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VERDIER-RIEMANN-ROCH FOR CHERN CLASS AND MILNOR CLASS*

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1. Introduction. In [BFM] Baum, Fulton and MacPherson formulated *a singular Riemann-Roch*, extending the Grothendieck-Riemann-Roch (e.g., see [BoSe]) to possibly singular varieties. That is the unique natural transformation from the K-theory of coherent sheaves to the rational homology theory

$$\text{(BFM)} \qquad \tau : \mathbf{K}_0 \to H_{\mathbb{Q}}$$

satisfying the normalization condition that $\tau(\mathcal{O}_X) = td(TX) \cap [X]$ for a non-singular variety X with $\mathcal{O}_X$ the structure sheaf and $td(TX)$ the total Todd class of the tangent bundle TX. This is a covariant aspect of the two theories $\mathbf{K}_0$ and $H_{\mathbb{Q}}$. As to the contravariant aspect of these two theories, we have the *Verdier-Riemann-Roch*, which was conjectured in [BFM] and proved affirmatively by J.-L. Verdier [V, Theorem 18.2 (3), p. 349], i.e., the following commutative diagram for a *local complete intersection morphism* $f : X \to Y$:

$$\text{(Verdier)} \qquad \begin{array}{ccc} \mathbf{K}_0(Y) & \xrightarrow{\tau_Y} & H_*(Y)_{\mathbb{Q}} \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle td(T_f)\cap f^*} \\ \mathbf{K}_0(X) & \xrightarrow[\tau_X]{} & H_*(X)_{\mathbb{Q}} \end{array}$$

where $td(T_f)$ is the total Todd class of the virtual relative tangent bundle T_f of the morphism f and $f^* : H_*(Y)_{\mathbb{Q}} \to H_*(X)_{\mathbb{Q}}$ is the Gysin homomorphism (see [F, Example 19.2.1], [BFM, IV.4] and [FM]).

In [SGA 6] the original Grothendieck-Riemann-Roch (for a morphism between nonsingular varieties) is generalized to the following Riemann-Roch for a local complete intersection morphism $f : X \to Y$:

$$\text{(SGA 6)} \qquad \begin{array}{ccc} \mathbf{K}^0(X) & \xrightarrow{ch} & H^*(X)_{\mathbb{Q}} \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_!(td(T_f)\cup\ \ )} \\ \mathbf{K}^0(Y) & \xrightarrow[ch]{} & H^*(Y)_{\mathbb{Q}} \end{array}$$

Here $\mathbf{K}^0$ is the K-theory of vector bundles and $f_! : H^*(X)_{\mathbb{Q}} \to H^*(Y)_{\mathbb{Q}}$ is the Gysin homomorphism (see [F, Example 19.2.1], [BFM, IV.4] and [FM]).

The above three Riemann-Roch theorems, i.e., Baum-Fulton-MacPherson's Riemann-Roch (BFM), Verdier-Riemann-Roch (Verdier) and the Riemann-Roch in

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[SGA6], follow from the Grothendieck transformation (see [FM,II, §1])

$$\tau : \mathbb{K}_{\mathrm{alg}} \to \mathbb{H}_{\mathbb{Q}}$$

from the bivariant algebraic K-theory to the bivariant homology theory with rational coefficients and the bivariant-theoretic "Riemann-Roch formula" ([FM, I, §1.4])

$$\tau(\mathcal{O}_f) = td(T_f) \bullet U_f$$

of the canonical orientations $\mathcal{O}_f \in \mathbb{K}_{\mathrm{alg}}(X \xrightarrow{f} Y)$ and $U_f \in \mathbb{H}(X \xrightarrow{f} Y)_{\mathbb{Q}}$ for a local complete intersection morphism $f : X \to Y$ (see [FM, II, 0.2]).

As remarked in [BFM, (0.3)], the motivation of BFM's Riemann-Roch is the *Chern-Schwartz-MacPherson class theory*, i.e., the unique natural transformation from the covariant functor F of constructible functions to the covariant homology functor

$$c_* : F \to H_*$$

satisfying the normalization condition that $c_*(\mathbb{1}_X) = c(TX) \cap [X]$ for a nonsingular variety X with $\mathbb{1}_X$ being the characteristic function on X. The natural transformation $c_* : F \to H_*$ is nothing but "Grothendieck-Riemann-Roch" for the Chern class (cf. [G]). It is, therefore, quite natural and reasonable to think of the "contravariant aspect" of the Chern-Schwartz-MacPherson class theory; i.e., *"Verdier-Riemann-Roch" for Chern class*, since the original Verdier-Riemann-Roch is one for Todd class. Thus, first of all, a naïve or simple-minded question is to ask about the commutativity of the following diagram:

$$\begin{array}{ccc} F(Y) & \xrightarrow{c_*} & H_*(Y;\mathbb{Z}) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle c(T_f)\cap f^*} \\ F(X) & \xrightarrow[c_*]{} & H_*(X;\mathbb{Z}) \end{array} \tag{1.1}$$

Here, $f^* : F(Y) \to F(X)$ is the usual functional pullback of constructible functions, i.e., $(f^*\alpha)(x) = \alpha(f(x))$ for a constructible function $\alpha \in F(Y)$, $c(T_f)$ is the total Chern class of the bundle T_f and $f^* : H_*(Y;\mathbb{Z}) \to H_*(X;\mathbb{Z})$ is the Gysin homomorphism as above with integral coefficients.The above diagram (1.1) is commutative if f is smooth ([Y1]), otherwise it is not. Consider the very simple case when $f : X \to pt$ is a local complete intersection morphism from a singular variety X to a point, which means that the variety X is a singular local complete intersection in a smooth variety. Then the problem of the commutativity of the diagram (1.1) is equivalent to that of whether the Chern-Schwartz-MacPherson class $c_*(X)$ of X and the Fulton-Johnson class $c^{FJ}(X)$ of X ([F], [FJ]) are the same or not. Indeed, since $Y = pt$, let us consider the characteristic function $\mathbb{1}_{pt}$. Then the commutativity of (1.1) implies that

$$c_*(f^*(\mathbb{1}_{pt})) = c(T_f) \cap f^*(c_*(\mathbb{1}_{pt})).$$

The left-hand-side of this equality is $c_*(f^*(\mathbb{1}_{pt})) = c_*(\mathbb{1}_X) = c_*(X)$, the Chern-Schwartz-MacPherson class of X. On the other hand, as pointed out just above, since X is a local complete intersection in a smooth variety, say M, it follows that the right-hand-side is $c(T_f)\cap f^*(c_*(\mathbb{1}_{pt}) = c(TM|_X - N_X M)\cap[X])$, which is the Fulton-Johnson class of X ([F], [FJ]). Here $N_X M$ is the normal bundle of X in M. If X is singular, then these two classes are not the same and the difference of these two classes, which

is called *Milnor class*, has been recently studied well from different motivations (e.g., see [A3], [BLSS1,2], [PP3], [Su1], [Y2], etc.). Note that in the case of a singular plane curve, it is already implicitly observed in [P, §6, Comparaison des classes] that the difference of these two classes is the sum of Milnor numbers of the singularities (cf. [F, Example 4.2.6 (b)]).

From a bivariant-theoretic viewpoint, a more natural question to ask is then whether or not there exists a certain constructible function $\alpha \in F(X)$ such that the following diagram commutes:

$$\begin{array}{ccc} F(Y) & \xrightarrow{c_*} & H_*(Y;\mathbb{Z}) \\ {\scriptstyle \alpha\cdot f^*}\downarrow & & \downarrow{\scriptstyle c(T_f)\cap f^*} \\ F(X) & \xrightarrow[c_*]{} & H_*(X;\mathbb{Z}). \end{array} \tag{1.2}$$

The starting point of the present work is the observations that the constructible function functor F itself can be a bivariant theory and furthermore that there are several bivariant theories of constructible functions [Y4, 5]. For example, roughly speaking, a constructible function $\alpha \in F(X)$ making the diagram (1.2) commutative, with $c(T_f) \cap f^*$ replaced by a certain homomorphism $\theta_f : H_*(Y;\mathbb{Z}) \to H_*(X;\mathbb{Z})$, is also "bivariant" (see Theorem 2.8 below for more details).

