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A PRODUCT FORMULA FOR THE SEIBERG-WITTEN INVARIANT ALONG CERTAIN SEIFERT FIBERED MANIFOLDS*

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1. Introduction. Arguably the most important smooth invariant of 4-manifolds known to date is the monopole invariant introduced by Seiberg and Witten. For the definition and basic properties of the Seiberg-Witten invariant we refer the reader to [KM1], [M] and [W]. The most general method of computing the SW-invariant is to cut up the 4-manifold in question into manageable pieces, compute the relative SW-invariants of the individual pieces and then deduce the SW-invariant of the original 4-manifold using some kind of a product formula along the common boundary 3-manifold. This involves solving the 3-dimensional analogue of the Seiberg-Witten equations on the boundary and studying the qualitative behavior of the solutions on an infinite cylinder. This approach fits the general framework of "topological quantum field theory" (TQFT) as outlined in [At].

There has been a flurry of research activities along this line, most notably [FS2], [KM2], [MMS], [MST], [MOY], [OS1] and [OS2]. A horde of new results were obtained as byproducts of the knowledge gained by computing the SW-invariant in this fashion. To give the reader some flavor of these recent applications, we cite the works on the geography problem (e.g. [PS]), the existence of exotic smooth structures (e.g. [FS3], [P1], [P2], [Sz]), and the complete positive resolution of the symplectic Thom conjecture ([OS1]).

In this paper we give a product formula for the Seiberg-Witten invariant of the 4-manifolds that can be gotten by gluing along certain 3-dimensional Seifert fibered spaces. This new product formula will then be used to derive some interesting applications.

2. Solutions over the 3-manifold. For the sake of concreteness, we begin by selecting a particularly 'nice' 3-manifold and analyzing what happens in this special case. Later on we shall indicate how to generalize the results to other 3-manifolds. The 3-manifold Y we study is a Seifert fibered space. We keep the notational conventions in the paper [MOY]. Let Σ be the 2-dimensional orbifold of genus 0, with four marked points, each of which has multiplicity 2. Y is the unit circle bundle of the canonical orbifold bundle K_Σ over Σ. Note that the Seifert invariant of K_Σ is $(-2; 1, 1, 1, 1)$ and $\deg(K_\Sigma) = 0$. We have $H^1(Y) \cong \mathbb{Z}$ and $H^2(Y) \cong \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$. We choose a free generator γ of $H_1(Y)$, i.e. γ generates $H_1(Y)/\mathrm{Tor} \cong \mathbb{Z}$.

Choose a constant curvature connection on the unit circle bundle Y and let $i\zeta$ denote the corresponding connection form. Let g_Σ be a constant curvature metric on the orbifold Σ, normalized so that the area of Σ is equal to one. We endow Y with the metric

$$h_Y = \zeta^2 + \pi^*(g_\Sigma)\,,$$

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where $\pi : Y \to \Sigma$ is the bundle projection map. Note that the tangent bundle TY has an orthogonal splitting

$$TY \cong \mathbb{R} \oplus \pi^*(T\Sigma)\,.$$

Hence the global 1-form ζ allows a reduction in the structure group of TY to $SO(2)$, and the Levi-Civita connection on Σ induces a reducible connection on Y which respects this splitting. We study the moduli space of solutions to the Seiberg-Witten equations over Y, using the above metric and connection on TY.

We consider the following perturbed Seiberg-Witten equations on Y corresponding to a $\mathrm{Spin}^c(3)$ structure W:

$$\begin{aligned} F_A &= \tau(\Psi) + ir\pi^*\mu_\Sigma\,, \\ \not\partial_A \Psi &= 0\,, \end{aligned} \tag{2.1}$$

where $r \in \mathbb{R}$ is a fixed real parameter, μ_Σ is the volume form on Σ, and $\pi : Y \to \Sigma$ is the projection map. Here, $\tau : \Gamma(Y, W) \to \Omega^2(Y, i\mathbb{R})$ is the quadratic map adjoint to the Clifford multiplication.

As in [MOY] (§5.5–5.7), we identify the Seiberg-Witten moduli space with the moduli space of Kähler vortices on Σ. (Contrary to the hypothesis in [MOY], Y has degree zero but the identification is still valid.) In the notation of [MOY], the vortex equations read

$$\begin{gathered} 2F_{B_0} - F_{K_\Sigma} = i(r + |\alpha_0|^2 - |\beta_0|^2)\mu_\Sigma\,, \\ \overline{\partial}_{B_0}\alpha_0 = 0 \quad \text{and} \quad \overline{\partial}^*_{B_0}\beta_0 = 0\,, \\ \alpha_0 \equiv 0 \quad \text{or} \quad \beta_0 \equiv 0\,. \end{gathered} \tag{2.2}$$

Here, B_0 is a connection on a Hermitian orbifold line bundle E_0 over Σ. α_0 and β_0 are orbifold sections in $\Gamma(\Sigma, E_0)$ and $\Gamma(\Sigma, K_\Sigma^{-1} \otimes E_0)$, respectively.

For generic r, we immediately see that there is no reducible solution to the perturbed SW-equations above. For generic negative values of r with $|r|$ very small, there is only one $\mathrm{Spin}^c(3)$ structure on Y for which the corresponding SW-moduli space of irreducible solutions is not empty. This is because we must have, by virtue of vortex equations, $\deg(E_0) = 0$, $\beta_0 \equiv 0$, and $\alpha_0 =$ constant. Thus the canonical $\mathrm{Spin}^c(3)$ structure, $\mathbb{C} \oplus K_\Sigma^{-1}$, is the only $\mathrm{Spin}^c(3)$ structure that has nonempty SW solution space. We denote this $\mathrm{Spin}^c(3)$ structure by $\mathcal{L}_0$. The connections in this solution space correspond to constant sections of the trivial line bundle over S^2, and hence $\mathcal{M}(Y) = \mathcal{M}_{sw}(Y, \mathcal{L}_0, r\pi^*\mu_\Sigma) = \mathrm{Sym}^0(\Sigma) = \{\text{point}\}\,.$

LEMMA 2.1. *The single point set $\mathcal{M}(Y)$ is smooth (non-degenerate), in the sense that it is transversally cut out by the Seiberg-Witten equations* (2.1) *modulo gauge.*

Proof. π^* induces a natural identification between the kernels of the linearizations of the Seiberg-Witten equations on Y and the Kähler vortex equations on Σ (cf. [MOY], §5.6). As shown in [B] and [JT], the moduli space of vortices is always a smooth manifold. The argument is algebro-geometric in nature. Namely, one first identifies the moduli space of vortices with the moduli space of "holomorphic divisors", which consist of pairs $\{\, (B_0, \alpha_0) \mid \overline{\partial}_{B_0}\alpha_0 = 0 \,\}$ modulo the action of maps $\Gamma(\Sigma, \mathbb{C}^*)$.

One then observes that the obstruction cokernel in the linearization (being the first sheaf cohomology group over a zero-dimensional divisor) must vanish, and hence the moduli space of divisors is always cut out transversally. □

For generic small positive values of r, we similarly have $\deg(E_0) = 0$, $\alpha_0 \equiv 0$, and $\beta_0 =$ constant. As in the negative case, the SW-moduli space $\mathcal{M}(Y)$ consists of a single smooth point.

Now suppose that Y is the boundary of some smooth 4-manifold M. Then we get a distinguished subgroup of the gauge group $\mathcal{G}_0(M) \subset \mathcal{G}(Y)$ which consists of maps u that can be extended to $u : M \to U(1)$. Dividing out by the action of $\mathcal{G}_0(M)$ instead of the full gauge group, we obtain another moduli space $\widetilde{\mathcal{M}}(Y)$. Of course this moduli space depends on M. Note that dividing by $\mathcal{G}(Y)$ gives a covering $p : \widetilde{\mathcal{M}}(Y) \to \mathcal{M}(Y)$. The fiber of p is $H^1(Y;\mathbb{Z})/i^*(H^1(M;\mathbb{Z}))$, where $i : Y \hookrightarrow M$ is the inclusion map.

3. Moduli space for a cylindrical end manifold. First we consider the case of the cylinder $Y \times \mathbb{R}$. Given a Spin^c structure on $Y \times \mathbb{R}$, let W^+ and W^- be the associated Spin^c bundles. Clifford multiplication defines a linear map

$$\rho : i\Lambda^2 \to \mathrm{End}_{\mathbb{C}}(W^+),$$

whose kernel is $i\Lambda^-$. Here $\Lambda^- \subset \Lambda^2$ is the subbundle of anti-self-dual (ASD) 2-forms with respect to the metric $h_Y + dt^2$. We denote $L = \det(W^+)$ and write $\mathcal{A}(L)$ for the affine space of connections on L. We pull back the perturbing form on Y of the previous section and get the following 4-dimensional Seiberg-Witten equations for a pair $(A, \phi) \in \mathcal{A}(L) \times \Gamma(W^+)$:

$$\partial\!\!\!/_A \phi = 0, \tag{3.1}$$

$$\rho(F_A - ir\pi_1^*\pi^*\mu_\Sigma) = q(\phi) = \phi^* \otimes \phi - \frac{|\phi|^2}{2}\mathrm{Id}, \tag{3.2}$$

where $\pi_1 : Y \times \mathbb{R} \to Y$ is the projection map. We identify $L = L_0 \times \mathbb{R}$, where L_0 is a complex line bundle over Y. Similarly, $W^+ = W_0 \times \mathbb{R}$, where W_0 is the Spin^c bundle over Y with respect to the Spin^c structure inherited from $Y \times \mathbb{R}$. As shown in [KM1], the Equations (3.1) and (3.2) then become the gradient flow equation for the Chern-Simons-Dirac functional $C : \mathcal{A}(L_0) \times \Gamma(W_0) \to \mathbb{R}$, given by

$$C(A,\phi) = \int_Y (F_{A_0} + \xi) \wedge a + \frac{1}{2}\int_Y a \wedge da + \int_Y \langle \phi, \partial\!\!\!/_A \phi \rangle dvol,$$

where $\xi = -ir\pi_1^*\pi^*\mu_\Sigma$, A_0 is a fixed connection on L_0, and $a = A - A_0$. Note that there is an ambiguity up to a constant, made necessary by the fact that $\mathcal{A}(L_0)$ has no preferred base point in general.

Let X be a compact smooth 4-manifold whose boundary is Y. Assume that the 2-form $ir\pi^*\mu_\Sigma$ on Y extends to a closed 2-form on X. Then C descends to a real-valued function on the space $\widetilde{\mathcal{B}} := (\mathcal{A}(L_0) \times \Gamma(W_0))/\mathcal{G}_0(X)$, where $\mathcal{G}_0(X) \subset \mathcal{G}(Y)$ is as in the previous section. From now on we shall always view C as a functional on $\widetilde{\mathcal{B}}$. Note that the set of critical points of C is the moduli space $\widetilde{\mathcal{M}}(Y)$ of the previous section.

