j(C_3 s)|ds \geq L_0 - Cw - Cw\max_{i\neq 0,k}|g_j(C_4 w)|.$$

Then by the comparison principle,

$$E(\underline{\ell}(t_0)) \geq E(\underline{\ell}'(t_0)) \geq L_0 - Cw\max_{i\neq 0,k}|g_j(C_4 w)| \geq E(\underline{\ell}(0)) - Cw\max_{i\neq 0,k}|g_j(C_4 w)|.$$

On the other hand

$$E(\underline{\ell}(t_0)) - E(\underline{\ell}(0)) = \int_0^{t_0}\sum_i c_i g(\ell_i(s))ds,$$

so it follows that there exists $t_1 \in [0, t_0]$ such that

$$\sum_i c_i|g(\ell_i(t_1))| \leq C\max_{i\neq 0,k}|g_i(C_4 w)|.$$

By Lemma 3.3 it follows that

$$\sup_{\theta\in S^1}\left\{\tilde{g}(\theta,t_1)^2 + \tilde{g}_\theta(\theta,t_1)^2\right\} \leq C\max_{i\neq 0,k}|g_i(C_4 w)|.$$

Now consider the time t_2 at which L reaches $L_0/2$. Such a time exists since degenerate pinching is assumed not to occur, and $t_2 > t_0$ in view of the estimates above on $E(\ell(t_0))$. However, $t_2 \leq Cw$ using the evolution equation (15) for w. On the time interval $[0, t_2]$,

$$cw - L_0/2 \geq E(\underline{\ell}(t_2)) - E(\underline{\ell}(0)) = \int_0^{t_2}\sum_i c_i g_i(\ell_i(t))dt$$

and therefore for some $t_3 \in (t_0, t_2)$,

$$\max_i |g_i(\ell_i(t_3))| \geq \frac{CL_0}{w}.$$

LEMMA 4.6. *For any $\varepsilon > 0$ there is a constant $W_3(\varepsilon)$ such that for any $z < W_3$,*

$$z \max_{i \neq 0,k} |g_i(z)| \leq \varepsilon.$$

Proof. Otherwise, there exists a sequence $z_j \to 0$ such that

$$z_j \max_{i \neq 0,k} |g_i(z_j)| \geq \varepsilon.$$

Without loss of generality, $z_{k+1} \leq z_k/2$. Then on the interval $z \in [z_k/2, z_k]$ the monotonicity of g_i implies

$$\max_{i \neq 0,k} |g_i(z)| \geq \max_{i \neq 0,k} |g_i(z_k)|.$$

Therefore

$$\max_{i \neq 0,k} \int_{z_m/2}^{z_1} |g_i(z)| dz \geq \sum_{j=1}^{m} \int_{z_j/2}^{z_j} \frac{\varepsilon}{z}\, dz \geq m\varepsilon \log 2.$$

Taking $m \to \infty$ contradicts the growth restriction (23). □

It follows that CL_0/w is larger than $C \max_{i \neq 0,k} |g_i(C_4 w)|$ for w sufficiently small, and so by Proposition 3.2,

$$\sup_{\theta \in S^1} \{\tilde{g}(\theta, t_2)^2 + \tilde{g}_\theta(\theta, t_2)^2\} = \sup_{\theta \in S^1} \tilde{g}(\theta, t_2)^2.$$

Also, $L(\underline{\ell}(t_2)) \geq L_0/2$ and $w(\underline{\ell}(t_2)) \leq W_0$. Therefore the conditions of Proposition 4.2 are satisfied for the initial condition $\underline{\ell}(t_2)$ provided $W_0 \leq W_1(L_0/2)$. Proposition 4.2 implies that a degenerate pinching singularity occurs at the final time. This is a contradiction which completes the proof. □

5. Entropy. In this section the entropy associated with the homogeneous crystalline flow (2) is defined and proved to be nondecreasing in time.

For the flow (2) the associated entropy $\mathcal{Z} : \mathcal{L} \to \mathbb{R}$ is defined by

$$\mathcal{Z} = \begin{cases} \left(\sum_{i=0}^{N-1} f_i \ell_i^{1-\alpha} \right)^{\frac{1}{1-\alpha}} & \text{if } \alpha \neq 1; \\ \left(\prod_{i=0}^{N-1} \ell_i^{f_i} \right)^{\frac{1}{\sum_j f_j}} & \text{if } \alpha = 1. \end{cases}$$

In the case $\alpha = 1$ these functionals were defined in [S2]. For other α the above definitions are natural generalisations of those that work in the smooth case (see [A1] and [A3]). The basic result concerning entropy is the following:

PROPOSITION 5.1. *For any solution of Eq.* (2),

$$\frac{d}{dt}\left(\mathcal{Z} A^{-1/2}\right) \geq 0$$

with equality only for homothetically contracting solutions.