In this paper we show that for a trivial fiber bundle with the fiber being a local complete intersection in a smooth variety there does exist a constructible function $\alpha \in F(X)$ such that the diagram (1.2) is commutative, and thus we speculate that it would be true even for fiber bundles both in the Zariski topology and in the usual topology. We also show that for a blow-up map (which is a non-smooth local complete intersection morphism) there does not exist any constructible function $\alpha \in F(X)$ such that the diagram (1.2) is commutative.

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2. Bivariant Theories of Constructible Functions. First we recall a general theory of bivariant theory due to Fulton and MacPherson (see [FM] for full details). A bivariant theory $\mathbb{B}$ on a category $\mathcal{C}$ with values in an abelian category is an assignment to each morphism

$$X \xrightarrow{f} Y$$

in the category $\mathcal{C}$ a graded abelian group

$$\mathbb{B}(X \xrightarrow{f} Y)$$

which is equipped with the following three basic operations:
(Product operations): For morphisms $f : X \to Y$ and $g : Y \to Z$, the product operation

$$\bullet : \mathbb{B}(X \xrightarrow{f} Y) \otimes \mathbb{B}(Y \xrightarrow{g} Z) \to \mathbb{B}(X \xrightarrow{gf} Z)$$

is defined.
(Pushforward operations): For morphisms $f : X \to Y$ and $g : Y \to Z$ with f proper, the pushforward operation

$$f_* : \mathbb{B}(X \xrightarrow{gf} Z) \to \mathbb{B}(Y \xrightarrow{g} Z)$$

is defined.
(Pullback operations): For a fiber square (which will be sometimes simply denoted by $X' = X \times_Y Y'$)

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y, \end{array}$$

the pullback operation

$$g^* : \mathbb{B}(X \xrightarrow{f} Y) \to \mathbb{B}(X' \xrightarrow{f'} Y')$$

is defined.

And these three operations are required to satisfy the following seven axioms (see [FM, Part I, §2.2] for details):

(B-1) product is associative,
(B-2) pushforward is functorial,
(B-3) pullback is functorial,
(B-4) product and pushforward commute,
(B-5) product and pullback commute,
(B-6) pushforward and pullback commute, and
(B-7) projection formula.

Let $\mathbb{B}, \mathbb{B}'$ be two bivariant theories on a category C. Then a *Grothendieck transformation* from $\mathbb{B}$ to $\mathbb{B}'$

$$\gamma : \mathbb{B} \to \mathbb{B}'$$

is a collection of homomorphisms

$$\mathbb{B}(X \to Y) \to \mathbb{B}'(X \to Y)$$

for a morphism $X \to Y$ in the category C, which preserves the above three basic operations.

Fulton and MacPherson also introduced the notion of *operational bivariant theory associated to a homology theory* ([FM, Part I, §8]). Let T_* be a covariant functor (or sometimes called a homology theory) on the category $\mathcal{C}$. Then the associated *operational bivariant theory* $\mathbb{OT}$ of T_* is defined as follows. For a morphism $f : X \to Y$, an element $c \in \mathbb{OT}(X \xrightarrow{f} Y)$ is defined to be a collection of homomorphisms

$$c(g) : T_*(Y') \to T_*(X')$$

for all $g : Y' \to Y$ and the fiber square $X' = X \times_Y Y'$. And these homomorphisms

$c(g)$ are required to be compatible with proper pushforward, i.e., for a fiber diagram

$$\begin{array}{ccccc} X'' & \xrightarrow{h'} & X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y'' & \xrightarrow[h]{} & Y' & \xrightarrow[g]{} & Y, \end{array}$$

the following diagram must commute:

$$\begin{array}{ccc} T_*(Y'') & \xrightarrow{c(goh)} & T_*(X'') \\ {\scriptstyle h_*}\downarrow & & \downarrow{\scriptstyle h'_*} \\ T_*(Y') & \xrightarrow[c(g)]{} & T_*(X'). \end{array}$$

If $\mathcal{C}$ has a final object pt and $T_*(pt)$ has a distinguished element 1, then the homomorphism $\mathrm{ev} : \mathbb{OT}(X \to pt) \to T_*(X)$ defined by $\mathrm{ev}(c) := (c(\mathrm{id}_{pt}))(1)$ is called the *evaluation homomorphism.*

Let $\mathbb{B}$ be a bivariant theory. Then the *associated operational bivariant theory* $\mathbb{B}^{\mathrm{op}}$ of $\mathbb{B}$ is defined to be the operational bivariant theory constructed from the covariant functor $B_*(X) = \mathbb{B}(X \to pt)$. Then we have the following canonical Grothendieck transformation

$$\mathrm{op} : \mathbb{B} \to \mathbb{B}^{\mathrm{op}}$$

defined by, for each $\alpha \in \mathbb{B}(X \to Y)$,

$$\mathrm{op}(\alpha) := \{(g^*\alpha)\bullet : \mathbb{B}(Y' \to pt) \to \mathbb{B}(X' \to pt) | g : Y' \to Y\}$$

where $X' = X \times_Y Y'$ is the fiber square.

Now we discuss some bivariant theories of constructible functions. First, the abelian group $F(X)$ of a given analytic variety X consists of all the constructible functions on X. The association $X \longmapsto F(X)$ becomes a contravariant functor with the usual pullback and at the same time a covariant functor with the pushforward f_* which takes the topological Euler-Poincaré characteristic of the fibers weighted by constructible functions.

The constructible function functor F itself can be a bivariant theory without any geometric or topological requirement on constructible functions as follows:

PROPOSITION (2.1). *([Y5, Proposition (3.1)]) For any morphism $f : X \to Y$ the group $s\mathbb{F}(X \to Y)$ is defined by*

$$s\mathbb{F}(X \xrightarrow{f} Y) := F(X).$$

Then this is a bivariant theory with the following operations of product, pushforward and pullback, i.e., they satisfy the seven axioms of the bivariant theory.
(i): the product operation

$$\bullet : s\mathbb{F}(X \xrightarrow{f} Y) \otimes s\mathbb{F}(Y \xrightarrow{g} Z) \to s\mathbb{F}(X \xrightarrow{gf} Z)$$

is defined by:

$$\alpha \bullet \beta := \alpha \cdot f^*\beta.$$

(ii): the pushforward operation

$$f_* : s\mathbb{F}(X \xrightarrow{gf} Z) \to s\mathbb{F}(Y \xrightarrow{g} Z)$$

is the pushforward

$$f_* : F(X) \to F(Y).$$

(iii): For a fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y, \end{array}$$

the pullback operation

$$g^* : s\mathbb{F}(X \xrightarrow{f} Y) \to s\mathbb{F}(X' \xrightarrow{f'} Y')$$

is the pullback

$$g'^* : F(X) \to F(X').$$

This bivariant group is called the simple bivariant group of constructible functions.

It is clear that Axioms **(B-2)** and **(B-3)** hold, and to see that these three operations satisfy the other five axioms, we use the following three properties:

(2.2) for the above fiber square in (iii) the following diagram commutes (e.g., see [Er, Proposition 3.5], [FM, Axiom (A_{23})])

$$\begin{array}{ccc} F(Y') & \xrightarrow{f'^*} & F(X') \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g'_*} \\ F(Y) & \xrightarrow{f^*} & F(X), \end{array}$$

(2.3) for a morphism $f : X \to Y$ and constructible functions $\alpha, \beta \in F(Y)$

$$f^*(\alpha \cdot \beta) = f^*\alpha \cdot f^*\beta,$$

(2.4) (**projection formula**): for a morphism $f : X \to Y$ and constructible functions $\alpha \in F(Y)$ and $\beta \in F(X)$

$$f_*(f^*\alpha \cdot \beta) = \alpha \cdot f_*\beta.$$

Let $\mathbb{H}$ be the Fulton-MacPherson's bivariant homology theory, constructed from the cohomology theory. For a morphism $f : X \to Y$, choose a morphism $\phi : X \to \mathbb{R}^n$ such that $\Phi := (f, \phi) : X \to Y \times \mathbb{R}^n$ is a closed embedding. Then the i-th bivariant homology group $\mathbb{H}^i(X \xrightarrow{f} Y)$ is defined by

$$\mathbb{H}^i(X \xrightarrow{f} Y) := H^{i+n}(Y \times \mathbb{R}^n, Y \times \mathbb{R}^n \setminus X_\phi),$$

where X_ϕ is defined to be the image of the morphism $\Phi = (f, \phi)$. The definition is independent of the choice of ϕ. See [FM, §3.1] for more details of $\mathbb{H}$.