Now we further perturb Equation (3.2) using a method due to Frøyshov. We briefly recall the necessary definitions from [Fr]. Let $f_1 : \mathbb{R} \to [0, \infty)$ be a smooth function supported in the interval $[-1,1]$ and satisfying $\int f_1 = 1$. Let $f_2 : \mathbb{R} \to \mathbb{R}$ be a smooth function with compact support such that $f_2(t) = t$ on some interval containing all critical values of C. If A is any connection on L and ϕ a section of W^+, we let $S = (A, \phi)$ and define a smooth function $h_S : \mathbb{R} \to \mathbb{R}$ by

$$h_S(T) = \int_{\mathbb{R}} f_1(t_1 - T) f_2\Big(\int_{\mathbb{R}} f_1(t_2 - t_1) C(S_{t_2}) dt_2\Big) dt_1,$$

where $S_t = S(t)$ is the restriction of S to $Y \times \{t\}$.

We choose a compactly supported 2-form $\omega \in \Omega^2(Y \times \mathbb{R})$ such that $||\omega||$ is very small. We require the support of ω to lie in a set $Y \times \Xi$, where Ξ is the result of removing from $\mathbb{R}$ a small open interval around each critical value of C. Let $h_S^*(\omega)$ denote the pull-back of ω by the map $(\mathrm{id}_Y \times h_S) : Y \times \mathbb{R} \to Y \times \mathbb{R}$. We study the following translation invariant equations for $S = (A, \phi)$:

$$\partial\!\!\!/_A \phi = 0, \tag{3.3}$$

$$\rho(F_A - ir\pi_1^* \pi^* \mu_\Sigma + ih^*_{(A,\phi)}(\omega)) = q(\phi). \tag{3.4}$$

LEMMA 3.1. ([Fr]) *Let $S = (A, \phi)$ be any smooth solution to the Equations* (3.3) *and* (3.4) *satisfying a pointwise bound $|\phi| \leq B$ for some constant B. Then*

(i) *Either $\frac{\partial}{\partial t} C(S_t) > 0$ for all t, or $[S_t] \equiv x$ for some critical point x.*

(ii) *If $C(S_t)$ is bounded in t, then there are critical points x_+, x_- of C such that the gauge equivalence class $[S_t]$ converges to $x_\pm$ as $t \to \pm\infty$.*

Proof. Here $[S_t]$ denotes the image of the restriction $S(t)$ in the configuration space $\widetilde{\mathcal{B}}$. Although Frøyshov only concentrates on the case when the 3-manifold Y is an oriented rational homology sphere, the proof in [Fr] (Appendix A) still goes through with very little modification. Note that our Chern-Simons-Dirac functional C is the negative of the one used by Frøyshov. □

Now we choose our Sobolev spaces and let $\mathcal{B} = L^2_4(Y \times \mathbb{R}, i\Lambda^1 \oplus W^+)$ and

$$\mathcal{G} = \{u : Y \times \mathbb{R} \to U(1) \,|\, u \in L^2_{5,\mathrm{loc}}\}.$$

Let $x, y \in \widetilde{\mathcal{M}}(Y)$ be critical points of C, i.e. solutions to (2.1), the perturbed Seiberg-Witten equations on Y. We define the space of "flowlines" on the cylinder between x and y to be the set

$$\mathcal{F}_\omega(x,y) = \{S \in \mathcal{B} \text{ satisfying (3.3) and (3.4)} \,|\, \lim_{t \to -\infty} [S_t] = x \,;\, \lim_{t \to \infty} [S_t] = y\}/\mathcal{G}.$$

Note that the elements of $\mathcal{F}_\omega(x,y)$ satisfy the gradient flow equation for C outside a compact subset of $Y \times \mathbb{R}$. We show that there are no nontrivial flowlines in the following

LEMMA 3.2. *For generic small $\omega \in C^k$, $\mathcal{F}_\omega(x,x)$ consists of a single smooth point, and $\mathcal{F}_\omega(x,y)$ is empty when $x \neq y$.*

Proof. The first statement follows immediately from Lemma 3.1. Let Ω^2_Ξ denote the space of C^k 2-forms on $Y \times \mathbb{R}$ with compact support contained in $Y \times \Xi$. As in [Fr] (Proposition 5), one can show that the linearization of Equations (3.3) and (3.4) at a point (ω, A, ϕ),

$$F = F_{(\omega,A,\phi)} : \Omega^2_\Xi \times \mathcal{B} \longrightarrow L^2_3(Y \times \mathbb{R},\, i\Lambda^0 \oplus i\Lambda^+ \oplus W^-),$$

is Fredholm on the slices $\{\omega\} \times \mathcal{B}$, and surjective whenever (ω, A, ϕ) is a solution to Equations (3.3) and (3.4). Thus Smale-Sard theorem implies that $\mathcal{F}_\omega(x, y)$ is a smooth manifold for generic ω.

Now we let $\mathcal{P} = \mathcal{P}(0, \omega)$ to be the space of L^2_1 maps,

$$\nu : [0,1] \longrightarrow L^2_3(Y \times \mathbb{R}, \Lambda^2),$$

satisfying $\nu(0) = 0$ and $\nu(1) = \omega$. We define a map

$$G : \mathcal{B} \times \mathcal{P}(0, \omega) \times [0,1] \longrightarrow L^2_3(Y \times \mathbb{R}, \ i\Lambda^+ \oplus W^-)$$

by

$$G(a, \phi, \nu, t) = (F^+_{A_0+a} + \eta(t)^+ - \tau(\phi), \ \partial\!\!\!/_{A_0+a}(\phi)),$$

where $\eta(t) = ih^*_{(A_0+a,\phi)}(\nu(t)) - ir\pi_1^*\pi^*\mu_\Sigma$. One can show that the differential DG is surjective at every point (a, ϕ, ν, t) for which G vanishes. Let $\mathbf{M}$ denote the zero set $G^{-1}(0)$ modulo the L^2_5 gauge transformations. Let $\mathbf{F}$ be a generic fiber of the projection, $\mathbf{M} \to \mathcal{P}$, onto the second factor. Note that the boundary of $\mathbf{F}$ consists of two ends, one of which is cut out by the gradient of the Chern-Simons-Dirac functional C (whose linearization always has index zero on the critical set). Thus the corresponding boundary components have *expected* dimension zero. It follows that $\mathbf{F}$ is a 1-dimensional smooth manifold with boundary. Consequently the expected dimension of the space of "flowlines" *modulo* the L^2_5 gauge transformations has to be zero. But recall from [Fr] that the solutions to Equations (3.3) and (3.4) are translation invariant in the $\mathbb{R}$-direction. Hence we conclude that the expected (or virtual) dimension of $\mathcal{F}_\omega(x, y)$ is (-1), which implies that $\mathcal{F}_\omega(x, y)$ is empty for generic ω. □

Now suppose that M is a smooth oriented 4-manifold, and that the end of M is diffeomorphic to $Y \times [0, \infty)$. Assume that the intersection form of M is not negative definite, and that the end perturbation $\xi = -ir\pi_1^*\pi^*\mu_\Sigma$ extends to a closed 2-form $\overline{\xi}$ over the whole manifold M. Fix a Riemannian metric h on Y as in the previous section, and a Riemannian metric g on M such that g is equal to $h + dt^2$ at the cylindrical end of M. We look at the perturbed SW-equations:

$$\partial\!\!\!/_A \phi = 0,$$

$$\rho(F_A + \eta) = q(\phi), \tag{3.5}$$

$$\eta = f \cdot (ih^*_{(A,\phi)}(\omega) - ir\pi_1^*\pi^*\mu_\Sigma),$$

where $f : M \to [0,1]$ is a suitable cut-off function that vanishes away from the cylindrical end of M. Note that the perturbing 2-form η depends on the unknowns (A, ϕ). We shall write $\eta = \eta^{[r]}$ or $\lceil \eta \rceil = r$ to emphasize the dependence on the parameter r. Similarly, we write ξ_r and $\overline{\xi}_r$ to accentuate the parameter r.

We require our configuration (A, ϕ) to lie in $\mathcal{A}_{L^2_4}(\det \mathcal{L}) \times L^2_4(M, W^+(\mathcal{L}))$, where $\mathcal{A}_{L^2_4}(\det \mathcal{L})$ denotes the space of L^2_4 unitary connections on the line bundle $\det \mathcal{L}$, and $W^+(\mathcal{L})$ is the positive spinor bundle for the Spin^c structure $\mathcal{L}$. The *energy* of a solution (A, ϕ) is defined to be the total variation of the Chern-Simons-Dirac functional C over the cylindrical end $Y \times [0, \infty)$. The cylindrical end moduli space

$\mathcal{M}_M(\mathcal{L}, g, \eta)$ is defined by dividing the space of finite energy solutions by the action of the $L^2_{5,\mathrm{loc}}$ gauge group $\mathcal{G}(M)$. Note that every solution to (3.5) is irreducible, i.e. $\phi \not\equiv 0$. We say that a solution (A, ϕ) is *δ-confined* if

$$\int_{Y\times[0,\infty)} F_A \wedge (-ir\pi_1^*\pi^*\mu_\Sigma) \ < \ \delta \, .$$

Let $\mathcal{M}_M^\delta(\mathcal{L}, g, \eta)$ denote the space of δ-confined finite energy solutions modulo the action of $\mathcal{G}(M)$. Note that $\mathcal{M}_M^\delta(\mathcal{L}, g, \eta)$ is an open subspace of $\mathcal{M}_M(\mathcal{L}, g, \eta)$ (since $\int F_A \wedge \xi$ is a continuous function on $\mathcal{M}_M(\mathcal{L}, g, \eta)$), and that $\varinjlim \mathcal{M}_M^\delta(\mathcal{L}, g, \eta) = \mathcal{M}_M(\mathcal{L}, g, \eta)$. We shall only be interested in the cases when $\delta \gg 0$.

LEMMA 3.3. *If $\mathcal{L}$ does not restrict to $\mathcal{L}_0$ on the slice Y, then $\mathcal{M}_M(\mathcal{L}, g, \eta)$ is empty. If $\mathcal{L}|_Y = \mathcal{L}_0$, then $\mathcal{M}_M(\mathcal{L}, g, \eta)$ is a smooth oriented manifold of dimension*

$$d = d(\mathcal{L}) = \frac{1}{4}(c_1(\det \mathcal{L})^2 - 2\mathrm{e}(M) - 3\mathrm{sign}(M)).$$

Moreover, by taking limits at the open non-compact end of the infinite cylinder $Y \times [0, \infty)$, we have a continuous map

$$\partial_\infty : \mathcal{M}_M(\mathcal{L}, g, \eta) \to \widetilde{\mathcal{M}}(Y).$$

For each point $x \in \widetilde{\mathcal{M}}(Y)$, the preimage $\partial_\infty^{-1}(x)$ is compact. There is a constant $v_r > 0$ such that every solution $[(A, \phi)] \in \partial_\infty^{-1}(x)$ decays exponentially to x with exponent at least v_r, i.e. the L^2_4 distance between x and the restriction $(A(t), \phi(t))$ is less than $\exp(-v_r t)$ for all t large.