Proof. By Equation (12),

$$\frac{d}{dt} A = -\sum_i f_i \ell_i^{-\alpha}$$

and (for $\alpha \neq 1$)

$$\frac{d}{dt}\mathcal{Z} = -\mathcal{Z}^{\alpha}\sum_i \frac{f_i}{\ell_i^{\alpha}}\ell_i\left(\frac{f}{\ell^{\alpha}}\right).$$

Therefore

$$\frac{d}{dt}\left(\mathcal{Z}A^{-1/2}\right) = -\frac{\mathcal{Z}^{\alpha}}{A^{1/2}}\left(\sum_i \frac{f_i}{\ell_i^{\alpha}}\ell_i\left(\frac{f}{\ell^{\alpha}}\right) - \frac{1}{2A}\left(\sum_i \frac{f_i}{\ell_i^{\alpha}}\ell_i\right)^2\right) \geq 0$$

by the Brunn-Minkowksi theorem (inequality (11)), and equality holds only if

$$\ell_i\left(\frac{f}{\ell^{\alpha}}\right) = c\ell_i \tag{28}$$

for some $c > 0$. But a curve satisfying (28) evolves homothetically under (3), with solution given explicitly by $\ell_i(t) = (1 - c(1+\alpha)t)^{\frac{1}{1+\alpha}}\ell_i(0)$. □

6. The homogeneous case $\alpha > 1$. In the case $\alpha > 1$ of Eq. (2) considerations of entropy allow a complete description of the asymptotic behaviour of convex solutions of crystalline curvature flows: These always shrink to points while asymptotically approaching a homothetically shrinking solution.

6.1 Isoperimetric ratio bound. Let $\underline{\ell}(t)$ be a solution of Eq. (2). Then the entropy ratio $\mathcal{Z}A^{-1/2}$ is bounded below by its initial value. This provides an isoperimetric ratio bound, since

$$\mathcal{Z} = \left(\sum_i f_i \ell_i^{1-\alpha}\right)^{\frac{1}{1-\alpha}} \leq f_{\min}^{\frac{1}{1-\alpha}}\ell_{\min}$$

and so $\ell_{\min} \geq CA^{1/2}$. Since $A \geq C\ell_{\min}\ell_{\max}$ by (16), this implies $\ell_{\max} \leq C\ell_{\min}$. It follows that the solution continues to exist and remains smooth while the maximum edge length remains positive, and therefore the solution converges to a point $p \in \mathbb{R}^2$.

It follows that

$$\begin{aligned}\frac{dA}{dt} &= -\sum_i f_i \ell_i^{1-\alpha}\\ &\sim -CA^{\frac{1-\alpha}{2}}\end{aligned}$$

and therefore $A \sim (T-t)^{\frac{2}{1+\alpha}}$ and $\ell_i(t) \sim (T-t)^{\frac{1}{1+\alpha}}$ for every i.

6.2 Convergence. Now consider any sequence of times t_k approaching the final time T at which the solution contracts to $p \in \mathbb{R}^2$. Then consider the rescaled solutions $\underline{\ell}^{(k)}(t)$ defined by $\ell_i^{(k)}(t) = (T - t_k)^{-\frac{1}{1+\alpha}}\ell_i(t_k + t(T - t_k))$. For each k this defines a solution of Eq. (2) for $t \in [0,1)$, with $C_1(1-t)^{\frac{1}{1+\alpha}} \leq \ell_i^{(k)}(t) \leq C_2(1-t)^{\frac{1}{1+\alpha}}$, C_1 and C_2 independent of k.

It follows that there is a subsequence on which these rescaled solutions converge (uniformly on compact subintervals) to a limit $\underline{\ell}^{\infty}(t)$ which is again a solution. Proposition 3.1 guarantees that $\mathcal{Z}A^{-1/2}$ is non-decreasing on the limit $\underline{\ell}^{\infty}(t)$. It is in fact constant, for the following reason: If not, then $\mathcal{Z}A^{-1/2}\big|_{\underline{\ell}^{\infty}(1/2)} \geq \mathcal{Z}A^{-1/2}\big|_{\underline{\ell}^{\infty}(0)} + \varepsilon$ for some $\varepsilon > 0$. But for k large, $\left|\mathcal{Z}A^{-1/2}\big|_{\underline{\ell}^{(k)}(t)} - \mathcal{Z}A^{-1/2}\big|_{\underline{\ell}^{\infty}(t)}\right| \leq \frac{\varepsilon}{4}$ for $0 \leq t \leq 1/2$.