A key feature of the simple bivariant group $s\mathbb{F}$ is the following result, which gives a counterexample to [FM, I, §8.2, pp.90-91]:

THEOREM 2.5. *([Y5, Remark 3.3]) Let $\mathbb{H}^{\mathrm{op}}$ be the operational bivariant homology theory associated to the homology theory H_*. There does not exist a Grothendieck transformation*

$$\gamma : s\mathbb{F} \to \mathbb{H}^{\mathrm{op}}$$

such that for each morphism $X \to pt$ the associated homomorphism followed by the evaluation homomorphism

$$\mathrm{ev} \circ \gamma : F(X) = s\mathbb{F}(X \to pt) \to \mathbb{H}^{\mathrm{op}}(X \to pt) \to H_*(X)$$

is the Chern-Schwartz-MacPherson class homomorphism c_*.

And similarly we can show the following:

THEOREM 2.6. *([Y5, Theorem 3.2]) There does not exist a Grothendieck transformation*

$$\gamma^{\mathrm{s}} : s\mathbb{F} \to \mathbb{H}$$

such that $\gamma^{\mathrm{s}}(\mathbb{1}_\pi) = c(TX) \cap [X]$ for X smooth, where $\pi : X \to pt$ and $\mathbb{1}_\pi = \mathbb{1}_X$.

The proof of these theorems tells us that the simple bivariant group, i.e., the constructible function group is certainly too large for the possible existence of a Grothendieck transformation from this bivariant theory of constructible functions to the bivariant homology theory. And there are more finer bivariant theories of constructible functions (see Theorem 2.7) below and also see [Y5]), but the most interesting bivariant theory of constructible functions, which is conjecturally best-fit, for the possible existence of a Grothendieck transformation, as supported by the Brasselet's theorem [B1], is Fulton-MacPherson's bivariant theory of constructible functions, i.e., $\mathbb{F}(X \xrightarrow{f} Y)$ consists of all the constructible functions on X which satisfy the local Euler condition with respect to f (see [B1], [FM], [Sa], [Z]). Here a constructible function $\alpha \in F(X)$ is said to satisfy the *local Euler condition with respect to f* if for any point $x \in X$ and for any local embedding $(X, x) \to (\mathbf{C}^N, 0)$ the following equality holds

$$\alpha(x) = \chi\big(B_\epsilon \cap f^{-1}(z); \alpha\big),$$

where B_ϵ is a sufficiently small open ball of the origin 0 with radius ϵ and z is any point close to $f(x)$ (cf. [B1], [Sa]). The three operations on $\mathbb{F}$ are the same as above in $s\mathbb{F}$ and it is known that these three operations are well-defined for $\mathbb{F}$ (e.g., see [BY], [Sa], [Z]). Note that $\mathbb{F}(X \xrightarrow{\mathrm{id}_X} X)$ consists of all locally constant functions and $\mathbb{F}(X \to pt) = F(X)$.

Suppose that there exists a Grothendieck transformation

$$\gamma : \mathbb{F} \to \mathbb{H}$$

satisfying the normalization condition that $\gamma(\mathbb{1}_\pi) = c(TX) \cap [X]$ for X smooth, where $\pi : X \to pt$ and $\mathbb{1}_\pi = \mathbb{1}_X$. In [B1], J.-P. Brasselet constructed such a Grothendieck transformation $\gamma^{\mathrm{Br}} : \mathbb{F} \to \mathbb{H}$ in the category whose objects are complex analytic varieties and whose morphisms are *cellular*. Then, since Grothendieck transformations preserve product, pushforward and pullback, and since the Chern-Schwartz-MacPherson class transformation is unique, it follows that for any bivariant constructible function $\alpha \in \mathbb{F}(X \to Y)$ we get the following commutative diagram (*Verdier-*

Riemann-Roch for Chern classes associated to the bivariant constructible function α):

$$\begin{array}{ccc} F(Y) & \xrightarrow{c_*} & H_*(Y;\mathbb{Z}) \\ {\scriptstyle \alpha\bullet}\downarrow & & \downarrow{\scriptstyle \gamma(\alpha)\bullet} \\ F(X) & \xrightarrow[c_*]{} & H_*(X;\mathbb{Z}). \end{array}$$

Here we emphasize that $\alpha\bullet$ means $\alpha \cdot f^*$ for $f : X \to Y$.

Note that no matter which bivariant theory of constructible functions we consider, as long as we assume the existence of a Grothendieck transformation from that to the homology theory satisfying the above normalization condition, we get the above commutative diagram.

Such a commutative diagram as above, without requiring $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$ and with $\gamma(\alpha)\bullet$ replaced simply by a certain homomorphism, already requires some strong condition on the constructible function $\alpha \in F(X)$, as observed below.

OBSERVATION 2.7. *Suppose that for a morphism* $f : X \to Y$ *we have the following commutative diagram:*

$$\begin{array}{ccc} F(Y) & \xrightarrow{c_*} & H_*(Y;\mathbb{Z}) \\ {\scriptstyle \alpha\cdot f^*}\downarrow & & \downarrow{\scriptstyle \theta_f} \\ F(X) & \xrightarrow[c_*]{} & H_*(X;\mathbb{Z}). \end{array} \tag{2.7.1}$$

with a constructible function $\alpha \in F(X)$ *(not necessarily* $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$*) and a homomorphism* $\theta_f : H_*(Y;\mathbb{Z}) \to H_*(X;\mathbb{Z})$. *Then we can see that this commutative diagram always implies the following things:*

(i) If $f : X \to Y$ *is not surjective, then* $\alpha \equiv 0$.

(ii) The Chern-Schwartz-MacPherson classes $c_*(f^{-1}(y);\alpha) := c_*(\alpha|_{f^{-1}(y)})$ *of the fiber weighted by the constructible function* α *are locally constant, considered as the homology classes in the total variety* X. *In particular, the pushforward* $f_*\alpha \in F(Y)$ *is locally constant, or equivalently, the Euler-Poincaré characteristics* $\chi(f^{-1}(y);\alpha) := \chi(\alpha|_{f^{-1}(y)})$ *of the fiber weighted by the constructible function* α *are locally constant.*

(iii) If $f : \widetilde{X} \to X$ *is a blow-up of* X *along a subvariety* $V \subset X$ *and let* E *be the exceptional divisor, then* α *is constant on* $\widetilde{X} \setminus E = f^{-1}(X \setminus V)$.

Finally we note the following theorem:

THEOREM 2.8. *([Y5, Theorem 3.6]) For a morphism* $f : X \to Y$, *we define* $v\mathbb{F}(X \xrightarrow{f} Y)$ *to be the set of all constructible functions* $\alpha \in F(X)$ *satisfying the following condition: for any morphism* $g : Y' \to Y$ *and fiber square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y, \end{array} \tag{2.8.1}$$

we have the following commutative diagram

$$\begin{array}{ccc} F(Y') & \xrightarrow{c_*} & H_*(Y';\mathbb{Z}) \\ {\scriptstyle g'^*\alpha\cdot f'^*}\downarrow & & \downarrow{\scriptstyle \theta_{f'}} \\ F(X') & \xrightarrow[c_*]{} & H_*(X';\mathbb{Z}). \end{array} \tag{2.8.2}$$

with a certain homomorphism $\theta_{f'} : H_*(Y';\mathbb{Z}) \to H_*(X';\mathbb{Z})$. *Then we have*

(i) $v\mathbb{F}$ *becomes a bivariant theory with the same operations of product, pushforward and pullback as in* $s\mathbb{F}$,

(ii) $v\mathbb{F}(X \xrightarrow{\mathrm{id}_X} X)$ *consists of all locally constant functions, and*

(iii) $v\mathbb{F}(X \to pt) = F(X)$.