Proof. Suppose that $[(A, \phi)] \in \mathcal{M}_M(\mathcal{L}, g, \eta)$. Since (A, ϕ) has finite energy, it follows from [KM1] (Proposition 8) that $\mathcal{M}(Y, \mathcal{L}|_Y)$ is not empty. Now the results from Section 2 imply that $\mathcal{L}|_Y = \mathcal{L}_0$. The smoothness of $\mathcal{M}_M(\mathcal{L}, g, \eta)$ for a small generic 2-form ω follows from what is now a 'standard' argument which we choose to omit. As in the closed case, the homology orientation of the pair $(M, \partial M)$ induces an orientation of $\mathcal{M}_M(\mathcal{L}, g, \eta)$. The existence of the continuous map ∂_∞ follows from the arguments in [MMR] (pp. 63–70), which rely on the basic analytic results in [Si]. Given a point $x \in \widetilde{\mathcal{M}}(Y)$, we can calculate the formal dimension of $\partial_\infty^{-1}(x)$, and hence of $\mathcal{M}_M(\mathcal{L}, g, \eta)$, by the index formula of [APS], which gives

$$\dim(\partial_\infty^{-1}(x)) = \dim \mathcal{M}_M(\mathcal{L}, g, \eta) = \frac{1}{4}(c_1(\det \mathcal{L})^2 - 2\mathrm{e}(M) - 3\mathrm{sign}(M)) \, .$$

Note that the eta invariant (or rho invariant) of the linearization of (2.1) on Y is zero. (This is because Y admits an orientation reversing self-diffeomorphism and $\mathrm{eta}(-Y) = -\mathrm{eta}(Y)$.) Lemma 3.2 implies that every finite energy flowline over the cylinder $Y \times \mathbb{R}$ is *static*, i.e. pulls back from Y. Hence the arguments in [KM1] (Lemma 4) imply that the preimage $\partial_\infty^{-1}(x)$ is compact. The statement about exponential decay can be proved as in [MMR] (Chapter 5). □

We introduce the notation $N = Y \times [0, \infty)$ and for any $\ell > 0$ we let $N_\ell = Y \times [0, \ell]$. We let $Y_0 = Y \times \{0\}$ and $Y_\ell = Y \times \{\ell\}$.

LEMMA 3.4. *There is a constant $K > 0$ depending only on M, g and η such that*

$$-K - 2\int_{N_\ell} \xi \wedge F_A \ \le \ \int_{N_\ell} F_A \wedge F_A \ \le \ K$$

holds for all ℓ sufficiently large and every $[(A,\phi)] \in \mathcal{M}_M(\mathcal{L},g,\eta)$ for any Spin^c *structure $\mathcal{L}$.*

Proof. From the dimension formula in Lemma 3.3, $\mathcal{M}_M(\mathcal{L},g,\eta)$ is not empty only if

$$c_1(\det\mathcal{L})^2 \geq 2\mathrm{e}(M) + 3\mathrm{sign}(M)\,.$$

For every $[A,\phi] \in \mathcal{M}_M(\mathcal{L},g,\eta)$, we have

$$\|F_A^+\|^2 - \|F_A^-\|^2 = -\int_{M-N} F_A \wedge F_A \;=\; 4\pi^2 c_1(\det\mathcal{L})^2 + \int_N F_A \wedge F_A\,,$$

where $\|\cdot\|$ denotes the L^2-norm on the compact manifold $M-N$. As in the closed case, we have

$$\sup_M |\phi|^2 \;\leq\; \sup_M \left(4\sqrt{2}|\eta| - s\right)\,,$$

where s denotes the scalar curvature. Therefore there is an upper bound on $\|F_A^+\|^2$ independent of the Spin^c structure. Since

$$\int_N F_A \wedge F_A \;\leq\; \|F_A^+\|^2 - 4\pi^2 c_1(\det\mathcal{L})^2\,,$$

we get an upper bound on $\int_N F_A \wedge F_A$.

Next let S_t denote the restriction of (A,ϕ) to the slice $Y \times \{t\}$. By Stokes' theorem we have

$$C(S_\ell) - C(S_0) = \frac{1}{2}\int_{N_\ell} F_A \wedge F_A + \int_{N_\ell} \xi \wedge F_A + \int_{Y_\ell} \langle\phi, \partial\!\!\!/_A \phi\rangle - \int_{Y_0} \langle\phi, \partial\!\!\!/_A \phi\rangle\,.$$

From Lemma 3.1 (i), we can assume that

$$C(S_\ell) - C(S_0) \geq 0\,.$$

Moreover, there is a universal bound depending only on the Riemannian manifold M and the perturbation η to both

$$\int_{Y_\ell} \langle\phi, \partial\!\!\!/_A \phi\rangle \qquad \text{and} \qquad \int_{Y_0} \langle\phi, \partial\!\!\!/_A \phi\rangle\,,$$

which is gotten from the usual *a priori* pointwise bound on $|\phi|$ and the L^2-bound on $\nabla\phi$. Our claim now follows easily. □

REMARK 3.5. Note that the above lemma also gives a lower bound on the integral $\int_{N_\ell} \xi \wedge F_A$, namely $(-K)$, that is independent of the Spin^c structure.

4. Relative SW-invariant. Let X be a smooth oriented compact 4-manifold with boundary, and suppose that ∂X is diffeomorphic to Y. Recall that there is a well-defined bilinear pairing $H^2(X;\mathbb{Z}) \otimes H^2(X,\partial X;\mathbb{Z}) \to \mathbb{Z}$. The associated quadratic form, which is given by the composition with the inclusion-induced homomorphism $H^2(X,\partial X;\mathbb{Z}) \to H^2(X;\mathbb{Z})$, is the intersection form on $H^2(X,\partial X;\mathbb{Z})$. Let $b_2^+(X) = \dim H^2_{\geq 0}(X,\partial X;\mathbb{R})$, i.e. the dimension of the maximal subspace of $H^2(X,\partial X;\mathbb{R})$ on which the intersection form is positive semi-definite. (A more standard notation would be $b_2^{\geq 0}(X)$, but for future applications we want to synchronize our notation with the closed case.) Let $\mathcal{K}(X)$ denote the set of isomorphism classes of Spin^c structures on X that restrict to $\mathcal{L}_0$ on ∂X. We define the corresponding non-compact cylindrical end manifold $M := X \cup_Y Y \times [0,\infty)$, and choose a cylindrical end metric g on M. (Sometimes we shall denote such M by $\widehat{X}$.) The goal of this section is to define the relative Seiberg-Witten invariant

$$SW_X : \mathcal{K}(X) \times \widetilde{\mathcal{M}}(\partial X) \longrightarrow \mathbb{Z}$$

using moduli spaces over M. Given $\mathcal{L} \in \mathcal{K}(X)$, we continue to denote the corresponding Spin^c structure on M by $\mathcal{L} \to M$. Let $\mathcal{M}_M(\mathcal{L}, g, \eta)$ be the cylindrical end moduli space of the previous section. Now suppose that $d(\mathcal{L}) \equiv 0 \pmod 2$. We take a geometric representative D of $\mu(pt)^{d/2}$, and define

$$\mathcal{N}_X(\mathcal{L}, x, g, \omega, D) := \mathcal{M}_M(\mathcal{L}, g, \eta) \cap D \cap \partial_\infty^{-1}(x).$$

Note that D is a generic d-codimensional stratified set in the space of configurations, where we can choose D to be supported in a small neighborhood of the base fibration point. For the definition and properties of the μ map, we refer the reader to the last section of [OS2].

DEFINITION 4.1. *Let X, M, $\mathcal{L}$, g, η be as above. Then for a generic D, $\mathcal{N}_X(\mathcal{L}, x, g, \omega, D)$ is a compact oriented 0-dimensional manifold, and by counting its points with signs, we define*

$$SW_X(\mathcal{L}, x) := \#(\mathcal{N}_X(\mathcal{L}, x, g, \omega, D)).$$

If $d(\mathcal{L}) \equiv 1 \pmod 2$, then we define $SW_X(\mathcal{L}, x) = 0$. As in the closed case, we say that X is of simple type when $SW_X(\mathcal{L}, x) \neq 0$ only if $d(\mathcal{L}) = 0$.

DEFINITION 4.2. *Similarly, we can define the δ-confined relative Seiberg-Witten invariant*

$$SW_X^\delta : \mathcal{K}(X) \times \widetilde{\mathcal{M}}(\partial X) \longrightarrow \mathbb{Z}$$

by substituting the δ-confined cylindrical end moduli space $\mathcal{M}_M^\delta(\mathcal{L}, g, \eta)$ in the place of $\mathcal{M}_M(\mathcal{L}, g, \eta)$ in the definitions above.

THEOREM 4.3. *If $b_2^+(X) > 1$, then SW_X is independent of g and D. We have $SW_X^\delta(\mathcal{L}, \cdot) = 0$ for all but finitely many $\mathcal{L} \in \mathcal{K}(X)$. Furthermore, for any orientation preserving self-diffeomorphism $f : X \to X$, we have*

$$SW_X(\mathcal{L}, x) = (-1)^\epsilon SW_X(f^*(\mathcal{L}), f^*(x)),$$

where $\epsilon \in \mathbb{Z}/2$ is the sign of the action of f^* on the cohomology orientation of the pair $(X, \partial X)$.

Proof. From the dimension formula in Lemma 3.3, $\mathcal{M}_M^\delta(\mathcal{L}, g, \eta)$ is not empty only if

$$c_1(\det \mathcal{L})^2 \geq 2\mathrm{e}(M) + 3\mathrm{sign}(M).$$

For every $[A, \phi] \in \mathcal{M}_M^\delta(\mathcal{L}, g, \eta)$, we have

$$\|F_A^+\|^2 - \|F_A^-\|^2 = -\int_X F_A \wedge F_A = 4\pi^2 c_1(\det \mathcal{L})^2 + \int_N F_A \wedge F_A,$$

where $\|\cdot\|$ denotes the L^2-norm on the compact manifold X. It follows that there is a universal constant, $K_0 = K + 2\delta - 4\pi^2(2\mathrm{e}(M) + 3\mathrm{sign}(M))$, independent of the Spin^c structure such that

$$\|F_A^-\|^2 - \|F_A^+\|^2 \leq K_0 .$$

Here K is the constant found in Lemma 3.4. As before, we have an estimate

$$\|F_A^+\|^2 \leq K_1$$

for a suitable constant K_1, and we can conclude that

$$\|F_A\|^2 \leq K_0 + 2K_1 .$$

Hence there are only finitely many values of $c_1(\det \mathcal{L}) \in H^2(X;\mathbb{Z})$ for which the corresponding δ-confined SW-moduli space is nonempty. Since for any $c \in H^2(X;\mathbb{Z})$ there are only finitely many Spin^c structures $\mathcal{L}$ with $c_1(\det \mathcal{L}) = c$, the function SW_X^δ has finite support in the first factor $\mathcal{K}(X)$. The rest of the statements can also be proved exactly the same way as in the closed case. □

As in the closed case, one has to worry about the chamber structures in the auxiliary space of parameters when $b_2^+(X) = 1$. Let $\mathfrak{M}$ denote the space of Riemannian metrics on X, and $\mathfrak{M}_0$ the subspace of metrics on X that restrict to h_Y on the boundary Y. Note that $\mathfrak{M}_0$ is of infinite codimension in $\mathfrak{M}$. Let $H_{\geq 0}$ denote the image of the relative cohomology $H^2_{\geq 0}(X, \partial X; \mathbb{R})$ in the absolute cohomology group $H^2(X;\mathbb{R})$. We have $H_{\geq 0} \cong \mathbb{R}$, since we are always assuming that the intersection form of X is not negative definite. Let $p^+ : H^2(X;\mathbb{R}) \to H_{\geq 0}$ be the projection map. Let ϖ_g be the unique g-harmonic L^2 real 2-form on $M = X \cup_Y Y \times [0,\infty)$ that has L^2 norm 1 and corresponds to a generator of $H_{\geq 0}$ with the chosen orientation (cf. Proposition (4.9) in [APS]). Given $\mathcal{L} \in \mathcal{K}(X)$, we define a function $\varepsilon_{\mathcal{L}} : \mathfrak{M} \times i\Omega^{2,+}(M) \to \mathbb{R}$ by

$$\varepsilon_{\mathcal{L}}(g,\eta) = -\int_M \langle i\eta, \varpi_g \rangle dvol_g - 2\pi \frac{c_1(\det \mathcal{L})}{[\varpi_g]} \quad ,$$

where $p^+(c_1(\det \mathcal{L})) = a[\varpi_g]$, and $a = (c_1(\det \mathcal{L})/[\varpi_g])$.