Therefore on the sequence of times t_k, $\mathcal{Z}A^{-1/2}$ is at most $\mathcal{Z}A^{-1/2}\big|_{\underline{\ell}^\infty(0)} + \frac{\varepsilon}{4}$, and on the sequence $t_k + 1/2(T - t_k)$, $\mathcal{Z}A^{-1/2}$ is at least $\mathcal{Z}A^{-1/2}\big|_{\underline{\ell}^\infty(0)} + \frac{3\varepsilon}{4}$. But both of these sequences approach T as $k \to \infty$, so $\mathcal{Z}A^{-1/2}$ cannot be nondecreasing. This contradicts Proposition 6.1.

It follows from the second part of Proposition 5.1 that the limit solution is homothetically contracting. Finally, subsequential convergence can be improved to give uniform convergence of the rescaled solutions to the homothetic limit (this uses the Łojasiewicz inequality, via an adaptation of the argument in [A2]).

In the case of a symmetric flow, there is a unique homothetic solution, which attracts all other convex solutions. This follows from the result just proved (which gives existence of a symmetric homothetic solution), together with a slight modification of the argument in [S1].

7. The homogeneous case $\alpha = 1$. This last section deals with the homogeneous case $\alpha = 1$, which turns out to allow a remarkable range of different singularity behaviour. The main result is the following:

PROPOSITION 7.1. *Under equation* (2) *with* $\alpha = 1$*:*

(1) *If there are no parallel pairs of edges, or if every pair of parallel edges (i.e. i, j such that $\theta_j = \theta_i + \pi$) satisfies $f_i + f_j < \sum_{m \neq i,j} f_m$, then for any initial data in $\mathcal{L}$, the solution is asymptotic to a homothetically contracting solution. In particular, if the flow is symmetric (i.e. $\theta_{i+k} = \theta_i + \pi$ and $f_{i+k} = f_i$ for every i) then there exists a unique homothetically contracting solution, and all solutions are asymptotic to this. In this case*

$$\min_m \ell_m(t) \sim \max_m \ell_m(t) \sim \sqrt{T - t}$$

(2) *If there exist edges i, j such that $\theta_j = \theta_i + \pi$, and $f_i + f_j > \sum_{m \neq i,j} f_m$, then there exist solutions for which the isoperimetric ratio becomes unbounded as the final time is approached, in such a way that*

$$\min_m \ell_m(t) \sim (T - t)^{\frac{f_i + f_j}{\sum_r f_r}}, \qquad \max_m \ell_m(t) \sim (T - t)^{1 - \frac{f_i + f_j}{\sum_r f_r}}.$$

If the flow is symmetric then this occurs for all solutions.

(3) *If the flow is symmetric with $N > 4$ and there is a pair of parallel edges i, j such that $f_i + f_j = \sum_{m \neq i,j} f_m$, then for every solution the isoperimetric ratio becomes unbounded as $t \to T$, in such a way that*

$$\min_m \ell_m(t) \sim \sqrt{\frac{T - t}{|\log(T - t)|}}, \qquad \max_m \ell_m(t) \sim \sqrt{(T - t)|\log(T - t)|}.$$

If the flow is symmetric with $N = 4$ and $f_1 = f_2$, then every solution is homothetically contracting.

This result does not cover the case of non-symmetric flows in the critical case where there is a pair of parallel edges carrying half the total of the weights f_j. Examples of this kind will be provided below where all solutions are aymptotic to homothetic solutions, as well as others showing divergent behaviour of the same kind as part (3) of the Proposition, and others showing divergent behaviour where $\ell_{\min} \sim (T - t)^{1/2}/|\log(T - t)|^{1/4}$. It seems probable that for any positive integer k

there should be examples where $\ell_{\min} \sim (T-t)^{1/2}/|\log(T-t)|^{1/2k}$, but the criteria distinguishing these possibilities in terms of the weights f_j and the angles θ_j are probably very complicated.

Proof. In the case where there are no parallel edges, there is an automatic isoperimetric ratio bound, so the argument of Section 6 shows that the solution is asymptotic to a homothetic solution. For symmetric flows it suffices to start with symmetric initial data and deduce convergence to a symmetric homothetic solution, since the globally attracting nature of symmetric homothetic solutions was established in [S1].

The proof of part (1) of the Proposition can now be completed by establishing an isoperimetric ratio bound under the assumption that every pair of parallel edges carries less than half the total of the weights f_j. Take any such pair of parallel edges, and parametrise such that these are in directions $\theta_0 = 0$ and $\theta_k = \pi$. For simplicity one can also reparametrise time to make $\sum_i f_i = 1$.