REMARK 2.9. In general $v\mathbb{F}(X \to Y) \neq \mathbb{F}(X \to Y)$. Indeed, consider a blow-up $\pi : \widetilde{X} \to X$ of X ($\dim X > 1$) along a nonsingular subvariety $V \subset X$ whose codimension is > 1 and let E be the exceptional divisor. Let v_0 be a point in V and let x and y be two distinct points in the fiber $\pi^{-1}(v_0)$. Note that $\dim \pi^{-1}(v_0) \geq 1$. And we set $\alpha = \mathbb{1}_x - \mathbb{1}_y \in F(\widetilde{X})$. Then it is clear that we have the following commutative diagram with θ_π being the zero homomorphism

$$\begin{array}{ccc} F(X) & \xrightarrow{c_*} & H_*(X;\mathbb{Z}) \\ {\scriptstyle \alpha\cdot\pi^*}\downarrow & & \downarrow{\scriptstyle \theta_\pi} \\ F(\widetilde{X}) & \xrightarrow[c_*]{} & H_*(\widetilde{X};\mathbb{Z}). \end{array}$$

In fact, we can see that for any morphism $g : X' \to X$ and the fiber square

$$\begin{array}{ccc} \widetilde{X}' & \xrightarrow{g'} & \widetilde{X} \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{g} & X, \end{array}$$

we have the following commutative diagram with $\theta_{\pi'}$ being the zero homomorphism

$$\begin{array}{ccc} F(X') & \xrightarrow{c_*} & H_*(X';\mathbb{Z}) \\ {\scriptstyle g'^*\alpha\cdot\pi'^*}\downarrow & & \downarrow{\scriptstyle \theta_{\pi'}} \\ F(\widetilde{X}') & \xrightarrow[c_*]{} & H_*(\widetilde{X}';\mathbb{Z}). \end{array}$$

Hence $\alpha \in v\mathbb{F}(\widetilde{X} \xrightarrow{\pi} X)$. On the other hand $\alpha \notin \mathbb{F}(\widetilde{X} \xrightarrow{\pi} X)$, because α does not satisfy the local Euler condition with respect to the blow-up π exactly at these two points x and y.

REMARK 2.10. One might be tempted to think or guess that even if in the definition of $v\mathbb{F}$ we just require the constructible function $\alpha \in F(X)$ to satisfy the commutativity of the diagram (2.7.1) instead of considering all the fiber squares as above we would get a bivariant theory, but it is not clear or obvious at all whether the pullback is well-defined or not. Certainly, the product and pushforward are both well-defined. Indeed, for morphisms $f : X \to Y$ and $g : Y \to Z$, let us suppose

that for constructible functions $\alpha \in F(X)$ and $\beta \in F(Y)$ there exist homomorphisms $\theta_f : H_*(Y;\mathbb{Z}) \to H_*(X;\mathbb{Z})$ and $\theta_g : H_*(Z;\mathbb{Z}) \to H_*(Y;\mathbb{Z})$ such that the following diagrams commute:

$$\begin{array}{ccc} F(Z) & \xrightarrow{c_*} & H_*(Z;\mathbb{Z}) \\ {\scriptstyle \beta\cdot g^*}\downarrow & & \downarrow{\scriptstyle \theta_g} \\ F(Y) & \xrightarrow{c_*} & H_*(Y;\mathbb{Z}) \\ {\scriptstyle \alpha\cdot f^*}\downarrow & & \downarrow{\scriptstyle \theta_f} \\ F(X) & \xrightarrow[c_*]{} & H_*(X;\mathbb{Z}). \end{array}$$

This diagram implies the following commutative diagram

$$\begin{array}{ccc} F(Z) & \xrightarrow{c_*} & H_*(Z;\mathbb{Z}) \\ {\scriptstyle \alpha\cdot f^*(\beta\cdot g^*)}\downarrow & & \downarrow{\scriptstyle \theta_f\circ\theta_g} \\ F(X) & \xrightarrow[c_*]{} & H_*(X;\mathbb{Z}). \end{array}$$

Noticing that it follows from (2.3) that $\alpha \cdot f^*(\beta \cdot g^*) = \alpha \cdot (f^*\beta \cdot f^*g^*) = (\alpha \bullet \beta) \cdot (gf)^*$, one can see that the product $\alpha \bullet \beta$ is well-defined. To see the well-definedness of the pushforward, let us suppose that for $\alpha \in F(X)$ there exists a homomorphism $\theta_{gf} : H_*(Z;\mathbb{Z}) \to H_*(X;\mathbb{Z})$ such that the following diagram commutes:

$$\begin{array}{ccc} F(Z) & \xrightarrow{c_*} & H_*(Z;\mathbb{Z}) \\ {\scriptstyle \alpha\cdot(gf)^*}\downarrow & & \downarrow{\scriptstyle \theta_{gf}} \\ F(X) & \xrightarrow{c_*} & H_*(X;\mathbb{Z}) \end{array}$$

Then this diagram together with MacPherson's theorem, i.e., $c_*f_* = f_*c_*$, implies the following commutative diagram

$$\begin{array}{ccc} F(Z) & \xrightarrow{c_*} & H_*(Z;\mathbb{Z}) \\ {\scriptstyle f_*(\alpha\cdot(gf)^*)}\downarrow & & \downarrow{\scriptstyle f_*\circ\theta_{gf}} \\ F(Y) & \xrightarrow{c_*} & H_*(Y;\mathbb{Z}) \end{array}$$

Noticing that it follows from the projection formula (2.4) that $f_*(\alpha \cdot (gf)^*) = f_*(\alpha \cdot f^*g^*) = f_*\alpha \cdot g^*$, one can see that the pushforward is well-defined. However, as to the pullback operation, it is not clear at all whether for the fiber square (2.8.1) a constructible function $\alpha \in F(X)$ making the diagram (2.7.1) commutative with a certain homomorphism $\theta_f : H_*(Y;\mathbb{Z}) \to H_*(X;\mathbb{Z})$ automatically implies that the pullback $g^*\alpha (= g'^*\alpha)$ makes the diagram (2.8.2) commutative with a certain homomorphism $\theta_{f'} : H_*(Y';\mathbb{Z}) \to H_*(X';\mathbb{Z})$. The well-definedness of the pullback operation is therefore left as an open problem.

3. Results. In this section we show positive results on Verdier-Riemann-Roch for Chern classes in the case of trivial fiber bundles and negative results in the case of blow-ups.

Let X be a possibly singular variety embeddable into a nonsingular variety M. Then the Fulton's canonical class $c^F(X)$ of X ([F, Example 4.2.6]) is defined by

$$c^F(X) := c(TM|_X) \cap s(X, M),$$

where $s(X, M)$ is the relative Segre class of X in M ([F, §4.2]). If X is a local complete intersection in a smooth variety M, then the Fulton-Johnson class $c^{FJ}(X)$ of X ([FJ]) and the Fulton's canonical class of X ([F, Example 4.2.6]) are the same:

$$c^{FJ}(X) = c^F(X) = c(TM|_X)c(N_X M)^{-1} \cap [X].$$

Since the Chern-Schwartz-MacPherson class homomorphism $c_* : F(W) \to A_*(W)$ is always surjective for any variety W, any polynomial or power series of Chern classes of vector bundles acts on the Chow homology group and the action commutes with the cycle map (see [F, Remark 3.2.2, Proposition 19.1.2]), there exists a constructible function $\xi^{FJ} \in F(X)$ (which shall be called a *Fulton-Johnson constructible function*) such that

$$c^{FJ}(X) = c_*(\xi^{FJ}).$$

Note that there are of course infinitely many such constructible functions. To obtain the constructible function ξ^{FJ} in a canonical form, we note that the difference between the two classes $c^{FJ}(X)$ and $c_*(X)$ is supported on the singular locus (e.g., see [Su1] for a rigorous proof of this). Hence a reasonable canonical constructible function ξ^{FJ} is of the form

$$\xi^{FJ} = \mathbb{1}_X + \varepsilon_{X_{\mathrm{Sing}}},$$

with a constructible function $\varepsilon_{X_{\mathrm{Sing}}}$ supported on the singular locus X_{Sing} of X, and up to sign, the Chern-Schwartz-MacPherson class of the extra constructible function $\varepsilon_{X_{\mathrm{Sing}}}$, considered as the homology class of the ambient variety X, is the so-called *Milnor class* of X and usually denoted by $\mathcal{M}(X)$. More precisely,

$$\mathcal{M}(X) = (-1)^{\dim X} c_*(\varepsilon_{X_{\mathrm{Sing}}}).$$

See [A3], [BLSS1, 2], [PP3], [Su1], etc., for some fundamental results and formulas of the Milnor class. (See [PP3, §5] for a formula for ξ^{FJ}.) With this definition we have

LEMMA 3.1. *Let X be a singular local complete intersection in a smooth variety M, Y any possibly singular variety and $X \xleftarrow{q} X \times Y \xrightarrow{p} Y$ the projections to each factor. Then the following diagram commutes:*

$$\begin{array}{ccc} F(Y) & \xrightarrow{c_*} & H_*(Y;\mathbb{Z}) \\ {\scriptstyle \alpha \cdot p^*}\downarrow & & \downarrow{\scriptstyle c(T_p)\cap p^*} \\ F(X \times Y) & \xrightarrow[c_*]{} & H_*(X \times Y;\mathbb{Z}), \end{array}$$

where $\alpha = q^*\xi^{FJ}$.