Using the standard cobordism argument (cf. [Sa]), one can show that for any pair of triples $(g,\omega,r), (g',\omega',r') \in \mathfrak{M}_0 \times \Omega^2_{\Xi} \times \mathbb{R}$ with $\|\omega\|, \|\omega'\|, |r|$ and $|r'|$ all sufficiently small and non-zero,

$$\#(\mathcal{N}_X(\mathcal{L}, x, g, \omega, D)) = \#(\mathcal{N}_X(\mathcal{L}, x, g', \omega', D))$$

holds whenever $\varepsilon_{\mathcal{L}}(g, \overline{\xi}_r)$ and $\varepsilon_{\mathcal{L}}(g', \overline{\xi}_{r'})$ have the same sign. Thus we can define the function $SW_X^+(\mathcal{L}, x)$ to be the number $\#(\mathcal{N}_X(\mathcal{L}, x, g, \omega, D))$ for any generic triple $(g,\omega,r) \in \mathfrak{M}_0 \times \Omega^2_{\Xi} \times \mathbb{R}$ for which $\|\omega\|, |r|$ are extremely small and $\varepsilon_{\mathcal{L}}(g, \overline{\xi}_r) > 0$. Similarly, we define $SW_X^-(\mathcal{L}, x)$ using any generic triple of parameters $(g,\omega,r) \in \mathfrak{M}_0 \times \Omega^2_{\Xi} \times \mathbb{R}$ for which $\|\omega\|, |r|$ are small and $\varepsilon_{\mathcal{L}}(g, \overline{\xi}_r) < 0$. As a convention, we shall henceforth let $SW_X \equiv SW_X^+$ when $b_2^+(X) = 1$, for both closed and non-closed cases.

REMARK 4.4. As a consequence of the wall-crossing formula, the function SW_X may well have infinite support in the variable $\mathcal{L}$ when $b_2^+(X) = 1$. We shall see later that SW_X actually depends on the sign of the parameter r in the perturbation.

5. Orientation. In this short section we establish the orientation and sign conventions that will be used throughout the next section and beyond. Let X be as in the previous section. Instead of attaching $Y \times [0, \infty)$ to X and forming the cylindrical end manifold $\widehat{X}$, we can attach $Y \times (-\infty, 0]$ and form another cylindrical end manifold $\overleftarrow{X} := Y \times (-\infty, 0] \cup_Y X$. Consider the moduli space $\mathcal{M}_{\overleftarrow{X}}$ of solutions to the perturbed Seiberg-Witten equations (3.5) on $\overleftarrow{X}$. This is easily seen to be diffeomorphic to the SW-moduli space for $\widehat{X}$ corresponding to the perturbation $\eta^{[-r]}$. In other words, $\mathcal{M}_{\overleftarrow{X}}(\eta^{[r]}) \cong \mathcal{M}_{\widehat{X}}(\eta^{[-r]})$. In what follows we shall be looking at a smooth closed 4-manifold M that can be decomposed as $M = M_1 \cup M_2$ such that $M_1 \cap M_2 = Y$. We will compare the SW-moduli spaces over M, $\widehat{M_1}$ and $\overleftarrow{M}_2$. Just as in (3.5) we perturb the standard SW-equations on M by a 2-form that depends on a real parameter r. Let r_1, r_2 denote the parameters in the perturbing 2-forms η_1, η_2 over $\widehat{M_1}$ and $\overleftarrow{M}_2$ respectively. We shall *always* choose $r_1 = r_2 = r$. This means that when we actually evaluate SW_{M_2}, we are computing with moduli spaces over $\widehat{M_2}$ corresponding to the parameter $-r_2$.

Now suppose $M_2 \cong M_1$. Once we choose orientations for M and Y, we obtain the induced orientations on M_j and we have $\partial M_2 = -\partial M_1$. Let $\mathcal{L}$ be a Spin^c structure on M_1 that restricts to $\mathcal{L}_0$ on Y. Then $SW_{M_1}(\mathcal{L}, x) \neq 0$ implies that

$$SW_{M_2}(-\mathcal{L}, \epsilon x) = \pm SW_{M_1}(\mathcal{L}, x) \neq 0 ,$$

where the sign $\epsilon = \pm 1$ depends on the action of the actual identification of the boundaries. Here $-\mathcal{L}$ denotes the Spin^c structure on M_2 that is the "reflection" of $\mathcal{L}$ along Y. Finally, we refer the reader to § 9.1 of [MST] for the way in which the (co)homology orientations of M, $(M_1, \partial M_1)$ and $(M_2, \partial M_2)$ fit together in general.

6. The product formula. Let M_j ($j = 1, 2$) be a smooth compact oriented 4-manifold with boundary $\partial M_j = Y$. For any orientation reversing self-diffeomorphism $\varphi : Y \to Y$, we define a closed oriented 4-manifold $M(\varphi) = M_1 \cup_\varphi M_2$. Let $(i_j)_* : H_1(Y) \to H_1(M_j)$ be the homomorphism induced by the inclusion map. From this moment on, we assume that $\gamma \in \mathrm{Ker}(i_j)_*$ for $j = 1, 2$. This assumption implies, via the Mayer-Vietoris sequence, that $(i_j)^* : H^1(M_j) \to H^1(Y)$ is the zero homomorphism. (For emphasis and future reference we shall say that such $M(\varphi)$ and M_j satisfy Condition (A).) Under such assumption, we can choose $b_j \in H_2(M_j, \partial M_j)$ such that $\partial b_1 = \gamma$ and $\partial b_2 = \varphi(\gamma)$. Let $\Sigma_j \subset M_j$ be a smoothly embedded surface with boundary, representing b_j. Let $\hat{b}_j$ denote the dual element in $H^2(M_j)$.

Let $\mu \in H^2(M(\varphi); \mathbb{Z})$ denote the cohomology class Poincaré dual to the homology class represented by the smooth surface $(\Sigma_1 \cup_\gamma \Sigma_2)$ in $M(\varphi)$. We have $(\iota_j)^*(\mu) = \hat{b}_j$, where $\iota_j : M_j \hookrightarrow M(\varphi)$ is the inclusion map. Note that $i^*(\mu) = \pi^*([\mu_\Sigma])$ inside the group $H^2(Y; \mathbb{Z})$, where $i : Y \hookrightarrow M(\varphi)$ denotes the inclusion map and $[\mu_\Sigma] \in H^2(\Sigma; \mathbb{Z})$ is the integral cohomology class represented by the volume form of the orbifold Σ as in Section 2. In what follows, we shall abuse the notation somewhat and use $[\mu_\Sigma]$ to denote the cohomology class $\mu \in H^2(M(\varphi); \mathbb{Z})$.

We define a family of metrics on $M(\varphi)$ as follows. First we have the decomposition

$$M(\varphi) \cong M_1 \cup Y \times [-1, 1] \cup M_2 .$$

Suppose we are given a metric g on $M(\varphi)$ that is of the form $h + dt^2$ on the neck $Y \times [-1, 1]$, where h is a metric on Y as in Section 2. For each $\ell \geq 1$, let $\lambda_\ell(t)$ be a

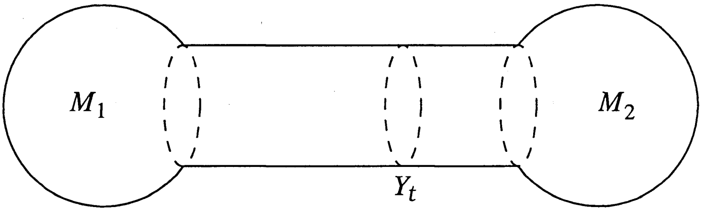

FIG. 6.1.

positive smooth function on $[-1,1]$ which is identically equal to one on $[-1,-1/2] \cup [1/2,1]$ and satisfies

$$\int_{-1}^{1} \lambda_\ell(t)dt = 2\ell .$$

We define a metric g_ℓ to be g on the two ends $X = M_1 \cup M_2$, and $h + \lambda_\ell(t)^2 dt^2$ along the neck $Y \times [-1,1]$. One should think of the family $\{g_\ell\}$ as stretching out the neck $Y \times [-1,1]$ isometrically into $T_\ell = Y \times [-\ell, \ell]$. We denote the Riemannian manifold $(M(\varphi), g_\ell)$ by $M(\varphi)_\ell$.

Next we construct a family of perturbing 2-forms that are supported on the neck T_ℓ . As in Section 3, we choose a compactly supported 2-form $\omega \in \Omega^2(Y \times \mathbb{R})$ such that $||\omega||_{L^2_k}$ is very small. Let $k_\ell : T_\ell \hookrightarrow Y \times \mathbb{R}$ be the inclusion map. Let W_0 denote the Spin^c bundle over Y corresponding to $\mathcal{L}_0$ and let $L_0 = \det \mathcal{L}_0 = \det W_0$. As in Section 3, we let $W^+ = W_0 \times \mathbb{R}$ and $L = \det W^+$. Suppose $\mathcal{L}$ is a Spin^c structure on $M(\varphi)$ that restricts to $\mathcal{L}_0$ on Y and $W^\pm(\mathcal{L})$ are the associated Spin^c bundles. Given a pair $(A,\phi) \in \mathcal{A}(\det \mathcal{L}) \times \Gamma(W^+(\mathcal{L}))$, we define the "push-forward" $(k_\ell)_*(A,\phi) \in \mathcal{A}(L) \times \Gamma(W^+)$ as follows. We extend the restriction $(A,\phi)|_{T_\ell}$ over the whole infinite cylinder $Y \times \mathbb{R}$ by constants, i.e.

$$(k_\ell)_*(A,\phi)\,|_{Y\times\{t\}} = \begin{cases} (A,\phi)|_{Y\times\{-\ell\}} & \text{if } t \le -\ell\,, \\ (A,\phi)|_{Y\times\{t\}} & \text{if } -\ell \le t \le \ell\,, \\ (A,\phi)|_{Y\times\{\ell\}} & \text{if } t \ge \ell\,. \end{cases}$$

Using the same notation as before, we define

$$\eta_\ell = f_\ell \cdot (k_\ell)^*\Big(ih^*_{(k_\ell)_*(A,\phi)}(\omega) - ir\pi_1^*\pi^*\mu_\Sigma\Big) ,$$

where $f_\ell : M(\varphi) \to [0,1]$ is a suitable cut-off function that vanishes away from the interior of the neck T_ℓ .