Define $w = \sum_{i=1}^{k-1} \ell_i \sin\theta_i = \sum_{i=k+1}^{N-1} \ell_i |\sin\theta_i|$. Then observe that $|\ell_0 - A/w| \le Cw$ and $|\ell_k - A/w| \le Cw$, so that

$$\begin{aligned}\mathcal{Z}A^{-1/2} &= A^{-1/2}\ell_0^{f_0}\ell_k^{f_k}\prod_{i=1}^{k-1}\ell_i^{f_i}\prod_{j=k+1}^{N-1}\ell_j^{f_j}\\ &\le A^{-1/2}\left(\frac{A}{w}\right)^{f_0+f_k}\left(1+C\frac{w^2}{A}\right)\prod_{i=1}^{k-1}\ell_i^{f_i}\prod_{j=k+1}^{N-1}\ell_j^{f_j}.\end{aligned}$$

The two products can be estimated as follows:

$$\begin{aligned}\prod_{i=1}^{k-1}\ell_i^{f_i} &= \exp\left\{\sum_{i=1}^{k-1} f_i \log \ell_i\right\}\\ &= \exp\left\{\sum_{i=1}^{k-1} f_i\log\left(\frac{\ell_i|\sin\theta_i|}{f_i}\right)\right\}\exp\left\{\sum_{i=1}^{k-1} f_i\log\left(\frac{f_i}{|\sin\theta_i|}\right)\right\}\\ &\le \exp\left\{\sum_{i=1}^{k-1} f_i\log\left(\frac{f_i}{|\sin\theta_i|}\right)\right\}\left(\frac{1}{\sum_{i=1}^{k-1} f_i}\sum_{i=1}^{k-1}\ell_i|\sin\theta_i|\right)^{\sum_{i=1}^{k-1} f_i}\\ &= \prod_{i=1}^{k-1}\left(\frac{f_i}{|\sin\theta_i|\sum_{j=1}^{k-1} f_j}\right)^{f_i} w^{\sum_{j=1}^{k-1} f_j},\end{aligned}$$

and similarly

$$\prod_{i=k+1}^{N-1}\ell_i^{f_i} \le \prod_{i=k+1}^{N-1}\left(\frac{f_i}{|\sin\theta_i|\sum_{j=k+1}^{N-1} f_j}\right)^{f_i} w^{\sum_{j=k+1}^{N-1} f_j}.$$

It follows that

$$\mathcal{Z}A^{-1/2} \le C_1\left(1+C_2\frac{w^2}{A}\right)\left(\frac{w}{\sqrt{A}}\right)^{\sum_{i\neq 0,k} f_i - f_0 - f_k},$$

where

$$C_1 = \prod_{i=1}^{k-1} \left(\frac{f_i}{|\sin\theta_i| \sum_{j=1}^{k-1} f_j} \right)^{f_i} \prod_{i=k+1}^{N-1} \left(\frac{f_i}{|\sin\theta_i| \sum_{j=k+1}^{N-1} f_j} \right)^{f_i}.$$

In particular, since the entropy ratio is bounded below, a lower bound on $r = w^2/A$ follows (the exponent $\sum_{i\neq 0,k} f_i - f_0 - f_k$ is positive by hypothesis). Part (1) of Proposition 7.1 now follows as in Section 6.

The proof of part (2) of Proposition 7.1 proceeds as follows: The same argument as above gives an upper bound for w^2/A in terms of the entropy ratio, and in particular solutions with large entropy must have small w^2/A as long as they exist. The following initial data will give arbitrarily large values for the entropy ratio: Take

$$\begin{aligned} &\frac{\ell_0 + \ell_k}{2} = 1; \\ &\frac{f_i}{\ell_i} = \alpha \sin\theta_i, \quad i = 1, \dots, k-1; \\ &\frac{f_i}{\ell_i} = -\beta \sin\theta_i, \quad i = k+1, \dots, N-1; \\ &\sum_{i=1}^{k-1} \ell_i \sin\theta_i = -\sum_{i=k+1}^{N-1} \ell_i \sin\theta_i = \lambda. \end{aligned} \tag{29}$$

Thus $\alpha = \lambda^{-1} \sum_{i=1}^{k-1} f_i$ and $\beta = \lambda^{-1} \sum_{i=k+1}^{N-1} f_i$. ℓ_0 and ℓ_k are determined by these conditions using the identity (20). With this choice, $|A - \lambda| \leq C\lambda^2$, $|\ell_0 - 1| \leq C\lambda$, $|\ell_k - 1| \leq C\lambda$ and

$$\mathcal{Z} A^{-1/2} \geq C_1 \lambda^{\sum_{i\neq 0,k} f_i - f_0 - f_k} (1 - C\lambda).$$

Therefore the entropy ratio can be made arbitrarily large by taking λ small, and w^2/A can be made to remain as small as desired as long as the solution exists, by choosing this initial data with λ small.