Proof. The relative tangent bundle T_p is q^*T_X and the homology pullback $p^* : H_*(Y) \to H_*(X \times Y)$ is given by $p^*(b) = [X] \times b$, the homology cross product. Hence

for any homology class $b \in H_*(Y)$ we have

$$
\begin{aligned}
c(T_p) \cap p^*(b) &= c(q^*T_X) \cap ([X] \times b) \\
&= q^*c(T_X) \cap ([X] \times b) \\
&= (c(T_X) \times 1) \cap ([X] \times b) \\
&= (c(T_X) \cap [X]) \times b \\
&= c^{FJ}(X) \times b.
\end{aligned} \tag{3.5.1}
$$

Therefore, for any constructible function $\beta \in F(Y)$ we have

$$c(T_p) \cap p^*(c_*(\beta)) = c_*(\xi^{FJ}) \times c_*(\beta).$$

The cross product $\omega \times \zeta$ of two constructible functions $\omega \in F(W)$ and $\zeta \in F(Z)$ is defined to be $(\omega \times \zeta)(w, z) := \omega(w)\zeta(z)$. Then, the Kwieciński's cross product formula [K1, Théorème 1] (cf. [KY, Theorem 4]) says that $c_*(\omega \times \zeta) = c_*(\omega) \times c_*(\zeta)$. Therefore we have

$$c(T_p) \cap p^*(c_*(\beta)) = c_*(\xi^{FJ} \times \beta) = c_*(q^*\xi^{FJ} \cdot p^*\beta),$$

which is nothing but the commutativity of the above diagram. □

LEMMA 3.2. *Let X and Y be any varieties and let $X \xleftarrow{q} X \times Y \xrightarrow{p} Y$ be the projections to each factor. Let $\gamma : \mathbb{F} \to \mathbb{H}$ be any Grothendieck transformation satisfying the normalization condition that $\gamma(\mathbb{1}_\pi) = c(TX) \cap [X]$ for X smooth, where $\pi : X \to pt$ and $\mathbb{1}_\pi = \mathbb{1}_X$. Then for any constructible function $\alpha \in F(X) = \mathbb{F}(X \to pt)$ and $b \in H_*(Y; \mathbb{Z})$ we have*

$$c_*(\alpha) \times b = \gamma(q^*\alpha) \bullet b \in H_*(X \times Y; \mathbb{Z}).$$

Proof. Since $q^*\alpha$ is a bivariant constructible function, i.e., $q^*\alpha \in \mathbb{F}(X \times Y \xrightarrow{p} Y)$, we get the following commutative diagram (the Verdier-Riemann-Roch for Chern class associated to the bivariant constructible function $q^*\alpha$) from the given Grothendieck transformation $\gamma : \mathbb{F} \to \mathbb{H}$:

$$
\begin{array}{ccc}
F(Y) & \xrightarrow{c_*} & H_*(Y; \mathbb{Z}) \\
{\scriptstyle (q^*\alpha)\bullet}\downarrow & & \downarrow{\scriptstyle \gamma(q^*\alpha)\bullet} \\
F(X \times Y) & \xrightarrow[c_*]{} & H_*(X \times Y; \mathbb{Z}),
\end{array}
$$

which implies that for any constructible function $\beta \in F(Y)$ we have

$$c_*((q^*\alpha) \bullet \beta) = \gamma(q^*\alpha) \bullet c_*(\beta).$$

Since $(q^*\alpha) \bullet \beta = (q^*\alpha) \cdot p^*\beta = \alpha \times \beta$, we have

$$c_*(\alpha) \times c_*(\beta) = \gamma(q^*\alpha) \bullet c_*(\beta).$$

The lemma claims that in this equality any algebraic homology class of Y, i.e., the Chern-Schwartz-MacPherson class $c_*(\beta)$ of any constructible function β, can be replaced by any homology class of Y.

To show this, first we should note that it follows from the definition of bivariant product of the bivariant homology theory that the right-hand-side of the above equality

actually means

$$\mathcal{A}_{X\times Y}(\gamma(q^*\alpha)\bullet\mathcal{A}_Y^{-1}(b)),$$

where $\mathcal{A}_Z : H^*(N, N\backslash Z) \xrightarrow{\cong} H_*(Z)$ is the Alexander duality isomorphism for a variety Z embedded in a (in fact, any) smooth variety N. Also note that the Alexander duality isomorphisms commute with homology and cohomology cross products and furthermore for a morphism $X \to pt$, $c_* = \mathcal{A}_X \circ \gamma$, due to the normalization condition imposed on γ (see [BS]). Next, we recall the bivariant cross products ([FM, 2.4, p.24]): For morphisms $f : X_1 \to X_2$ and $g : Y_1 \to Y_2$, the bivariant-theoretic cross product $\times$

$$\times : \mathbb{B}(X_1 \xrightarrow{f} X_2) \otimes \mathbb{B}(Y_1 \xrightarrow{g} Y_2) \to \mathbb{B}(X_1 \times Y_1 \xrightarrow{f\times g} X_2 \times Y_2)$$

is defined by

$$\alpha \times \beta := h^*\alpha \bullet s^*\beta, \tag{3.2.1}$$

where $h : X_2 \times Y_1 \to X_2$ and $s : X_2 \times Y_2 \to Y_2$ are the projections. Thus we are dealing with the following two fiber squares

$$\begin{array}{ccccc}
 & & X_1 \times Y_1 & \longrightarrow & X_1 \\
 & & {\scriptstyle f\times \mathrm{id}_{Y_1}}\downarrow & & \downarrow{\scriptstyle f} \\
Y_1 & \longleftarrow & X_2 \times Y_1 & \xrightarrow[h]{} & X_2 \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{X_2}\times g} & & \\
Y_2 & \xleftarrow[s]{} & X_2 \times Y_2 & &
\end{array}$$

In our case we set $X_1 = X, X_2 = pt, Y_1 = Y$ and $Y_2 = pt$, which means that we consider the following two fiber squares:

$$\begin{array}{ccccc}
 & & X \times Y & \xrightarrow{q} & X \\
 & & {\scriptstyle p}\downarrow & & \downarrow \\
Y & \xleftarrow{\mathrm{id}_Y} & Y & \xrightarrow[h]{} & pt \\
\downarrow & & \downarrow & & \\
pt & \xleftarrow[s=\mathrm{id}_{pt}]{} & pt & &
\end{array}$$

Then the lemma can be seen as follows: first, as observed above, since the Alexander duality isomorphisms commute with homology cross products, we have

$$c_*(\alpha) \times b = \mathcal{A}_{X\times Y}\left(\mathcal{A}_X^{-1}(c_*(\alpha)) \times \mathcal{A}_Y^{-1}(b)\right).$$