Now we consider the following perturbed Seiberg-Witten equations on the closed manifold $M(\varphi)_\ell$:

$$\text{(6.1)} \qquad \begin{aligned} \partial\!\!\!/_A \phi &= 0\,, \\ \rho(F_A + \eta_\ell) &= q(\phi)\,. \end{aligned}$$

The corresponding moduli space, or the set of solutions divided by the action of the gauge group, will be denoted $\mathcal{M}_{M(\varphi)}(\mathcal{L}, g_\ell, \eta_\ell)$.

LEMMA 6.1. *There is a constant* $K > 0$ *depending only on* $M(\varphi)$, g *and* η *such that*

$$\int_{T_\ell} F_A \wedge F_A \ge -K - 4\pi r\,[\mu_\Sigma] \cdot c_1(\det \mathcal{L})$$

holds for all ℓ sufficiently large and every $[A,\phi] \in \mathcal{M}_{M(\varphi)}(\mathcal{L}, g_\ell, \eta_\ell)$ for any Spin^c *structure $\mathcal{L}$ on $M(\varphi)$.*

Proof. Let S_t denote the restriction of (A,ϕ) to the slice $Y_t = Y \times \{t\}$. By Stokes' theorem we have

$$C(S_\ell) - C(S_{-\ell}) = \frac{1}{2}\int_{T_\ell} F_A \wedge F_A + \int_{T_\ell} \xi \wedge F_A + \int_{Y_\ell} \langle \phi, \partial\!\!\!/_A \phi\rangle - \int_{Y_{-\ell}} \langle \phi, \partial\!\!\!/_A \phi\rangle . \tag{6.2}$$

From Lemma 3.1 (i), we can assume that

$$C(S_\ell) - C(S_{-\ell}) \geq 0 .$$

Over $M(\varphi)_\ell$ we have

$$\int_{M_1} \xi \wedge F_A + \int_{T_\ell} \xi \wedge F_A + \int_{M_2} \xi \wedge F_A = 2\pi r\,[\mu_\Sigma] \cdot c_1(\det \mathcal{L}) .$$

From the compactness of M_j and *a priori* pointwise bound on $|\phi|$, we obtain a universal bound on the integrals

$$\int_{M_1} \xi \wedge F_A \qquad \text{and} \qquad \int_{M_2} \xi \wedge F_A ,$$

that is independent of the neck-length ℓ and the Spin^c structure $\mathcal{L}$. Thus there is a constant $K_1 > 0$ such that

$$\int_{T_\ell} \xi \wedge F_A - 2\pi r\,[\mu_\Sigma] \cdot c_1(\det \mathcal{L}) \leq K_1 \tag{6.3}$$

holds for all ℓ. Moreover, there is a universal bound depending only on the Riemannian manifolds M_j and the perturbation η_ℓ to both

$$\int_{Y_\ell} \langle \phi, \partial\!\!\!/_A \phi\rangle \qquad \text{and} \qquad \int_{Y_{-\ell}} \langle \phi, \partial\!\!\!/_A \phi\rangle ,$$

which is gotten from the usual *a priori* pointwise bound on $|\phi|$ and the L^2-bound on $\nabla\phi$. Our claim now follows easily. □

LEMMA 6.2. *There is a constant $K' > 0$ independent of the neck-length ℓ and the* Spin^c *structures such that*

$$0 \leq C(S_\ell) - C(S_{-\ell}) \leq K' + 2\pi r\,[\mu_\Sigma] \cdot c_1(\det \mathcal{L})$$

holds for any solution $S = (A,\phi)$ to the perturbed Seiberg-Witten equations on $M(\varphi)_\ell$ corresponding to the Spin^c *structure $\mathcal{L}$.*

Proof. First we observe that

$$\frac{1}{4\pi^2}\int_{M(\varphi)_\ell} iF_A \wedge iF_A = \langle c_1(\det \mathcal{L})^2, [M(\varphi)]\rangle .$$

It follows that

$$\|F_A^+\|^2_{L^2(X)} - \|F_A^-\|^2_{L^2(X)} + \int_{T_\ell} iF_A \wedge iF_A = 4\pi^2 \langle c_1(\det \mathcal{L})^2, [M(\varphi)]\rangle . \tag{6.4}$$

Now there is a uniform pointwise bound for $|\phi|$ and hence for $|F_A^+|$ independent of ℓ and $\mathcal{L}$. Thus there is a universal constant $K_2 > 0$ such that

$$||F_A^+||^2_{L^2(X)} \leq K_2 . \tag{6.5}$$

From the dimension formula, $\mathcal{M}_{M(\varphi)}(\mathcal{L}, g_\ell, \eta_\ell)$ is not empty only if

$$\langle c_1(\det \mathcal{L})^2, [M(\varphi)]\rangle \geq 2\mathrm{e}(M(\varphi)) + 3\mathrm{sign}(M(\varphi)) .$$

From Equation (6.4) we see that

$$\begin{aligned} \int_{T_\ell} F_A \wedge F_A &\leq ||F_A^+||^2_{L^2(X)} - 4\pi^2 \langle c_1(\det \mathcal{L})^2, [M(\varphi)]\rangle \\ &\leq K_2 - 4\pi^2(2\mathrm{e}(M(\varphi)) + 3\mathrm{sign}(M(\varphi))) . \end{aligned} \tag{6.6}$$

From Equation (6.2) and Inequalities (6.3) and (6.6) we conclude that the difference $C(S_\ell) - C(S_{-\ell})$ is also bounded from above by a constant that doesn't depend on ℓ. □

REMARK 6.3. From Lemma 6.1 we get the following inequality

$$\int_{T_\ell} iF_A \wedge iF_A \leq K + 4\pi r\, [\mu_\Sigma] \cdot c_1(\det \mathcal{L}) . \tag{6.7}$$

It follows from Equation (6.4) and Inequalities (6.5) and (6.7) that $||F_A^-||^2_{L^2(X)}$ is bounded from above by a constant

$$K_3 = K_2 + K + 4\pi r\, [\mu_\Sigma] \cdot c_1(\det \mathcal{L}) - 4\pi^2(2\mathrm{e}(M(\varphi)) + 3\mathrm{sign}(M(\varphi)))$$

that doesn't depend on ℓ. Now both $||F_A^\pm||^2_{L^2(X)}$ are bounded by constants independent of the neck-length and hence Equation (6.4) tells us that

$$\left| \int_{T_\ell} iF_A \wedge iF_A \right|$$

is also bounded by a constant that depends only on the Riemannian manifolds M_j, η and the Spin^c structure.

COROLLARY 6.4. *If* $[\mu_\Sigma] \cdot c_1(\det \mathcal{L}) \gg 0$ *and* $r < 0$, *then there is no solution to the perturbed Seiberg-Witten equations* (6.1) *on* $M(\varphi)$ *corresponding to the* Spin^c *structure* $\mathcal{L}$. *Similarly for the case when* $[\mu_\Sigma] \cdot c_1(\det \mathcal{L}) \ll 0$ *and* $r > 0$. □

COROLLARY 6.5. *There is a constant* $K_0 > 0$ *independent of the neck-length* ℓ *(but depending on* $\mathcal{L}$*) such that for any solution* (A, ϕ) *to the perturbed* SW*-equations* (6.1), *the* L^2_4 *distance between the restriction* $(A(t), \phi(t))$ *and a static solution is less than*

$$K_0 \cdot \exp\left(-v_r \cdot \min\{t + \ell, \ell - t\}\right)$$

for every $t \in [-\ell, \ell]$, *where* v_r *is the constant in Lemma* 3.3.

Proof. We can argue exactly the same way as in the proof of Corollary 7.5 in [MST] (p.762). □

As before, let $\widetilde{\mathcal{M}}_j(Y)$ denote the SW-moduli space of Y that is gotten by dividing out the SW-solution space by the action of the restricted gauge group $\mathcal{G}_0(M_j)$. Note that $\widetilde{\mathcal{M}}_j(Y)$ is a $\mathbb{Z}$-affine space, i.e. there is a set-theoretic one-to-one correspondence between $\widetilde{\mathcal{M}}_j(Y)$ and $H^1(Y) \cong \mathbb{Z}$. Let $\mathcal{K}(M_j)$ denote the set of isomorphism classes of Spin^c structures on the compact manifold M_j that restrict to $\mathcal{L}_0$ on ∂M_j. For $\mathcal{L} \in \mathcal{K}(M_j)$, note that $c_1(\det \mathcal{L}) \in H^2(M_j)$ can be lifted to an element of $H^2(M_j, \partial M_j; \mathbb{R})$. Now one can show that there is a natural one-to-one correspondence between $\widetilde{\mathcal{M}}_j(Y)$ and the set

$$\Xi(M_j) := \{\, c \in \mathbb{R} \mid c = \langle c_1(\det \mathcal{L}), b_j \rangle \ \text{ for some } \mathcal{L} \in \mathcal{K}(M_j) \,\}.$$

Let $\mathrm{Supp}(SW_{M_j})$ denote the support of the function SW_{M_j}. Consider the real-valued function $\theta_j : \mathrm{Supp}(SW_{M_j}) \to \Xi(M_j)$ defined by

$$\theta_j(\mathcal{L}, x) = \langle c_1(\det \mathcal{L}), b_j \rangle.$$

Let $\mathrm{Im}(\theta_j)$ denote the image of the map θ_j in $\Xi(M_j)$.

LEMMA 6.6. *For a negative parameter r, the sets $\mathrm{Im}(\theta_j)$ are bounded from below.*

Proof. We shall often drop the subscript j to simplify our notation. Suppose (A, ϕ) is a solution to the perturbed SW-equations on the cylindrical end manifold $M \cup_Y Y \times [0, \infty)$. We have to find a lower bound on the integral

$$\frac{i}{2\pi} \int_{\Sigma_j} F_A \;=\; \frac{i}{2\pi} \int_{\Sigma_j} F_A^+ \;+\; \frac{i}{2\pi} \int_{\Sigma_j} F_A^- .$$

From the universal pointwise bound on $|F_A^+|$, we see that

$$\left| i \int_{\Sigma_j} F_A^+ \right| \;\le\; \mathrm{area}(\Sigma_j) \cdot \sup_{\Sigma_j} |F_A^+| \;\le\; K'' \tag{6.8}$$

for some positive constant K'' independent of the Spin^c structure.