Now compute the evolution of A using (12):

$$\frac{dA}{dt} = -\sum_i f_i,$$

so that $A(t) = \sum_i f_i (T - t)$. Equation (15) gives the evolution of w:

$$\frac{dw}{dt} = -\frac{f_0}{\ell_0} - \frac{f_k}{\ell_k} \leq -(f_0 + f_k) \frac{w}{A} \left(1 - C\frac{w^2}{A}\right) \leq -\frac{(f_0 + f_k - \varepsilon) w}{\sum_i f_i (T - t)},$$

for any $\varepsilon > 0$, provided λ is sufficiently small. This inequality integrates to give

$$w(t) \leq C(T - t)^{\frac{f_0 + f_k - \varepsilon}{\sum_i f_i}}.$$

The exponent here is greater than $1/2$ for ε small. Substituting this estimate for w back in the evolution equation for w gives

$$\frac{d \log w}{dt} \leq -\frac{f_0 + f_k}{\sum_i f_i} \left(\frac{1}{(T - t)} - \frac{C}{(T - t)^{1 - \sigma}} \right)$$

for some $\sigma > 0$, and therefore

$$w(t) \leq C(T-t)^{\frac{f_0+f_k}{\sum_i f_i}}.$$

This provides an example of initial conditions where the isoperimetric ratio becomes unbounded in the way claimed in the Proposition.

In the case of symmetric flows, a different proof applies, and this also gives the result of part (3): The methods of section 6 imply that any solution either has isoperimetric ratio approaching infinity, or converges to a homothetically shrinking solution (if there is any sequence of times approaching the final time for which the isoperimetric ratio remains bounded, the methods of Section 6 imply convergence to a homothetically shrinking limit). The latter possibility will be excluded by showing that there do not exist homothetically shrinking solutions:

Suppose $\underline{\ell}$ is a homothetically shrinking solution. Then $\ell_i s_i = \lambda f_i$, which means geometrically that the area subtended by the ith edge is proportional to the weight f_i. By scaling the curve, one can assume that the area subtended by the ith edge is equal to f_i, and without loss of generality one can assume $\sum_i f_i = 1$. The hypotheses of the Proposition imply $f_0 > 1/4$ in case (2) and $f_0 = 1/4$ in case (3).

Consider the area subtended by the edges 0 and k: These are each equal to f_0. It follows that the perpendicular distance of edge 0 from the origin is equal to $2f_0/\ell_0$, and the perpendicular distance of edge k from the origin is equal to $2f_0/\ell_k$. Therefore the width w of the curve is equal to $2f_0(1/\ell_0 + 1/\ell_k)$. By identity (16), the total area satisfies

$$A \geq Lw = f_0\frac{(\ell_0+\ell_k)^2}{\ell_0\ell_k} \geq 4f_0 \geq 1,$$

where the last inequality is strict in case (2), and the first inequality is strict in case (3) unless $N = 4$. This contradicts the fact that the total area is equal to $\sum_i f_i = 1$.

It remains to show that the singularity is always of the type claimed. The next results will show that the solutions asymptotically approach curves similar to those defined in Equation (29).

LEMMA 7.2. *For any solution of Eq.* (2),

$$\ell_i\left(\frac{f}{\ell}\right) + \frac{\ell_i}{2t} \geq 0$$

for every i and every $0 < t < T$.

This result also holds for other flows of the form (2) with $\alpha > 0$, if the last term is replaced by $\alpha\ell_i/(1+\alpha)t$.

Proof. This is true for small times. Consider the evolution equation for $\ell_i(f/\ell)$:

$$\frac{d}{dt}\ell_i = -\ell_i\left(\frac{f}{\ell}\right),$$

so that

$$\frac{d}{dt}\left(\frac{f_i}{\ell_i}\right) = \frac{f_i}{\ell_i^2}\ell_i\left(\frac{f}{\ell}\right),$$

and

$$\frac{d}{dt}\left(\ell_i\left(\frac{f}{\ell}\right)\right) = \ell_i\left(\frac{f}{\ell^2}\ell\left(\frac{f}{\ell}\right)\right)$$

$$= \frac{f_{i+1}\ell_{i+1}\left(\frac{f}{\ell}\right)}{\ell_{i+1}^2\sin(\theta_{i+1}-\theta_i)} + \frac{f_{i-1}\ell_{i-1}\left(\frac{f}{\ell}\right)}{\ell_{i-1}^2\sin(\theta_i-\theta_{i-1})} - \frac{f_i\ell_i\left(\frac{f}{\ell}\right)\sin(\theta_{i+1}-\theta_{i-1})}{\ell_i^2\sin(\theta_{i+1}-\theta_i)\sin(\theta_i-\theta_{i-1})}$$