Which continues as follows

$$\begin{aligned}
&= \mathcal{A}_{X\times Y}\big(\gamma(\alpha)\times \mathcal{A}_Y^{-1}(b)\big) \quad (\text{since } c_* = \mathcal{A}_X \circ \gamma)\\
&= \mathcal{A}_{X\times Y}\big(h^*\gamma(\alpha)\bullet \mathcal{A}_Y^{-1}(b)\big) \quad (\text{by the bivariant cross product (3.2.1)})\\
&= \mathcal{A}_{X\times Y}\big(\gamma(h^*\alpha)\bullet \mathcal{A}_Y^{-1}(b)\big) \quad (\text{since } \gamma \text{ commutes with pullbacks})\\
&= \mathcal{A}_{X\times Y}\big(\gamma(q^*\alpha)\bullet \mathcal{A}_Y^{-1}(b)\big) \quad (h^*\alpha := q^*\alpha \text{ by definition})\\
&= \gamma(q^*\alpha)\bullet b.
\end{aligned}$$

□

Now it is easy to see the following

COROLLARY 3.3. *Let the situation be as in Lemma 3.1. If we let* $p^\bigstar := \gamma(\mathbb{1}_{X\times Y})\bullet : H_*(Y;\mathbb{Z}) \to H_*(X\times Y;\mathbb{Z})$ *for any Grothendieck transformation* $\gamma : \mathbb{F} \to \mathbb{H}$ *satisfying the normalization condition, then for any homology class* $b \in H_*(Y;\mathbb{Z})$ *we have*

$$p^\bigstar(b) = c(T_p)\cap p^*(b) - (-1)^{\dim X}\mathcal{M}(X)\times b,$$

or equivalently

$$p^\bigstar(b) = c(T_p)\cap p^*(b) - \gamma(\varepsilon_{X_{\mathrm{Sing}}}\times \mathbb{1}_Y)\bullet b.$$

□

Note that Corollary 3.3 is a solution to [Y1, Problem (3.4)] or [BY, Problem (4.5)] in the case of trivial fiber bundles.

Now, motivated or hinted by the well-known fact (e.g., see the recent articles [Cr, §1.3] and [Gö, Remark 4.1]) that if $f : Z \to Y$ is a Zariski locally trivial fiber bundle with fiber F, then

$$[Z] = [Y][F]$$

in the Grothendieck ring of complex algebraic (or analytic) varieties, we speculate that just like in the case of $Z = Y\times F$ the following conjecture would be correct:

CONJECTURE 3.4. *Let* X *be a local complete intersection in a smooth variety* M *and* $p : X\widetilde{\times}Y \to Y$ *be a Zariski locally trivial fiber bundle over a possibly singular variety* Y *with fiber* X. *Furthermore we assume that* $X\widetilde{\times}Y \to Y$ *is a subbundle of a Zariski locally trivial fiber bundle* $M\widetilde{\times}Y \to Y$ *with fiber* M. *Then we have the following commutative diagram*

$$\begin{array}{ccc}
F(Y) & \xrightarrow{c_*} & H_*(Y;\mathbb{Z})\\
{\scriptstyle \alpha\cdot p^*}\big\downarrow & & \big\downarrow{\scriptstyle c(T_p)\cap p^*}\\
F(X\widetilde{\times}Y) & \xrightarrow[c_*]{} & H_*(X\widetilde{\times}Y;\mathbb{Z}),
\end{array}$$

where $\alpha = \mathbb{1}_{X\widetilde{\times}Y} + \varepsilon_{X_{\mathrm{Sing}}}\widetilde{\times}\mathbb{1}_Y$ *and* $(\varepsilon_{X_{\mathrm{Sing}}}\widetilde{\times}\mathbb{1}_Y)(x,y) := \varepsilon_{X_{\mathrm{Sing}}}(x)$.

In the earlie version of the paper, the above conjecture was stated as a theorem, but Jörg Schürmann pointed out an error in its proof. Here, however, the earlier "proof" is given as a reference for the reader, who might be able to come up with a correct and slick proof.

"Proof". Since $p : X\widetilde{\times}Y \to Y$ is a Zariski locally trivial fiber bundle, there exists a stratification $\{Y_i\}$ of Y by constructible sets Y_i's such that the restriction of p to

each stratum Y_i

$$p_i := p|_{p^{-1}(Y_i)} : X \times Y_i \to Y_i$$

is a trivial bundle. Note that $F(Y) = \bigoplus F(Y_i)$, i.e., any constructible function of Y can be expressed as a direct sum of constructible functions of the constructible sets Y_i. Let $\eta_i : Y_i \to Y$ and $\rho_i : X \times Y_i \to X \tilde{\times} Y$ be the inclusions. Let $\alpha_i \in F(Y_i)$. Then it follows from Lemma 3.1 that the following diagram is commutative

$$\begin{array}{ccc} F(Y_i) & \xrightarrow{c_*} & H_*(Y_i;\mathbb{Z}) \\ {\scriptstyle (\xi^{FJ}\times \mathbf{1}_{Y_i})\cdot p_i^*}\downarrow & & \downarrow{\scriptstyle c(T_{p_i})\cap p_i^*} \\ F(X\times Y_i) & \xrightarrow[c_*]{} & H_*(X\times Y_i;\mathbb{Z}), \end{array}$$

i.e., for any constructible function $\beta_i \in F(Y_i)$

$$c(T_{p_i}) \cap p_i^*(c_*(\beta_i)) = c_*((\xi^{FJ} \times \mathbb{1}_{Y_i}) \cdot p_i^*(\beta_i)).$$

Now, let $\beta \in F(Y)$. Then we can express β as follows:

$$\beta = \sum_i \eta_{i*}\beta_i, \qquad \beta_i \in F(Y_i).$$

Consider the following fiber square

$$\begin{array}{ccc} X \times Y_i & \xrightarrow{\rho_i} & X \tilde{\times} Y \\ {\scriptstyle p_i}\downarrow & & \downarrow{\scriptstyle p} \\ Y_i & \xrightarrow[\eta_i]{} & Y. \end{array}$$

Since p is flat, it follows from [F, Proposition 1.7] that the following diagram is commutative

$$\begin{array}{ccc} H_*(X \times Y_i) & \xrightarrow{\rho_{i*}} & H_*(X \tilde{\times} Y) \\ {\scriptstyle p_i^*}\uparrow & & \uparrow{\scriptstyle p^*} \\ H_*(Y_i) & \xrightarrow[\eta_{i*}]{} & H_*(Y). \end{array}$$

(Here is an error; since η_i is not necessarily proper !)

Here it should be noted that the "constructible function" version of this commutative diagram always holds for any fiber square, namely the formula (2.2) given in

§2; thus we do not require the flatness of p. Therefore we get

$$\begin{aligned}
&\sum_i \rho_{i*}\left(c(T_{p_i}) \cap p_i^*(c_*(\beta_i))\right) \\
&= \sum_i \rho_{i*}\left(c(\rho_i^* T_p) \cap p_i^*(c_*(\beta_i))\right) \\
&= \sum_i c(T_p) \cap \rho_{i*} p_i^*\left(c_*(\beta_i)\right) \\
&= \sum_i c(T_p) \cap p^* \eta_{i*}\left(c_*(\beta_i)\right) \\
&= \sum_i c(T_p) \cap p^*\left(c_*(\eta_{i*}\beta_i)\right) \\
&= c(T_p) \cap p^*\left(c_*(\sum_i \eta_{i*}\beta_i)\right) \\
&= c(T_p) \cap p^*(c_*(\beta)).
\end{aligned}$$

On the other hand, we get that

$$\begin{aligned}
&\sum_i \rho_{i*} c_*\left((\xi^{FJ} \times \mathbb{1}_{Y_i}) \cdot p_i^* \beta_i\right) \\
&= \sum_i c_*\left(\rho_{i*}\left((\xi^{FJ} \times \mathbb{1}_{Y_i}) \cdot p_i^* \beta_i\right)\right) \\
&= \sum_i c_*\left((\xi^{FJ} \widetilde{\times} \mathbb{1}_Y) \cdot \rho_{i*} p_i^* \beta_i\right) \\
&= \sum_i c_*\left((\xi^{FJ} \widetilde{\times} \mathbb{1}_Y) \cdot p^* \eta_{i*} \beta_i\right) \\
&= c_*\left((\xi^{FJ} \widetilde{\times} \mathbb{1}_Y) \cdot p^*(\sum_i \eta_{i*}\beta_i)\right) \\
&= c_*\left((\xi^{FJ} \widetilde{\times} \mathbb{1}_Y) \cdot p^*(\beta)\right).
\end{aligned}$$