Next we let $M_\ell = M \cup_Y (Y \times [0, \ell])$. Let $\hat{b}$ denote a closed 2-form on M_ℓ whose de Rham cohomology class is dual to $[\Sigma_j]$. Without loss of generality, we can assume that (A, ϕ) is in a temporal gauge, i.e. the dt component of A vanishes along the neck $N_\ell = Y \times [0, \ell]$. Then we have

$$-i \int_{\Sigma_j} F_A^- \;-\; \int_{N_\ell} (i \pi_1^* \pi^* \mu_\Sigma) \wedge * F_A^- \;\le\; \left| \langle i\hat{b}, F_A^- \rangle_{L^2} \right| \;\le\; \| i\hat{b} \| \cdot \| F_A^- \| ,$$

where $\|\cdot\|$ denotes the L^2-norm on the imaginary valued 2-forms on M_ℓ. If $\|F_A^-\| < 1$,

$$i \int_{\Sigma_j} F_A^- \;\ge\; - \| i\hat{b} \| - \frac{1}{r} \int_{N_\ell} \xi \wedge F_A^- .$$

From Inequality (6.8) we easily obtain a universal bound on $\left| \int_{N_\ell} \xi \wedge F_A^+ \right|$. Thus Remark 3.5 now gives a universal lower bound (independent of the Spin^c structure) to the integral $\int_{N_\ell} \xi \wedge F_A^-$. In the case when $\|F_A^-\| \ge 1$, we get

$$i \int_{\Sigma_j} F_A^- \;\ge\; - \| i\hat{b} \| \cdot \| F_A^- \|^2 \;-\; \frac{1}{r} \int_{N_\ell} \xi \wedge F_A^- .$$

Now recall that

$$\|F_A^-\|^2 = \|F_A^+\|^2 - 4\pi^2 c_1(\det \mathcal{L})^2 - \int_{Y\times[\ell,\infty)} F_A \wedge F_A$$
$$\leq (\text{constant}) + 2\int_{Y\times[\ell,\infty)} \xi \wedge F_A ,$$

where $\mathcal{L}$ is the corresponding Spin^c structure on the cylindrical end manifold $M \cup_Y Y \times [0,\infty)$. It follows that

$$i\int_{\Sigma_j} F_A^- \geq (\text{constant}) - 2\|i\hat{b}\| \cdot \int_{Y\times[\ell,\infty)} \xi \wedge F_A - \frac{1}{r}\int_{N_\ell} \xi \wedge F_A^- .$$

Since $|r|$ is very small, we can choose ℓ large enough such that the right side of the above inequality is dominated by the last term, which is indeed bounded from below independent of the Spin^c structure. □

Now consider the non-compact surface $\widehat{\Sigma}_j := \Sigma_j \cup (\gamma \times [0,\infty))$ inside the cylindrical end manifold $\widehat{M}_j = M_j \cup_Y Y \times [0,\infty)$. We define a map $\vartheta^j_{\mathcal{L}} : \mathcal{M}_{\widehat{M}_j}(\mathcal{L}, g_j, \eta_j) \to \mathbb{R}$ by

$$\vartheta^j_{\mathcal{L}}(A,\phi) = \frac{i}{2\pi}\int_{\widehat{\Sigma}_j} F_A .$$

Note that the above integral makes sense since we have exponential decay along the cylinder. There is a natural one-to-one correspondence between $\mathrm{Im}(\vartheta^j_{\mathcal{L}})$ and the image $\partial_\infty(\mathcal{M}_{\widehat{M}_j}(\mathcal{L}, g_j, \eta_j)) \subset \widetilde{\mathcal{M}}_j(Y)$.

COROLLARY 6.7. *For a negative parameter r, the sets* $\mathrm{Im}(\vartheta^j_{\mathcal{L}})$ *are bounded from below independent of the* Spin^c *structure* $\mathcal{L}$.

Proof. From the previous lemma and Remark 3.5, we get a lower bound on

$$\vartheta^j_{\mathcal{L}}(A,\phi) = \frac{i}{2\pi}\int_{\Sigma_j} F_A + \frac{i}{2\pi}\int_{\gamma\times[0,\infty)} F_A$$
$$= \frac{i}{2\pi}\int_{\Sigma_j} F_A - \frac{1}{2\pi r}\int_{Y\times[0,\infty)} \xi \wedge F_A$$

independent of the Spin^c structure. □

REMARK 6.8. By modifying some of the signs in the above proofs, one can easily show that for small positive values of r the sets $\mathrm{Im}(\theta_j)$ and $\mathrm{Im}(\vartheta^j_{\mathcal{L}})$ are bounded from above (independent of the Spin^c structure).

The gluing map φ induces an identification map, $\varphi^* : \widetilde{\mathcal{M}}_2(Y) \to \widetilde{\mathcal{M}}_1(Y)$. More precisely we fix, once and for all, a particular Spin^c structure $\mathcal{L}$ on the closed manifold $M(\varphi)$ that restricts to $\mathcal{L}_0$ on Y. This choice specifies a base point on each $\mathbb{Z}$-affine set $\widetilde{\mathcal{M}}_j(Y)$ via the evaluation on b_j of the first Chern class of the restriction of the Spin^c structure to M_j, and hence an identification $\widetilde{\mathcal{M}}_j(Y) \cong H^1(Y)$. Now we can use the induced automorphism on the cohomology $\varphi^* : H^1(Y) \to H^1(Y)$. Hence using the map φ^*, we can define the 'graph' set

$$G_{\mathcal{L}}(\varphi) := \{ (x,y) \in \widetilde{\mathcal{M}}_1(Y) \times \widetilde{\mathcal{M}}_2(Y) \mid x = \varphi^*(y) \}.$$

Let $\mathcal{K}(M(\varphi))$ denote the set of Spin^c structures on $M(\varphi)$ that restrict to $\mathcal{L}_0$ on Y. There is the obvious gluing map

$$P : \mathcal{K}(M_1) \times \mathcal{K}(M_2) \longrightarrow \mathcal{K}(M(\varphi)) .$$

For every $(\mathcal{L}_1, \mathcal{L}_2) \in P^{-1}(\mathcal{L})$, we have $d(\mathcal{L}_1) + d(\mathcal{L}_2) = d(\mathcal{L})$, where

$$d(\mathcal{L}) = \frac{1}{4}(c_1(\det \mathcal{L})^2 - 2\mathrm{e}(M(\varphi)) - 3\mathrm{sign}(M(\varphi))) .$$

THEOREM 6.9 (Product Formula I). *Let M_j, φ and $M(\varphi)$ be as above. Suppose that $b_2^+(M_j) \geq 1$ and that $\gamma \in \mathrm{Ker}(i_j)_*$. Then for every* Spin^c *structure $\mathcal{L} \in \mathcal{K}(M(\varphi))$, we have*

$$SW_{M(\varphi)}(\mathcal{L}) = \sum_{P^{-1}(\mathcal{L})} \sum_{G_{\mathcal{L}}(\varphi)} SW_{M_1}(\mathcal{L}_1, x) \cdot SW_{M_2}(\mathcal{L}_2, y) ,$$

where the outer sum on the right side is taken over all pairs $(\mathcal{L}_1, \mathcal{L}_2)$ in the preimage $P^{-1}(\mathcal{L})$, and the inner sum is taken over all points $(x, y) \in G_{\mathcal{L}}(\varphi)$.

Proof. First we form the cylindrical end manifolds $\widehat{M_j} = M_j \cup_Y Y \times [0, \infty)$. Given $\mathcal{L}_j \in \mathcal{K}(M_j)$, let $\mathcal{N}_j(\mathcal{L}_j) = \mathcal{M}_{\widehat{M_j}}(\mathcal{L}_j, g_j, \eta_j)$, where $\lceil \eta_1 \rceil = - \lceil \eta_2 \rceil$. From Lemma 3.3, we have maps $\partial^j_\infty : \mathcal{N}_j(\mathcal{L}_j) \to \widetilde{\mathcal{M}}_j(Y)$. Define $\mathcal{N}(\mathcal{L}_1, \mathcal{L}_2)$ to be the set

$$\{ ([A_1, \phi_1], [A_2, \phi_2]) \in \mathcal{N}_1(\mathcal{L}_1) \times \mathcal{N}_2(\mathcal{L}_2) \mid \partial^1_\infty [A_1, \phi_1] = \varphi^* \partial^2_\infty [A_2, \phi_2] \} .$$

Applying the estimates in previous lemmas to the standard gluing results and limiting arguments as in [MM], [T1] or [T2] shows that there is a diffeomorphism

$$\mathcal{M}_{M(\varphi)}(\mathcal{L}, g_\ell, \eta_\ell) \xrightarrow{\cong} \coprod_{P^{-1}(\mathcal{L})} \mathcal{N}(\mathcal{L}_1, \mathcal{L}_2) , \tag{6.9}$$

for all ℓ sufficiently large, where the right side is the disjoint union taken over all pairs $(\mathcal{L}_1, \mathcal{L}_2)$ in the preimage $P^{-1}(\mathcal{L})$.

Next we show that the moduli space $\mathcal{M}_{M(\varphi)}(\mathcal{L}, g_\ell, \eta_\ell)$ can be used to calculate the Seiberg-Witten invariant of $M(\varphi)_\ell$. Let $\mathcal{P}$ be as in the proof of Lemma 3.2. We consider the map

$$H : \mathcal{A}_{L^2_4}(\det \mathcal{L}) \times L^2_4(W^+(\mathcal{L})) \times \mathcal{P} \times [0,1] \longrightarrow L^2_3(M(\varphi)_\ell, i\Lambda^+ \oplus W^-(\mathcal{L}))$$

defined by

$$H(A, \phi, \nu, s) = (\rho(F_A + \eta_\ell(s)) - q(\phi) \, , \, \partial\!\!\!/_A \phi) ,$$

where $\eta_\ell(s)$ is the imaginary valued 2-form gotten by replacing ω in the definition of η_ℓ with $\nu(s) \in \mathcal{P}$. Note that $\eta_\ell(0) = f_\ell \cdot k_\ell^* \xi$ is a *constant* 2-form. Now let $\mathbf{M}$ denote the zero set $H^{-1}(0)$ modulo the L^2_5 gauge transformations. Let $\mathbf{F}$ denote the generic fiber of the projection, $\mathbf{M} \to \mathcal{P}$, onto the third factor. One can easily show that $\mathbf{F}$ is a smooth compact manifold with boundary

$$\partial \mathbf{F} = \mathcal{M}_{M(\varphi)}(\mathcal{L}, g_\ell, \eta_\ell(0)) \coprod \mathcal{M}_{M(\varphi)}(\mathcal{L}, g_\ell, \eta_\ell) .$$

Thus we have the desired cobordism between moduli spaces.

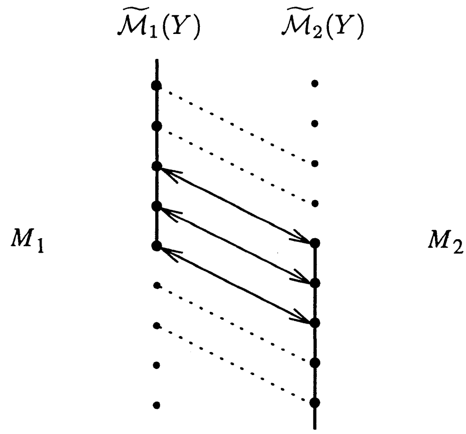

FIG. 6.2.