If $\ell_i(f/\ell) + \ell_i/2t$ first reaches zero at some positive time t then

$$\ell_{i+1}(f/\ell) \geq -\ell_{i+1}/2t$$

and

$$\ell_{i-1}(f/\ell) \geq -\ell_{i-1}/2t.$$

It follows that

$$\begin{aligned}\frac{d}{dt}\left(\ell_i\left(\frac{f}{\ell}\right)+\frac{\ell_i}{2t}\right) &\geq -\frac{f_{i+1}}{2t\ell_{i+1}\sin(\theta_{i+1}-\theta_i)} - \frac{f_{i-1}}{2t\ell_{i-1}\sin(\theta_i-\theta_{i-1})}\\ &\quad + \frac{f_i\sin(\theta_{i+1}-\theta_{i-1})}{2t\ell_i\sin(\theta_{i+1}-\theta_i)\sin(\theta_i-\theta_{i-1})}\\ &\quad - \frac{1}{2t}\ell_i\left(\frac{f}{\ell}\right) - \frac{\ell_i}{2t^2}\\ &= -\frac{1}{t}\ell_i\left(\frac{f}{\ell}\right) - \frac{\ell_i}{2t^2}\\ &= -\frac{1}{t}\left(\ell_i\left(\frac{f}{\ell}\right)+\frac{\ell_i}{2t}\right)\\ &= 0.\end{aligned}$$

The Lemma follows by the maximum principle (Proposition 2.1). □

LEMMA 7.3. *For any solution of a symmetric flow of the form* (2) *with* $\alpha = 1$, *there exists* $t_* < T$ *and* $C > 0$ *such that for all* $t_* \leq t < T$,

$$\left|\ell_i - \frac{wf_i}{\sin\theta_i\sum_{j=1}^{k-1}f_j}\right| \leq C\frac{w^3}{A}$$

for $i = 1, \ldots, k-1$, *and*

$$\left|\ell_i - \frac{wf_i}{|\sin\theta_i|\sum_{j=k+1}^{N-1}f_j}\right| \leq C\frac{w^3}{A}$$

for $i = k+1, \ldots, N-1$, *while*

$$\left|\ell_0 - \frac{A}{w} + C_1 w\right| \leq C\frac{w^3}{A}, \qquad \left|\ell_k - \frac{A}{w} + C_1 w\right| \leq C\frac{w^3}{A}$$

where

$$C_1 = \frac{1}{2}\sum_{0<i<j<k}\frac{f_if_j\sin(\theta_j-\theta_i)}{(\sum_{r=1}^{k-1}f_r)^2\sin\theta_i\sin\theta_j} + \frac{1}{2}\sum_{k<i<j<N}\frac{f_if_j\sin(\theta_j-\theta_i)}{(\sum_{r=k+1}^{N-1}f_r)^2\sin\theta_i\sin\theta_j}.$$

Lemma 7.3 has the following nice interpretation: As the solutions contract to points, they become very long and thin, and their asymptotic shape at each of the two ends is that of a curve that evolves purely by translation.

Proof. To deduce estimates on ℓ_i for $1 \leq i \leq k-1$, the estimates from Lemma 7.2 are applied: First note that for $t > T/2$,

$$\begin{aligned} 0 &\leq \ell_1 \left(\frac{f}{\ell}\right) + \frac{\ell_1}{2t} \\ &= \frac{f_2}{\ell_2 \sin(\theta_2 - \theta_1)} + \frac{f_0}{\ell_0 \sin\theta_1} - \frac{f_1 \sin\theta_2}{\ell_1 \sin(\theta_2-\theta_1)\sin\theta_1} + \frac{\ell_1}{2t}. \end{aligned}$$

Rearranging this and applying the estimate for ℓ_0 and the bound $\ell_1 \leq Cw$, one obtains (if necessary choosing t_* sufficiently large to make w^2/A small for $t > t_*$, and noting A approaches zero near the final time)

$$\frac{f_2}{\ell_2 \sin\theta_2} \geq \frac{f_1}{\ell_1 \sin\theta_1} - C\frac{w}{A}.$$