Hence, for any constructible function $\beta \in F(Y)$, we have

$$c(T_p) \cap p^*(c_*(\beta)) = c_*\left((\xi^{FJ} \widetilde{\times} \mathbb{1}_Y) \cdot p^*(\beta)\right),$$

thus the theorem holds. □

Let $p^\bigstar = \gamma(\mathbb{1}_{X \widetilde{\times} Y}) \bullet : H_*(Y;\mathbb{Z}) \to H_*(X \widetilde{\times} Y;\mathbb{Z})$ for any Grothendieck transformation $\gamma : \mathbb{F} \to \mathbb{H}$ satisfying the normalization condition. Then for any homology class $b \in H_*(Y)$ determined by an algebraic or analytic cycle, i.e., the Chern-Schwartz-MacPherson class of a constructible function of Y, we have

$$p^\bigstar(b) = c(T_p) \cap p^*(b) - \gamma(\varepsilon_{X_{\mathrm{Sing}}} \widetilde{\times} \mathbb{1}_Y) \bullet b,$$

which is just another way of putting Conjecture 3.4. Furthermore we speculate that

$$p^\bigstar = c(T_p) \cap p^* - \gamma(\varepsilon_{X_{\mathrm{Sing}}} \widetilde{\times} \mathbb{1}_Y) \bullet . \tag{3.5}$$

Now it is furthermore reasonable to make the following

CONJECTURE 3.6. *Even if the Zariski topology is replaced by the usual topology in Conjecture 3.4, (i) the diagram in Conjecture 3.4 is commutative and (ii) (3.5) also holds.*

We hope to address ourselves to these conjectures in a different paper later.

Note that by the induction on dimensions of subvarieties of Y we can see that the statement (i) of the above conjecture holds if and only if for the above bundle $X \widetilde{\times} Y \to Y$

$$c_*(\xi^{FJ} \widetilde{\times} \mathbb{1}_Y) = c(T_p) \cap p^*(c_*(Y)).$$

Furthermore we note that

$$c_*(\xi^{FJ} \widetilde{\times} \mathbb{1}_Y) = \gamma(\xi^{FJ} \widetilde{\times} \mathbb{1}_Y) \bullet c_*(Y).$$

REMARK 3.7. Since $c(T_p) \cap p^*(b) = c(T_p) \bullet U_p \bullet b$, we speculate that

$$\gamma(\xi^{FJ} \widetilde{\times} \mathbb{1}_Y) = c(T_p) \bullet U_p.$$

Otherwise this simple example would show the non-uniqueness of the Grothendieck transformation $\gamma : \mathbb{F} \to \mathbb{H}$ satisfying the normalization condition.

Now we recall that a local complete intersection morphism $f : X \to Y$ is the composite $f = p \circ r$ of a regular embedding $r : X \to M$ (i.e., X is a local complete intersection in M) and a smooth morphism $p : M \to Y$. Local complete intersection morphisms which we now deal with are blowups (see [F, 6.7] and [FM, 9.2.2]). Let Z be a regularly embedded closed subscheme of a scheme Y, and let $\widetilde{Y}$ be the blow-up of Y along Z, then the projection $f : \widetilde{Y} \to Y$ is a local complete intersection morphism of relative codimension 0.

THEOREM 3.8. *Let $f : \widetilde{\mathbb{P}^n} \to \mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$ at a point P. Then there is no constructible function $\alpha \in F(\widetilde{\mathbb{P}^n})$ such that the diagram (1.2) is commutative.*

Proof. Let us suppose that there exist some constructible function $\alpha \in F(\widetilde{\mathbb{P}^n})$ such that the following diagram commutes:

$$\begin{array}{ccc} F(\mathbb{P}^n) & \xrightarrow{c_*} & H_*(\mathbb{P}^n; \mathbb{Z}) \\ {\scriptstyle \alpha \cdot f^*}\downarrow & & \downarrow{\scriptstyle c(T_f) \cap f^*} \\ F(\widetilde{\mathbb{P}^n}) & \xrightarrow[c_*]{} & H_*(\widetilde{\mathbb{P}^n}; \mathbb{Z}). \end{array}$$

Then it follows from the commutativity of the above diagram that $\alpha = 1$ on $\widetilde{\mathbb{P}^n} \setminus E$, where $E(\cong \mathbb{P}^{n-1})$ denotes the exceptional divisor. To see this, let Q be any point different from P and set $Q' = f^{-1}(Q)$. Then we have $\alpha(Q') = c_*(\alpha(Q') \cdot \mathbb{1}_{Q'}) = c_*(\alpha \cdot f^*(\mathbb{1}_Q)) = c(T_f) \cap f^*(c_*(\mathbb{1}_Q)) = c(T_f) \cap f^*([Q]) = c(T_f) \cap [Q'] = [Q'] = 1$. Hence α can be expressed as

$$\alpha = \mathbb{1}_{\widetilde{\mathbb{P}^n} \setminus E} + \beta_E$$

with some constructible function β_E supported on E. Then we have

$$c_*(\alpha \cdot f^*(\mathbb{1}_P)) = c_*(\alpha \cdot \mathbb{1}_E) = c_*(\beta_E).$$

On the other hand, by the same argument as above, we get that

$$c_*(\alpha \cdot f^*(\mathbb{1}_P)) = c(T_f) \cap f^*(c_*(\mathbb{1}_P)) = c(T_f) \cap f^*([P]) = 1.$$

Thus it follows that $c_*(\beta_E) = 1$. Now, since any constructible function can be expressed as a linear combination of some characteristic functions of subvarieties, it follows from the irreducibility of E that

$$\beta_E = b \cdot \mathbb{1}_E + \delta$$

where b is some integer and δ is another constructible function supported on subvarieties of lower dimensions, i.e., of dimension $< n-1$. If $b \neq 0$, then

$$c_*(\beta_E) = b[E] + \cdots,$$

since $[E]$ cannot vanish, because $H_{2n-2}(\widetilde{\mathbb{P}^n}) \cong H_{2n-2}(\mathbb{P}^n \setminus P) \oplus H_{2n-2}(E)$ by the Mayer-Vietoris sequence. Hence $c_*(\beta_E) \neq 1$. Therefore $b = 0$, and since $c_*(\beta_E) = 1$ it follows that $\delta \neq 0$ and that $c_*(\delta) = 1$. Now consider any line L going through the blown-up point P, and let $\widetilde{L}$ be the proper transform of L and let $[[L]]$ be the corresponding point in the exceptional divisor E. Then we have

$$\begin{aligned} \alpha \cdot f^*(\mathbb{1}_L) &= \left(\mathbb{1}_{\widetilde{\mathbb{P}^n} \setminus E} + \delta\right) \cdot \left(\mathbb{1}_E + \mathbb{1}_{\widetilde{L}} - \mathbb{1}_{[[L]]}\right) \\ &= \delta + \mathbb{1}_{\widetilde{L}} - \mathbb{1}_{[[L]]} \end{aligned}$$

Therefore we have

$$\begin{aligned} c_*(\alpha \cdot f^*(\mathbb{1}_L)) &= c_*(\delta) + c_*(\widetilde{L}) - c_*([[L]]) \\ &= c_*(\delta) + c_*(\widetilde{L}) - 1 \\ &= c_*(\widetilde{L}) \quad (\text{since } c_*(\delta) = 1) \\ &= [\widetilde{L}] + 2 \quad (\text{since } \widetilde{L} \cong \mathbb{P}^1). \end{aligned}$$

On the other hand it follows from [F, Corollary 6.7.1] and the commutativity of the pullbacks with the cycle map [F, Example 19.2.1] that we have

$$f^*([L]) = [\widetilde{L}] + [L'],$$

where L' is a projective line in $E \cong \mathbb{P}^{n-1}$. Therefore we get that

$$\begin{aligned} c(T_f) \cap f^*(c_*(L)) &= c(T_f) \cap f^*([L] + 2) \quad (\text{since } L \cong \mathbb{P}^1) \\ &= c(T_f) \cap f^*([L]) + 2 \quad (\text{since } c(T_f) = 1 + \cdots) \\ &= c(T_f) \cap ([\widetilde{L}] + [L']) + 2 \\ &= [\widetilde{L}] + [L'] + \text{some integer}. \end{aligned}$$

Here we note that $[L']$ does not vanish since $H_2(\widetilde{\mathbb{P}^n}) \cong H_2(\mathbb{P}^n \setminus P) \oplus H_2(E)$ by the Mayer-Vietoris sequence. Then the above commutative diagram implies that

$$c_*(\alpha \cdot f^*(\mathbb{1}_L)) = c(T_f) \cap f^*(c_*(L)),$$

namely we have to have that

$$[\widetilde{L}] + 2 = [\widetilde{L}] + [L'] + \text{some integer}.$$

Thus, in particular $[L']$ has to vanish, which is a contradiction. □

More generally, we can construct such an example from any variety V of dimension $n > 2$.