Finally let Θ be the first Chern class of the *based* moduli space over $\mathcal{M}_{M(\varphi)}(\mathcal{L}, g_\ell, \eta_\ell)$ when $d(\mathcal{L}) = 2n > 0$. From the diffeomorphism (6.9), we see easily that the Chern class Θ on $\mathcal{M}_{M(\varphi)}(\mathcal{L}, g_\ell, \eta_\ell)$ can be decomposed as $\Theta_1 + \Theta_2$ on the component $\mathcal{N}_x := (\partial^1_\infty \times \partial^2_\infty)^{-1}(\varphi^*(x), x) \subset \mathcal{N}(\mathcal{L}_1, \mathcal{L}_2)$, from which the contribution to $SW_{M(\varphi)}(\mathcal{L})$ is

$$\int_{\mathcal{N}_x} (\Theta_1 + \Theta_2)^{d(\mathcal{L})/2} = \binom{d(\mathcal{L})/2}{d(\mathcal{L}_1)/2} \int_{(\partial^1_\infty)^{-1}(\varphi^* x)} \Theta_1^{d(\mathcal{L}_1)/2} \int_{(\partial^2_\infty)^{-1}(x)} \Theta_2^{d(\mathcal{L}_2)/2} .$$

Now each moduli space $(\partial^j_\infty)^{-1}(x)$ is compact and every configuration (A, ϕ) in it is asymptotic at infinity to the same irreducible solution x on Y. This implies that the base point fibration over $(\partial^j_\infty)^{-1}(x)$ is trivial. Hence there will be a non-zero contribution only when $d(\mathcal{L}_1) = d(\mathcal{L}_2) = 0$. Note that by Lemma 6.6, Corollary 6.7, Remark 6.8 and the orientation conventions in Section 5, the sums occuring in the product formula are actually finite (cf. Figure 6.2). □

REMARK 6.10. The above proof and Lemma 3.3 imply that $SW_{M(\varphi)}(\mathcal{L}) = 0$ unless $\mathcal{L} \in \mathcal{K}(M(\varphi))$. Also note that $M(\varphi)$ and M_j are of simple type.

Now for every Spin^c structure $\mathcal{L}$ on $M(\varphi)$ that restricts to $\mathcal{L}_0$ on Y, we define a subset $K(\mathcal{L}) \subset \mathcal{K}(M_1) \times \mathcal{K}(M_2)$ as follows: $(\mathcal{L}_1, \mathcal{L}_2) \in K(\mathcal{L})$ if and only if

$$\langle c_1(\det \mathcal{L}), b_1 + b_2 \rangle = \langle c_1(\det \mathcal{L}_1), b_1 \rangle + \langle c_1(\det \mathcal{L}_2), b_2 \rangle .$$

Note that the definition of $K(\mathcal{L})$ doesn't depend on the choice of b_j.

THEOREM 6.11 (Product Formula II). *Let M_j, φ and $M(\varphi)$ be as above. Suppose that $b_2^+(M_j) \geq 1$ and that $\gamma \in \mathrm{Ker}(i_j)_*$. Then for every* Spin^c *structure $\mathcal{L} \to M(\varphi)$ that restricts to $\mathcal{L}_0$ on Y, we have*

$$\sum_{\{\mathcal{L}' \,|\, c_1(\mathcal{L}') = c_1(\mathcal{L})\}} SW_{M(\varphi)}(\mathcal{L}') = \sum_{K(\mathcal{L})} \sum_{G_{\mathcal{L}'}(\varphi)} SW_{M_1}(\mathcal{L}_1, x) \cdot SW_{M_2}(\mathcal{L}_2, y) ,$$

where the sum on the left side is taken over all the elements of $\mathcal{K}(M(\varphi))$ whose determinant line bundle has $c_1(\det \mathcal{L})$ as its first Chern class, and the outer sum on the right side is taken over all pairs $(\mathcal{L}_1, \mathcal{L}_2)$ in $K(\mathcal{L})$.

Proof. Define a map $Q : \mathcal{K}(M(\varphi)) \to \mathbb{Z}$ by $Q(\mathcal{L}') = \langle c_1(\det \mathcal{L}'), b_1 + b_2 \rangle$. We also define $R : \mathcal{K}(M_1) \times \mathcal{K}(M_2) \to \mathbb{Z}$ by

$$R(\mathcal{L}_1, \mathcal{L}_2) = \langle c_1(\det \mathcal{L}_1), b_1 \rangle + \langle c_1(\det \mathcal{L}_2), b_2 \rangle .$$

Note that $R = Q \circ P$. If κ denotes the constant $Q(\mathcal{L}) = \langle c_1(\det \mathcal{L}), b_1 + b_2 \rangle$, then we easily get a diffeomorphism

$$\coprod_{Q^{-1}(\kappa)} \Big(\coprod_{P^{-1}(\mathcal{L}')} \mathcal{N}(\mathcal{L}_1, \mathcal{L}_2) \Big) \xrightarrow{\cong} \coprod_{K(\mathcal{L})} \mathcal{N}(\mathcal{L}_1, \mathcal{L}_2) ,$$

where $K(\mathcal{L}) = R^{-1}(\kappa)$ and $\mathcal{N}(\mathcal{L}_1, \mathcal{L}_2)$ are as in the previous proof. □

DEFINITION 6.12. *Given a smooth compact oriented 4-manifold M with a possibly nonempty boundary, let $\mathcal{C}(M)$ denote the set of characteristic cohomology classes,*

$$\{L \in H^2(M, \partial M; \mathbb{Z}) \mid L \equiv w_2(M, \partial M) \pmod 2\}.$$

When M is closed, Seiberg-Witten invariant defines, in the usual manner, a function $SW_M : \mathcal{C}(M) \to \mathbb{Z}$ by

$$SW_M(L) := \sum_{\{\mathcal{L} \mid c_1(\mathcal{L}) = L\}} SW_M(\mathcal{L}) .$$

For the case when ∂M is not empty, we define $SW_M : \mathcal{C}(M) \times \widetilde{\mathcal{M}}(\partial M) \to \mathbb{Z}$ by

$$SW_M(L, x) := \sum_{\{\mathcal{L} \mid c_1(\mathcal{L}) = L\}} SW_M(\mathcal{L}, x) .$$

We shall say that an element $L \in H^2(M, \partial M; \mathbb{Z})$ is a (relative) SW-basic class if $SW_M(L, \cdot) \neq 0$.

DEFINITION 6.13. *Suppose M is a closed smooth oriented 4-manifold with $b_2^+(M) > 0$ and the SW-basic classes $\{L_i\}_{i \in I} \subset H^2(M; \mathbb{Z})$. We then define the formal series*

$$\overline{SW}_M := \sum_{i \in I} SW_M(L_i) \cdot \exp(2L_i) .$$

Similarly, given a smooth compact oriented 4-manifold X with boundary $\partial X = Y$ such that $\gamma = \partial b$ for some $b \in H_2(X, \partial X)$, we define

$$\overline{SW}_X := \sum_{\widetilde{\mathcal{M}}(Y)} \sum_{j \in J} SW_X(K_j, x) \cdot \exp(K_j) ,$$

where the first sum is taken over the boundary values $x \in \widetilde{\mathcal{M}}(\partial X)$ and the second sum runs over the relative SW-basic classes $\{K_j\}_{j \in J}$ of X.

COROLLARY 6.14 (Product Formula III). *When $M(\varphi)$ and M_j satisfy Condition* (A), *we have the following equality of formal series:*

$$\overline{SW}_{M(\varphi)} = \overline{SW}_{M_1} \cdot \overline{SW}_{M_2} . \tag{6.10}$$

Here, sums in the exponents are given by Poincaré duality and the Mayer-Vietoris sequence.

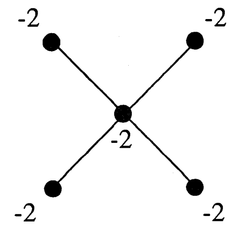

FIG. 7.1. *Plumbing graph for* $E(1/2)$.

7. Examples and computations. First we check our product formula against the well-known examples of elliptic surfaces. Recall that $E(n)$ is a simply-connected elliptic surface with no multiple fibers and with geometric genus $p_g = n-1$. Let F denote the Poincaré dual of the regular fiber in $E(n)$. We simplify the notation by letting $t := \exp(F)$. We look at the case when $r < 0$.

Y embeds inside $E(n)$ in the following way. First we recall the half-Kummer surface description of $E(1)$. Let $L \cong \mathbb{Z}^2 \subset \mathbb{C}$ be a lattice and form the quotient of $\mathbb{C}/L \times \mathbb{CP}^1$ by the involution $(z,x) \mapsto (-z,-x)$, where $z \in \mathbb{C}/L$ and $x \in \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ is an inhomogeneous coordinate. This quotient space has eight singular points corresponding to the fixed points of the involution. One can resolve the singularities by replacing each singular point by a nonsingular rational curve with self-intersection number (-2). The resulting nonsingular surface is diffeomorphic to the rational elliptic surface $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$. The projection onto the second factor, $\mathbb{C}/L \times \mathbb{CP}^1 \to \mathbb{CP}^1$, induces a fibration $p : E(1) \to \mathbb{CP}^1/(x \sim -x) = \mathbb{CP}^1$. This fibration has exactly two singular fibers, $p^{-1}(0)$ and $p^{-1}(\infty)$, and each singular fiber is the union of five nonsingular rational curves of self-intersection (-2). See Figure 7.1 for the linking diagram for these rational curves in each singular fiber.

Now choose two disjoint open disks D_0 and D_∞ in $\mathbb{CP}^1$ centered at points 0 and ∞, respectively. Suppose we have the hemisphere decomposition $\mathbb{CP}^1 = \overline{D}_0 \cup \overline{D}_\infty$. We let $E(1/2)$ to be a regular neighborhood of the singular fiber, i.e.

$$E(1/2) := p^{-1}(D_0) = p^{-1}(D_\infty)\,.$$

We see easily that our Seifert fibered space Y occurs as the boundary of $E(1/2)$, i.e.

$$Y = p^{-1}(\partial D_0) = p^{-1}(\partial D_\infty)\,.$$

Consequently we have the following decomposition of the rational elliptic surface along Y

$$E(1) = E(1/2) \cup_Y E(1/2)\,.$$

Note that Y is a T^2 bundle over S^1, with monodromy given by the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We can choose the generator γ to be a section of this torus bundle Y. For more details on $E(1/2)$ we refer the reader to [HKK].