Next an induction argument will be given to show that

$$\frac{f_{j+1}}{\ell_{j+1}\sin\theta_{j+1}} \geq \frac{f_j}{\ell_j \sin\theta_j} - C\frac{w}{A}$$

for $j = 1, \dots, k-2$. The case $j = 1$ is proved above. Suppose it holds for $j = 1, \dots, m$ for some $m < k-2$. Then Lemma 7.2 with $i = m$ gives

$$\begin{aligned} \frac{f_{m+1}}{\ell_{m+1}\sin\theta_{m+1}} &\geq \frac{f_m}{\ell_m}\left(\frac{\cos(\theta_{m+1}-\theta_m)}{\sin\theta_{m+1}} + \frac{\cos(\theta_m - \theta_{m-1})\sin(\theta_{m+1}-\theta_m)}{\sin(\theta_m - \theta_{m-1})\sin\theta_{m+1}}\right) \\ &\quad - \frac{f_{m-1}\sin(\theta_{m+1}-\theta_m)}{\ell_{m-1}\sin(\theta_m-\theta_{m-1})\sin\theta_{m+1}} - \frac{\ell_m \sin(\theta_{m+1}-\theta_m)}{2t\sin\theta_{m+1}} \\ &\geq \frac{f_m}{\ell_m}\left(\frac{\sin(\theta_{m+1}-\theta_{m-1})\sin\theta_m - \sin(\theta_{m+1}-\theta_m)\sin\theta_{m-1}}{\sin\theta_m \sin(\theta_m-\theta_{m-1})\sin\theta_{m+1}}\right) - C\frac{w}{A} \\ &= \frac{f_m}{\ell_m \sin\theta_m} - C\frac{w}{A}, \end{aligned}$$

where the induction hypothesis for $j = m$ was applied to get the second inequality, and the identity

$$\sin B \sin(C - A) = \sin A \sin(C - B) + \sin C \sin(B - A)$$

was used to get the last equality. This completes the induction. The same argument starting with $i = k-1$ and decreasing shows that

$$\frac{f_{j-1}}{\ell_{j-1}\sin\theta_{j-1}} \geq \frac{f_j}{\ell_j \sin\theta_j} - C\frac{w}{A}$$

for $j = 2, \dots, k-1$. It follows that

$$\left|\frac{f_i}{\ell_i \sin\theta_i} - \frac{f_j}{\ell_j \sin\theta_j}\right| \leq C\frac{w}{A}$$

for $1 \leq i, j \leq k-1$. A similar argument applies for $k+1 \leq i, j \leq N-1$. The first two identities of the Lemma now follow from the expressions (14) for w.

The last two identities of the Lemma can now be deduced: Identity (16) gives (using the estimates already proved)

$$|L - A/w + C_1 w| \le C(w^3/A).$$

Then the identity (20) gives (since $f_{k+i} = f_i$ and $\theta_{k+i} = \theta_i + \pi$)

$$|\ell_0 - \ell_k| = \left|\sum_{i\neq 0,k} \ell_i \cos\theta_i\right| = \frac{w}{\sum_{r=1}^{k-1} f_r}\left|\sum_{i=1}^{k-1}\left(\frac{f_i\cos\theta_i}{\sin\theta_i} + \frac{f_i\cos(\theta_i+\pi)}{|\sin(\theta_i+\pi)|}\right)\right| + O\left(\frac{w^3}{A}\right)$$

and the terms in the bracket cancel. The last two identities of Lemma 7.3 follow. □

In the case (2), the previous argument applies starting from the time t_*.

Finally, the proof of part (3) of the Proposition can be completed: It has been shown that all solutions must have isoperimetric ratio becoming unbounded as $t \to T$. Lemma 7.3 implies that for t close to T,

$$\frac{1}{\ell_0} = \frac{w}{A} + C_1\frac{w^3}{A^2} + O\left(\frac{w^5}{A^3}\right),$$

and similarly for $1/\ell_k$. Therefore if $q = w^2/A$ then

$$\begin{aligned}\frac{dq}{dt} &= \frac{w^2}{A^2}\left(\sum_i f_i - 2(f_0 + f_k)\right) + C_1\frac{w^4}{A^3} + O\left(\frac{w^6}{A^4}\right)\\ &= C_1\frac{q^2}{A} + O\left(\frac{q^3}{A}\right).\end{aligned}$$

and so (since $A = \sum_i f_i(T-t)$)

$$\frac{w}{\sum_i f_i\sqrt{\frac{(T-t)}{C_1|\log(T-t)|}}} \to 1$$

as $t \to T$. Asymptotics for each of the lengths $\ell_i(t)$ follow from Lemma 7.3. □

The following example illustrates that those cases omitted from Proposition 7.1 can still be quite complicated.