COROLLARY 3.9. *Let $\widetilde{V}$ be the blow-up of V at any smooth point of V and $f : \widetilde{\widetilde{V}} \to \widetilde{V}$ the blow-up of $\widetilde{V}$ at any point of the exceptional divisor ($\cong \mathbb{P}^{n-1}$) of $\widetilde{V}$. Then there is no constructible function $\alpha \in F(\widetilde{\widetilde{V}})$ such that the diagram (1.2) is commutative.*

This can be seen using the above case of the blow-up $f : \widetilde{\mathbb{P}^{n-1}} \to \mathbb{P}^{n-1}$.

As we can see from the above argument, as to the cohomology class $c(T_f)$, we use only the fact that the 0-dimensional part of the cohomology class $c(T_f)$ is equal to 1, thus as a corollary we get

COROLLARY 3.10. *Let $f : \widetilde{X} \to X$ be a blow-up map as above. Then there are no constructible function $\alpha \in F(\widetilde{X})$ and no total cohomology class $c\ell(\widetilde{X})$ of $\widetilde{X}$ whose 0-dimensional part is 1, i.e., no total cohomology class $c\ell(\widetilde{X}) \in 1 + \bigoplus_{i>0} H^{2i}(\widetilde{X}; \mathbb{Z})$, such that the following diagram is commutative :*

$$\begin{array}{ccc} F(X) & \xrightarrow{c_*} & H_*(X;\mathbb{Z}) \\ {\scriptstyle \alpha\cdot f^*}\downarrow & & \downarrow{\scriptstyle c\ell(\widetilde{X})\cap f^*} \\ F(\widetilde{X}) & \xrightarrow[c_*]{} & H_*(\widetilde{X};\mathbb{Z}). \end{array}$$

REMARK 3.11. For a blow-up map $\pi : \widetilde{X} \to X$ of X along any subvariety of X there is no bivariant constructible function $\alpha \in \mathbb{F}(\widetilde{X} \to X)$ whose value is generically equal to 1, because of the local Euler condition imposed on the constructible function α (e.g., see [Sa, (1.3) Remarque]). As to the bivariant-theoretic "Riemann-Roch formula", it is therefore obvious that for any blow-up map $\pi : \widetilde{X} \to X$ there is no canonical orientation $\alpha_f \in \mathbb{F}(X \to Y)$ and no total cohomology class $c\ell(\widetilde{X})$ of $\widetilde{X}$ whose 0-dimensional part is 1 such that

$$\gamma(\alpha_f) = c\ell(\widetilde{X}) \bullet U_f$$

for *any* Grothendieck transformation $\gamma : \mathbb{F} \to \mathbb{H}$ satisfying the normalization condition that $\gamma(\mathbb{1}_\pi) = c(TX) \cap [X]$ for X smooth with $\pi : X \to pt$ and $\mathbb{1}_\pi := \mathbb{1}_X$.

REMARK 3.12. In this remark we discuss the extra constructible function $\varepsilon_{X_{\mathrm{Sing}}}$ a bit more.

Let E be a vector bundle of rank k over a nonsingular variety M of dimension $n+k$ and $s : M \to E$ a regular section, and let $X := s^{-1}(0)$ be the zero of the section, which is a local complete intersection of dimension n. Let $i : X \to M$ be the inclusion. Then the Fulton-Johnson class $c^{FJ}(X)$ is nothing but $\frac{i^*c(TM)}{i^*c(E)} \cap [X]$ and the Milnor class is by defintion

$$\mathcal{M}(X) := (-1)^{\dim X}\left(\frac{i^*c(TM)}{i^*c(E)} \cap [X] - c_*(X)\right) = (-1)^{\dim X} c_*(\varepsilon_{X_{\mathrm{Sing}}}).$$

When X has isolated singularities $x_1, x_2, \cdots, x_r$, it follows from [Su1] that we can set

$$\varepsilon_{X_{\mathrm{Sing}}} = \sum_{i+1}^{r} \mu_{x_i} \mathbb{1}_{x_i},$$

where μ_{x_i} denoting the Milnor number of X at the isolated singularity x_i.

In [A3, Theorem I.5] P. Aluffi expressed the Milnor class in terms of his μ-class [A2] in the case of hypersurfaces and it can be also expressed as follows ([A3, Theorem I.4]):

$$\mathcal{M}(X) = (-1)^{\dim X} \frac{1}{c(L)} \cap (s(Y, M)^{\vee} \otimes L),$$

where $E = L$ is a line bundle since we are now dealing with the hypersurface case, Y is the singular locus of X and the class $s(Y, M)^{\vee} \otimes L$ is supported on the singular locus Y. See [A1], [A2] and [A3] for the other notation.

In [PP3] A. Parusiński and P. Pragacz described the Milnor class in the case of hypersurfaces, using some data coming from Whitney stratifications of the hypersurface X. They described it as follows:

$$\mathcal{M}(X) = \frac{1}{c(L)} \cap \sum_{\substack{S \in \mathcal{S}_X \\ S \subset \mathrm{Sing}(X)}} \alpha_S c_*(\overline{S}).$$

Here $\mathcal{S}_X$ is a (in fact, any) Whitney stratification of X and α_S is a certain integer attached to each stratum S obtained by using the Milnor numbers of the strata. (The degree 0-part of this formula, i.e., a formula for the Euler-Poincaré characteristic $\chi(X)$ of X was obtained in [PP2, Theorem 4].)

In [BLSS2] (see [BLSS1] for its summary) J.-P. Brasselet, D. Lehmann, J. Seade and T. Suwa have expressed the Milnor class by the so-called localized Milnor classes of the connected components of the singular locus. In the special case when the connected components are all nonsingular, they describe it very explicitly involving some kind of cohomology classes "α" [BLSS2, Lemma 7.5, Theorem 7.6 and Corollary 7.7]; which is, roughly speaking, as follows:

$$\mathcal{M}(X) = \sum_{S} \frac{(\text{some cohomology class})}{c(E)} \cap c_*(S),$$

where S's are the connected components of the singular locus and assumed to be nonsingular.

So one could expect that a general formula would be of the following form

$$\mathcal{M}(X) = \sum_{\substack{S \in \mathcal{S}_X \\ S \subset \mathrm{Sing}(X)}} \frac{\Theta_S}{c(E)} \cap (i_{\overline{S},X})_* c_*(\overline{S}),$$

where $i_{A,B} : A \to B$ is the inclusion and $\mathcal{S}_X$ is a Whitney stratification of X and Θ_S is a certain cohomology class involving not only the bundle E but also the subvariety $\overline{S}$, and the problem is of course to determine the cohomology class Θ_S. (Note that in general Θ_S cannot be expressed as a polynomial in the individual Chern classes of E [OY] and surely involves some classes of the variety $\overline{S}$ itself as seen in [BLSS2].) Thus, we could express the extra constructible function $\varepsilon_{X_{\mathrm{Sing}}}$, as a "cohomology valued" constructible function, as follows:

$$\sum_{\substack{S \in \mathcal{S}_X \\ S \subset \mathrm{Sing}(X)}} \frac{\Theta_S}{c(E)} \cdot \mathbb{1}_{\overline{S}}.$$

Certainly the value of c_* on a $H^*(X)$-valued constructible function is defined to be

$$c_*(\sum_W \Theta_W \cdot \mathbb{1}_W) := \sum_W \Theta_W \cap c_*(W),$$

with Θ_W denoting a cohomology class of X.

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