Since the elliptic surface $E(n)$ is the fiber sum of $E(n-1)$ and $E(1)$, we immediately get n embeddings of Y into $E(n)$. See Figure 7.2 for some possible decompositions of $E(n)$ along Y. The dotted lines in the figure indicate the fiber sum

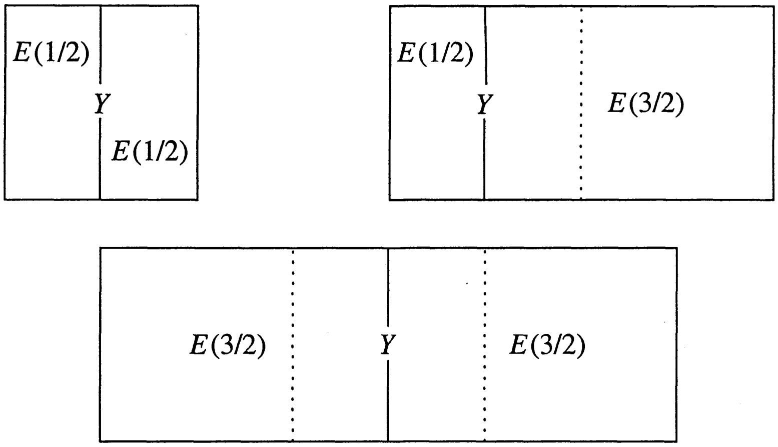

FIG. 7.2. *Decomposition of* $E(1)$, $E(2)$ *and* $E(3)$.

operation. We will let $E((2m+1)/2)$ denote the fiber sum, $E(m)\#_F E(1/2)$, of $E(m)$ and $E(1/2)$. To compute the relative Seiberg-Witten invariant of $E((2m+1)/2)$, we shall need the following

LEMMA 7.1 (Adjunction Inequality). *Let X be a smooth compact oriented 4-manifold with boundary $\partial X = Y$. Let $\Sigma_0 \hookrightarrow X$ be a smoothly embedded surface with boundary $\partial\Sigma_0 = \gamma$. Suppose V is a smoothly embedded surface inside X of genus $g(V) > 0$, whose regular neighborhood lies in the interior of X and is disjoint from Σ_0. If L is a relative SW-basic class of X, then*

$$|\langle L, V\rangle| + V\cdot V \leq 2g(V) - 2.$$

Proof. The argument for the closed case goes through with very little change, once we substitute the cylindrical end moduli space $\mathcal{M}_{\hat{X}}(\mathcal{L}, g, \eta)$ in the place of the ordinary Seiberg-Witten moduli space over a closed manifold. We just have to observe that the boundary map ∂_∞ on $\mathcal{M}_{\hat{X}}(\mathcal{L}, g, \eta)$ takes values in the perturbed SW-moduli space over Y, which is everywhere both nondegenerate *and* irreducible. From the hypothesis we can assume that the support of $\overline{\xi}$ (and hence η) is disjoint from V. This means that we are allowed to freely modify the metric around a small neighborhood of V without effecting the perturbation of the Seiberg-Witten equations (3.5) nor the corresponding moduli spaces over the cylindrical end. For the proof in the closed case, we refer the reader to [FS1], [KM1], [MST], [OS1] and [Sa]. □

Now suppose $L = c_1(\det\mathcal{L})$ is a SW-basic class of $E((2m+1)/2)$. From the above lemma and the complete knowledge of the embedded surfaces representing the generators of $H_2(E(n))$, we easily deduce that L is a multiple of F. The assumption in the beginning of Section 6 is readily seen to be satisfied. Note that $[\mu_\Sigma] \in H^2(E(n))$ is the class Poincaré dual to the section of the elliptic fibration. Similarly, $b_j \in H_2(E((2m+1)/2), Y)$ is represented by the "half-section". We start out by looking at the "doubling" decomposition $E(3) = E(3/2) \cup_Y E(3/2)$. Since $E(3)$ has only finitely many SW-basic classes, the moduli space $\mathcal{M}_{E(3)}(\mathcal{L}, g_\ell, \eta_\ell)$ is empty for all but finitely many Spin^c structures $\mathcal{L}$, provided that $\|\omega\|_{L^2_k}$ is sufficiently small. (This follows from a cobordism argument as in the proof of Theorem 6.9. Otherwise, there would be a sequence of 2-forms, $\omega_i \to 0$, such that the corresponding solutions to (6.1) in turn converge to a solution corresponding to the *constant* perturbation.) Hence $E(3)$ possesses a uniform energy bound on the solutions along the neck (namely, $K' - 2\pi r$; see Lemma 6.2), so we can choose to work with δ-confined moduli spaces for some $\delta \gg 0$. That is, we can change for $SW \equiv SW^\delta$ in the product formulae. Just how large of a δ is needed for such substitution depends on the choice of the metric g_j on the end manifold $E(3/2)$.

It follows that all the terms appearing in Equation (6.10) for $E(3)$ are finite sums. Keeping in mind the orientation conventions set down in Section 5, we can express the right side of Equation (6.10) as

$$\overline{SW}_{E(3/2)} \cdot \overline{SW}_{E(3/2)} = \left(\sum_{k=-l}^{m} a_k t^k \right) \cdot \left(\sum_{n=-m}^{l} \pm a_{-n}\, t^n \right) , \tag{7.1}$$

where $k \equiv n \equiv l \pmod 2$. Since this has to equal $\overline{SW}_{E(3)} = t^{-2} - t^2$, we must have $m = -l+2$ and $a_k = \pm 1$ for all k. Hence (7.1) becomes (up to sign)

$$\left(t^{-l} + \epsilon\, t^{-l+2}\right) \cdot \left(t^{l-2} - \epsilon\, t^{l}\right) , \tag{7.2}$$

where $\epsilon = \pm 1$. Since $\overline{SW}_{E(2)} = 1$, we conclude from the decomposition $E(2) = E(1/2) \cup_Y E(3/2)$ that

$$\begin{aligned} \overline{SW}_{E(1/2)} &= \overline{SW}_{E(3/2)}^{-1} = \left(t^{l-2} - \epsilon\, t^{l}\right)^{-1} \\ &= t^{2-l}\left(1 + \epsilon\, t^2 + t^4 + \epsilon\, t^6 + \cdots\right) . \end{aligned}$$

Now both $E(1)$ and $E(1/2)$ admit metrics of positive scalar curvature. We choose a metric g of positive scalar curvature on $E(1)$ such that the corresponding self-dual harmonic 2-form ϖ_g is the Kähler form with respect to which $\mathbb{CP}^1$ has area 1. We can further arrange metric g so that $\varepsilon_{\mathcal{L}}(g, \bar{\xi}) = -2\pi[\varpi_g] \cdot L$ (cf. Section 4). It follows that for odd $k > 0$, $SW^-_{E(1)}(kF) \equiv 0 \equiv SW_{E(1)}(-kF)$. (This is because g has positive scalar curvature.) From the wall crossing formula in [LL], we conclude that $SW_{E(1)}(kF) \equiv 1$. Analogous argument for $E(1/2)$ shows that $|l|$ has to be small. Indeed we easily see that $l = 0$ or 2, depending on a suitable choice of the homology orientation. (Note that the case $l=1$ violates our orientation convention in Section 5 considering (7.2).)

We summarize our computations so far: For the decomposition $E(3) = E(3/2) \cup_Y E(3/2)$, Equation (6.10) reads (up to sign)

$$t^{-2} - t^2 = (t^{-2} + 1) \cdot (1 - t^2) .$$

For the decomposition $E(2) = E(1/2) \cup_Y E(3/2)$, Equation (6.10) reads

$$1 = (1 + t^2 + t^4 + t^6 + \cdots)(1 - t^2) .$$

For the decomposition $E(1) = E(1/2) \cup_Y E(1/2)$, Equation (6.10) reads

$$\begin{aligned} \sum_{i=0}^{\infty} t^{2(2i+1)} &= \frac{1}{1-t^2} \cdot \frac{1}{1+t^{-2}} \\ &= (1 + t^2 + t^4 + t^6 + \cdots)(1 - t^{-2} + t^{-4} - t^{-6} + \cdots) . \end{aligned}$$

Since we already know from [FS2] the Seiberg-Witten invariants of $E(n)$, we can inductively calculate the relative Seiberg-Witten invariants of the halves $E((2m+1)/2)$ for all non-negative integers m.

THEOREM 7.2. *If* $t = \exp(F)$, *then for any* $m \geq 0$ *we have (up to sign)*

$$\overline{SW}_{E((2m+1)/2)} = \begin{cases} \left(t^{-2} - t^2\right)^{m-1}(1 + t^{-2}) & \text{if } r < 0\,, \\ \left(t^{-2} - t^2\right)^{m-1}(1 - t^2) & \text{if } r > 0\,. \end{cases}$$

Proof. We recall that $\overline{SW}_{E(n)} = \left(t^{-2} - t^2\right)^{n-2}$. It follows that

$$\begin{aligned} \overline{SW}_{E(2m)} &= \left(t^{-2} - t^2\right)^{m-1}(1 + t^{-2}) \cdot (1 - t^2)\left(t^{-2} - t^2\right)^{m-2} \\ &= \overline{SW}_{E((2m+1)/2)} \cdot \overline{SW}_{E((2m-1)/2)} \end{aligned}$$

holds up to a prescribed sign convention. □

8. Adjunction inequality for manifolds with boundary Y. Using the results of previous sections, we can come up with a new adjunction inequality valid for a smoothly embedded surface V inside an open 4-manifold X with $\partial V \subset \partial X = Y$. More precisely, we have the following

THEOREM 8.1 (Adjunction Inequality II). *Let X be a smooth compact oriented 4-manifold with boundary $\partial X = Y$. Suppose that $b_2^+(X) > 1$ and $V \hookrightarrow X$ is a smoothly embedded surface with boundary $\partial V \hookrightarrow \partial X$ such that $\partial V = \gamma$. If L is a relative SW-basic class for X, then we have*

$$|\langle L, V\rangle| + 2V \cdot V \leq 4g(V)\,.$$

Proof. Form the closed manifold $Z = X \cup_Y E(3/2)$. Then there is a SW-basic class $\bar{L}$ of Z such that $\bar{L}|_X = L$ and $\bar{L}|_{E(3/2)} = 2F$. Let $\bar{V} = V \cup_\gamma \Sigma_0$, where Σ_0 is the half-section of $E(3/2)$. From the adjunction inequality for Z, we have

$$|\langle \bar{L}, \bar{V}\rangle| + \bar{V} \cdot \bar{V} \leq 2g(\bar{V}) - 2\,.$$

But note that

$$|\langle \bar{L}, \bar{V}\rangle| = \frac{1}{2}|\langle L, V\rangle + 2| \geq \frac{1}{2}|\langle L, V\rangle| - 1\,,$$

$\bar{V} \cdot \bar{V} = V \cdot V - 1$, and $g(\bar{V}) = g(V)$. □

9. Extension to other 3-manifolds. It is easy to extend our results to other Seifert fibered 3-manifolds of degree zero, e.g. $\Sigma(2,3,6)$. In fact, let Y be the unit circle bundle corresponding to an orbifold complex line bundle N over a 2-dimensional orbifold Σ. Then the following two properties

$$\text{(i)}\quad \deg(N) = 0$$
$$\text{(ii)}\quad b_1(Y) = 1$$

are all that we need in order to formally extend our arguments of the previous sections to Y. In particular, all the statements and formulae in Section 6 continue to be valid for this special class of Seifert fibered 3-manifolds.

We shall give more applications and further generalizations of the product formula for other 3-manifolds with $b_1 = 1$ in the future work [P3]. There, we plan to take a more refined approach and define our relative SW-invariant to take values in an infinitely generated Floer-type homology. The corresponding product formulae will then take place inside a suitably defined Novikov ring. We also study an analogous product formula for the three-torus in [P4].

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