Consider the case $N = 4$, with $\theta_0 = 0$, $\theta_1 = \theta$, $\theta_2 = \pi$ and $\theta_3 = 2\pi - \theta$. The cases omitted from the theorem are then $\theta \neq \pi/2$, with $f_0 + f_2 = f_1 + f_3$. Take $f_0 = \alpha$, $f_1 = \beta$, $f_2 = 1 - \alpha$ and $f_3 = 1 - \beta$. The geometric constraints imply that $\ell_1 = \ell_3$ and $\ell_2 = \ell_0 + 2\ell_1\cos\theta$, so there are two independent variables ℓ_0 and ℓ_1. These evolve according to the equations

$$\begin{aligned}\frac{d\ell_0}{dt} &= \frac{2\alpha\cos\theta}{\ell_0\sin\theta} - \frac{1}{\ell_1\sin\theta};\\ \frac{d\ell_1}{dt} &= -\frac{1}{\sin\theta}\left(\frac{\alpha}{\ell_0} + \frac{1-\alpha}{\ell_0 + 2\ell_1\cos\theta}\right).\end{aligned}$$

The ratio $r = \frac{\ell_0}{\ell_1}$ evolves according to

$$\frac{dr}{dt} = \frac{2}{r\ell_1^2\sin\theta\left(1 + \frac{2\cos\theta}{r}\right)}\left((2\alpha - 1)\cos\theta + \frac{2\alpha\cos^2\theta}{r}\right).$$

This gives the following types of behaviour: If $\cos\theta = 0$, then $dr/dt = 0$ for any value of r, reflecting the fact that every solution is homothetically shrinking in this case. If $\cos\theta > 0$, then $dr/dt < 0$ for large r when $\alpha > 1/2$, so in this case all solutions converge to the homothetic solution with $r = 2\alpha\cos\theta/(2\alpha - 1)$. However if $\alpha < 1/2$, then $dr/dt > 0$ for large r, so in this case r approaches infinity as the final time is approached, and $\ell_0 \sim \sqrt{(T-t)|\log(T-T)|}$ and $\ell_1 \sim \sqrt{(T-t)/|\log(T-t)|}$. If $\alpha = 1/2$, then $dr/dt > 0$ for large r (for any $\cos\theta \neq 0$), so r approaches infinity as the final time is approached, and $\ell_0 \sim \sqrt{T-t}|\log(T-t)|^{1/4}$ and $\ell_1 \sim \sqrt{T-t}/|\log(T-t)|^{1/4}$.

Examples can be constructed with six edges at equal angles, such that A/w^2 approaches infinity as the final time is approached, with $w \sim \sqrt{T-t}/|\log(T-t)|^{1/6}$. It seems unlikely that a simple criterion can be found in terms of the angles θ_i and the weights f_i which distinguish these more and more extreme cases of 'slow blow-up' from the case where all solutions are asymptotic to homothetically shrinking solutions.

REFERENCES

[A1] B. ANDREWS, *Entropy estimates for evolving hypersurfaces*, Comm. Analysis and Geometry, 2 (1994), pp. 53–64.

[A2] ———, *Monotone quantities and unique limits for evolving convex hypersurfaces*, Int. Math. Res. Not., 20 (1997), pp. 1001–1031.

[A3] ———, *Evolving convex curves*, Calc. Var., 7 (1998), pp. 315–371.

[AG] S. ANGENENT AND M. GURTIN, *Multiphase thermodynamics with interfacial structure 2. Evolution of an isothermal interface*, Arch. Rat. Mech. Anal., 108 (1989), pp. 323–391.

[GG] M.-H. GIGA AND Y. GIGA, *Crystalline and level set flow – Convergence of a crystalline algorithm for a general curvature flow in the plane*, GAKUTO International Series Mathematical Sciences and Applications, 13 (2000), pp. 64–79.

[S1] A. STANCU, *Uniqueness of self-similar solutions for a crystalline flow*, Indiana Univ. Math. J., 45 (1996), pp. 1157–1174.

[S2] ———, *Asymptotic behaviour of solutions to a crystalline flow*, Hokkaido Math. J., 27 (1998), pp. 303–320.

[T1] J. TAYLOR, *Constructions and conjectures in crystalline nondifferential geometry*, in Differential Geometry, B. Lawson and K. Tannenblat, eds., Pitman Monographs in Pure and Applied Math., 1991, pp. 417–438.

[TCH] J. TAYLOR, J. CAHN AND C. HANDWERKER, *Geometric models of crystal growth*, Acta Metallica, 40 (1992), pp. 1443–1474.